T&D Coordination - V 1.3

Etinosa Ekomwenrenren & Ilyas Farhat

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Chapter 1

Iteration #1

1.1 Introduction and Motivation

The goal behind this write-up is to develop a controller for distribution system network with DERs, for which the controller would utilize the available energy resources to support the transmission network power demand/request, in a quick and stable manner.

1.2 Distribution System - Power System setup

- $\mathcal{N} = \{0, 1, \cdots, k\}$ is the set of buses in the distribution system of study, where bus 0 is the slack bus
- $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges (branches) of the network.
- let $\mathcal{G} := (\mathcal{N}, \mathcal{L})$ be acyclic graph representation of the radial system
- Let $\mathcal{D}$, with $|\mathcal{D}| = r$, be the set of DERs in the network, which are contributing to power response to transmission network signal
- Let $\hat{p}_i$ be the power output of the i$^{\text{th}}$ DER device, where $\hat{p}_i > 0$ indicates that the device is injecting power into the network and $\hat{p}_i < 0$ indicates that it is consuming power. We define the vector $\hat{\mathbf{p}} = [\hat{p}_1, \ldots, \hat{p}_r]^\top \in \mathbb{R}^r$.
- Let $M \in \mathbb{R}^{k \times r}$ be the mapping matrix of DERs and buses, where $M_{ij} = 1$ if the j$^{\text{th}}$ DER is connected at bus i, and 0 otherwise.
- let $z_{ij} = r_{ij} + \mathbf{j}x_{ij}$ be the impedance of the branches, mapped following $(i, j) \in \mathcal{L}$, where r_{ij} is the resistance and x_{ij} is the impedance of the branch.

- let $\bar{v}_n = v_n \angle \arg(\bar{v}_n) \in \mathbb{C}$ and $\bar{i}_n = i_n \angle \arg(\bar{i}_n) \in \mathbb{C}$ denote the bus-to-ground voltage and current injected at each bus $n \in \mathcal{N}$
- $\bar{\mathbf{v}} = [\bar{v}_1, \cdots, \bar{v}_k]^\top \in \mathbb{C}^k$ and $\bar{\mathbf{i}} = [\bar{i}_1, \cdots, \bar{i}_k]^\top \in \mathbb{C}^k$ be the buses voltages and injection currents.
- Let $\overline{\mathbf{V}}_d = \mathsf{diag}(\overline{\mathbf{v}})$ be a diagonal matrix of non-slack bus voltages.
- Let $\overline{v}_0$ be the slack bus voltage, then $\overline{v}_0 = v_0$ and $arg(\overline{v}_0) = 0$. Also let $\mathbf{v}_s = [v_0, \ldots, v_0]^\top \in \mathbb{R}^k$.
- Let $\mathbf{p}$ and $\mathbf{q} \in \mathbb{R}^k$ be the vectors containing the active and reactive power demands (load - generation) at non-slack buses (i.e. the power extracted at the buses [1])
- For our system, we will receive a power demand signal from the transmission network to be supplied at the slack bus (main bus). Let $\Delta P^\star$ be the change in power requested by the transmission network, where $\Delta P^\star > 0$ indicates an increase in power flow to the distribution network (more power consumption –> DER decreases generation or increases consumption), and $\Delta P^\star < 0$ indicates a decrease in power flow to the distribution network (less power consumption –> DER decreases consumption or increases generation).
- We let p_s and q_s be the active and reactive power injections at the slack bus to the distribution network. We define $\overline{p}_s$ and $\overline{q}_s$ as the pre-request powers at the slack bus (i.e before the request from the transmission network is received).
- The controller needs to satisfy and supply the needed power as fast as possible by adjusting the p (and potentially q) values at "contributing" buses (buses with DERs).
- Let t_s denote the sampling time of the discretized system.

1.3 Distribution System Model - LinDistFlow [1]

Following the defined parameters and variables in section 2, we can get the following relations:

$$\bar{\mathbf{i}}_b = \mathrm{N}_r \bar{\mathbf{i}} \tag{1.1}$$

where $\bar{\mathbf{i}}_b = [\bar{i}_{b_1}, \bar{i}_{b_2}, \ldots, \bar{i}_{b_k}]^\top \in \mathbb{C}^k$ is the vector of branch currents, $\mathrm{N}_r \in \mathbb{R}^{k \times k}$ is the bus injection to branch current matrix with binary values (it is the inverse of the reduced bus incidence matrix).

The branch voltage drops can be calculated using

$$\overline{\mathbf{v}}_b = (\mathbf{R}_{db} + \mathbf{j}\mathbf{X}_{db})\, \bar{\mathbf{i}}_b \tag{1.2}$$

Hence, we can now relate the buses voltages with the slack bus using the following equation:

$$\overline{\mathbf{v}} = \mathbf{v}_s - \mathrm{N}_r^\top \overline{\mathbf{v}}_b \tag{1.3}$$

Here, $\mathrm{N}_r^\top \overline{\mathbf{v}}_b$ would calculate the voltage drops from the slack bus to each bus in the network. Hence, we can express the bus voltages as follows:

$$\overline{\mathbf{v}} = \mathbf{v}_s - \mathrm{N}_r^\top \left(\mathbf{R}_{db} + \mathbf{j}\mathbf{X}_{db}\right) \bar{\mathbf{i}}_b \tag{1.4}$$

We can rewrite Eq. 1.1, using the pre-request voltages as follows:

$$\bar{\mathbf{i}}_b = \mathrm{N}_r \overline{\mathbf{V}}_d^{-1} \left(\mathbf{p} + \mathbf{j}\mathbf{q}\right)^* \tag{1.5}$$

Then, substituting Eq. 1.5 in Eq. 1.4, we get

$$\overline{\mathbf{v}} = \mathbf{v}_s - \mathrm{N}_r^\top \left(\mathbf{R}_{db} + \mathbf{j}\mathbf{X}_{db}\right) \mathrm{N}_r \overline{\mathbf{V}}_d^{-1} \left(\mathbf{p} - \mathbf{j}\mathbf{q}\right) \tag{1.6a}$$

$$\overline{\mathbf{v}} = \mathbf{v}_s - \left(\underbrace{\mathrm{N}_r^\top \mathbf{R}_{db} \mathrm{N}_r \overline{\mathbf{V}}_d^{-1}}_{\overline{\mathbf{B}_{\mathbf{Rv}}}} \mathbf{p} - \mathbf{j} \underbrace{\mathrm{N}_r^\top \mathbf{R}_{db} \mathrm{N}_r \overline{\mathbf{V}}_d^{-1}}_{\overline{\mathbf{B}_{\mathbf{Rv}}}} \mathbf{q} + \mathbf{j} \underbrace{\mathrm{N}_r^\top \mathbf{X}_{db} \mathrm{N}_r \overline{\mathbf{V}}_d^{-1}}_{\overline{\mathbf{B}_{\mathbf{Xv}}}} \mathbf{p} + \underbrace{\mathrm{N}_r^\top \mathbf{X}_{db} \mathrm{N}_r \overline{\mathbf{V}}_d^{-1}}_{\overline{\mathbf{B}_{\mathbf{Xv}}}} \mathbf{q} \right) \tag{1.6b}$$

$$\overline{\mathbf{v}} = \mathbf{v}_s - \left(\overline{\mathbf{B}_{\mathbf{Rv}}} + \mathbf{j}\overline{\mathbf{B}_{\mathbf{Xv}}}\right) \mathbf{p} - \left(\overline{\mathbf{B}_{\mathbf{Xv}}} - \mathbf{j}\overline{\mathbf{B}_{\mathbf{Rv}}}\right) \mathbf{q} \tag{1.6c}$$

$$\overline{\mathbf{v}} = \mathbf{v}_s - \left[\left(\overline{\mathbf{B}_{\mathbf{Rv}}}\mathbf{p} + \overline{\mathbf{B}_{\mathbf{Xv}}}\mathbf{q}\right) - \mathbf{j}\left(\overline{\mathbf{B}_{\mathbf{Rv}}}\mathbf{q} - \overline{\mathbf{B}_{\mathbf{Xv}}}\mathbf{p}\right)\right] \tag{1.6d}$$

Now, let's consider the power flows in the network we have (acyclic graph / radial circuit). For each bus i, let $\overline{s}_i = p_i + \mathbf{j}q_i$ be the net power demand (demand - supply) at the bus. Then, for a radial network, we can define the power flow between the buses as follows. Let $P_i + \mathbf{j}Q_i$ be the power flowing out of bus i to bus $i+1$. Then, the power flowing from bus $i+1$ can be obtained as follows:

$$P_{i+1} = P_i - \frac{P_i^2 + Q_i^2}{v_i^2} r_i - p_{i+1} \tag{1.7a}$$

$$Q_{i+1} = Q_i - \frac{P_i^2 + Q_i^2}{v_i^2} x_i - q_{i+1} \tag{1.7b}$$

$$v_{i+1}^2 = v_i^2 - 2(r_i P_i + x_i Q_i) + (r_i^2 + x_i^2)\frac{P_i^2 + Q_i^2}{v_i^2} \tag{1.7c}$$

