Every graph is $(2,3)$-choosable

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Abstract

A total weighting of a graph G is a mapping ϕ that assigns to each element $z \in V(G) \cup E(G)$ a weight $\phi(z)$. A total weighting ϕ is proper if for any two adjacent vertices u and v, $\sum_{e \in E(u)} \phi(e) + \phi(u) \neq \sum_{e \in E(v)} \phi(e) + \phi(v)$. This paper proves that if each edge e is given a set $L(e)$ of 3 permissible weights, and each vertex v is given a set $L(v)$ of 2 permissible weights, then G has a proper total weighting ϕ with $\phi(z) \in L(z)$ for each element $z \in V(G) \cup E(G)$.

Key words: Total weighting, edge weighting, (k, k')-choosable, permanent

1 Introduction

A *total weighting* of a graph G is a mapping $\phi : V(G) \cup E(G) \to R$. A total weighting ϕ is *proper* if for any edge uv of G,

$$\sum_{e \in E(u)} \phi(e) + \phi(u) \neq \sum_{e \in E(v)} \phi(e) + \phi(v),$$

where $E(v)$ is the set of edges incident to v. A total weighting ϕ with $\phi(v) = 0$ for all vertices v is also called an *edge weighting*.

Karonski, Luczak and Thomason [7] first studied edge weighting of graphs. If G has an isolated edge, then it is obvious G has no proper edge weighting. On the other hand,

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if G has no isolated edges, then G has a proper k-edge weighting for some k, i.e., a proper edge weighting ϕ with $\phi(e) \in \{1, 2, \ldots, k\}$ for all edges e. The problem is to determine the smallest such integer k. There are many graphs (for example, odd cycles) that have no proper 2-edge weighting. However, we do not know any graph without isolated edges which has no proper 3-edge weighting. Karonski, Luczak and Thomason [7] conjectured that $k = 3$ is enough for every graph with no isolated edges. This conjecture received considerable attention, and is called the 1-2-3 conjecture. A constant bound $k = 30$ was proved by Addario-Berry, Dalal, McDiarmid, Reed and Thomason in 2007 [2]. This bound was improved to $k = 16$ by Addario-Berry, Dalal and Reed in [1] and to $k = 13$ by Wang and Yu in [12]. A breakthrough on this conjecture was obtained by Kalkowski, Karosnki and Pfender in 2010 [9], where the bound is reduced to $k = 5$.

Total weighting of graphs was first studied by Przybyło and Woźniak in [10], where they defined $\tau(G)$ to be the least integer k such that G has a proper total weighting ϕ with $\phi(z) \in \{1, 2, \ldots, k\}$ for $z \in V(G) \cup E(G)$. They proved that $\tau(G) \leq 11$ for all graphs G, and conjectured that $\tau(G) = 2$ for all graphs G. This conjecture is called the 1-2 conjecture. A breakthrough on 1-2 conjecture was obtained by Kalkowski in [8], where it was proved that every graph G has a proper total weighting ϕ with $\phi(v) \in \{1, 2\}$ for $v \in V(G)$ and $\phi(e) \in \{1, 2, 3\}$ for $e \in E(G)$.

The list version of edge weighting of graphs was introduced by Bartnicki, Grytczuk and Niwczyk in [6], and the list version of total weighting of graphs was introduced independently by Wong and Zhu in [15] and by Przybyło and Woźniak [11]. Suppose $\psi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \}$ is a mapping which assigns to each vertex and each edge of G a positive integer. A ψ-list assignment of G is a mapping L which assigns to $z \in V(G) \cup E(G)$ a set $L(z)$ of $\psi(z)$ real numbers. Given a total list assignment L, a proper L-total weighting is a proper total weighting ϕ with $\phi(z) \in L(z)$ for all $z \in V(G) \cup E(G)$. We say G is *total weight ψ-choosable* if for any ψ-list assignment L, there is a proper L-total weighting of G. We say G is (k, k')-choosable if G is ψ-total weight choosable, where $\psi(v) = k$ for $v \in V(G)$ and $\psi(e) = k'$ for $e \in E(G)$.

