Weight Choosability of Graphs

Tomasz Bartnicki,[1] Jarosław Grytczuk,[2] and Stanisław Niwczyk[1]

[1] *FACULTY OF MATHEMATICS, COMPUTER SCIENCE, AND ECONOMETRICS
UNIVERSITY OF ZIELONA GÓRA
65-516 ZIELONA GÓRA, POLAND
E-mail: t.bartnicki@wmie.uz.zgora.pl; s.niwczyk@wmie.uz.zgora.pl*

[2] *FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
THEORETICAL COMPUTER SCIENCE DEPARTMENT
JAGIELLONIAN UNIVERSITY, 30-387 KRAKÓW, POLAND
E-mail: grytczuk@tcs.uj.edu.pl*

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Abstract: Suppose the edges of a graph G are assigned 3-element lists of real weights. Is it possible to choose a weight for each edge from its list so that the sums of weights around adjacent vertices were different? We prove that the answer is positive for several classes of graphs, including complete graphs, complete bipartite graphs, and trees (except K_2). The argument is algebraic and uses permanents of matrices and Combinatorial Nullstellensatz. We also consider a directed version of the problem. We prove by an elementary argument that for digraphs the answer to the above question is positive even with lists of size two. © 2008 Wiley Periodicals, Inc. J Graph Theory 60: 242–256, 2009

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1. INTRODUCTION

Let S be a subset of the field of real numbers $\mathbb{R}$ and let G be a simple graph. We say that G is *weight colorable* by S if there is an edge weighting $w : E \to S$ such that for any two adjacent vertices $u, v \in V(G)$, the sum of weights of the edges incident with u is different than the sum of weights of the edges incident with v.

Conjecture 1. *Every connected graph $G \neq K_2$ is weight colorable by* $\{1, 2, 3\}$.

The problem was posed by Karoński et al. in [11], as a variation of the *irregularity strength* of a graph (cf. [10]). They proved that every connected graph (except K_2) is weight colorable by any set $S \subset \mathbb{R}$ of size at least 183, provided S is independent over the field of rational numbers. With this restriction the bound was reduced to 4 by Addario-Berry et al. [1]. Notice that this does not imply that there is a finite set of *integers* doing the same job. However in [2] it was proved that 30 integer weights $\{1, \dots, 30\}$ are enough. The proof uses spanning subgraphs with appropriate vertex degrees. Currently best bound is 16 and was obtained in [3]. It was also proved there that for any fixed $p \in (0, 1)$ the random graph $G_{n,p}$ is almost surely $\{1, 2\}$-weight colorable.

In this article we make a different approach, based on the algebraic method of Alon, known as Combinatorial Nullstellensatz [4]. The basic idea goes as follows. We associate a multivariable polynomial P_G with a graph G, so that a non-zero substitution for variables of P_G gives a desired weighting of G. Then we look at the exponents of variables in the expansion of P_G into a linear combination of monomials. If there is a non-vanishing monomial with the highest exponent of a variable less than 3, then by Combinatorial Nullstellensatz, G is weight colorable by *any* set of three real weights. Actually, conclusion is much stronger, and leads to the following *list version* of the problem. Suppose that each edge e of a graph G is assigned a set of real numbers L_e. We say that G is *weight colorable* from the lists L_e if there is an edge weighting $w : E \to \bigcup_{e \in E(G)} L_e$ such that for each edge e, $w(e) \in L_e$, and the sums of weights on the ends of e are different. A graph G is *k-weight choosable* if it is weight colorable from any collection of lists of size k.

Consider for instance the *diamond D*—a cycle on four vertices with a chord. This graph is weight colorable by the set $\{1, 2\}$ as shown in Figure 1.

In fact D is weight colorable by any set of two different weights. However, it is not 2-weight choosable, as can be seen by putting the list $\{1, 2\}$ on every edge of the cycle and the list $\{1, -1\}$ on the chord.