Assumptions for System Model

In radial distribution networks with $R/X > 2$, the following assumptions are valid [2, 3]:

a) $\overline{\mathbf{v}} \approx \mathbf{v} = |\overline{\mathbf{v}}|$, which implies that $\overline{\mathbf{V}}_d \approx \mathbf{V}_{dm} = \mathsf{diag}(\mathbf{v})$ (this is the measured buses voltages magnitudes at each time step)

b) Linearized power flow equations holds for each time-step

Hence, we can rewrite Eq. 1.6d as follows:

$$\mathbf{v} \approx \mathbf{v}_s - \underbrace{\mathrm{N}_r^\top \mathbf{R}_{db} \mathrm{N}_r \mathbf{V}_{dm}^{-1}}_{\mathbf{B_{Rv}}} \mathbf{p} - \underbrace{\mathrm{N}_r^\top \mathbf{X}_{db} \mathrm{N}_r \mathbf{V}_{dm}^{-1}}_{\mathbf{B_{Xv}}} \mathbf{q} \tag{1.8a}$$

$$\approx \mathbf{v}_s - \mathbf{B_{Rv}} \mathbf{p} - \mathbf{B_{Xv}} \mathbf{q} \tag{1.8b}$$

Also, we can rewrite Eq. 1.7 as follows, considering a lossless network[1]:

$$P_{i+1} \approx P_i - p_{i+1} \tag{1.9a}$$

$$Q_{i+1} \approx Q_i - q_{i+1} \tag{1.9b}$$

$$v_{i+1}^2 \approx v_i^2 - 2(r_i P_i + x_i Q_i) \tag{1.9c}$$

Constraints

◇ at any time, $v_n \forall n \in \mathcal{N}, v_n \in [\underline{v}, \overline{v}]$

◇ at any time, $\hat{p}_i \ \forall i \in \mathcal{D},\ \hat{p}_i \in [\underline{\hat{p}_i}, \overline{\hat{p}_i}]$. This is the power range for the i[th] DER ($\underline{\hat{p}_i}$ can be negative value, and $\underline{\hat{p}_i} < \overline{\hat{p}_i}$))

◇ at any time, $|p_i - p_j| \forall (i,j) \in \mathcal{L}, |p_i - p_j| \leq p_{ij}^{\text{max}}$

◇ $|\dot{\hat{p}}_i| \leq r_i$ (max rate of change in power limit)

Controller Assumptions

- For the time being, we are going to exclude delays from the development.
- Let $T_i(s)$ be the model of the i$^{\text{th}}$ DER, representing the input-output relationship from the power set point to the electrical power output

DER Generic Model

$$T_i(s) = \begin{bmatrix} \frac{1}{\tau_{p,i}s+1} & 0 \\ 0 & \frac{1}{\tau_{q,i}s+1} \end{bmatrix} \tag{1.10}$$

where $\tau_{p,i}$ and $\tau_{q,i}$ are the time-constants that represent how fast the DER responds to their input signals.

1.4 Problem Formulation

$$\min_{\mathbf{u}} J = \sum_{j=t}^{t+n-1} |\Delta p^j - \Delta P^{\star}|^2 \tag{1.11}$$

subject to

Power Balance Equations

$$\Delta p^j = p_s^j - \overline{p}_s \tag{1.12a}$$

$$p_s^j = \mathbb{1}_k^\top \mathbf{p}^j \tag{1.12b}$$

$$q_s^j = \mathbb{1}_k^\top \mathbf{q}^j \tag{1.12c}$$

$$\mathbf{p}^j = \left[p_1^j, \ldots, p_k^j\right] \tag{1.12d}$$

$$\mathbf{q}^j = \left[q_1^j, \ldots, q_k^j\right] \tag{1.12e}$$

$$\mathbf{p}^{j+1} = \mathbf{p}^j + (\hat{\mathbf{p}}^{j+1} - \hat{\mathbf{p}}^j)M \tag{1.12f}$$

$$\mathbf{q}^{j+1} = \mathbf{q}^j + (\hat{\mathbf{q}}^{j+1} - \hat{\mathbf{q}}^j)M \tag{1.12g}$$

The above equations describe the power balance conditions following the adopted model discussed earlier.

Bus Voltages with p and q linear constraint

$$\mathbf{v}^j = \mathbf{v}_s^j - \mathbf{B_{Rv}}\mathbf{p}^j - \mathbf{B_{Xv}}\mathbf{q}^j \tag{1.12h}$$

Bus Voltage and Branch Current Limits

The bus voltages has to be within the limits defined by $\underline{v}$ and $\overline{v}$. In addition, branch currents limits are defined as $-\mathbf{i}_{b,max}$ and $\mathbf{i}_{b,max}$. We define $\mathbf{B}_m = k_m \mathrm{N}_r \mathbf{V}_{dm}^{-1}$, where k_m is a scaling factor based on the branch power factor [2].

$$\underline{\mathbf{v}} \leq \mathbf{v}_n \leq \overline{\mathbf{v}} \tag{1.12i}$$

$$-\mathbf{i}_{b,\max} \leq \mathbf{B}_m \mathbf{p} \leq \mathbf{i}_{b,max} \tag{1.12j}$$

DER Constraints

The general transfer function provided in Eq. 1.10 can represented in state space representation (continuous-time) as follows:

$$\underbrace{\begin{bmatrix} \dot{\hat{x}}_{p,i} \\ \dot{\hat{x}}_{q,i} \end{bmatrix}}_{\dot{\mathcal{X}}_i} = \begin{bmatrix} -\frac{1}{\tau_{p,i}} & 0 \\ 0 & -\frac{1}{\tau_{q,i}} \end{bmatrix} \underbrace{\begin{bmatrix} \hat{x}_{p,i} \\ \hat{x}_{q,i} \end{bmatrix}}_{\mathcal{X}_i} + \begin{bmatrix} \frac{1}{\tau_{p,i}} & 0 \\ 0 & \frac{1}{\tau_{q,i}} \end{bmatrix} \underbrace{\begin{bmatrix} u_{p,i} \\ u_{q,i} \end{bmatrix}}_{\mathcal{U}_i}$$

$$\begin{bmatrix} \hat{p}_i \\ \hat{q}_i \end{bmatrix} = I_2 \begin{bmatrix} \hat{x}_{p,i} \\ \hat{x}_{q,i} \end{bmatrix}$$

This in turn can be discretized to achieve the following

$$\dot{\mathcal{X}}_i^{j+1} = \hat{A}\mathcal{X}_i^j + \hat{B}\,\mathcal{U}_i^j \tag{1.12k}$$

$$\begin{bmatrix} \hat{p}_i^j \\ \hat{q}_i^j \end{bmatrix} = \hat{C}\mathcal{X}_i^j \tag{1.12l}$$

In addition, we have other bounds for the DER operation as follows:

$$\underline{\hat{p}_i} \leq \hat{p}_i^j \leq \overline{\hat{p}_i} \tag{1.12m}$$

$$|\hat{p}_i^j - \hat{p}_i^{j-1}| \leq r_{p,i} t_s \tag{1.12n}$$

$$\underline{\hat{q}_i} \leq \hat{q}_i^j \leq \overline{\hat{q}_i} \tag{1.12o}$$

$$|\hat{q}_i^j - \hat{q}_i^{j-1}| \leq r_{q,i} t_s \tag{1.12p}$$

Implementation Algorithm

Algorithm 1 RHC-MPC for T & D Coordination Algorithm

1: Get the measured or estimated ($\mathbf{p}$, $\overline{p}_s$, $\mathbf{q}$, $\overline{q}_s$, $\mathbf{v}$, $\hat{\mathbf{p}}$, $\Delta P^{\star}$)
2: Calculate $\mathbf{B_{Rv}}$ and $\mathbf{B_{Xv}}$ using measured or estimated $\mathbf{v}$
3: Solve the optimization problem for the DER input u (Eq. 1.11) for the horizon n
4: Pass the set points from the optimal solution to the DER devices controllers and go to step 1

Summary of Parameters in this problem formulation

Known parameters

- Impedance of the branches
- Network structure
- Pre-request power injection at the slack bus $\overline{p}_s$
- The requested change in power from the transmission network $\Delta P^{\star}$

measured parameters

- voltages (magnitudes) at all buses

  OR estimation of bus voltages using the LinDistFlow model

  OR use slack bus voltage as a simplification

- DERs power injections ($\hat{\mathbf{p}}$)
- Power demand at all the buses

optimization arguments/parameters

- DER devices power set points (u_i in Eq. 1.12k)

Questions

- **We should include q injection from DERs, do we include two state space models of each DER, one for p and one for q?**
- **How do we choose n in Eq. 1.11, because we will use the 2nd iteration value, should not we choose n to be 2 for example?**
- **why this has to be linear in the first place? in the actual implementation, how it is actually calculated? (i know about gradient decent and other similar techniques, but this how it will be solved?)**

Chapter 2

Iteration #2

2.1 Multi-phase Distribution Network Model

The multi-phase unbalanced power system is adopted from [4, 5].