As strengthening of the 1-2-3 conjecture and the 1-2 conjecture, it was conjectured in [15] that every graph with no isolated edges is $(1, 3)$-choosable and every graph is $(2, 2)$-choosable. Some special graphs are shown to be $(1, 3)$-choosable, such as complete graphs, complete bipartite graphs, trees [6], Cartesian product of an even number of even cycles, of a path and an even cycle, of two paths [14]. Some special graphs are shown to be $(2, 2)$-choosable, such as complete graphs, generalized theta graphs, trees [15], subcubic graphs, Halin graphs [16], complete bipartite graphs [13]. However, before this paper, it was unknown if there are constants k, k' such that every graph is (k, k')-choosable. The existence of such constants is proposed as a conjecture in [15]. In this paper, we prove that every graph is $(2, 3)$-choosable.

2 Proof of the main result

The proof of the main result uses polynomial method. For each $z \in V(G) \cup E(G)$, let x_z be a variable associated to z. Fix an arbitrary orientation D of G. Consider the polynomial

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e=uv \in E(D)} \left(\left(\sum_{e \in E(u)} x_e + x_u \right) - \left(\sum_{e \in E(v)} x_e + x_v \right) \right).$$

Assign a real number $\phi(z)$ to the variable x_z, and view $\phi(z)$ as the weight of z. Let $P_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$. Then ϕ is a proper total weighting of G if and only if $P_G(\phi) \neq 0$. The question is under what condition one can find an assignment ϕ for which $P_G(\phi) \neq 0$.

An *index function* of G is a mapping η which assigns to each vertex or edge z of G a non-negative integer $\eta(z)$. An index function η of G is *valid* if $\sum_{z \in V \cup E} \eta(z) = |E|$. Note that $|E|$ is the degree of the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$. For a valid index function η, let c_η be the coefficient of the monomial $\prod_{z \in V \cup E} x_z^{\eta(z)}$ in the expansion of P_G. It follows from the Combinatorial Nullstellensatz [3, 5] that if $c_\eta \neq 0$, and L is a list assignment which assigns to each $z \in V(G) \cup E(G)$ a set $L(z)$ of $\eta(z) + 1$ real numbers, then there exists a mapping ϕ with $\phi(z) \in L(z)$ such that

$$P_G(\phi) \neq 0.$$

An index function η of G is called *non-singular* if there is a valid index function $\eta' \leq \eta$ (i.e., $\eta'(z) \leq \eta(z)$ for all $z \in V(G) \cup E(G)$) such that $c_{\eta'} \neq 0$. The following is the main result of this paper.

Theorem 1 *Every graph G has a non-singular index function η with $\eta(v) \leq 1$ for $v \in V(G)$ and $\eta(e) \leq 2$ for $e \in E(G)$.*

As observed above, this implies that every graph G is $(2, 3)$-choosable.

We write the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$ as

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e \in E(D)} \sum_{z \in V(G) \cup E(G)} A_G[e, z] x_z.$$

It is straightforward to verify that for $e \in E(G)$ and $z \in V(G) \cup E(G)$, if $e = (u, v)$ (oriented from u to v), then

$$A_G[e, z] = \begin{cases} 1 & \text{if } z = v, \text{ or } z \neq e \text{ is an edge incident to } v, \\ -1 & \text{if } z = u, \text{ or } z \neq e \text{ is an edge incident to } u, \\ 0 & \text{otherwise.} \end{cases}$$

Now A_G is a matrix, whose rows are indexed by the edges of G and the columns are indexed by edges and vertices of G. Given a vertex or edge z of G, let $A_G(z)$ be the column of A_G indexed by z. For an index function η of G, let $A_G(\eta)$ be the matrix, each of its column is a column of A_G, and each column $A_G(z)$ of A_G occurs $\eta(z)$ times as a column of $A_G(\eta)$. It is known [4] and easy to verify that for a valid index function η of G, $c_\eta \neq 0$ if and only if $\mathrm{per}(A_G(\eta)) \neq 0$, where $\mathrm{per}(A)$ denotes the permanent of the square matrix A. Thus a valid index function η of G is non-singular if and only if $\mathrm{per}(A_G(\eta)) \neq 0$. Recall that if A is an $m \times m$ matrix, then

$$\mathrm{per}(A) = \sum_{\sigma \in S_m} A[i, \sigma(i)],$$

where S_m is the symmetric group of order m.

