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Degree constrained subgraphs☆

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Abstract

In this paper, we present new structural results about the existence of a subgraph where the degrees of the vertices are pre-specified. Further, we use these results to prove a 16-edge-weighting version of a conjecture by Karoński, Łuczak and Thomason, an asymptotic 2-edge-weighting version of the same conjecture, and a $\frac{7}{8}$ version of Louigi's Conjecture.

Keywords: Vertex colouring; Edge weighting; Degree constrained subgraphs

1. Introduction

A *k-edge-weighting* of a graph G is an assignment of an integer weight, $w(e) \in \{1, \ldots, k\}$ to each edge e. The edge-weighting is *vertex-colouring* if for every edge uv, $\sum_{e \ni u} w(e) \neq \sum_{e \ni v} w(e)$. Let us say that a graph is *nice* if it does not contain a connected component which has only one edge. Note that only nice graphs have vertex-colouring edge-weightings.

In [9], Karoński et al. initiated the study of vertex-colouring edge-weightings as defined here. (See also [3,5,7] for alternate notions that combine ideas from vertex and edge colouring.) In particular, [9] conjectures that every nice graph permits a vertex-colouring 3-edge-weighting and proves the conjecture for graphs G with $\chi(G) \leqslant 3$. For general graphs, the first finite bound was shown in [2], where it is proved that nice graphs always permit a vertex-colouring 30-edge-weighting. In this paper, we substantially improve this result to prove the following:

Theorem 1. *Every nice graph permits a vertex-colouring* 16-*edge-weighting*.

To get a feeling for our approach, note that if it were possible to find a spanning subgraph H of G such that $d_H(v) \neq d_H(w)$ for any edge vw of $E(G)$, then giving the edges of H weight 1 and all other edges weight 0 would yield a vertex-colouring edge-weighting with weights in $\{0, 1\}$. In general, such a subgraph H may not exist, e.g., for K_3. However using this idea we shall prove the following result (which [9] found evidence for experimentally):

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Theorem 2. *Let G be a random graph chosen from $G_{n,p}$ for constant $p \in (0,1)$. Then, asymptotically almost surely, there exists a vertex-colouring 2-edge-weighting for G. In fact, there exists a 2-edge-weighting such that the colours of two adjacent vertices are distinct mod $2\chi(G)$.*

When dealing with an arbitrary graph, our approach is to find an intermediate weighting of the edges in which no vertex has many neighbours of the same weight, then find a subgraph H which allows us to distinguish such neighbours without creating new conflicts. Our tool will be Theorem 5, a result on when it is possible to find a subgraph H in which each vertex has degree $d_H(v)$ in some target set D_v. For arbitrary D_v, this problem is known as the *generalized f-factor problem* and has been well studied (see, e.g., [1,2,8,10,11,13]). In [2], we find the following conjecture.

Conjecture 3 (*Louigi's conjecture*). *Given $G=(V,E)$ and, for each $v \in V$, a list $D_v \subseteq \{0,1,\dots,d(v)\}$ satisfying $|D_v| > \lceil d(v)/2 \rceil$, there exists a spanning subgraph H of G so that for all v, $d_H(v) \in D_v$.*

We will use a result of Sebö [13] to show that this conjecture holds if we additionally require that $\{0,1,\dots,d(v)\} - D_v$ contains no two consecutive integers (see Theorem 9). Also, if we weaken the conjecture by replacing $d(v)/2$ with $7d(v)/8$, the result follows easily from Theorem 5 (see Corollary 6). This is an improvement over the $11d(v)/12$ version found in [2].

In Section 2, we prove two theorems on when, given G, it is possible to find a subgraph H such that every vertex v has $d_H(v)$ in one of two small intervals (Theorems 5 and 7). In addition, we prove the above statements about Louigi's Conjecture. Finally, in Section 3 we prove Theorems 1 and 2.

2. Degree constrained subgraphs

In this section we strengthen results from [2] in Theorems 5 and 7. The backbone of our results is the following strengthening by Heinrich et al. [8] of a lemma of Lovász [10]. This lemma can be viewed as a special case of the *f*-factor theorem (see e.g. [11,12]).