- $\mathcal{N} = \{1, 2, \ldots, N\}$: the set of buses (all considered as PQ buses)
- $\mathbf{s}_j^Y = \left[s_j^a, s_j^b, s_j^c\right]^\top \in \mathbb{C}^3$: the complex power injections at each phase at bus j for Y grounded sources
- $\mathbf{s}_j^\Delta = \left[s_j^{ab}, s_j^{bc}, s_j^{ca}\right]^\top \in \mathbb{C}^3$: the complex power injections for delta-connected sources.
- $\mathbf{v}_j = \left[v_j^a, v_j^b, v_j^c\right]^\top \in \mathbb{C}^3$, $\mathbf{i}_j = \left[i_j^a, i_j^b, i_j^c\right]^\top \in \mathbb{C}^3$ and $\mathbf{i}_j^\Delta = \left[i_j^{ab}, i_j^{bc}, i_j^{ca}\right]^\top \in \mathbb{C}^3$: phase-to-ground voltages, phase net current injections and phase-to-phase currents at bus j.
- $\mathbf{v}_0 = \left[v_0^a, v_0^b, v_0^c\right]^\top \in \mathbb{C}^3$: voltages at the slack bus (interface bus)
- $\mathbf{v} = \left[\mathbf{v}_1^\top, \ldots, \mathbf{v}_N^\top\right]^\top$: voltages at PQ buses (network buses)
- $\mathbf{i} = \left[\mathbf{i}_1^\top, \ldots, \mathbf{i}_N^\top\right]^\top$: current injections PQ buses
- $\mathbf{i}^\Delta = \left[\mathbf{i}_1^{\Delta\top}, \ldots, \mathbf{i}_N^{\Delta\top}\right]^\top$: phase-to-phase currents at PQ buses
- $\mathbf{s}^Y = \left[\mathbf{s}_1^{Y\top}, \ldots, \mathbf{s}_N^{Y\top}\right]^\top$: wye sources at PQ buses
- $\mathbf{s}^\Delta = \left[\mathbf{s}_1^{\Delta\top}, \ldots, \mathbf{s}_N^{\Delta\top}\right]^\top$: delta sources at PQ buses

- $\mathbf{Y}$ is three-phase admittance matrix

$$\mathbf{Y} := \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{0L} \\ \mathbf{Y}_{L0} & \mathbf{Y}_{LL} \end{bmatrix} \in \mathbb{C}^{3(N+1)\times 3(N+1)} \tag{2.1}$$

- $\mathbf{Y}_{00} \in \mathbb{C}^{3\times 3}$, $\mathbf{Y}_{L0} \in \mathbb{C}^{3N\times 3}$, $\mathbf{Y}_{0L} \in \mathbb{C}^{3\times 3N}$ and $\mathbf{Y}_{LL} \in \mathbb{C}^{3N\times 3N}$
- $\mathbf{H}$ is $3N \times 3N$ block diagonal matrix (phase-to-ground to phase-to-phase voltage 3-phase conversion matrix)

$$\mathbf{H} := \begin{bmatrix} \mathbf{\Gamma} & & \\ & \ddots & \\ & & \mathbf{\Gamma} \end{bmatrix}, \qquad \mathbf{\Gamma} := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \tag{2.2}$$

- $\mathbf{w}$ is zero load voltage profile
- $\mathbf{p}^Y, \mathbf{q}^Y, \mathbf{p}^\Delta, \mathbf{q}^\Delta$: active and reactive power injections (from $\mathbf{s}^Y$ & $\mathbf{s}^\Delta$)
- $\mathbf{x}^Y = \left[\mathbf{p}^{Y\top}, \mathbf{q}^{Y\top}\right]^\top$
- $\mathbf{x}^\Delta = \left[\mathbf{p}^{\Delta\top}, \mathbf{q}^{\Delta\top}\right]^\top$

2.1.1 Voltage Phasors

$$\tilde{\mathbf{v}} = \mathbf{M}^Y\mathbf{x}^Y + \mathbf{M}^\Delta\mathbf{x}^\Delta + \mathbf{a} \tag{2.3a}$$

where

$$\mathbf{M}^Y = \left[\mathbf{Y}_{LL}^{-1}\mathsf{diag}(\overline{\widehat{\mathbf{v}}})^{-1}, -\jmath\mathbf{Y}_{LL}^{-1}\mathsf{diag}(\overline{\widehat{\mathbf{v}}})^{-1}\right] \tag{2.3b}$$

$$\mathbf{M}^\Delta = \left[\mathbf{Y}_{LL}^{-1}\mathbf{H}^\top\mathsf{diag}\left(\mathbf{H}\overline{\widehat{\mathbf{v}}}\right)^{-1}, -\jmath\mathbf{Y}_{LL}^{-1}\mathbf{H}^\top\mathsf{diag}\left(\mathbf{H}\overline{\widehat{\mathbf{v}}}\right)^{-1}\right] \tag{2.3c}$$

$$\mathbf{a} = \mathbf{w} \tag{2.3d}$$

2.1.2 Voltage magnitudes

$$|\tilde{\mathbf{v}}| = \mathbf{K}^Y \mathbf{x}^Y + \mathbf{K}^\Delta \mathbf{x}^\Delta + \mathbf{b} \tag{2.4a}$$

where

$$\mathbf{W} = \mathsf{diag}(\mathbf{w}) \tag{2.4b}$$

$$\mathbf{K}^Y = |\mathbf{W}| \, \mathsf{Re}\{\mathbf{W}^{-1}\mathbf{M}^Y\} \tag{2.4c}$$

$$\mathbf{K}^\Delta = |\mathbf{W}| \, \mathsf{Re}\{\mathbf{W}^{-1}\mathbf{M}^\Delta\} \tag{2.4d}$$

$$\mathbf{b} = |\mathbf{w}| \tag{2.4e}$$

2.1.3 Power flow at the interface bus/substation

$$\tilde{\mathbf{s}}_0 = \mathbf{G}^Y \mathbf{x}^Y + \mathbf{G}^\Delta \mathbf{x}^\Delta + \mathbf{c} \tag{2.5a}$$

where

$$\mathbf{G}^Y = \mathsf{diag}(\mathbf{v}_0)\overline{\mathbf{Y}}_{0L}\overline{\mathbf{M}}^Y \tag{2.5b}$$

$$\mathbf{G}^\Delta = \mathsf{diag}(\mathbf{v}_0)\overline{\mathbf{Y}}_{0L}\overline{\mathbf{M}}^\Delta \tag{2.5c}$$

$$\mathbf{c} = \mathsf{diag}(\mathbf{v}_0)\left(\overline{\mathbf{Y}}_{00}\overline{\mathbf{v}}_0 + \overline{\mathbf{Y}}_{0L}\overline{\mathbf{a}}\right) \tag{2.5d}$$

2.1.4 Branch currents

$$\tilde{\mathbf{i}}_{ij} = \mathbf{J}^Y_{ij}\mathbf{x}^Y + \mathbf{J}^\Delta_{ij}\mathbf{x}^\Delta + \mathbf{c}_{ij} \tag{2.6a}$$

where

$$\mathbf{Z}_{ij} \in \mathbb{C}^{3\times 3} \text{ phase impedance matrix of line } (i,j) \tag{2.6b}$$

$$\mathbf{Y}_{ij}^{(s)} \in \mathbb{C}^{3\times 3} \text{ shunt admittance matrix of line } (i,j) \tag{2.6c}$$

$$\mathbf{E}_i = \left[\mathbf{0}_{3\times 3(i-1)}, \mathbf{I}_3, \mathbf{0}_{3\times 3(N-i)}\right] \tag{2.6d}$$

$$\mathbf{J}_{ij}^{Y} = \left[\left(\mathbf{Y}_{ij}^{(s)} + \mathbf{Z}_{ij}^{-1}\right)\mathbf{E}_i - \mathbf{Z}_{ij}^{-1}\mathbf{E}_j\right]\mathrm{M}^{Y} \tag{2.6e}$$

$$\mathbf{J}_{ij}^{\Delta} = \left[\left(\mathbf{Y}_{ij}^{(s)} + \mathbf{Z}_{ij}^{-1}\right)\mathbf{E}_i - \mathbf{Z}_{ij}^{-1}\mathbf{E}_j\right]\mathrm{M}^{\Delta} \tag{2.6f}$$

$$\mathbf{c}_{ij} = \left[\left(\mathbf{Y}_{ij}^{(s)} + \mathbf{Z}_{ij}^{-1}\right)\mathbf{E}_i - \mathbf{Z}_{ij}^{-1}\mathbf{E}_j\right]\mathbf{w} \tag{2.6g}$$

With this, we update our problem formulation as follows:

2.2 Problem Formulation - Iteration #2

$$\min_{\mathbf{u}} J = \sum_{j=t}^{t+n-1} |\Delta p^j - \Delta P^\star|^2 \tag{2.7}$$

subject to

Power Balance Equations

$$\Delta p^j = p_s^j - \overline{p}_s \tag{2.8a}$$

$$p_s^j = \mathsf{Re}\{\tilde{\mathrm{s}}_0\} \tag{2.8b}$$

$$q_s^j = \mathsf{Im}\{\tilde{\mathrm{s}}_0\} \tag{2.8c}$$

$$\tilde{\mathrm{s}}_0 = \mathbf{G}^Y\mathbf{x}^Y + \mathbf{G}^\Delta\mathbf{x}^\Delta + \mathbf{c} \tag{2.8d}$$

