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Note

1, 2 Conjecture—the multiplicative version

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Abstract

Let us assign positive integers to the edges and vertices of a simple graph G. We consider the colouring of G obtained by assigning to vertex v the product of its weight and those of its adjacent edges. Can we obtain a proper colouring using only weights 1 and 2 for an arbitrary graph G?

We give a positive answer when G is a 3-colourable or complete. We also show that it is enough to use weights 1, 2 and 3 for an arbitrary graph G.

Keywords: Graph algorithms; Total weighting; Vertex colouring

A k-total-weighting of a simple graph G is assignment of an integer weight, $weight(e), weight(v) \in \{1, \dots, k\}$ to each edge e and each vertex v of G. A k-total-weighting is a multiplicative vertex-colouring if for every edge uv,

$$weight(u) \cdot \prod_{e \ni u} weight(e) \neq weight(v) \cdot \prod_{e \ni v} weight(e).$$

If such a colouring exists, we say that G permits a multiplicative vertex-colouring k-total-weighting.

A similar problem, namely, sum vertex-colouring k-total-weighting, where instead of products the sums are considered, is introduced by J. Przybyło and M. Woźniak in [5]. They show that every simple graph G permits a sum vertex-colouring 11-total-weighting and prove that every complete or 3-colourable graph permits a sum vertex-colouring 2-total-weighting.

The study of sum vertex-colouring edge-weightings (not total) was initiated by Karoński, Łuczak and Thomason (see [4]). They conjectured that every connected, non-trivial (with at least three vertices) graph permits a sum vertex-colouring 3-edge-weighting (we call this the 1, 2, 3 Conjecture) and proved this conjecture for 3-colourable graphs. The sum vertex-colouring k-edge weightings were investigated by many authors in [1–5]. The best result is given by Addario-Berry, Dalal and Reed (see [2]) which proved that every connected, non-trivial graph G permits a sum vertex-colouring 16-edge-weighting.

In a multiplicative version of vertex-colouring edge-weighting (not total), instead of sums, the vertices are coloured by the products of the incident edge weights. It can be deduced from [1] that every, non-trivial graph G permits multiplicative vertex-colouring 5-edge-weighting and the multiplicative 1, 2, 3 Conjecture holds in the case the graph G is complete or 3-colourable.

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Note that if a graph permits a (sum or product) vertex-colouring k-edge-weighting, then it also permits a (sum or product) vertex-colouring k-total-weighting (it is enough to put ones at all vertices). If the $1, 2, 3$ Conjecture can be true (in sum or multiplicative version), then it is possible that the weights 1 and 2 are enough in the case of a total-weighting. In this paper only a multiplicative version of the $1, 2$ Conjecture is considered.

Conjecture 1. *Every simple graph permits multiplicative vertex-colouring* 2*-total-weighting.*

Proposition 2. *Every complete graph permits multiplicative vertex-colouring* 2*-total-weighting.*

Proof. We will assign to all vertices and all edges of K_n the weights 1 and 2 in such a way that for all vertices v the product colours

$$p(v) = weight(v) \cdot \prod_{e \ni v} weight(e)$$

are distinct and take the values from the set $\{2, 2^2, 2^3, \ldots, 2^n\}$. Put the vertices $v_1, v_2, \ldots, v_n$ of K_n on n-polygon (in the clockwise ordering). In the first step assign to v_1 the weight 1, the edge $v_1 v_n$ we weight with 2 and all remaining edges incident with v_1 we weight with 1. In the second step put on v_2 the weight 1, the two edges $v_2 v_{n-1}$ and $v_2 v_n$ weight with 2 and all remaining edges incident to v_2 weight with 1. In the third step assign to v_3 the number 1, the three edges $v_3 v_{n-2}$, $v_3 v_{n-1}$ and $v_3 v_n$ weight with 2 and all remaining edges incident to v_3 weight with number 1. We finish this procedure at the vertex v_k, $k = \lfloor n/2 \rfloor$, to which we assign 1, all k edges $v_k v_{n-(k-1)}$, $v_k v_{n-(k-2)}, \ldots, v_k v_{n-1}$, $v_k v_n$ we weight with 2 and all remaining edges incident to v_k we weight with 1. We get that the vertex v_i has the product $p(v_i) = 2^i$ for all $i \leqslant \lfloor n/2 \rfloor$. Next, all the remaining vertices v_j where $\lfloor n/2 \rfloor < j \leqslant n$ and all not weighted edges we must weight with the number 2 to achieve the product $p(v_j) = 2^j$, $1 \leqslant j \leqslant n$. □

Proposition 3. *Every* 3*-colourable, simple graph G permits multiplicative vertex-colouring* 2*-total-weighting.*

Proof. Notice, that every bipartite graph permits multiplicative vertex-colouring 2-total-weighting, so, without lost for generality we can assume, that G is not bipartite. If G is a 3-colourable graph coloured properly with colours c_1, c_2, c_3, then the set $V(G)$ of all vertices is partitioned into three disjoint subsets V_1, V_2, V_3 such that every vertex in V_i has a colour c_i, $1 \leqslant i \leqslant 3$. Of course, every two vertices in a given set V_i are not adjacent. Now, if a vertex $u \in V_2$ has no neighbour in V_3 then we colour u with c_3 and move it from V_2 to V_3. In the consequence, we get the perfect partition, in which every vertex u of V_2 has at least one neighbour in V_3. Now, let V_1, V_2, V_3 be such perfect partition. Put the weight 1 on all vertices in V_1 and on all edges with the end vertex in V_1. Then for every $v \in V_1$ the product colour $p(v) = 1$.