Lemma 4 (*Heinrich et al. [8] Lovász [10]*). *Given a graph $G=(V,E)$ and, for all $v \in V$, integers a_v, b_v such that $0 \leqslant a_v \leqslant b_v \leqslant d(v)$, if G is bipartite or $a_v \neq b_v$ for all v, then there exists a spanning subgraph H of G such that $d_H(v) \in [a_v, b_v]$ for all $v \in V$ if and only if for all disjoint sets of vertices A and B,*

$$\sum_{v \in A} (a_v - d_{G-B}(v)) \leqslant \sum_{v \in B} b_v. \tag{1}$$

We now prove

Theorem 5. *Given a graph $G=(V,E)$ and, for all $v \in V$, integers a_v^-, a_v^+ such that $a_v^- \leqslant \lfloor d(v)/2 \rfloor \leqslant a_v^+ < d(v)$, and*

$$a_v^+ \leqslant \min\left(\frac{d(v)+a_v^-}{2}+1, 2a_v^- + 3\right), \tag{2}$$

there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$ for all $v \in V$.

Proof. Given a set of integers $\{a_v \| v \in V\}$ and a subgraph H of G, we define the *deficiency* of H with respect to the integers a_v to be the quantity

$$\sum_v \max(0, a_v - d_H(v)).$$

Suppose the desired subgraph H does not exist. Choose $a_v \in \{a_v^-, a_v^+\}$, $b_v = a_v + 1$ and a spanning subgraph H of G such that for all $v \in V$, $d_H(v) \leqslant b_v$ so that the deficiency is minimized over all such choices. Necessarily, there is a vertex $v \in V$ such that $d_H(v) < a_v$, so the deficiency of H is positive.

Let $A_0 = \{v : d_H(v) < a_v\}$. An *H-alternating walk* is a walk $P = v_0 v_1 \dots v_k$ with $v_0 \in A_0$ and $v_i v_{i+1} \in G - H$ for i even, $v_i v_{i+1} \in H$ for i odd. We let $A = \{v :$ there is an even H-alternating walk ending in $v\}$, and $B = \{v :$

there is an odd H-alternating walk ending in v}. (Note that $A_0 \subseteq A$.) For $v \in A$, $d_H(v) \leqslant a_v$, or else by reversing which edges are in H along an even alternating walk ending in v, we decrease the deficiency. Similarly, for $v \in B$, $d_H(v) = b_v$ or else we can likewise decrease the deficiency by reversing which edges are in H, this time along an odd alternating walk ending in v. Since $b_v > a_v$ this implies A and B are disjoint. Furthermore note that for $v \in A$, if $vw \in E$ and $w \notin B$ then $vw \in H$ by the definition of B. Similarly if $v \in B$, $vw \in E$ and $w \notin A$ then $vw \notin H$. By these observations we have that

$$\sum_{v \in A} a_v > \sum_{v \in A} d_H(v) = \sum_{v \in A} d_{G-B}(v) + \sum_{v \in B} d_H(v) = \sum_{v \in A} d_{G-B}(v) + \sum_{v \in B} b_v,$$

which implies that (1) of Lemma 4 fails for these A and B.

We make the following two claims:

$$\forall v \in A, \quad a_v - d_{G-B}(v) \leqslant d_B(v)/2 \tag{3}$$

and

$$\forall v \in B, \quad b_v \geqslant d_A(v)/2. \tag{4}$$

These two statements together with the fact that $\sum_{v \in A} d_B(v) = \sum_{v \in B} d_A(v)$ imply (1) holds for these A and B, completing the proof of Theorem 5 by contradiction.

Consider $v \in A$ and assume that $d_H(v) < a_v$. (Note that reversing which edges are in H along an even alternating walk does not change the deficiency or the sets A and B: we may thus ensure that any single vertex $v \in A$ satisfies $d_H(v) < a_v$.) We may assume $a_v = a_v^+ > d(v)/2$ or else (3) holds automatically. We may further assume that $d_{G-B}(v) > a_v^- + 1$ or else by setting $a_v = a_v^-$ and removing from H some of the edges from v to B, we can reduce the deficiency. Now, by (2)

$$a_v \leqslant \frac{d(v)}{2} + \frac{a_v^-}{2} + 1 < \frac{d(v)}{2} + \frac{d_{G-B}(v)}{2} + \frac{1}{2} = d_{G-B}(v) + \frac{d_B(v)}{2} + \frac{1}{2},$$

so since a_v is an integer and $v \in A$ was arbitrary, (3) holds.