Bus Voltages with p and q linear constraint

$$\tilde{\mathbf{v}} = \mathbf{M}^Y\mathbf{x}^Y + \mathbf{M}^\Delta\mathbf{x}^\Delta + \mathbf{a} \tag{2.8e}$$

Bus Voltage and Branch Current Limits

The bus voltages has to be within the limits defined by $\underline{v}$ and $\overline{v}$. In addition, branch currents limits are defined as $-\mathbf{i}_{b,max}$ and $\mathbf{i}_{b,max}$. We define $\mathbf{B}_m = k_m \mathrm{N}_r \mathbf{V}_{dm}^{-1}$, where k_m is a scaling factor based on the branch power factor [2].

$$\underline{\mathbf{v}} \leq \mathbf{v}_n \leq \overline{\mathbf{v}} \tag{2.8f}$$

$$-\mathbf{i}_{ij,\max} \leq \mathbf{i}_{ij} \leq \mathbf{i}_{ij,max} \tag{2.8g}$$

Questions about DERs integration

- **Do we consider single phase controlled, three-phase DERs**?
- **Since we have aggregation, how does the leader/follower DERs would work, I did not understand that logic yet**
- **Usually how do we consider the communication delay, how is it modeled?**
- **Uncertainty if to be included, usually how do we represent it and deal with it? do we try to model it as Gaussian noise for example?**

Chapter 3

Iteration #3

We build iteration #3 based on the multi-phase distribution model presented in iteration #2. However, we consider a different problem setup in this iteration, and we follow a regularized Lagrangian optimization function, similar to what presented in [6].

3.1 Multi-level controller illustration

In this section, we describe the formulation of a multi-level controller for management of DERs in large distribution system, with a focus on T&D coordination. In particular, we will explain how the sub-controllers within the multi-level scheme communicate in order to meet the the power set-point provided by the TNO.

Does this need to be allocated to the phases?

In Fig. 3.1, we illustrate a multi-feeder distribution network that is connected with the transmission network at the interface bus. The Level 1 controller receives the power set-point $\mathbf{P}^\star$ from the transmission network, and then generates the set-point $P_0^{f_i}$ for the ith feeder such that:

$$\mathbf{P}^\star = \sum_i P_0^{f_i}. \tag{3.1}$$

As a simple example, the Level 1 controller could define $P_0^{f_i}$ via a set of paritcipation factors for each distribution feeder, i.e.,

$$P_0^{f_i} = \alpha_i \mathbf{P}^\star$$

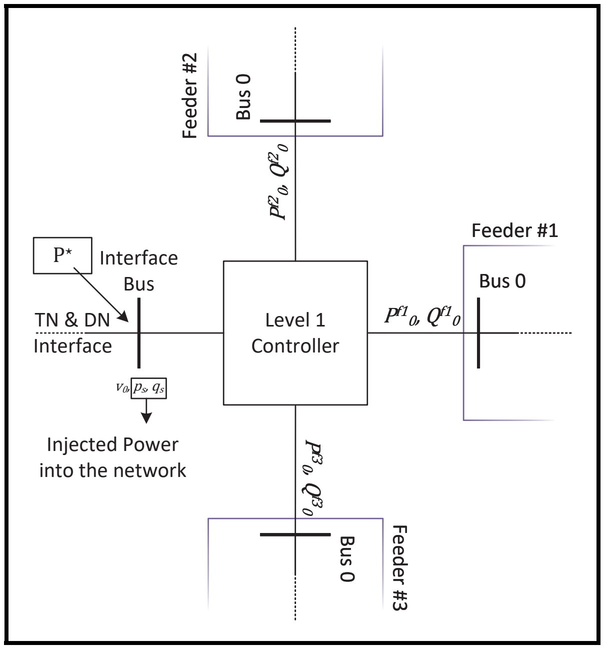

Figure 3.1: Level 1 controller in Distribution Network and feeders networks diagram.

where $\sum_i \alpha_i = 1$ and $\alpha_i > 0$. The participation factors could be defined a priori, or could be computed as a result of an optimization problem. One approach could be based on the available DER-powers (reserves) in each feeder. For such implementation, a power estimation problem can be considered following the approach in [7]. With this approach, once can estimate the available power (flexibility) in a feeder, which can be function running all the time. This function updates the flexibility estimation periodically (every 15 mins for example). Then, Level 1 controller can use this information, along with cost function for example (assuming different operators and different costs). The Level 1 controller would change the participation factors for each feeder based on the information it receives. The development of such controller is beyond the scope of this research problem considered here.

3.2 Feeder multi-level control structure

In each feeder, we consider a multi-level control structure that divides the feeder into smaller control areas. Fig. 3.2 illustrates the generic structure of this multi-level control scheme, consisting of two types of controllers: (i) Level 2 controllers and (ii) lower level controllers.

3.2.1 Level 2 Controller

The Level 2 controller is the highest level controller for each feeder. This controller is responsible for providing set-points to the lower level controllers within the feeder, and is ultimately responsible for tracking the set-point $P_{0,\text{set}} := P_0^{f_i}$ provided by Level 1 controller. The available inputs to the Level 2 controller are any available voltage and current measurements within its "control area", including current and voltage measurements at the feeder head. The controller generates updated "dual variables" and broadcasts them to the DERs and DER aggregators within its control area. Each DER and DER aggregator evaluates and updates its P and Q set-points. The definition of "control area" and "DER aggregator" will be discussed in Section 3.2.3.

3.2.2 Lower Level Controller

Lower level controllers are similar to Level 2 controllers in their structure and operation; however, they differ in the power set-point provided to them. Lower level controllers need to track two power set-points, P and Q. Like the Level 2 controller, the lower level controllers receive voltage and current measurements from within their control areas, and analogously update their dual variables. DERs and DER aggregators within the control area update their P and Q set-points accordingly.

3.2.3 Feeder Example

We now describe the meaning of "control areas", DER aggregators, and how the feeder is divided into control areas. The following example is used to help clarify these concepts.

Consider the example feeder shown in Figure 3.3. This feeder consists of 24 buses. At each bus

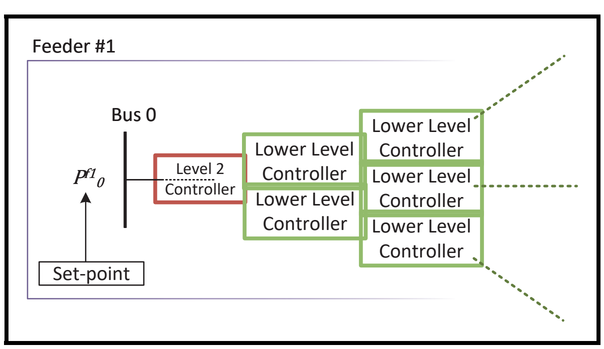

Figure 3.2: Feeder control structure.

we assume that we have a non-controllable load, a DER device, or both. The feeder is partitioned into layered control areas, shown in red, green, and blue boxes. These control areas may represent contractual arrangements for the management of DER resources via aggregators, or may be defined based on other operational criteria.

The resources within the red box are managed by the Level 2 controller. This controller has visibility over the system within the red box, but does not have visibility within the green boxes or beyond into the blue boxes. The boundary of the red box is defined by the two interface buses, Bus 3 and Bus 4, which connect to new areas that are independently managed by DER aggregators. The Level 2 controller will provide P and Q set-points for each interface bus. The green boxes indicate control areas under the responsibility of DER aggregators. These DER aggregators are responsible for managing DER resources and maintaining operational constraints within the control area, and