Conjecture 2. *Every graph without an isolated edge is* 3*-weight choosable.*

We prove Conjecture 2 for several classes of graphs, including cliques, complete bipartite graphs, and trees, by providing general recursive constructions preserving the desired algebraic properties. We also consider a natural oriented version of the problem. By an elementary argument we show that any directed graph is 2-weight

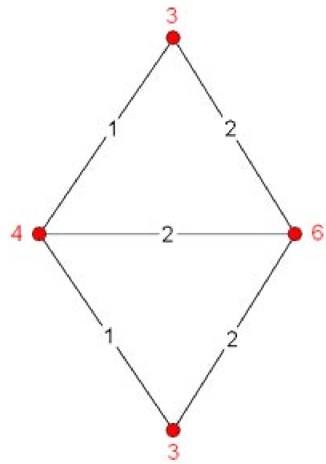

FIGURE 1. Weight coloring of the diamond.

choosable. At the end of the article we discuss other versions of the problem and pose several open questions.

2. POLYNOMIALS AND PERMANENTS

Let $E(G) = \{e_1, \ldots, e_m\}$ be the set of edges of a simple graph G. Let x_i be variables assigned to the edges e_i. For each vertex $u \in V$, let E_u be the set of edges incident to u, and let $X_u = \sum_{e_j \in E_u} x_j$. Now fix any orientation of G and define a polynomial P_G in variables $x_1, \ldots, x_m$ by

$$P_G(x_1, \ldots, x_m) = \prod_{(u,v) \in E(G)} (X_u - X_v).$$

Notice that if $P_G(s_1, \ldots, s_m) \neq 0$ for some $s_i \in S$, then G is S-weight colorable.

We will use the following theorem of Alon [4] (cf. [5,7], for earlier versions).

Theorem 1 (Alon [4]). *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, \ldots, x_m)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_m]$. Suppose the degree* $\deg(P)$ *of P is $\sum_{i=1}^{n} k_i$, where each k_i is a non-negative integer, and suppose the coefficient of $x_1^{k_1} \cdots x_m^{k_m}$ in P is non-zero. Then, if $S_1, \ldots, S_m$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, \ldots, s_m \in S_m$ so that $P(s_1, \ldots, s_m) \neq 0$.*

Suppose $M = cx_1^{k_1} \cdots x_m^{k_m}$ is a monomial in the expansion of P with $c \neq 0$. Let $h(M)$ be the highest exponent of a variable in M. Define the *monomial index* of P, denoted by mind(P), as the minimum of $h(M)$ taken over all non-vanishing monomials M in P. Since different orientations of a graph G give the same polynomial P_G up to the sign, we may consider mind(P_G) as a graph invariant.

Therefore we will speak about the monomial index of a graph G and write simply mind(G) instead of mind(P_G). Notice that mind(G) is not defined for graphs with an isolated edge (as in this case $P_G \equiv 0$). Otherwise P_G is a homogenous polynomial of degree m and Theorem 1 implies the following statement.

Corollary 1. *If* $\mathrm{mind}(G) \leq k$ *then* G *is* $(k+1)$*-weight choosable.*

Hence to prove Conjecture 2 it would be enough to show that $\mathrm{mind}(G) \leq 2$ for every graph G. But in general it is not easy to determine which monomials will survive in the expansion of P_G. One way to attempt it is to look at the permanent rank of associated matrices. Let $A = (a_{ij})$ be a square matrix of size m. Recall that the *permanent* of a A is defined by

$$\mathrm{per} A = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)}$$

where σ runs through all permutations of the set $\{1, \ldots, m\}$. The *permanent rank* of a matrix A is the size of a largest square submatrix with non-zero permanent. Now, let $A^{(k)} = [A, \ldots, A]$ be a matrix formed of k copies of a matrix A. Define the *permanent index* of A, denoted by pind(A), as the minimum k for which $A^{(k)}$ has the permanent rank equal to the size of A. If there is no such k, we put $\mathrm{pind}(A) = \infty$. Equivalently, we may define pind(A) as follows. Let $K = (k_1, \ldots, k_m)$ be any sequence of non-negative integers and let A be any matrix with m columns $A_1, \ldots, A_m$. We denote by $A(K)$ a matrix obtained from A by repeating k_j times each column A_j. Thus, if A is a square matrix then pind(A) is the minimum k such that $\mathrm{per}\, A(K) \neq 0$, where K is a sequence such that $k_1 + \cdots + k_m = m$, and $k_i \leq k$ for $i = 1, \ldots, m$. We will make use of the following simple lemma (cf. [6]).