To prove (4), consider $v \in B$. We may assume $a_v = a_v^- < \lfloor d(v)/2 \rfloor$ or the statement is automatic. Suppose for a contradiction that the statement fails—so $2b_v < d_A(v)$, i.e. $d_A(v) \geqslant 2a_v + 3$, and thus $d_A(v) \geqslant a_v^+$ by (2). There are $d_A(v) - b_v$ edges from v to A that are not in H—in particular there is a $w \in N(v) \cap A$, $vw \notin H$. As noted above, we can ensure that $d_H(w) < a_w$. This will not change the fact that $vw \notin H$. Setting $a_v = a_v^+$ and adding $a_v^+ - d_H(v)$ edges from v to A into H (including the edge vw), we decrease the deficiency. □

Corollary 6. *Given $G = (V, E)$ and, for all $v \in V$, a list $D_v \subseteq \{0, 1, \ldots, d(v)\}$ satisfying $|D_v| > \lceil 7d(v)/8 \rceil$, there exists a spanning subgraph $H \subseteq G$ so that for all v, $d_H(v) \in D_v$.*

Proof. It is easy to see that for all v there exists $a_v \in [\lfloor d(v)/4 \rfloor, \lfloor d(v)/2 \rfloor]$ such that $\{a_v, a_v+1, a_v+\lfloor d(v)/4 \rfloor+1, a_v+\lfloor d(v)/4 \rfloor+2\} \subset D_v$. Note that for such a_v, $a_v + \lfloor d(v)/4 \rfloor + 1 = a_v/2 + (a_v/2 + \lfloor d(v)/4 \rfloor) + 1 \leqslant (d(v) + a_v)/2 + 1$. Further, $a_v + \lfloor d(v)/4 \rfloor + 1 \leqslant a_v + a_v + 1 < 2a_v + 3$. Thus, setting $a_v^- = a_v$ and $a_v^+ = a_v + \lfloor d(v)/4 \rfloor + 1$, these choices satisfy (2) and $a_v^+ \geqslant \lfloor d(v)/2 \rfloor \geqslant a_v^-$. Thus the subgraph H guaranteed by Theorem 5 satisfies the requirements of this corollary. □

Theorem 7. *Given a bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$. For $v \in X$ let $a_v^- = \lfloor d(v)/2 \rfloor$ and set $a_v^+ = a_v^- + 1$. For $v \in Y$, choose a_v^-, a_v^+ such that $a_v^- \leqslant \lfloor d(v)/2 \rfloor \leqslant a_v^+$ and*

$$a_v^+ \leqslant \min\left(\frac{d(v) + a_v^-}{2} + 1, 2a_v^- + 1\right). \tag{5}$$

Then there is a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^+\}$ for all $v \in V$.

Proof. As in the proof of Theorem 5, for a given set of choices of the a_v^- and a_v^+, suppose such a subgraph does not exist. Choose $b_v = a_v \in \{a_v^-, a_v^+\}$ for all $v \in Y$ and a subgraph H to minimize the deficiency. Let A_0, A, B be defined as in Theorem 5—it is not hard to see using the bipartiteness condition that A and B are indeed disjoint. All the results on which edges are and are not in H from Theorem 5 clearly hold in this setting. Let $A_X = A \cap X$ and define A_Y, B_X, and B_Y similarly. Also as above, for $v \in A_X$, $d_H(v) = a_v^-$ and for $v \in B_X$, $d_H(v) = a_v^+$. It must be the case that either

$$\sum_{v \in A_X} (a_v^- - d_{G-B_Y}(v)) - \sum_{v \in B_Y} a_v > 0, \tag{6}$$

or

$$\sum_{v \in A_Y} (a_v - d_{G-B_X}(v)) - \sum_{v \in B_X} a_v^+ > 0, \tag{7}$$

or else, since there are no edges from A_X to B_X or from A_Y to B_Y, the negations of these two equations give us that the deficiency is in fact zero. We now show that in fact neither of these equations hold, proving the theorem by contradiction. The proof parallels that of Theorem 5. Let $v \in A_X$. By the definition of a_v^-, $a_v^- - d_{G-B_Y}(v) \leqslant \lfloor d_{B_Y}(v)/2 \rfloor$. We claim that for $v \in B_Y$, $a_v \geqslant d_{A_X}(v)/2$, which completes the proof that (6) does not hold. This is clear if $a_v = a_v^+$, so we may assume $a_v = a_v^-$. Assume for a contradiction that $2a_v < d_{A_X}(v)$—then as in the proof of Theorem 5 we may set $a_v = a_v^+$ and add some edges from v to A_X into H to reduce the deficiency, contradicting its minimality. A similar proof shows (7) does not hold. □

2.1. A theorem of Sebö

As mentioned in the introduction, a result of Sebö [13] implies the special case of Louigi's Conjecture where we require that for all v, $\{0, 1, \ldots, d(v)\} - D_v$ contains no two consecutive integers; in this case we say the sets $\{D_v : v \in V\}$ are *dense*. In order to state Sebö's result, the following definitions are required.