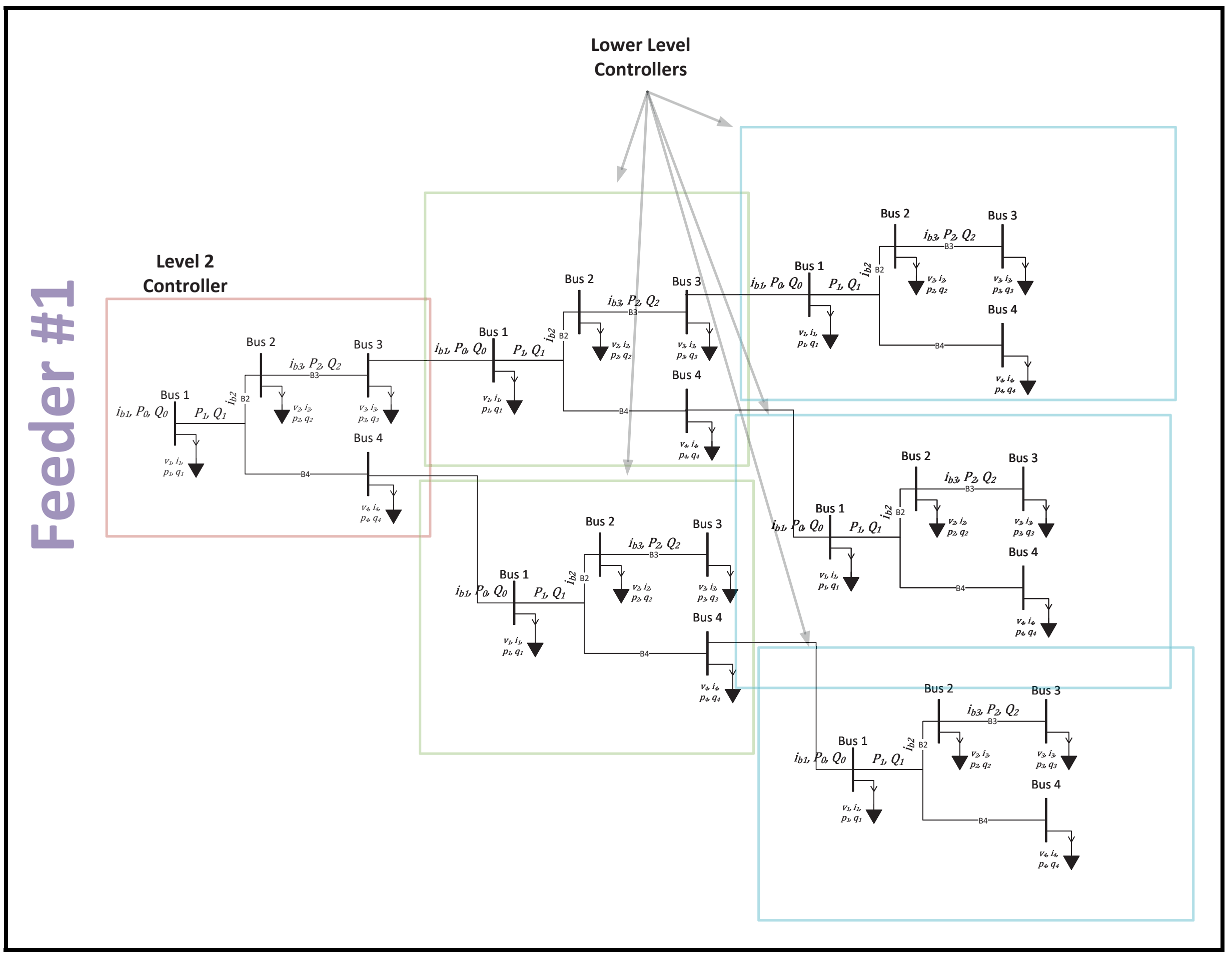

Figure 3.3: Feeder Example.

for tracking the set-points provided by the Level 2 controller. This hierarchical structure can then be further nested, with the blue boxes indicating an additional layer of DER aggregators; this structure can be repeated as required based on operational/information constraints, until all DER and DER aggregators are covered in the control-areas. The green box DER aggrgeators do not have visibility into the blue control areas, but provide set-points to the interface buses between the green and blue control areas.

The DER aggregators in each control area would be represented by their Thevenin Equivalent circuit/components. This is necessary for DN modeling equation. The Thevenin equivalent will be provided by the DER aggregator to the controller. The DER aggregator shall update its Thevenin equivalent model with each iteration, or periodically. More information about this implementation can be found in [8, 9].

To summarize the role of each controller and the DER/DER aggregators, we list them below:

- Level 2 Controller
  - Receive $p_{0,\text{set}}$
  - Acquire measurements from the visible block ($V_{\text{meas}}, i_{\text{meas}}, p_0$)
  - Generates updated dual variables $\mathbf{d}$ and broadcast them to connected DERs (DER devices and aggregators)
- Lower Level Controllers*
  - Receive $p_{0,\text{set}}$ and $q_{0,\text{set}}$
  - Acquire measurements from the visible block ($V_{\text{meas}}, i_{\text{meas}}, p_0, q_0$)
  - Generates updated dual variables $\mathbf{d}$ and broadcast them to connect DERs (DER devices and aggregators)
- DERs (devices and aggregators)
  - Receive updated dual variables $\mathbf{d}$
  - Acquire output power $\mathbf{x}_i$ measurements
  - Generates updated set points $p_{0,\text{set}}$ and $q_{0,\text{set}}$**

** if DER aggregator, the updated set-points will be fed to the lower level controller as the new/updated set-points

3.3 DER Parameters

- $\mathcal{P} := \{a, b, c\} \cup \{ab, bc, ca\}$ set of possible connections ($\{a, b, c\}$ representing wye-connection (line-to-ground) and $\{ab, bc, ca\}$ representing delta-connection (phase-to-phase))
- Let $\mathcal{D}$ represent the set of DERs that are controlled directly by the main controller (the set includes DERs and group of DERs that controlled by one controller)
- Let $\boxed{\mathbf{x}_i = [P_i, Q_i]^\top \in \mathbb{R}^2}$ be the active and reactive power set-points of the i^{th} DER.
- We define $\mathcal{P}_i \subset \mathcal{P}$ as the set that collects the phases where DER i is connected.
- For any DER $i \in \mathcal{D}$, we define $\boxed{\mathcal{X}_i \subset \mathbb{R}^2 \text{ as the set of possible power set-points } \mathbf{x}_i}$.
- We define a $\boxed{\text{cost function } f_i(\mathbf{x}_i)}$ for each DER.

Remove k; meaning of weights?

examples of $f(\mathbf{x})$

$$f_i(\mathbf{x_i}) = \begin{cases} \left(p_{i,av}^{(k)} - p_i^{(k)}\right)^2 + \left(q_i^{(k)}\right)^2 & \text{, for three-phase PV systems, where } p_{i,av} \text{ is} \\ & \text{max. real power available} \\ 100\left(p_{i,av}^{(k)} - p_i^{(k)}\right)^2 + 10\left(q_i^{(k)}\right)^2 & \text{, for single-phase PV systems} \\ \left(p_{i,\phi}^{(k)}\right)^2 + \left(q_{i,\phi}^{(k)}\right)^2 & \text{, for batteries} \\ 100\left(p_i^{(k)} - p_{i,max}\right)^2 & \text{, for EVs, where } p_{i,max} \text{ is the max. charging rate} \end{cases} \tag{3.2}$$

3.4 Distribution Network Model

We now describe the distribution network model which will be integrated into the controller within each control area.

3.4.1 Basic Notation

- $\boxed{\mathcal{N} = \{1, 2, \dots, N\}\text{: the set of buses (all considered as PQ buses)}}$. We consider node "0" as the point of connection with the rest of the electrical system (for example $\mathbf{p}_0$ represents the power injection at the interface bus/node)

- $\mathbf{s}_j^Y = \left[s_j^a, s_j^b, s_j^c\right]^\top \in \mathbb{C}^3$: the complex power injections at each phase at bus j for Y grounded sources

- $\mathbf{s}_j^\Delta = \left[s_j^{ab}, s_j^{bc}, s_j^{ca}\right]^\top \in \mathbb{C}^3$: the complex power injections for delta-connected sources.

- $\mathbf{v}_j = \left[v_j^a, v_j^b, v_j^c\right]^\top \in \mathbb{C}^3$, $\mathbf{i}_j = \left[i_j^a, i_j^b, i_j^c\right]^\top \in \mathbb{C}^3$ and $\mathbf{i}_j^\Delta = \left[i_j^{ab}, i_j^{bc}, i_j^{ca}\right]^\top \in \mathbb{C}^3$: phase-to-ground voltages, phase net current injections and phase-to-phase currents at bus j.

- $\mathbf{v}_0 = \left[v_0^a, v_0^b, v_0^c\right]^\top \in \mathbb{C}^3$: voltages at the slack bus (interface bus)

- $\mathbf{v} = \left[\mathbf{v}_1^\top, \dots, \mathbf{v}_N^\top\right]^\top$: voltages at PQ buses (network buses)

- $\mathbf{i} = \left[\mathbf{i}_1^\top, \dots, \mathbf{i}_N^\top\right]^\top$: current injections PQ buses

- $\mathbf{i}^\Delta = \left[\mathbf{i}_1^{\Delta\top}, \dots, \mathbf{i}_N^{\Delta\top}\right]^\top$: phase-to-phase currents at PQ buses

- $\mathbf{s}^Y = \left[\mathbf{s}_1^{Y\top}, \dots, \mathbf{s}_N^{Y\top}\right]^\top$: wye sources at PQ buses

- $\mathbf{s}^\Delta = \left[\mathbf{s}_1^{\Delta\top}, \dots, \mathbf{s}_N^{\Delta\top}\right]^\top$: delta sources at PQ buses

- $\mathbf{Y}$ is three-phase admittance matrix

$$\boxed{\mathbf{Y} := \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{0L} \\ \mathbf{Y}_{L0} & \mathbf{Y}_{LL} \end{bmatrix} \in \mathbb{C}^{3(N+1)\times 3(N+1)}} \tag{3.3}$$

- $\mathbf{Y}_{00} \in \mathbb{C}^{3\times 3}$, $\mathbf{Y}_{L0} \in \mathbb{C}^{3N\times 3}$, $\mathbf{Y}_{0L} \in \mathbb{C}^{3\times 3N}$ and $\mathbf{Y}_{LL} \in \mathbb{C}^{3N\times 3N}$

- $\mathbf{G}$ is $3N \times 3N$ block diagonal matrix (phase-to-ground to phase-to-phase voltage 3-phase conversion matrix). Note that this matrix was called $\mathbf{H}$ in iteration #2).

