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Note

Edge weights and vertex colours

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Abstract

Can the edges of any non-trivial graph be assigned weights from $\{1,2,3\}$ so that adjacent vertices have different sums of incident edge weights?

We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

Keywords: Edge weighting; Vertex colouring

A weighting of the edges of a graph with integer weights gives rise to a weighting of the vertices, the weight of a vertex being the sum of the weights of its incident edges. The number of consecutive integer weights needed so that all vertices receive different weights has been called the irregularity strength of a graph (cf. [4]).

It is natural to consider edge weightings where we require only that *adjacent* vertices have different weights—that is, the induced vertex weighting is a proper colouring of the vertices of the graph. We remark that in the literature one can find a number of results on proper *edge-colourings* in which either all vertices (cf. [2]), or just adjacent ones (cf. [1]), get different sets of colours. However, we believe that our setting is quite different from the above, which is best exemplified by the following question we would like to raise in this context. Here and below a non-trivial graph is a connected graph with at least three vertices.

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Question. *Is it possible to weight the edges of any non-trivial graph with the integers* $\{1, 2, 3\}$ *such that the resultant vertex weighting is a proper colouring*?

The answer is yes if the graph is complete or is small (say at most 10 vertices). In fact, experiment suggests that, for almost all graphs, perhaps just two weights $\{1, 2\}$ suffice. But we do not have any satisfactory bound on the number of different integer edge weights needed in general.

In this note we offer two pieces of information in relation to the question: firstly, that a weighting is possible for 3-colourable graphs, and secondly that, if real, rather than just integer, weights are permitted, then a finite number of weights suffices for all graphs.

The fact that the answer is affirmative if the graph is bipartite, or more generally is 3-colourable, follows from the fact that for these graphs the weights can be reduced modulo 3 and the conclusion still holds. More generally, we can consider graphs whose edges are weighted by elements of an abelian group Γ.

Theorem. *Let* Γ *be a finite abelian group of odd order and let* G *be a non-trivial* $|\Gamma|$*-colourable graph. Then there is a weighting of the edges of* G *with the elements of* Γ *such that the resultant vertex weighting is a proper colouring.*

Proof. Let $\Gamma = \{g_1, \dots, g_k\}$. Fix a colouring of G using at most k colours and let there be $n_i \geqslant 0$ vertices of colour i, $1 \leqslant i \leqslant k$. Since k is odd, for every element $g \in \Gamma$ there is an $h \in \Gamma$ with $g = 2h$, and so there is an $h \in \Gamma$ such that $n_1 g_1 + \cdots + n_k g_k = 2h$. Give one edge weight h and the rest zero, so the sum of vertex weights is $2h$. We now try to modify the weighting, maintaining the sum of vertex weights, until all the vertices coloured i have weight g_i, $1 \leqslant i \leqslant k$. Suppose there is a vertex u of colour i with the wrong weight $g \neq g_i$; since $n_1 g_1 + \cdots + n_k g_k = 2h$ there must be another vertex $v \in G$ whose weight is also wrong. Choose a walk of even length from u to v, which is always possible unless G is bipartite. Traverse this walk, adding $g_i - g, g - g_i, g_i - g, \dots$ alternately to the edges as they are encountered. This operation maintains the sum of vertex weights, leaves the weights of all but u and v unchanged, and yields one more vertex of correct weight. Hence, repeated applications give the desired weighting.

In the case that G is bipartite, to achieve vertex weights constant on colour classes would require $n_1 g_1 = n_2 g_2$, which is not always soluble with $g_1 \neq g_2$. Instead, choose colour 1 so that there is a vertex x of colour 1 with degree at least 2 (which is possible since G is non-trivial). Now choose $2h = g_1 \neq 0 = g_2$. Begin with weights 0 on all edges and pursue the previous strategy of modifying weights of edges by updating them on paths with both ends of colour 1, maintaining weight sum 0; this can be done so that if there is a wrong vertex of colour 1, it is x. But now all vertices of class 2 have weight 0, and the only vertex of class 1 which, perhaps, is not of weight g_1, is x, whose weight is $g = (1 - n_1) g_1$. If $g \neq 0$ we are done; if not we can finish by adding weight h to any two edges incident to x. $\square$

The theorem can fail if $|\Gamma|$ is even; for example, not all bipartite graphs can be weighted with integers modulo 2. An elementary "parity argument" shows also that K_6 can be weighted neither with $\mathbf{Z}_6$ nor with $(\mathbf{Z}_2)^3$. But one can modify the proof above to verify that G can still be weighted with Γ if G is $(|\Gamma|-3)$-colourable.

Returning to the original question, we do not even know whether a finite number of integer weights is in general enough to achieve a vertex colouring. However, if we allow real weights, then we can show that a finite number of them will do. Note that we have to have at least three weights even in the bipartite case, since one cannot divide the vertices of a cycle of length six into two colour classes using just two edge weights.