We say that a set S is *odd* (*even*) if it consists of only odd (even) integers, and that vertex v is odd (even) if D_v is odd (even). If S or v is odd or even, it has *fixed parity*. Denote by V_0^{o} (V_0^{e}) the set of odd (even) vertices of G, and suppose $\{w_1, \ldots, w_k\}$ is an ordering of the vertices in $V \backslash (V_0^{\mathrm{o}} \cup V_0^{\mathrm{e}})$.

Given the sets V_{i-1}^{o} and V_{i-1}^{e}, let $\mathbf{C_i}$ be the set of components C of $G - w_i$ such that $C \subseteq V_{i-1}^{\mathrm{o}} \cup V_{i-1}^{\mathrm{e}}$. For such a component, define $d(w_i, C)$ as the number of edges joining w_i to some vertex of C.

Let ℓ_i be the number of components $C \in \mathbf{C_i}$ for which $|C \cap V_{i-1}^{\mathrm{o}}|$ is odd. Let t_i be the number of $C \in \mathbf{C_i}$ for which $|C \cap V_{i-1}^{\mathrm{o}}|$ has a different parity from $d(w_i, C)$, and set $u_i = d(w_i) - t_i$. If $[\ell_i, u_i] \cap D_{w_i}$ does not have fixed parity, V_i^{o} and V_i^{e} are undefined. If $[\ell_i, u_i] \cap D_{w_i}$ is odd, let $V_i^{\mathrm{o}} = V_{i-1}^{\mathrm{o}} \cup \{w_i\}$ and let $V_i^{\mathrm{e}} = V_{i-1}^{\mathrm{e}}$; if it is even, let $V_i^{\mathrm{o}} = V_{i-1}^{\mathrm{o}}$ and let $V_i^{\mathrm{e}} = V_{i-1}^{\mathrm{e}} \cup \{w_i\}$. (If $[\ell_i, u_i] \cap D_{w_i}$ is empty it may be viewed as either even or odd.)

We say $\{w_1, \ldots, w_k\}$ is a *parity trace* if the sets V_k^{e} and V_k^{o} are defined. We have:

Theorem 8. *Let $G = (V, E)$ be a graph and let the lists $\{D_v : v \in V\}$ be dense. Then there exists a spanning subgraph H of G so that $d_H(v) \in D_v$ for all v if and only if there is no parity trace $\{w_1, \ldots, w_k\}$ with $|V_k^{\mathrm{o}}|$ odd.*

Unfortunately, the simplest examples which give an intuition for this theorem are already nontrivial; as we are only using this theorem as a technical result we refer the interested reader to [13] for more details. On the bright side, it is now not difficult to prove the following:

Theorem 9. *Let G be a graph and let the lists $\{D_v : v \in V\}$ be dense and satisfy $|D_v| > \lceil d(v)/2 \rceil$ for all v. Then there exists a spanning subgraph H* of *G so that $d_H(v) \in D_v$ for all v.*

Proof. Suppose $w_1, \ldots, w_k$ is a parity trace. We shall show by induction that for all i, V_i^{o} is empty, and all vertices in V_i^{e} have even degree.

By the condition that $|D_v| > \lceil (d(v)/2)) \rceil$, there are no odd vertices and if a vertex v is even, $d(v)$ is also even. Thus $V_0^{\mathrm{o}} = \emptyset$ and V_0^{e} only contains vertices with even degree, satisfying the base case for the induction.