- Let $\mathcal{M}_v$ denote the locations/buses where voltage measurements (phase-to-ground) are available, and let $\boxed{\mathbf{v}_{\mathcal{M}_v} \text{ be the vector of measurements}}$. $\boxed{\text{We denote } |\mathcal{M}_v| = r_v}$

- Let $\boxed{\mathbf{i}_{L,\mathcal{M}_i} \text{ denotes the vector of line currents for a subset of monitored distribution lines } \mathcal{M}_i}$ (this can be measured or given by pseudo-measurements, possibly relying on network model and voltage measurements, or using previous measurements). $\boxed{\text{We denote } |\mathcal{M}_i| = r_i}$

- Let $\boxed{\mathbf{p}_0 \in \mathbb{R}^3 \text{ denotes the real-power entering interface bus}}$ at phases $\{a, b, c\}$.

- Let $\boxed{\mathbf{q}_0 \in \mathbb{R}^3 \text{ denotes the real-power entering interface bus}}$ at phases $\{a, b, c\}$. Note that this would be utilized in the lower level controllers.

3.4.2 Modified Equations of distribution network model

Our goal is to obtain a linearized model describing the distribution feeder of the following form

$$|\tilde{\mathbf{v}}_{\mathcal{M}_v}(\mathbf{x})| = \sum_{i \in \mathcal{D}} \mathbf{A}_i \mathbf{x}_i + \mathbf{a} \qquad = \mathbf{A}\mathbf{x} + \mathbf{a} \tag{3.4a}$$

$$|\tilde{\mathbf{i}}_{L,\mathcal{M}_i}(\mathbf{x})| = \sum_{i \in \mathcal{D}} \mathbf{B}_i \mathbf{x}_i + \mathbf{b} \qquad = \mathbf{B}\mathbf{x} + \mathbf{b} \tag{3.4b}$$

$$\tilde{\mathbf{p}}_0(\mathbf{x}) = \sum_{i \in \mathcal{D}} \mathbf{M}_i \mathbf{x}_i + \mathbf{m} \qquad = \mathbf{M}\mathbf{x} + \mathbf{m} \tag{3.4c}$$

$$\tilde{\mathbf{q}}_0(\mathbf{x}) = \sum_{i \in \mathcal{D}} \mathbf{H}_i \mathbf{x}_i + \mathbf{h} \qquad = \mathbf{H}\mathbf{x} + \mathbf{h} \tag{3.4d}$$

where $\mathbf{x} = \mathrm{col}(\mathbf{x}_1, \ldots, \mathbf{x}_{|\mathcal{D}|})$ is the vector of all DER/DER-aggregator set-points and all other symbols are constants defined based on the network information. The $\sim$ over the variables indicates that these are not the true values, but the values obtained via a linearized model.

Integrate the equations from Iteration #2

Mapping $\mathbf{A}, \mathbf{B}, \mathbf{M}, \mathbf{H}, \mathbf{a}, \mathbf{b}, \mathbf{m}$ & $\mathbf{h}$ to multi-phase distribution network model matrices (Eqs. 2.4a, 2.5a & 2.6a in Iteration #2 network model equations (2.1)

Important note: $\mathbf{x}$ in Section 2.1 model are defined as follows:

- $\mathbf{p}^Y, \mathbf{q}^Y, \mathbf{p}^\Delta, \mathbf{q}^\Delta$: active and reactive power injections (from $\mathbf{s}^Y$ & $\mathbf{s}^\Delta$)

- $\mathbf{x}^Y = \left[\mathbf{p}^{Y\top}, \mathbf{q}^{Y\top}\right]^\top$
- $\mathbf{x}^\Delta = \left[\mathbf{p}^{\Delta\top}, \mathbf{q}^{\Delta\top}\right]^\top$

(1) **A matrices**

From Eq. 2.4a, we have that

$$\begin{aligned}
|\tilde{\mathbf{v}}| &= \mathbf{K}^Y \mathbf{x}^Y + \mathbf{K}^\Delta \mathbf{x}^\Delta + \mathbf{b} \\
&= \begin{bmatrix} \mathbf{K}^Y & \mathbf{K}^\Delta \end{bmatrix} \begin{bmatrix} \mathbf{x}^Y \\ \mathbf{x}^\Delta \end{bmatrix} + \mathbf{b} \\
&= \underbrace{\begin{bmatrix} \mathbf{K}^Y & \mathbf{K}^\Delta \end{bmatrix}}_{3N \times 12N} \underbrace{\begin{bmatrix} \mathbf{p}^Y \\ \mathbf{q}^Y \\ \mathbf{p}^\Delta \\ \mathbf{q}^\Delta \end{bmatrix}}_{12N \times 1} + \mathbf{b}
\end{aligned} \tag{3.5}$$

To extract only the measured values $|\mathbf{v}_{\mathcal{M}_\mathbf{v}}|$ out of $|\tilde{\mathbf{v}}|$, and to express the right-hand side as a function only of the DER point setpoints $\mathbf{x}$, we do the following manipulations. First, we split the powers p and q to DER and non-DER components, and then second, we extract the required components. To do this, let $\mathbf{Q}_L^v \in \mathbb{R}^{3D \times 3N}$ select the desired measured voltages from the vector of all voltages, with ith row defined by

$$(Q_L^v)_{ij} = \begin{cases} 1 & \text{if the } i\text{th voltage measurement is from bus } j \\ 0 & \text{otherwise} \end{cases}$$

and then set $\mathbf{Q}_L^v = Q_L^v \otimes I_3$. With this notation, we have that

$$|\tilde{\mathbf{v}}_{\mathcal{M}_v}| = \mathbf{Q}_L^v |\tilde{\mathbf{v}}|.$$

Next, we wish to express $|\tilde{\mathbf{v}}|$ in terms of $\mathbf{x}$, and we will define a matrix $\mathbf{Q}_R$ such that

$$\begin{bmatrix} \mathbf{p}^Y \\ \mathbf{q}^Y \\ \mathbf{p}^\Delta \\ \mathbf{q}^\Delta \end{bmatrix} = \mathbf{Q}_R \mathbf{x} + \mathbf{c}$$

Effectively, $\mathbf{Q}_R$ places DER powers at the correct phases, accounting for their method of interconnection. The elements are defined as

$$\underbrace{|\mathbf{v}_{\mathcal{M}_v}|}_{3r_v \times 1} = \overbrace{\underbrace{\mathbf{Q}_L^v}_{3r_v \times 3N} \underbrace{\begin{bmatrix} \mathbf{K}^Y & \mathbf{K}^\Delta \end{bmatrix}}_{3N \times 12N} \underbrace{\mathbf{Q}_R}_{12N \times 2D}}^{\mathbf{A}} \underbrace{\overbrace{\begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_D \\ q_D \end{bmatrix}}^{\mathbf{x}}}_{2D \times 1} + \overbrace{\mathbf{Q}_L^v \left(\begin{bmatrix} \mathbf{K}^Y & \mathbf{K}^\Delta \end{bmatrix} \begin{bmatrix} \mathbf{p}^Y_{\text{non-DER}} \\ \mathbf{q}^Y_{\text{non-DER}} \\ \mathbf{p}^\Delta_{\text{non-DER}} \\ \mathbf{q}^\Delta_{\text{non-DER}} \end{bmatrix} + \mathbf{b} \right)}^{\mathbf{a}} \tag{3.6a}$$

where

Fix indexing for e_js

- $\mathbf{Q}_R \in \mathbb{R}^{12N \times 2D}$, for DER j connect at bus i (here $e_j \in \mathbb{R}^{2D}$)
  - ◇ Y-connected balanced DER

$$\begin{bmatrix} \mathbf{Q}_{R,\text{rows}(i:i+2)} \\ \mathbf{Q}_{R,\text{rows}(3N+i:3N+i+2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\mathbb{1}_3 {e_{2j-1}}^\top \\ \frac{1}{3}\mathbb{1}_3 {e_{2j}}^\top \end{bmatrix} \tag{3.6b}$$

  - ◇ Single-phase connected DER

$$\begin{bmatrix} \mathbf{Q}_{R,\text{rows}(i:i+2)} \\ \mathbf{Q}_{R,\text{rows}(3N+i:3N+i+2)} \end{bmatrix} = \begin{cases} \begin{bmatrix} e_j^p & \mathbb{0} & \mathbb{0} & e_j^q & \mathbb{0} & \mathbb{0} \end{bmatrix}^\top & \text{, if connected at phase } a \\ \begin{bmatrix} \mathbb{0} & e_j^p & \mathbb{0} & \mathbb{0} & e_j^q & \mathbb{0} \end{bmatrix}^\top & \text{, if connected at phase } b \\ \begin{bmatrix} \mathbb{0} & \mathbb{0} & e_j^p & \mathbb{0} & \mathbb{0} & e_j^q \end{bmatrix}^\top & \text{, if connected at phase } c \end{cases} \tag{3.6c}$$

◇ Δ-connected balanced DER

$$\begin{bmatrix} \mathbf{Q}_{R,\text{rows}(6N+i:6N+i+2)} \\ \mathbf{Q}_{R,\text{rows}(9N+i:9N+i+2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\mathbb{1}_3 {e_{2j-1}}^\top \\ \frac{1}{3}\mathbb{1}_3 {e_{2j}}^\top \end{bmatrix} \tag{3.6d}$$

◇ phase-to-phase connected DER

$$\begin{bmatrix} \mathbf{Q}_{R,\text{rows}(6N+i:6N+i+2)} \\ \mathbf{Q}_{R,\text{rows}(9N+i:9N+i+2)} \end{bmatrix} = \begin{cases} \begin{bmatrix} e_j^p & \mathbb{0} & \mathbb{0} & e_j^q & \mathbb{0} & \mathbb{0} \end{bmatrix}^\top & \text{, if } ab\text{-phase connected} \\ \begin{bmatrix} \mathbb{0} & e_j^p & \mathbb{0} & \mathbb{0} & e_j^q & \mathbb{0} \end{bmatrix}^\top & \text{, if } bc\text{-phase connected} \\ \begin{bmatrix} \mathbb{0} & \mathbb{0} & e_j^p & \mathbb{0} & \mathbb{0} & e_j^q \end{bmatrix}^\top & \text{, if } ca\text{-phase connected} \end{cases} \tag{3.6e}$$

◇ Other connections - treat similarly to what presented here. We can have two phase-to-phase, or phase-to-ground connected DERs (“ab” and “bc”, or “a” and “b” for example). In both cases, we consider their corresponding e_j vectors, and we scale them by $1/2$.