Lemma 1. *Let* $A = (a_{ij})$ *be a square matrix of size* m *and finite permanent index. Let* $P(x_1, \ldots, x_m) = \prod_{i=1}^{m} (a_{i1}x_1 + \cdots + a_{im}x_m)$. *Then* $\mathrm{mind}(P) = \mathrm{pind}(A)$.

Proof. It is straightforward to check that coefficient of the monomial $x_1^{k_1} \cdots x_m^{k_m}$ in the expansion of $\prod_{i=1}^{m} (a_{i1}x_1 + \cdots + a_{im}x_m)$ is equal to $\frac{\mathrm{per} A(K)}{k_1! \cdots k_m!}$. ■

For a fixed orientation of a graph G, define a matrix $A_G = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with the } \textit{head} \text{ of } e_i, \\ -1 & \text{if } e_j \text{ is incident with the } \textit{tail} \text{ of } e_i, \\ 0 & \text{if } e_i \text{ and } e_j \text{ are not incident.} \end{cases}$$

Clearly, $P_G(x_1, \ldots, x_m) = \prod_{i=1}^{m} (a_{i1}x_1 + \cdots + a_{im}x_m)$. Hence by Lemma 1 we have $\mathrm{pind}(A_G) = \mathrm{mind}(G)$, and we may state another consequence of Theorem 1.

Corollary 2. *If* $\mathrm{pind}(A_G) \le k$ *then G is* $(k+1)$*-weight choosable. In particular, if* $\mathrm{per}A_G \neq 0$ *then G is* 2*-choosable.*

The following example illustrates the introduced notation.

Example 1. *Orient the diamond graph D as in Figure 2.*
Then the polynomial P_D

$$P_D = (x_2 - x_4 - x_5)(x_3 + x_5 - x_1)(x_4 - x_2 - x_5)(x_1 + x_5 - x_3)(x_1 + x_4 - x_2 - x_3)$$

has five linear factors, one for each edge of D. The associated matrix has five rows corresponding to these factors:

$$A_D = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

It turns out that $\mathrm{mind}(D) = 2$*. Indeed,* $\mathrm{per}A_D = 0$*, but for* $K = (1, 1, 1, 2, 0)$ *we get the matrix*

$$A_D(K) = \begin{bmatrix} 0 & 1 & 0 & -\mathbf{1} & -\mathbf{1} \\ -1 & 0 & 1 & \mathbf{0} & \mathbf{0} \\ 0 & -1 & 0 & \mathbf{1} & \mathbf{1} \\ 1 & 0 & -1 & \mathbf{0} & \mathbf{0} \\ 1 & -1 & -1 & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

with $\mathrm{per}A_D(K) = 12$*. Thus D is* 3*-choosable.*

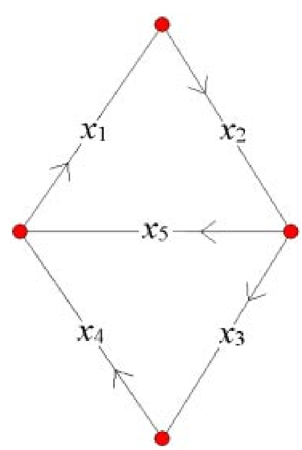

FIGURE 2. Orientation of the diamond.

3. CONSTRUCTIONS PRESERVING LOW MONOMIAL INDEX

We start with two simple lemmas on permanents.

Lemma 2. *Let A and L be square matrices of size n such that each column of L is a linear combination of columns of A. Let n_j be the number of those columns of L in which the j-th column of A appears with a non-zero coefficient. If $n_j \leq r$ and* $\mathrm{per}L \neq 0$, *then* $\mathrm{pind}(A) \leq r$.