Let $i > 0$, and assume that our hypotheses are true for all $i' < i$. Since $V_{i-1}^{\mathrm{o}} = \emptyset$, $l_i = 0$ by definition, and t_i is the number of components $C \subseteq V_{i-1}^{\mathrm{e}}$ for which $d(w_i, C)$ has a different parity from $|C \cap V_{i-1}^{\mathrm{o}}| = 0$, i.e., for which $d(w_i, C)$ is odd. For such a C, for all $v \in C$, $d(v)$ is even, so $\sum_{v \in C} d_C(v) = \left(\sum_{v \in C} d(v)\right) - d(w_i, C)$ is odd, which is not possible. Therefore, $t_i = 0$ and $u_i = d(w_i)$.

Since $\{w_1, \dots, w_k\}$ is a parity trace, $D_{w_i} \cap [l_i, u_i]$ has fixed parity which implies that w_i is actually even. Thus $V_i^{\mathrm{e}} = V_{i-1}^{\mathrm{e}} \cup \{w_i\}$ and thus only contains even vertices and $V_i^{\mathrm{o}} = V_{i-1}^{\mathrm{o}} = \emptyset$, thereby completing the induction. □

3. Proof of Theorems 1 and 2

We will need the following technical lemma whose proof is an easy modification of the proof of Theorem 1 from [9].

Lemma 10. *Given a connected, non-bipartite graph G, a set of target colours t_v for all $v \in V(G)$, and an integer k, where k is odd or $\sum_{v \in V} t_v$ is even, there exists a k-edge-weighting of G such that for all $v \in V(G)$, $\sum_{e \ni v} w(e) \equiv t_v (\mathrm{mod}\ k)$.*

We now proceed to:

Theorem 1. *Every nice graph permits a vertex-colouring* 16*-edge-weighting.*

Proof. Without loss of generality, assume that G is connected and nonbipartite. (If G is bipartite then by Theorem 1 of [9], there exists a vertex-colouring 3-edge-weighting.)

For any ordering of a set of vertices, let $F(v_i) = \{v_j | v_j \in N(v_i) \text{ and } j > i\}$ and call this set the *forward* neighbours of v_i. Define $B(v_i)$ and the *backward* neighbours of v_i similarly. Choose an ordering of $V(G)$ that maximizes $k = \max\{j : \forall i \leqslant j, |F(v_i)| > |B(v_i)|\}$. Place the first k vertices into V_1 and the remainder into a temporary set T. Note that k does not decrease if T is re-ordered. Also observe that for all $v \in T$, $d_T(v) \leqslant d_{V_1}(v)$. (Otherwise, we could move v to the $(k+1)$st position of the ordering and thereby create an ordering with a larger value of k.)

Next, place all bipartite components of the graph induced by T into a set L and then apply the prefix finding procedure to $T - L$ to generate V_2, then V_3, then V_4, and let V_5 be the remaining vertices. Note that each vertex in L (which may be empty) only has edges to vertices in L and V_1. Also, observe that each component of the graph induced by $V_2 \cup V_3 \cup V_4 \cup V_5$ must have at least one vertex in V_2 (since singleton components are bipartite) and that we can order the vertices in V_2, V_3, V_4, and V_5 such that each $v \in V_i$ has strictly fewer backward neighbours in V_i than forward neighbours. In addition, for all $v \in V_5$, by three applications of the observation at the end of the previous paragraph, we have that $|N(v) \cap V_1| \geqslant 8|N(v) \cap V_5|$.

Consider the edges from V_5 to V_1. Since every vertex v in V_5 has at least $8d_{V_5}(v)$ edges to V_1, we can choose a subset where each $v \in V_5$ has exactly $8d_{V_5}(v)$ edges to V_1. Let B be the bipartite graph spanned by this reduced set of edges. If $v \in V_1$ has an even (resp. odd) number of edges in B, then place v into the set $V_{1\mathrm{e}}$ (resp. $V_{1\mathrm{o}}$). Also, partition L into two sets L_a and L_b based on a 2-colouring of L.

We will weight the edges so that the colour of each vertex has an arity mod 8 as specified in Table 2. The arities of the vertices ensure that there will be no cross partition conflicts because vertices in L have no neighbours in V_2.

To begin, we assign weights between 1 and 8 to the edges within $V_1 \cup L$ so that every vertex that has no neighbours outside $V_1 \cup L$ has the arity mod 8 specified in Table 1. We can do so by applying Lemma 10 to $E(G)$ and discarding the weights of edges outside of $V_1 \cup L$. (Note that if $V_2 \cup \cdots \cup V_5$ is empty, the conditions of Lemma 10 may not hold. In this case it is easy to construct a vertex-colouring 6-edge-weighting for G such that vertices in V_1, L_1, and L_2 receive distinct arities mod 3; the details are left to the interested reader.)