If we allow real weights, then we might as well take the weights to be linearly independent over the rationals. In this case, two vertices will have equal weight only if the multiset of weights appearing at their incident edges is the same.

Since, if we use linearly independent real weights, two vertices can have equal weight only if their degrees are equal, it might seem hardest to prove our assertion for regular graphs. In fact, a standard application of the local lemma [3] shows that, for regular graphs, six weights suffice. The complication of our argument comes from accommodating graphs with widely differing degrees, for which we need a few more weights. We shall make use of the next straightforward lemma, in whose statement for example $d_F(u)$ denotes the degree of the vertex u in the graph F.

Lemma. *Let $0 \leqslant \alpha \leqslant 1$ and let F be a bipartite graph with vertices classes U and V. Then F has a spanning subgraph H, such that $d_H(u) = \lceil \alpha d_F(u) \rceil$ for $u \in U$ and $d_H(v) \geqslant \lfloor \alpha d_F(v) \rfloor$ for $v \in V$.*

Proof. Colour red $\lceil \alpha d_F(u) \rceil$ edges at each vertex $u \in U$ and colour the remaining edges blue. Let $r(v)$ be the number of red edges at $v \in V$. If $r(v) < \lfloor \alpha d_F(v) \rfloor$ we call v deficient, and we let $D \subset V$ be the set of deficient vertices. We call the quantity $\sum_{v \in D} \lfloor \alpha d_F(v) \rfloor - r(v)$ the deficiency.

Our aim is to reduce the deficiency to zero. Suppose that D is not empty and let $v \in D$. If we can find a path $v = v_1, u_1, v_2, u_2, \ldots, v_k$ such that $v_i u_i$ is blue, $u_i v_{i+1}$ is red and $r(v_k) > \lfloor \alpha d_F(v_k) \rfloor$, then we can recolour $v_i u_i$ red and $u_i v_{i+1}$ blue, $1 \leqslant i < k$, so reducing the deficiency, and if we keep on repeating this procedure the deficiency will vanish. The red edges will then form the graph H that we are after.

Suppose, then, that we get stuck, and that for some $v \in D$ no such alternating path exists. Let $A \subset U$ and $B \subset V$ be the set of vertices that can be reached by alternating blue–red paths from v (including length zero so $v \in B$). Then there are no red edges from A to $V - B$, nor are there any blue edges from B to $U - A$. Let E be the set of edges of F between A and B. Since each vertex $u \in A$ has at least $\alpha d_F(u)$ red edges and these all go to B, at least $\alpha|E|$ edges of E are red. But each vertex $w \in B$ has at least $(1-\alpha)d_F(w)$ blue edges and v has more than $(1-\alpha)d_F(v)$ blue edges, and all these blue edges go to A, so more than $(1-\alpha)|E|$ edges of E are blue. This is a contradiction and the proof is complete. □

Our second lemma gives a bound on the probability that if we weight edges randomly then a vertex gets a prescribed multiset of weights.

Lemma. *Let f be a function chosen uniformly from the set of all functions from D to R, where* $|D| = d$, $|R| = r$ *and* $d \geqslant r$. *Let integers* $d_i \geqslant 0$ *be given for* $1 \leqslant i \leqslant r$ *with* $\sum_{i=1}^{r} d_i = d$. *Then the probability that* $|f^{-1}(i)| = d_i$ *for* $1 \leqslant i \leqslant r$ *is at most* $3r^{r/2}(\pi d)^{(1-r)/2}$.

Proof. Let P be the probability sought; then $P = r^{-d} d!/(d_1! \cdots d_r!)$. This quantity is maximized if $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r \leqslant d_1 + 1$, which we assume holds. Stirling's formula states that $1 \leqslant n!(e/n)^n/\sqrt{2\pi n} \leqslant 2$ for $n \geqslant 1$. So, writing $d_i = \delta_i d \geqslant 1$, and using $2d_i \geqslant d_i + 1 \geqslant d/r$, we have

$$P \leqslant 2 \frac{\sqrt{2\pi d}}{\prod_{i=1}^{r} \sqrt{2\pi d_i}} \left[\frac{1}{r} \prod_{i=1}^{r} \delta_i^{-\delta_i} \right]^d \leqslant \frac{3r^{r/2}}{(\pi d)^{(r-1)/2}}$$

as claimed. □

Theorem. *There exists a finite set of real numbers which can be used to weight the edges of any non-trivial graph so that the resultant vertex weighting is a proper colouring.*

Proof. Let R and S be two sets of real numbers with $|R| = 14$ and $|S| = 16$, such that $R \cup S$ is linearly independent over the rationals. Our proof will show that the 30 numbers in $R \cup S$ are sufficient to weight all graphs of minimum degree at least $D = 10^{99}$; at the end of the proof we show how to handle any graph, provided a few extra weights are to hand.