Next, weight every edge between the sets V_2 and V_3 with the number 2. Let us assign to every vertex in V_2 and V_3 the appropriate weight 1 or 2 in order to obtain an even power of 2 in the product colour $p(u)$, if $u \in V_2$ and an odd power of 2, if $u \in V_3$.

In the final, the adjacent vertices have distinct colour products. □

To prove Theorem 5, we use the following Lemma 4 which was proved in [1]. In the statement of this lemma, addition on the indices $i \in \{0, 1, 2\}$ is considered modulo 3.

Lemma 4. *The vertices of every connected graph which is not* 3*-colourable can be partitioned into* 3 *sets V_0, V_1, V_2 such that*

(1) $\forall v \in V_i$, $|N(v) \cap V_{i+1}| \geqslant |N(v) \cap V_i|$ *and*
(2) *every vertex in V_i has a neighbour in V_{i+1}.*

Theorem 5. *Every simple graph G permits* 3*-total-weighting which is multiplicative vertex-colouring of G.*

Proof. Assume, that G is not 3-colourable. We can partition the set $V(G)$ of vertices into three sets $V_0 = A$, $V_1 = B$, $V_2 = C$ satisfying the conditions of the Lemma 4. We assign to edges within A weights 2 and to edges within B and C weights 3.

The edges between A and B will be weighted with numbers 2 or 1. The edges between B and C will be weighted with numbers 3 or 1. The edges between C and A will be weighted with numbers 3 or 1.

If $v \in A$ has no internal neighbours (i.e. $N(v) \cap A = \emptyset$) then all edges between v and vertices in B we weight with the number 2 (Lemma 4 ensures that there is at least one such edge). If $v \in B$ has no internal neighbours (i.e. $N(v) \cap B = \emptyset$) then all edges between v and vertices in C we weight with the number 3. If $v \in C$ has no internal neighbours (i.e. $N(v) \cap C = \emptyset$) then all edges between v and vertices in A we weight with the number 3.

If the vertex $v \in A$ has an internal neighbour belonging to A we assign to v some integer $t(v)$, where

$$\left|N(v) \cap A\right| \leqslant t(v) \leqslant 2\left|N(v) \cap A\right|$$

such that $t(v)$ is distinct from $t(u)$ for all $u \in N(v) \cap A$ for which we have already chosen $t(u)$.

According to Lemma 4 there are at least $|N(v) \cap A|$ edges between $v \in A$ and the set B. Weight $t(v) - |N(v) \cap A|$ of these edges with number 2 and the rest of edges weight with number 1.

Similarly, if the vertex $w \in B$ has an internal neighbour belonging to B we assign to w some integer $t(w)$, where

$$\left|N(w) \cap B\right| \leqslant t(w) \leqslant 2\left|N(w) \cap B\right|$$

such that $t(w)$ is distinct from $t(u)$ for all $u \in N(w) \cap B$ for which we have already chosen $t(u)$. There are at least $|N(w) \cap B|$ edges from $w \in B$ to C. Weight $t(w) - |N(w) \cap B|$ of these edges with number 3 and the rest of edges weight with number 1. The product colour $p(v)$ will finally contain only the factors equal to 2 or 3. Assign to every vertex $v \in A$ the weight 1 or 3 to obtain an even number of factors 2, 3 in the product $p(v)$ and to every vertex $w \in B$ the weight 1 or 2 to obtain an odd number of factors in $p(w)$.

Let $d_B(v)$ be the number of edges weighted with 3 between B and $v \in C$. If the vertex $v \in C$ has an internal neighbour in C we assign to v some integer $s(v)$, where

$$\left|N(v) \cap C\right| + d_B(v) \leqslant s(v) \leqslant d_B(v) + 2\left|N(v) \cap C\right| + 1$$

such that $s(v) \neq s(u)$ for all $u \in N(v) \cap C$ for which we have already chosen $s(u)$. There are at least $|N(v) \cap C|$ edges from $v \in C$ to A. Put exactly $s(v) - |N(v) \cap A| - d_B(v)$ weights equal to 3 on these edges and on the vertex v if necessary. The rest of edges between v and A weight with 1. Then the exponent in the power of 3 in the product $p(v)$, $v \in C$, is exactly equal to $s(v)$.

It can be occur $p(v_0) = p(u_0) = 3^k$ for some vertex $v_0 \in C$ adjacent to $u_0 \in B$ and for some odd integer k. In this case we have to change the weight of u_0 from 1 into 2. In the case u_0 has no neighbour in A with the same product, the proof is finished.

In the other case, there exists the neighbour $w_0 \in A$ of u_0 such that $p(u_0) = p(w_0) = 2 \cdot 3^k$. It means that w_0 has exactly one internal neighbour $x \in A$. Let us change the weight of w_0u_0 edge from 1 into 2. In the case there is exactly one edge between w_0 and B or $weight(w_0) = 1$ we change the weight of w_0 from 1 or 3 into 3 or 2 or 1 to achieve $p(w_0) \neq p(x)$ and to keep the even number of factors in $p(w_0)$ if necessary. If $weight(w_0) = 3$ and w_0 has at least two neighbours in B then we have two possibilities. In the first we change the weight of w_0 from 3 into 1. In the second possibility we change the weight of w_0 from 3 into 2 and weight also some edge from w_0 to B with new weight 2 (instead 1). Both these possibilities make the number of the factors in $p(w_0)$ even, just like we want, but we must choose this one, in which $p(w_0) \neq p(x)$ is also satisfied. After all such changes some vertex w in B can have wrong (not odd) number of factors in its product colour $p(w)$. In this case we fix the colour of w by changing its weight from 1 to 2 or vice-versa. This completes the proof. □

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