We will assign edge weights to the unweighted edges and modify some weighted edges to achieve the target arities from Table 2 and to ensure that there are no internal conflicts. The target arity choices and edge weighting steps are necessarily intermingled.

Process the vertices of V_1 in order. For each vertex v_i with current weighted degree w_{v_i}, if v_j is a backward neighbour of v_i, we say v_j *blocks* the range $[w_{v_j} - 2, w_{v_j} + 2]$. By giving weights to v_i's forward edges which are not yet weighted, and modifying the weights on some of v_i's remaining forward edges, we wish to change w_{v_i} to a new value which is not blocked and give it the right arity as specified in Table 1. Note that if v_i has d backward neighbours, it has at least $d+1$ forward edges. We allow forward edges of v_i to $V(G) - V_1 - L$ to take weights in the range [3, 14]. In addition, we allow ourselves to add 8 to an arbitrary subset of the forward edges to $V_1 \cup L$. By making such changes, there are at least $d+1$ distinct values with the right arity available to v_i. Choose one that is not blocked by any backward neighbour.

Table 1
Initial arity choices

$V_{1\mathrm{e}}$	$V_{1\mathrm{o}}$	L_a	L_b
0	2	1	2

Table 2
Target arity for partition elements

V_1	V_2	L_a	L_b	V_3	V_4	V_5
0 or 4	1 or 2	1	2	5	6	3 or 7

Remark. It might seem more natural to use either the discarded weight $w(e)$ or $w(e) + 8$ on an edge from V_1 to $V(G) - V_1 - L$, however, later we will need the fact that the edge weights in this set lie between 3 and 14. We note that our more complicated approach relies on the property that each $v_i \in V_1$ has strictly more forward neighbours than backward neighbours.

After processing the vertices of V_1, the weighted degrees of all vertices in V_1 and L have the arities specified in Table 1. Consider the subgraph induced by $V(G) - V_1 - L$ which, by construction, is simply a collection of non-bipartite components. We choose new target arities for the vertices in $V(G) - V_1 - L$ based on the arity difference between the sum of the edges from V_1 and the target arity from Table 2. We satisfy the requirements of Lemma 10 as each component has at least one vertex in V_2 which has both an even and odd choice for target arity. We then apply Lemma 10 to weight the edges of the graph induced by $V_2 \cup V_3 \cup V_4 \cup V_5$ to achieve the target arities. All edges of G are now weighted.

Process the vertices of V_2, V_3, V_4 in order. Distinguish $v \in V_i$ from previously processed neighbours $w \in V_i$ by adding 8 to a subset of v's forward edges. In our final step, we adjust the weight of edges in B to distinguish adjacent vertices in V_5 and ensure that the colour of all vertices in V_1 is either 0 or 4 mod 8, whilst preventing any new conflicts in V_1. We do this by using Theorem 7 where $X = V_1 \cap V(B)$ and $Y = V_5 \cap V(B)$ to determine a subgraph H. For each edge $e \in E(H)$, we will add 2 to its weight, and for each $e \notin E(H)$, we will subtract 2.

First, choose $\{a_v^-, a_v^+\}$ for each vertex in X as follows. For each $v \in X$, we choose $a_v^- = \lfloor d_B(v)/2 \rfloor$ and set $a_v^+ = a_v^- + 1$. Then, choose $\{a_v^-, a_v^+\}$ for each vertex in Y as follows. Process the vertices of Y in any order. For each $v \in Y$ in turn, we choose $a_v^- \in [d_B(v)/4, d_B(v)/2]$ (recall that 8 divides $d_B(v)$, so this range has integer endpoints), and set $a_v^+ = a_v^- + d_B(v)/4 + 1$. We make our choice to ensure that for any previously processed neighbour $u \in V_5$, for any $a_v \in \{a_v^-, a_v^+\}$, and for any $a_u \in \{a_u^-, a_u^+\}$, $w_v + 2a_v - 2(d_B(v) - a_v) \neq w_u + 2a_u - 2(d_B(u) - a_u)$. This is possible since each previously processed neighbour can prevent at most two choices for a_v^- and there are precisely $2d_{V_5}(v) + 1$ choices.