It is well-known (and follows easily from the definition) that the permanent of a matrix is multi-linear on its column vectors (as well as its row vectors): If a column C of A is a linear combination of two columns vectors $C = \alpha C' + \beta C''$, and A' (respectively, A'') is obtained from A by replacing the column C with C' (respectively, with C''), then

$$\mathrm{per}(A) = \alpha \mathrm{per}(A') + \beta \mathrm{per}(A''). \tag{1}$$

Assume A is a square matrix whose columns are expressed as linear combinations of columns of A_G. Define an index function $\eta_A : V(G) \cup E(G) \to \{0, 1, \ldots, \}$ as follows:

For $z \in V(G) \cup E(G)$, $\eta_A(z)$ is the number of columns of A in which $A_G(z)$ appears with nonzero coefficient.

Note that the column vectors of A_G are not linearly independent. A column of A may be written as the linear combination of columns of A_G in different ways. Thus the index function η_A is not uniquely determined by the matrix A itself, instead it is determined by how its columns are expressed as linear combinations of columns of A_G. For simplicity, we use the notation η_A. However, whenever the index function η_A is used, we refer to an explicit expression of its columns as linear combination of columns of A_G which is clear from the context.

To prove Theorem 1, it suffices to find a square matrix A whose columns are expressed as linear combinations of columns of A_G such that for each $v \in V(G)$, $\eta_A(v) \leq 1$, and for each edge e of G, $\eta_A(e) \leq 2$.

Another observation [15] we shall frequently use is that for an edge $e = uv$ of G,

$$A_G(e) = A_G(u) + A_G(v). \tag{2}$$

Now we are ready to prove Theorem 1. Indeed, we shall prove a slightly stronger result.

Theorem 2 *Assume G is a connected graph and F is a spanning tree of G. Then there is a matrix A whose columns are linear combinations of columns of A_G such that* $\mathrm{per}(A) \neq 0$ *and $\eta_A(v) \leq 1$ for $v \in V$, $\eta(e) = 0$ for $e \in E(F)$ and $\eta_A(e) \leq 2$ for $e \in E \setminus E(F)$.*

Proof. Observe that Theorem 2 is equivalent to the statement that G has a valid index function η such that $\mathrm{per}(A_G(\eta)) \neq 0$, and $\eta(v) \leq 1$ for $v \in V(G)$, $\eta(e) \leq 2$ for $e \in E(G)$, and moreover, for $e \in E(F)$, $\eta(e) = 0$.

Assume this theorem is not true, and G is a minimum counterexample. It is obvious that G is connected and $|V| \geq 3$.

Let u be a vertex of G, which is a leaf of F. Assume $N(u) = \{u_1, u_2, \dots, u_k\}$ and let $e_i = uu_i$ for $i = 1, 2, \dots, k$. Assume the edge $uv_k \in E(F)$. Let $G' = G - u$. By the minimality of G, G' has a valid index function η' such that $\mathrm{per}(A_{G'}(\eta')) \neq 0$, and $\eta'(v) \leq 1$ for $v \in V(G')$, $\eta'(e) \leq 2$ for $e \in E(G')$, and moreover, for $e \in E(F - u)$, $\eta(e) = 0$.