Proof. Let L_k be the kth column of L. By assumption, $L_k = \sum_{j=1}^{n} a_{kj} A_j$. Since permanent of a matrix is a multilinear function with respect to columns we may expand $\mathrm{per}L$ as a sum of terms of the form $c \times (\mathrm{per}[A_{j_1}, \ldots, A_{j_n}])$, where $c = \prod_{k=1}^{n} a_{kj_n}$. Since $\mathrm{per}L$ is non-zero at least one of these terms must be non-zero. The assertion follows by the assumption that $n_j \leq r$. ■

Let S denote a square matrix of size $2n+1$, $n \geq 1$, defined by $s_{11} = 0$, $s_{i1} = (-1)^i$ for $i = 2, \ldots, 2n+1$, $s_{i2} = s_{i1}$ for $i = 1, \ldots, 2n+1$, and $s_{ij} = 1$ everywhere else. For $n = 2$ we get

$$S = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

Lemma 3. *For every $n \geq 1$,* $\mathrm{per}S = -(2n)!$.

Proof. We apply Laplace expansion for permanents with respect to the first column of S. Assume for convenience that the rows of S were permuted so that those starting with -1 lay at the bottom. Let S_i denote the matrix obtained by deleting the ith row and the first column from S. Then we have

$$\mathrm{per}S = \sum_{i=1}^{2n+1} \mathrm{per}S_i = \sum_{i=1}^{n} \mathrm{per}S_{2i} - \sum_{i=1}^{n} \mathrm{per}S_{2i+1} = n\,\mathrm{per}A - n\,\mathrm{per}B,$$

where $A = S_{2i}$ and $B = S_{2i+1}$, for $i = 1, \ldots, n$. Now, the matrix A has ones everywhere, except the first column where it has one zero and n minus ones. Hence, applying Laplace expansion to the first column of A we get

$$\mathrm{per}A = (n-1-n)(2n-1)! = -(2n-1)!$$

Similarly for the matrix B, which has $n-1$ minus ones and one zero in the first column, we get

$$\operatorname{per} B = (n - n + 1)(2n - 1)! = (2n - 1)!$$

Hence, $\operatorname{per} S = -2n(2n-1)! = -(2n)!$ as asserted. ■

Our main result reeds as follows.

Theorem 2. *Let $G = (V, E)$ be a simple graph with* $\operatorname{mind}(G) \le 2$. *Let U be a nonempty subset of $V(G)$. Let F be a graph obtained by adding two new vertices u, v to G and joining them to each vertex of U. Let H be a graph obtained from F by joining u and v by an edge. Then* $\operatorname{mind}(F) \le 2$ *and* $\operatorname{mind}(H) \le 2$.

Proof. Let $U = \{u_1, \ldots, u_k\}$ be a subset of $V(G)$. Let $E_u = \{e_1, e_3 \ldots, e_{2k-1}\}$ and $E_v = \{e_2, e_4, \ldots, e_{2k}\}$ be the sets of edges incident to the vertices u and v, respectively. Assume these edges are oriented towards the set U and consider the matrix A_F, as defined in section 2 (see Fig. 3).

Let $A_1, \ldots, A_{2k}$ be the first $2k$ columns of A_F corresponding to new edges $e_1, \ldots, e_{2k}$, and let $A = [A_1, \ldots, A_{2k}]$. So, $A_F = [A, B]$, where $B = \begin{bmatrix} X \\ A_G \end{bmatrix}$. Consider a new matrix M given by

$$M = [M_1, M_1, M_2, M_2, \ldots, M_k, M_k, B(K)],$$

where $M_j = A_{2j-1} - A_{2j}$ for $j = 1, \ldots, k$, and K is a sequence justifying the assumption that $\operatorname{mind}(G) \le 2$. We claim that per M is non-zero. Notice that by Lemma 1 and 2, this implies that $\operatorname{mind}(F) \le 2$. To prove the claim take a closer look at the matrix A and write it in a block form $A = \begin{bmatrix} Y \\ Z \end{bmatrix}$, where Y is a square $2k \times 2k$ submatrix. First notice that any two columns A_{2i-1} and A_{2i} agree on Z. Indeed, the edges e_{2i-1} and e_{2i} dominate the same vertex u_i, while their tails u, v are