- Lastly, we define the matrix $\mathbf{A}$ as follows

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \cdots & \mathbf{A}_D \end{bmatrix} \tag{3.6f}$$

where $\mathbf{A}_i \in \mathbb{R}^{3r_v \times 2}, \forall i \in \{1, \ldots, D\}$

(2) B **matrices**

Let’s add selection of currents similar to the voltages

Following a similar approach, we can achieve the following. We start with Eq. 2.6a.

$$\tilde{\mathbf{i}}_{ij} = \mathbf{J}_{ij}^Y \mathbf{x}^Y + \mathbf{J}_{ij}^\Delta \mathbf{x}^\Delta + \mathbf{c}_{ij} \tag{3.7}$$

We combine the J matrices, and collect all constants in new $\mathbf{b}$, to get the updated equation of $\tilde{\mathbf{i}}_{ij}$

$$\tilde{\mathbf{i}}_{ij} = \begin{bmatrix} \mathbf{J}_{ij}^{Y} & \mathbf{J}_{ij}^{\Delta} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{Y} \\ \mathbf{x}^{\Delta} \end{bmatrix} + \mathbf{c}_{ij} \tag{3.8a}$$

$$\tilde{\mathbf{i}}_{ij} = \overbrace{\begin{bmatrix} \mathbf{J}_{ij}^{Y} & \mathbf{J}_{ij}^{\Delta} \end{bmatrix} \mathbf{Q}_R}^{\mathbf{B}} \overbrace{\begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_D \\ q_D \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} \mathbf{J}_{ij}^{Y} & \mathbf{J}_{ij}^{\Delta} \end{bmatrix} \begin{bmatrix} \mathbf{p}_{\text{non-DER}}^{Y} \\ \mathbf{q}_{\text{non-DER}}^{Y} \\ \mathbf{p}_{\text{non-DER}}^{\Delta} \\ \mathbf{q}_{\text{non-DER}}^{\Delta} \end{bmatrix} + \mathbf{c}_{ij}}^{\mathbf{b}} \tag{3.8b}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \dots & \mathbf{B}_D \end{bmatrix} \tag{3.8c}$$

where $\mathbf{B} \in \mathbb{R}^{3\times 2D}$ and $\mathbf{B}_i \in \mathbb{R}^{3\times 2}$

(3) M **& H matrices**

We start with Eq. 2.5a.

$$\tilde{\mathrm{s}}_0 = \mathbf{G}^{Y}\mathbf{x}^{Y} + \mathbf{G}^{\Delta}\mathbf{x}^{\Delta} + \mathbf{c} \tag{3.9}$$

We then combine G matrices and collect all constants in g as follows:

$$\tilde{\mathrm{s}}_0 = \begin{bmatrix} \mathbf{G}^{Y} & \mathbf{G}^{\Delta} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{Y} \\ \mathbf{x}^{\Delta} \end{bmatrix} + \mathbf{c} \tag{3.10a}$$

$$\tilde{\mathrm{s}}_0 = \overbrace{\begin{bmatrix} \mathbf{G}^{Y} & \mathbf{G}^{\Delta} \end{bmatrix} \mathbf{Q}_R}^{\overline{\mathbf{G}}} \overbrace{\begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_D \\ q_D \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} \mathbf{G}^{Y} & \mathbf{G}^{\Delta} \end{bmatrix} \begin{bmatrix} \mathbf{p}_{\text{non-DER}}^{Y} \\ \mathbf{q}_{\text{non-DER}}^{Y} \\ \mathbf{p}_{\text{non-DER}}^{\Delta} \\ \mathbf{q}_{\text{non-DER}}^{\Delta} \end{bmatrix} + \mathbf{c}}^{\mathbf{g}} \tag{3.10b}$$

(3.10c)

We then define $\mathbf{M}, \mathbf{H}, \mathbf{m}$ & $\mathbf{h}$ based on $\overline{\mathbf{G}}$ and $\mathbf{g}$ as follows:

$$\mathbf{M} = \mathsf{Re}(\overline{\mathbf{G}}) \tag{3.10d}$$

$$\mathbf{H} = \mathsf{Im}(\overline{\mathbf{G}}) \tag{3.10e}$$

$$\mathbf{m} = \mathsf{Re}(\mathbf{g}) \tag{3.10f}$$

$$\mathbf{h} = \mathsf{Im}(\mathbf{g}) \tag{3.10g}$$

3.5 Controller and optimization problem setup

3.5.1 Level 2 Controller

$$\min_{\mathbf{x}} \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \tag{3.11}$$

subject to

$$\mathbf{x}_i \in \mathcal{X}_i, \qquad \forall i \in \mathcal{D} \tag{3.12a}$$

$$s^{(k)} \mathbf{I}_3 \left(|\mathbf{p}_0^{(k)}(\mathbf{x}) - \mathbf{p}_{0,\mathsf{set}}^{(k)}| \right) \leq E^{(k)} \mathbb{1}_3 \tag{3.12b}$$

$$\underline{\mathbf{v}} \leq |\mathbf{v}^{(k)}(\mathbf{x})| \leq \overline{\mathbf{v}} \tag{3.12c}$$

$$|\mathbf{i}_L^{(k)}(\mathbf{x})| \leq \bar{\mathbf{i}} \tag{3.12d}$$

where:

- k is the time-step (discrete)
- $f_i^{(k)}$ is a time-varying convex function associated with the i^{th} DER (or DER agg.)
- $s^{(k)} \in \{0, 1\}$ enable/disable the controller from tracking the power set-point
- $\mathbf{p}_{0,\mathsf{set}}^{(k)}$ is the power set-point at the interface bus
- $E^{(k)}$ is the tolerance or accuracy

To solve this optimization problem, we follow the approach in [6]. We form the Lagrangian function. To build this, we re-write the constraints as follows:

$$\mathbf{x}_i \in \mathcal{X}_i, \qquad \forall i \in \mathcal{D} \tag{3.13a}$$

$$s^{(k)} \mathbf{I}_3 \left(\mathbf{p}_0^{(k)}(\mathbf{x}) - \mathbf{p}_{0,\mathsf{set}}^{(k)} \right) \leq E^{(k)} \mathbb{1}_3 \tag{3.13b}$$

$$s^{(k)} \mathbf{I}_3 \left(\mathbf{p}_{0,\mathsf{set}}^{(k)} - \mathbf{p}_0^{(k)}(\mathbf{x}) \right) \leq E^{(k)} \mathbb{1}_3 \tag{3.13c}$$

$$|\mathbf{v}^{(k)}(\mathbf{x})| \leq \overline{\mathbf{v}} \tag{3.13d}$$

$$\underline{\mathbf{v}} \leq |\mathbf{v}^{(k)}(\mathbf{x})| \tag{3.13e}$$

$$|\mathbf{i}_L^{(k)}(\mathbf{x})| \leq \bar{\mathbf{i}} \tag{3.13f}$$

DERs Assumptions

- $\mathcal{X}_i^{(k)}$ is convex and compact for all t_k
- $f_i^{(k)}(\mathbf{x}_i)$ is convex and continuously differentiable, and its gradient is Lipschitz continuous for all t_k

To build the Lagrangian function of our problem, we associate the dual variables $\boldsymbol{\lambda}^{(k)}$, $\boldsymbol{\mu}^{(k)}$, $\boldsymbol{\gamma}^{(k)}$, $\boldsymbol{\nu}^{(k)}$ and $\boldsymbol{\zeta}^{(k)}$ with constraints 3.13b, 3.13c, 3.13d, 3.13e and 3.13f, respectively. We will use $\boldsymbol{d}^{(k)}$ to represent the dual variables. At time t_k, the Lagrangian function associated with the optimization problem is given as

$$\begin{aligned} L^{(k)}(\mathbf{x}, \boldsymbol{d}) \coloneqq & \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\ & + \boldsymbol{\lambda}^\top \left(s^{(k)} \left(\mathbf{p}_0^{(k)}(\mathbf{x}) - \mathbf{p}_{0,\text{set}}^{(k)} \right) - E^{(k)} \mathbb{1}_3 \right) \\ & + \boldsymbol{\mu}^\top \left(s^{(k)} \left(\mathbf{p}_{0,\text{set}}^{(k)} - \mathbf{p}_0^{(k)}(\mathbf{x}) \right) - E^{(k)} \mathbb{1}_3 \right) \\ & + \boldsymbol{\gamma}^\top \left(|\mathbf{v}^{(k)}(\mathbf{x})| - \overline{\mathbf{v}} \right) \\ & + \boldsymbol{\nu}^\top \left(\underline{\mathbf{v}} - |\mathbf{v}^{(k)}(\mathbf{x})| \right) \\ & + \boldsymbol{\zeta}^\top \left(|\mathbf{i}_L^{(k)}(\mathbf{x})| - \bar{\mathbf{i}} \right) \end{aligned} \tag{3.14}$$