Assume $|E(G)| = m$ and $|E(G')| = m' = m - k$. We view η' as an index function of G, with $\eta'(z) = 0$ if $z \in (V(G) \cup E(G)) - (V(G') \cup E(G'))$. Then $A_G(\eta')$ is an $m \times m'$ matrix, consisting m' columns of A_G. Let $\eta = \eta'$ except that $\eta(u) = k$. Then $A_G(\eta)$ is an $m \times m$ matrix, which is obtained from $A_G(\eta')$ by adding k copies of the column $A_G(u)$. The added k columns has k rows (the rows indexed by edges incident to u) that are all 1's, and all the other entries of these k columns are 0. Therefore $\mathrm{per}(A_G(\eta)) = \mathrm{per}(A_{G'}(\eta'))k!$, and hence $\mathrm{per}(A_G(\eta)) \neq 0$.

Let $M_0 = A_G(\eta)$. For $i = 1, 2, \dots, k-1$, if $\eta'(u_i) = 0$, then let $M_i = M_{i-1}$. If $\eta'(u_i) = 1$, then let M_i be obtained from M_{i-1} by replacing $A_G(u_i)$ with $A_G(e_i)$.

Claim 1 *For $i = 1, 2, \dots, k$, $\mathrm{per}(M_i) = \mathrm{per}(M_{i-1})$.*

Proof. If $\eta'(u_i) = 0$, then $M_i = M_{i-1}$, there is nothing to prove. Assume $\eta'(u_i) = 1$ and M_i is obtained from M_{i-1} by replacing $A_G(u_i)$ with $A_G(e_i)$. Let M'_i be obtained from M_{i-1} by replacing $A_G(u_i)$ with $A_G(u)$. In M'_i, the column $A_G(u)$ occurs $k+1$ times. These $k+1$ columns have k rows (the rows indexed by edges incident to u) that are all 1's, and all the other entries of these k columns are 0. Therefore $\mathrm{per}(M'_i) = 0$. Since $A_G(e_i) = A_G(u_i) + A_G(u)$, by (1), we have $\mathrm{per}(M_i) = \mathrm{per}(M_{i-1}) + \mathrm{per}(M'_i) = \mathrm{per}(M_{i-1})$. ∎

Observe that $M_{k-1} = A_G(\tau)$ for an index function τ of G for which the following hold:

- $\tau(u_i) = 0$, $\tau(u) = k$, $\tau(v) \leq 1$ for other vertices v of G.
- $\tau(e_i) \leq 1$ for $i = 1, 2, \dots, k-1$, $\eta(e) = 0$ for edges in F, and $\tau(e) \leq 2$ for other edges of G.

Now we replace $k-1$ copies of $A_G(u)$ with $A_G(e_i) - A_G(u_i)$ ($i = 1, 2, \dots, k-1$). Denote the resulting matrix by A. The matrix A is indeed the same as $A_G(\tau)$, as $A_G(u) = A_G(e_i) - A_G(u_i)$ for each $i \in \{1, 2, \dots, k-1\}$. However, in this new format, we have $\eta_A(v) \leq 1$ for all vertices v of G, and $\eta_A(e) \leq 2$ for all edges of G, and $\eta(e) = 0$ for $e \in E(F)$. As $\mathrm{per}(A) = \mathrm{per}(A_G(\tau)) = \mathrm{per}(A_G(\eta)) \neq 0$, this completes the proof of Theorem 2. ∎

Corollary 1 *Every graph is $(2,3)$-choosable.*

A result slightly stronger than Corollary 1 follows from Theorem 2: Suppose G is a connected graph and F is a spanning tree of G. Let $\psi : V(G) \cup E(G) \to \{1, 2, 3\}$ be defined as $\psi(v) = 2$ for every vertex v, $\psi(e) = 1$ for $e \in E(F)$ and $\psi(e) = 3$ for $e \in E(G - F)$. Then G is total weight ψ-choosable. The non-list version of this result was obtained by Kalkowski et al [9].

The following two conjectures, which are weaker than the $(1, 3)$-choosability conjecture and the $(2, 2)$-choosability conjecture respectively, remain open.

Conjecture 1 *There is a constant k such that every graph with no isolated edges is $(1, k)$-choosable.*

Conjecture 2 *There is a constant k such that every graph is $(k, 2)$-choosable.*

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