The proof relies on carefully partitioning the vertices and the edges of the graph G and then weighting the edges in each part separately, so that the weightings of the separate parts cannot disrupt each other much. We partition G like this. Let $V_0 = \{u \in G : d_G(u) < D\}$ and $V_i = \{u \in G : 2^{i-1}D \leqslant d_G(u) < 2^i D\}$ for $i \geqslant 1$. For $j > i \geqslant 0$ let F_{ij} be the bipartite-induced subgraph of G with bipartition (V_i, V_j), and by F_i we mean the bipartite subgraph with vertex set $V_i \cup U_i$, where $U_i = \bigcup_{j>i} V_j$; thus, $E(F_i) = \bigcup_{j>i} E(F_{ij})$. We denote by H_i a subgraph of F_i as given by the first lemma, with $\alpha = \frac{1}{2}$, such that $d_{H_i}(v) \geqslant \lfloor d_{F_i}(v)/2 \rfloor$ for $v \in V_i$ and $d_{H_i}(u) = \lceil d_{F_i}(u)/2 \rceil$ for $u \in U_i$. Let G_i be the graph with vertex set $V(G)$ and edge set $E(G[V_i]) \cup E(H_i) \cup \bigcup_{j<i}(E(F_{ji}) \backslash E(H_j))$. In particular, every edge of G_i is incident with a vertex in V_i. Observe that every edge $e = uv$ in G lies in exactly one graph G_i; if $u \in V_i$ and $v \in V_j$, then $e \in E(G_i)$ if $i = j$, and if $i < j$ then $e \in E(G_i)$ or $e \in E(G_j)$ according as $e \in E(H_i)$ or $e \notin E(H_i)$.

The edges of G_i, which are each incident with a vertex in V_i, are of two kinds: those joining V_i to V_j for $j \leqslant i$, namely those in $E(G[V_i]) \cup \bigcup_{j<i} E(F_j - H_j)$, which we weight with R, and those joining V_i to V_j for $j > i$, namely those in $E(H_i)$, which we weight with S. Given a vertex $u \in V_i$, let there be $r(u)$ edges of the first kind incident with u and let there be $s(u)$ edges of the second kind. Observe that

$$r(u) + s(u) \geqslant d_{G[V_i]}(u) + \sum_{j<i} \lfloor d_{F_j}(u)/2 \rfloor + \lfloor d_{F_i}(u)/2 \rfloor \geqslant (d_G(u) - i)/2.$$

Notice too that, of the edges meeting u that are not in G_i, those that are in G_j for $j > i$ will receive weight from R, whereas those in G_j for $j < i$ will receive weight from S. The number of these edges weighted by R is at most $\lceil d_{F_i}(u)/2 \rceil \leqslant s(u) + 1$ and the number weighted by S is $\sum_{j<i} \lceil d_{F_j}(u)/2 \rceil < r(u) + i$.

We assign weights to the graphs G_i one at a time in order of decreasing suffix, namely $G_m, G_{m-1}, \ldots, G_1, G_0$ where $m = \max\{i : G_i \neq \emptyset\}$. Now two adjacent vertices u and v can get the same weight only if $d_G(u) = d_G(v)$, because the weights are linearly independent; in particular $e = uv$ must lie in $G[V_i] \subset G_i$ for some $i \geqslant 0$. Let us assume that weights have already been assigned to $G_m, \ldots, G_{i+1}$, and that weights are now assigned randomly to the edges of G_i in accordance with our earlier description. Let A_e be the event that, either u and v get the same weight in the random weighting of G_i, or there is some weighting of G_j for $j < i$ that can cause the weights of u and v to become equal. Our job is to show that the probability is positive of no A_e occurring for any $e \in E(G[V_i])$; a weighting of G_i in which no A_e occur is then fixed, and we move on to weight G_{i-1}. If we can do this for all $i \geqslant 0$ the proof will be finished.

We treat the cases $i \geqslant 1$ and $i = 0$ separately; suppose first that $i \geqslant 1$. Let $e = uv \in E(G[V_i])$ and let $d_G(u) = d_G(v) = d$, so $2^{i-1}D \leqslant d < 2^i D$. Now all the other events A_f, $f \in E(G[V_i])$, are independent of A_e, with the exception of at most $2(2^i D)^2 \leqslant 8d^2$ of them. The local lemma [3] then tells us it is enough to prove $P(A_e) \leqslant 1/32d^2$, for if this holds then the probability is positive that no A_e occur.