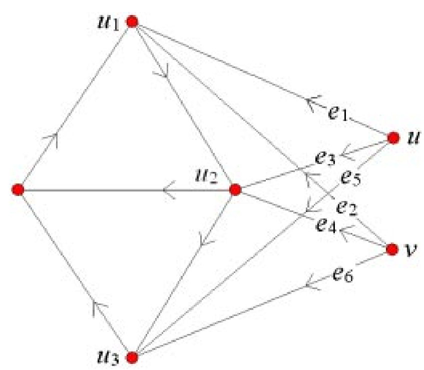

FIGURE 3. Orientation of the attached edges.

disjoint from the set $V(G)$. On the other hand, the submatrix Y consists of two types of 2×2 blocks, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, arranged so that B occupies the main diagonal, while C is everywhere else:

$$Y = \begin{bmatrix} \mathbf{0} & \mathbf{1} & -1 & 0 & & -1 & 0 \\ \mathbf{1} & \mathbf{0} & 0 & -1 & \cdots & 0 & -1 \\ -1 & 0 & \mathbf{0} & \mathbf{1} & & -1 & 0 \\ 0 & -1 & \mathbf{1} & \mathbf{0} & & 0 & -1 \\ & \vdots & & & \ddots & & \\ -1 & 0 & -1 & 0 & & \mathbf{0} & \mathbf{1} \\ 0 & -1 & 0 & -1 & & \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

This also follows from the definition of A_G and the initial orientation of the edges from E_u and E_v. Now, by this two properties of A, the matrix M has the following block form

$$M = \begin{bmatrix} R & X(K) \\ 0 & A_G(K) \end{bmatrix},$$

where the left upper corner R is a square matrix consisting of constant non-zero rows:

$$R = \begin{bmatrix} -1 & -1 & -1 & & -1 \\ 1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & -1 & & -1 \\ & \vdots & & \ddots & \\ 1 & 1 & 1 & & 1 \end{bmatrix}.$$

Hence $\mathrm{per}R \neq 0$. Since $\mathrm{per}M = \mathrm{per}R \times \mathrm{per}A_G(K)$, the assertion follows by the assumption that $\mathrm{per}A_G(K) \neq 0$.

The second assertion for the graph H can be proved similarly. Now we have one more edge e_0 between u and v, oriented from v to u, say (see Fig. 4).

The corresponding matrix A_H looks the same as before except the first row and column for the added edge e_0. It can be depicted in a block form as

$$A_H = \begin{bmatrix} Y' & X' \\ Z' & A_G \end{bmatrix},$$

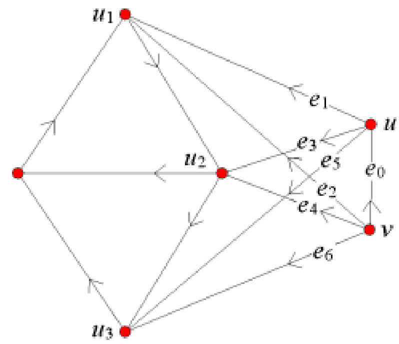

FIGURE 4. Orientation of the attached edges.

where

$$Y' = \begin{bmatrix} 0 & 1 & -1 & \cdots & 1 & -1 \\ -1 & & & & & \\ -1 & & & & & \\ \vdots & & & Y & & \\ -1 & & & & & \\ -1 & & & & & \end{bmatrix}, \qquad Z' = \begin{bmatrix} 0 & \\ 0 & \\ \vdots & Z \\ 0 & \\ 0 & \end{bmatrix}.$$

