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Note

Vertex-coloring edge-weightings: Towards the 1-2-3-conjecture

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ABSTRACT

A weighting of the edges of a graph is called vertex-coloring if the weighted degrees of the vertices yield a proper coloring of the graph. In this paper we show that such a weighting is possible from the weight set $\{1, 2, 3, 4, 5\}$ for all graphs not containing components with exactly 2 vertices.

All graphs in this note are finite and simple. For notation not defined here we refer the reader to [4].

For some $k \in \mathbb{N}$, let $\omega : E(G) \to \{1, 2, \dots, k\}$ be an integer weighting of the edges of a graph G. This weighting is called vertex-coloring if the weighted degrees $\omega(v) = \sum_{u \in N(v)} \omega(uv)$ of the vertices yield a proper vertex-coloring of the graph. It is easy to see that for every graph which does not have a component isomorphic to K^2, there exists such a weighting for some k.

In 2002, Karoński, Łuczak and Thomason (see [5]) conjectured that such a weighting with $k = 3$ is possible for all such graphs ($k = 2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4). A first constant bound of $k = 30$ was proved by Addario-Berry, Dalal, McDiarmid, Reed and Thomason in 2007 [1], which was later improved to $k = 16$ by Addario-Berry, Dalal and Reed in [2] and to $k = 13$ by Wang and Yu in [6].

In this note we show a completely different approach that improves the bound to $k = 5$, based on a method developed in [3].

Theorem 1. *For every graph G without components isomorphic to K^2, there is a weighting $\omega : E(G) \to \{1, 2, 3, 4, 5\}$, such that the induced vertex weights $\omega(v) := \sum_{u \in N(v)} \omega(uv)$ properly color $V(G)$.*

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Proof. We may assume that G is connected, otherwise we can argue component-wise. If $|G| \leqslant 1$, there is nothing to prove, so we may assume that $|G| \geqslant 3$ and G contains a vertex with degree $\geqslant 2$. Order the vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ so that $d(v_n) \geqslant 2$ and for every $1 \leqslant i \leqslant n-1$, v_i has a neighbor in $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$.

We start by assigning the provisional weight $f(e) = 3$ to every edge and adjust it at most twice while going through all vertices in order. To every vertex v_i with $i < n$, we will assign a set of two colors $W(v_i) = \{w(v_i), w(v_i) + 2\}$ with $w(v_i) \in \{0, 1\}$ mod 4, so that for every edge $v_j v_i \in E(G)$ with $1 \leqslant j < i$, we have $W(v_j) \cap W(v_i) = \emptyset$, and we will guarantee that $f(v_i) = \sum_{u \in N(v_i)} f(uv_i) \in W(v_i)$. Finally we will adjust the weights of the edges incident to v_n to make sure that $f(v_n)$ is different from $f(v_i)$ for all $v_i \in N(v_n)$.

To this end, let $f(v_1) = 3d(v_1)$, and pick $W(v_1) = \{w(v_1), w(v_1) + 2\}$ so that $f(v_1) \in W(v_1)$ and $w(v_1) \in \{0, 1\}$ mod 4. Let $2 \leqslant k \leqslant n-1$ and assume that we have picked $W(v_i)$ for all $i < k$ and

- $f(v_i) \in W(v_i)$ for $i < k$,
- $f(v_k v_j) = 3$ for all edges with $j > k$, and
- if $f(v_i v_k) \neq 3$ for some edge with $i < k$, then $f(v_i v_k) = 2$ and $f(v_i) = w(v_i)$ or $f(v_i v_k) = 4$ and $f(v_i) = w(v_i) + 2$.

If $v_i v_k \in E$ for some $i < k$ we can either add or subtract 2 to $f(v_i v_k)$ keeping $f(v_i) \in W(v_i)$. If v_k has d such neighbors, this gives us a total of $d + 1$ choices (all of the same parity) for $f(v_k)$. In addition to this we will allow to alter the weight $f(v_k v_j)$ by 1, where $j > k$ is smallest such that $v_k v_j \in E$. This way, $f(v_k)$ can take all values in an interval $[a, a + 2d + 2]$. We want to adjust the weights and assign $w(v_k)$ so that

(1) $f(v_i) \in W(v_i)$ for $1 \leqslant i \leqslant k$,
(2) $w(v_i) \neq w(v_k)$ for $v_i v_k \in E$ with $i < k$, and
(3) either $f(v_k) = w(v_k)$ and $f(v_k v_j) \in \{2, 3\}$ or $f(v_k) = w(v_k) + 2$ and $f(v_k v_j) \in \{3, 4\}$.

Condition (2) can block at most $2d$ values in $[a, a + 2d + 2]$, and condition (3) can block only the values a and $a + 2d + 2$ (for all other values $f(v_k)$ with $f(v_k v_j) \neq 3$, we have the choice between $f(v_k v_j) = 2$ and $f(v_k v_j) = 4$). At least one value remains open for $f(v_k)$.

This way, we can assign the sets $W(v_k)$ step by step for all $k \leqslant n-1$ without conflict. Note that the first time $f(v_k)$ may get changed by an adjustment of an edge $v_k v_i$ for $i > k$ is when $i = j$, so we don't run into problems with edges weighted 2 or 4.

As the final step, we have to find an open weight for v_n. This time, we don't have an extra edge $v_n v_j$ to work with, but we don't have to worry about later vertices. If $v_i v_n \in E$ for some $i < n$ we can again either add or subtract 2 to $f(v_i v_n)$ keeping $f(v_i) \in W(v_i)$. These possible adjustments give a total of $d(v_n) + 1 \geqslant 3$ options (all of the same parity) for $f(v_n)$. Hence if the smallest such option a has $a \in \{2, 3\}$ mod 4, then picking the lower possible weight on each edge incident to v_n gives a proper coloring of the vertices. If $a \in \{0, 1\}$ mod 4 and there is a $v_i \in N(v_n)$ with $w(v_i) \neq a$, then picking the higher weight on $v_i v_n$ and the lower weight on all other edges gives $f(v_n) = a + 2$ in a proper coloring. Finally, if $a \in \{0, 1\}$ mod 4 and $w(v_i) = a$ for all $v_i \in N(v_n)$, picking the higher weight on at least two edges gives a proper coloring. This finalizes f and finishes the proof by setting $\omega = f$. $\square$

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