We estimate $P(A_e)$ by two different methods, according to the sizes of $r(u)$ and $r(v)$. Let $k = \lfloor d^{1/3} \rfloor$. Suppose that either $r(u) > k$ or $r(v) > k$ holds, say the former. Assign weights to all edges at v and assign all weights but k of those from R to edges at u. Since the weighting of G_j for $j < i$ will alter the weights at u and v only by elements of S, the probability that A_e occurs is at most the probability that, after the final k weights at u have been assigned, then u acquires the same multiset of weights from R as v already has. By our second lemma, this probability is at most $3r^{r/2}(\pi k)^{(1-r)/2}$, where $r = |R| = 14$. Since $d \geqslant D$ this quantity is at most $1/32d^2$, as desired.

If, on the other hand, both $r(u) \leqslant k$ and $r(v) \leqslant k$, then $s(u) \geqslant (d - i)/2 - k$. Moreover, the number of edges at u that will receive weight from S when we weight G_j for $j < i$ is less than $r(u) + i \leqslant k + i$, and the same holds true for v, so these future weightings can change by at most $2k + 2i$ the difference between the number of occurrences of a given weight at u and the number of occurrences of it at v. Assign weights to all edges at v and to all edges at u except those to be weighted from S. There are now at most $(2k + 2i)^{|S|}$ weightings of the remaining edges at u for which some future weighting of G_j, $j < i$, can cause the weights of u and v to become equal; that is, for which A_e can occur. So, by our second lemma, $P(A_e)$ is at most

$(2k+2i)^s 3s^{s/2}[\pi(d/2-\frac{1}{2}-k)]^{(1-s)/2}$ where $s=|S|=16$. Once again this quantity is at most $1/32d^2$ because $d \geqslant D$ and $i \leqslant 1+\log_2(d/D)$.

If G has minimum degree at least D then $V_0=\emptyset$ and we are done. Otherwise we proceed as follows. We introduce two disjoint sets S' and T of real weights, disjoint from $R \cup S$, such that $R \cup S \cup S' \cup T$ is linearly independent over the rationals. S' is chosen with $|S'|=|S|=16$ and we set up a one-to-one correspondence between S and S'. Each edge incident with a vertex of V_0 is either not in G_0, in which case it is already weighted with an element of S, or it is in G_0, in which case it is currently unweighted. In the first case we might perhaps change the weight to the corresponding element of S', and in the second case we shall weight the edge with a random element of T. Note that no such weighting can cause an event A_e to occur for some $e=uv \in \bigcup_{i \geqslant 1} G[V_i]$, because for each such edge either the existing multisets of weights from R at u and v differ, or the existing multisets of weights from S differ.

Therefore, let $e=uv \in E(G[V_0])$ and let $d_G(u)=d_G(v)=d$, so $2 \leqslant d < D$, as G is non-trivial. The events A_f, $f \in E(G[V_0])$, are independent of A_e except for at most $2D^2$ of them, and we are home if we can show that $P(A_e) \leqslant 1/8D^2$. Now the number of unweighted edges at u, other than e itself, is $d_{G_0}(u)-1=d_{G[V_0]}-1+\lfloor d_{F_0}(u)/2 \rfloor$, which is positive unless $d_{G[V_0]}(u)=d_{F_0}(u)=1$. So if there is no unweighted edge incident with e, then $d=2$, and u and v are both incident with an edge weighted from S. In such a case, pick one of the two weighted edges and change its weight to the corresponding weight of S'; this means that A_e does not happen. In all other cases, there is at least one unweighted edge incident with e. Weight all but one of the unweighted edges with weights from T; the probability that the weighting of the final edge with an element of T causes the multisets of weights from T at u and v to be the same is at most $1/|T|$. Therefore it is enough to take $|T|=8D^2=8 \times 10^{198}$. $\square$

We note that the total number of weights needed can be reduced a little by, say, the following method for weighting G_0. Let $d_1=40$, $d_2=60$, $d_3=250$, $d_4=10^4$, $d_5=10^{10}$, $d_6=10^{30}$, and $d_7=D$. Partition V_0 into $V_0'=\{u \in G : d_G(u) < d_1\}$ and $V_i'=\{u \in G : d_i \leqslant d_G(u) < d_{i+1}\}$ for $1 \leqslant i \leqslant 6$. Form graphs G_i', $0 \leqslant i \leqslant 6$, much as before. Choose a fresh set S_i of 16 numbers to weight G_i' for $i \geqslant 1$; since $3 \times 16^8(\pi d_i)^{-15/2} \leqslant 1/8d_{i+1}^2$ the weighting is possible without any A_e occurring in G_i'. As for G_0, if it has isolated edges they can be handled as before by putting each S_i in one-to-one correspondence with S'. Finally, since G_0' is 41-colourable, the remaining edges can be weighted by 41 further numbers by the first theorem. In this way we use only 183 numbers.

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