Let $A_0, A_1, \ldots, A_{2k}$ denote the first $2k+1$ columns of the matrix A_H corresponding to the edges e_i, $i = 0, 1, \ldots, 2k$. Form a new matrix

$$N = [N_0, N_0, N_1, N_2, N_2, \ldots, N_k, N_k, B(K)]$$

so that $N_0 = A_0$ and $N_j = A_{2j-1} - A_{2j}$ for $j = 1, 2, \ldots, k$. Arguing as before we get

$$N = \begin{bmatrix} R' & X'(K) \\ 0 & A_G(K) \end{bmatrix},$$

where R' is a square matrix of the form

$$R' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{2} & \cdots & \mathbf{2} \\ \mathbf{-1} & \mathbf{-1} & -1 & -1 & & -1 \\ \mathbf{-1} & \mathbf{-1} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \ddots & \\ \mathbf{-1} & \mathbf{-1} & 1 & 1 & & 1 \end{bmatrix}.$$

Now, the matrix R' can be transformed to the matrix S from Lemma 3, by multiplying the first row by 1/2 and every second row by -1. So, by Lemma 3 per$R' \neq 0$ and the proof is complete. ■

The theorem allows for recursive constructions of many graphs with low monomial index. In particular, we get the following corollary.

Corollary 3. *If $G \neq K_2$ is a clique, complete bipartite graph, or a tree, then* $\mathrm{mind}(G) \leq 2$.

Proof. The first two cases follow immediately from Theorem 2, after checking that K_3, K_4 and $K_{1,2}$, $K_{1,3}$, $K_{2,2}$ have monomial index at most 2, respectively. In case of trees the following simple inductive argument will do. For the base of induction one checks that trees with two or three edges have monomial index at most 2. Let T be a tree with $m \geq 4$ edges and assume the assertion holds for all trees with less than m edges. If T has two leafs joined to the same vertex we are done by the theorem. Otherwise, let $e = uv$ be an edge such that u is a leaf and v has degree two. Let f be the unique edge joining e to the rest of the tree T. Then the matrix of T can be written as

$$A_T = \begin{bmatrix} A_{T-e-f} & X & \mathbf{0} \\ Y & 0 & 1 \\ \mathbf{0} & 1 & 0 \end{bmatrix}.$$

where the last two rows correspond to the edges f and e, respectively. By inductive assumption $\mathrm{mind}(T - e - f) \leq 2$, hence there is a sequence K such that per$A_{T-e-f}(K) \neq 0$. Taking $K' = (K, 1, 1)$ we get

$$A_T(K') = \begin{bmatrix} A_{T-e-f}(K) & X & \mathbf{0} \\ Y(K) & 0 & 1 \\ \mathbf{0} & 1 & 0 \end{bmatrix}.$$

Expanding the permanent of the last matrix with respect to the last row we see that it is non-zero. ■

Also the following corollary follows easily from Theorem 2.

Corollary 4. *Every graph G is an induced subgraph of a graph H such that* $\mathrm{mind}(H) \leq 2$ *and* $\chi(H) \leq \chi(G) + 1$.

Let G be a graph with $\mathrm{mind}(G) \leq 2$ and let $A_G(K)$ be a matrix with non-zero permanent, where $K = (k_1, \ldots, k_m)$, $k_1 + \cdots + k_m = m$, $0 \leq k_i \leq 2$. Let K_0 be the set of those $j \in \{1, \ldots, m\}$ for which $k_j = 0$ and let $E_0(G)$ be the set of corresponding edges.

Proposition 1. *Let G be a graph whose edge set can be partitioned into two subgraphs P, Q, both of monomial index at most* 2*, so that any edge $e \in P$ incident with some $f \in Q$ belongs to $E_0(P)$. Then* $\mathrm{mind}(G) \leq 2$.

Proof. Let $S = P - E_0(P)$. Assume the edges of G are ordered so that the associated matrix A_G takes the block form

$$A_G = \begin{bmatrix} A_S & \cdots & \mathbf{0} \\ \cdots & A_{E_0(P)} & Y \\ \mathbf{0} & X & A_Q \end{bmatrix}.$$

Let K' and K'' be sequences turning A_P and A_Q into matrices of non-zero permanent, respectively. Let $K = K'K''$ (where K'' is appropriately shifted). Then by the definition of $E_0(P)$ we get

$$A_G(K) = \begin{bmatrix} A_P(K') & \begin{matrix} \mathbf{0} \\ Y(K'') \end{matrix} \\ \mathbf{0} & A_Q(K'') \end{bmatrix}.$$