Next, we show that this set of degree choices satisfies the conditions of Theorem 7. The degree choices for X exactly match the theorem. Also, it is clear that for all $v \in Y$, $a_v^- \leqslant d_B(v)/2 \leqslant a_v^+$, so it only remains to show that for all $v \in Y$, (5) holds. Since $a_v^- \leqslant d_B(v)/2$, $a_v^+ = a_v^- + d_B(v)/4 + 1 = d_B(v)/4 + a_v^-/2 + a_v^-/2 + 1 \leqslant d_B(v)/2 + a_v^-/2 + 1$. Also, since $a_v^- \geqslant d_B(v)/4$, $a_v^+ = a_v^- + d_B(v)/4 + 1 \leqslant 2a_v^- + 1$. Thus, by Theorem 7, a subgraph H of B exists such that after performing the additions/subtractions described in the previous paragraph, all adjacent vertices in V_5 have different weights.

The weighted degrees of vertices in $V_{1\mathrm{e}}$ either stay the same or increase by 4, and thus are now either 0 or 4 mod 8. No conflicts exist within $V_{1\mathrm{e}}$ because adjacent vertices' weighted degrees were initially at least 8 apart. Similarly, the weighted degrees of vertices in $V_{1\mathrm{o}}$ are now either 0 or 4 mod 8, and there are no conflicts within $V_{1\mathrm{o}}$. Let $uv \in E(G)$ with $u \in V_{1\mathrm{e}}$ and $v \in V_{1\mathrm{o}}$. Prior to the final step, w_u and w_v were at least 3 apart. This implies, by a simple arity argument, that either w_u was at least 6 greater than w_v or w_u was at least 10 less than w_v. Since w_u can only increase by 4 and w_v can only change by two, no conflict is possible inside V_1.

Furthermore, the weighted degrees of all vertices in V_5 are either 3 or 7 mod 8 because these vertices have even degree in B. Thus, we have achieved the target arities from Table 2. It is easy to verify that all edges end up with a weight in the range of [1, 16] to complete the proof. $\square$

Theorem 2. *Let G be a random graph chosen from $G_{n,p}$ for a constant $p \in (0, 1)$. Then, asymptotically almost surely, there exists a vertex-colouring 2-edge-weighting for G. In fact, there exists a 2-edge-weighting such that the colours of two adjacent vertices are distinct mod $2\chi(G)$.*

Proof. Let G be a random graph with probability distribution $G_{n,p}$. Fix $\varepsilon > 0$. We have the following facts (see, e.g., [6, Chapter 11]):

- asymptotically almost surely $\min_v d(v) > (p - \varepsilon)n$;
- asymptotically almost surely $\chi(G) < (\log(1/(1 - p))/(2 - \varepsilon))/n/\log n$.

It follows from these two facts that asymptotically almost surely $2\chi(G) < \min_v d(v)/6$. Assuming this inequality holds, we construct a vertex-colouring 2-edge-weighting for G.

Let $\{V_1, \ldots, V_{\chi(G)}\}$ be a partition of $V(G)$ into stable sets. For each $v \in V_i$, choose $a_v^- \in [\lfloor d(v)/3\rfloor, \lfloor d(v)/2\rfloor]$, $a_v^+ \in [\lfloor d(v)/2\rfloor, \lfloor 2d(v)/3\rfloor]$ such that $a_v^- + d_G(v) \equiv a_v^+ + d_G(v) \equiv 2i \bmod 2\chi(G)$. Such choices for a_v^- and a_v^+ exist as the interval $[\lfloor d(v)/3\rfloor, \lfloor d(v)/2\rfloor]$ contains at least $2\chi(G)$ consecutive integers, as does $[\lfloor d(v)/2\rfloor, \lfloor 2d(v)/3\rfloor]$.

Furthermore, such choices of a_v^-, a_v^+ satisfy the conditions of Theorem 5, so there is an H such that for all v, $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$. Set $w(e) = 2$ for $e \in E(H)$ and $w(e) = 1$ for $e \in E(G) - E(H)$. If $v \in V_i$, we have

$$\sum_{e \ni v} w(e) = 2d_H(v) + d_{G-H}(v) = d_G(v) + d_H(v) \in \{2i, 2i + 1\} \bmod (2\chi(G)).$$

Thus adjacent vertices in different parts of $\{V_1, \ldots, V_{\chi(G)}\}$ have different arities. As each V_i is a stable set, these weights form a vertex-colouring 2-edge-weighting of G. □

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