Hence $\mathrm{per}A_G(K) = \mathrm{per}A_P(K') \times \mathrm{per}A_Q(K'') \neq 0$. ∎

Example 2. *Consider a path P_4 oriented as a directed path. Then the related matrix is*

$$A_{P_4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

For $K = (0, 2, 2, 0)$ *we have* $K_0 = \{1, 4\}$,

$$A_{P_4}(K) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix},$$

and $\mathrm{per}A_{P_4}(K) = 4$. *Therefore* $\mathrm{mind}(P_4) \leq 2$ *and* $E_0(K)$ *consists of the two end edges of* P_4.

By Proposition 1 and Example 2 we get the following corollary.

Corollary 5. *Every graph G can be transformed into a graph H with* $\mathrm{mind}(H) \leq 2$ *by subdividing each edge of G with at most three vertices.*

Proof. The assertion is true for trees by Corollary 3. So, assume G has $m \geq 3$ edges and suppose the statement holds for all connected graphs with less than m edges. Let $e = uv$ be an edge of some cycle in G and let H be a subdivision of $G - e$ such that $\mathrm{mind}(H) \leq 2$. Let F be a graph obtained by identifying the end vertices of a path P_4 with vertices u and v. By Proposition 1 and Example 2, $\mathrm{mind}(F) \leq 2$. Clearly, F is a subdivision of G, as asserted. ■

4. WEIGHT CHOOSABILITY OF DIRECTED GRAPHS

In much the same way one may consider weight choosability of digraphs. Let $G = (V, E)$ be a digraph and let $w : E \to S$ be an edge weighting of G. For each vertex $v \in V$, denote by $E^+(v)$ and $E^-(v)$ the sets of edges directed out of v and into v, respectively. Now, let $b(v)$ be a *balanced* sum around v defined by

$$b(v) = \sum_{e \in E^+(v)} w(e) - \sum_{f \in E^-(v)} w(f).$$

We say that G is *S-weight colorable* if $b(v) \neq b(u)$ whenever u and v are adjacent. The list version of weight colorability and weight choosability of digraphs is defined in the same way as in the undirected case.

In [8] it was proved that every digraph is weight colorable by the set of just two weights $\{1, 2\}$. The proof provided explicit algorithm for finding a desired weighting. We extend this argument here to get the following theorem.

Theorem 3. *Every digraph is* 2*-weight choosable.*

Proof. Let $L_i = \{s_i, t_i\}$ be the lists assigned to the edges e_i of a digraph G, where $s_i < t_i$ are any real numbers. Let $L = L_1 \cup \cdots \cup L_m$. We will define a desired weighting $w : E \to L_1 \cup \cdots \cup L_m$ as follows. Let $E(v) = E^+(v) \cup E^-(v)$. Let $X \subset E$ be a fixed subset of the edges of G and let $w_X : X \to L$ be a partial weighting of G. Let $V_X = \{v \in V : E(v) \setminus X \neq \emptyset\}$ and for each vertex $v \in V_X$, let $m_X(v)$ be a quantity defined by

$$m_X(v) = \left(\sum_{e_i \in E^+(v) \cap X} w_X(e_i) - \sum_{e_i \in E^-(v) \cap X} w_X(e_i) \right) + \left(\sum_{e_i \in E^+(v) \setminus X} s_i - \sum_{e_i \in E^-(v) \setminus X} t_i \right).$$

So, $m_X(v)$ is the minimum possible value of a balanced sum which v can get in any extension of a partial weighting w_X.

Now, we define recursively a sequence of subsets $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_r = E$, and the related partial weightings w_{X_i}, as follows. For $i \geq 1$ let $m_i = \min\{m_{X_{i-1}}(v) : v \in V_{X_{i-1}}\}$. Let v_i be any vertex v satisfying $m_{X_{i-1}}(v) = m_i$. Next define $X_i = X_{i-1} \cup E(v_i)$ and extend the weighting $w_{X_{i-1}}$ to X_i by assigning $w_{X_i}(e_k) = s_k$, for $e_k \in E^+(v_i) \setminus X_{i-1}$, and $w_{X_i}(e_k) = t_k$, for $e_k \in E^-(v_i) \setminus X_{i-1}$.

Clearly, after some number $r \leq |V|$ of steps we must get $X_r = E$ and a final weighting $w = w_E$. We claim that this weighting satisfies the assertion of the theorem. First notice that our procedure defines a partition of the vertex set V into two parts $V_1 = \{v_1, \ldots, v_r\}$ and $V_2 = V \setminus V_1$, where V_2 is an independent set. We may enumerate the vertices of V_2 arbitrarily as $v_{r+1}, \ldots, v_n$, where $n = |V|$, and obtain in this way a linear ordering of the vertices of G. This ordering satisfies the following property: for each vertex v_i, the sequence $m_{X_0}(v_i), m_{X_1}(v_i), \ldots, m_{X_r}(v_i)$ is nondecreasing and

$$m_{X_{i-1}}(v_i) = m_{X_i}(v_i) = \cdots = m_{X_r}(v_i) = m_E(v_i)$$

is the final balanced sum around vertex v_i. So, our goal is to show that $m_E(v_i) \neq m_E(v_j)$ whenever v_i and v_j are joined by an edge.

Suppose that $i < j$ and let $e_k = v_i v_j$ be a directed edge of G. Since V_2 is an independent set we have $i \leq r$. Now, $i < j$ implies $m_{X_{i-1}}(v_i) \leq m_{X_{i-1}}(v_j)$. But the edge e_k was not weighted yet, when these quantities were computed. So it was counted as s_k in $m_{X_{i-1}}(v_i)$ and as $-t_k$ in $m_{X_{i-1}}(v_j)$. Since its weight was finally fixed as s_k, it follows that $m_{X_{i-1}}(v_j) < m_{X_i}(v_j)$. Consequently, $m_{X_{i-1}}(v_i) < m_{X_i}(v_j)$, which implies that $m_E(v_i) < m_E(v_j)$. Similarly, if $e_k = v_j v_i$ then it was counted as $-t_k$ in $m_{X_{i-1}}(v_i)$ and as s_k in $m_{X_{i-1}}(v_j)$. Hence, assigning t_k to e_k increases $m_{X_i}(v_j)$, which again gives that $m_{X_{i-1}}(v_i) < m_{X_i}(v_j)$. This completes the proof. ■

Clearly, the result is best possible (as long as we consider real or complex numbers). It would be however interesting to determine whether the index of the associated matrix or polynomial always equal to 1.

Example 3. *Consider again the diamond D from Example* 1*, but this time as a directed graph.*

Its polynomial is

$$P_G = (2x_1 - x_2 - x_4 - x_5)(2x_2 - x_3 - x_5 - x_1)(2x_3 - x_4 - x_2 + x_5)$$
$$(2x_4 - x_1 + x_5 - x_3)(2x_5 - x_1 + x_4 - x_2 + x_3),$$

and the related matrix is

$$\begin{bmatrix} 2 & -1 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 1 \\ -1 & 0 & -1 & 2 & 1 \\ -1 & -1 & 1 & 1 & 2 \end{bmatrix}.$$

Its permanent equals 104.

Notice that matrices arising in this way will always be symmetric with 2's occupying the main diagonal.

5. FINAL REMARKS

The main question remaining open is whether there is a finite bound on the monomial index for graphs without an isolated edge.

Conjecture 3. *For every connected graph $G \neq K_2$,* $\text{mind}(G) \leq 2$.

This conjecture may look too optimistic, but there are some heuristic arguments on its side. In equivalent matrix version, it resembles the following Kahn's conjecture stating that $\text{pind}(A) \leq 2$ for any nonsingular matrix A over arbitrary field (cf. [9,12]). This conjecture is widely believed, though it is also not clear if there is a finite absolute bound here. Unfortunately, matrices A_G arising from graphs are far from being nonsingular (their rank is at most $|V|$). But this cannot be an obstacle in general, as evidenced by the case of cliques.

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