Using Eq. 3.4, we can re-write Eq. 3.14 as follows:

$$\begin{aligned} L^{(k)}(\mathbf{x}, \boldsymbol{d}) \coloneqq & \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\ & + \sum_{i \in \mathcal{D}} \left[s^{(k)} (\boldsymbol{\lambda} - \boldsymbol{\mu})^\top \mathbf{M}_i \mathbf{x}_i + (\boldsymbol{\gamma} - \boldsymbol{\nu})^\top \mathbf{A}_i \mathbf{x}_i + \boldsymbol{\zeta}^\top \mathbf{B}_i \mathbf{x}_i \right] \\ & + s^{(k)} (\boldsymbol{\lambda} - \boldsymbol{\mu})^\top (\mathbf{m}^{(k)} - \mathbf{p}_{0,\text{set}}) - (\boldsymbol{\lambda} + \boldsymbol{\mu})^\top E^{(k)} \mathbb{1} \\ & + \boldsymbol{\gamma}^\top (\mathbf{a}^{(k)} - \overline{\mathbf{v}}) + \boldsymbol{\nu}^\top (\underline{\mathbf{v}} - \mathbf{a}^{(k)}) + \boldsymbol{\zeta}^\top (\mathbf{b}^{(k)} - \bar{\mathbf{i}}) \end{aligned} \tag{3.15}$$

We then define the regularized Lagrangian function, with r_p and $r_d > 0$ (regularization factors):

$$\begin{aligned} L_r^{(k)}(\mathbf{x}, \boldsymbol{d}) =& L^{(k)}(\mathbf{x}, \boldsymbol{d}) \\ &+ \frac{r_p}{2}\|\mathbf{x}\|_2^2 - \frac{r_d}{2}\|\boldsymbol{d}\|_2^2 \\ =& L^{(k)}(\mathbf{x}, \boldsymbol{d}) \\ &+ \frac{r_p}{2}\|\mathbf{x}\|_2^2 - \frac{r_d}{2}\left(\|\boldsymbol{\lambda}\|_2^2 + \|\boldsymbol{\mu}\|_2^2 + \|\boldsymbol{\gamma}\|_2^2 + \|\boldsymbol{\nu}\|_2^2 + \|\boldsymbol{\zeta}\|_2^2\right) \end{aligned} \tag{3.16}$$

Using a step-size $\alpha > 0$, and

$$\alpha < \frac{\min\{r_p, r_d\}}{(L + r_p + 5G)^2 + 5(G + r_d)^2}, \tag{3.17}$$

where $L = \sup\{L^{(k)}\}$, $G = \max\{G_v, G_0, G_L\}$, $\|\nabla_{\mathbf{x}}|\mathbf{v}^{(k)}(\mathbf{x})|\|_2 \leq G_v$, $\|\nabla_{\mathbf{x}}|\mathbf{p}^{(k)}(\mathbf{x})|\|_2 \leq G_0$, $\|\nabla_{\mathbf{x}}|\mathbf{i}^{(k)}(\mathbf{x})|\|_2 \leq G_L$.

Algorithm 2 Real-time optimization algorithm

At each t_k

[Step 1]: Collect $|\mathbf{v}^{(k)}|$ at $\mathcal{M}_v$, $|\mathbf{i}^{(k)}|$ at $\mathcal{M}_i$ and $\mathbf{p}_0^{(k)}$, and perform the following updates to the dual variables:

$$\boldsymbol{\nu}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}} \left\{\boldsymbol{\nu}^{(k)} + \alpha \left(\underline{\mathbf{v}}\mathbb{1} - |\mathbf{v}^{(k)}| - r_d\boldsymbol{\nu}^{(k)}\right)\right\} \tag{3.18}$$

$$\boldsymbol{\gamma}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}} \left\{\boldsymbol{\gamma}^{(k)} + \alpha \left(|\mathbf{v}^{(k)}| - \overline{\mathbf{v}}\mathbb{1} - r_d\boldsymbol{\gamma}^{(k)}\right)\right\} \tag{3.19}$$

$$\boldsymbol{\zeta}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}} \left\{\boldsymbol{\zeta}^{(k)} + \alpha \left(|\mathbf{i}^{(k)}| - \overline{\mathbf{v}}\mathbb{1} - r_d\boldsymbol{\zeta}^{(k)}\right)\right\} \tag{3.20}$$

$$\boldsymbol{\lambda}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{3}} \left\{\boldsymbol{\lambda}^{(k)} + \alpha \left(\mathbf{p}_0^{(k)} - \mathbf{p}_{0,\mathsf{set}}^{(k)} - E^{(k)}\mathbb{1} - r_d\boldsymbol{\lambda}^{(k)}\right)\right\} \tag{3.21}$$

$$\boldsymbol{\mu}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{3}} \left\{\boldsymbol{\mu}^{(k)} + \alpha \left(\mathbf{p}_{0,\mathsf{set}}^{(k)} - \mathbf{p}_0^{(k)} - E^{(k)}\mathbb{1} - r_d\boldsymbol{\mu}^{(k)}\right)\right\} \tag{3.22}$$

[Step 2]: Each DER device $i \in \mathcal{D}$ measures the output powers $\mathbf{x}_i^{(k)}$ and updates the set-point as follows

$$\begin{aligned}\mathbf{x}_i^{(k+1)} =& \mathsf{proj}_{\mathcal{X}^{(k)}}\Bigg\{\mathbf{x}_i^{(k)} - \alpha\Bigg[\nabla_{\mathbf{x}_i} f_i^{(k)}(\mathbf{x}_i^{(k)}) \\ &+ s^{(k)}(\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\mu}^{(k+1)})^\top \mathbf{M}_i \\ &+ \boldsymbol{\zeta}^{(k+1)}\mathbf{B}_i + (\boldsymbol{\gamma}^{(k+1)} - \boldsymbol{\nu}^{(k+1)})^\top \mathbf{A}_i + r_d\mathbf{x}_i^{(k)}\Bigg]\Bigg\}\end{aligned} \tag{3.23}$$

3.5.2 Low-level controllers

Each controller at the lower-levels will receive two power set-points, $\mathbf{p}_{0,\mathsf{set}}$ and $\mathbf{q}_{0,\mathsf{set}}$. Therefore, we update the problem formulation for these controllers as follows:

$$\min_{\mathbf{x}} \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \tag{3.24}$$

subject to

$$\mathbf{x}_i \in \mathcal{X}_i, \qquad \forall i \in \mathcal{D} \tag{3.25a}$$

$$s^{(k)} \mathbf{I}_3 \left(\mathbf{p}_0^{(k)}(\mathbf{x}) - \mathbf{p}_{0,\mathsf{set}}^{(k)} \right) \le E_p^{(k)} \mathbb{1}_3 \tag{3.25b}$$

$$s^{(k)} \mathbf{I}_3 \left(\mathbf{p}_{0,\mathsf{set}}^{(k)} - \mathbf{p}_0^{(k)}(\mathbf{x}) \right) \le E_p^{(k)} \mathbb{1}_3 \tag{3.25c}$$

$$s^{(k)} \mathbf{I}_3 \left(\mathbf{q}_0^{(k)}(\mathbf{x}) - \mathbf{q}_{0,\mathsf{set}}^{(k)} \right) \le E_q^{(k)} \mathbb{1}_3 \tag{3.25d}$$

$$s^{(k)} \mathbf{I}_3 \left(\mathbf{q}_{0,\mathsf{set}}^{(k)} - \mathbf{q}_0^{(k)}(\mathbf{x}) \right) \le E_q^{(k)} \mathbb{1}_3 \tag{3.25e}$$

$$|\mathbf{v}^{(k)}(\mathbf{x})| \le \overline{\mathbf{v}} \tag{3.25f}$$

$$\underline{\mathbf{v}} \le |\mathbf{v}^{(k)}(\mathbf{x})| \tag{3.25g}$$

$$|\mathbf{i}_L^{(k)}(\mathbf{x})| \le \bar{\mathbf{i}} \tag{3.25h}$$

Similarly, we update the Lagrangian function. We associate two additional dual variables, $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$

with the new constraints Eq. 3.25d-3.25e.

$$
\begin{aligned}
L^{(k)}(\mathbf{x}, \boldsymbol{d^{(k)}}) \coloneqq & \sum_{i\in\mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\
& + \boldsymbol{\lambda}\top \left(s^{(k)} \left(\mathbf{p}_0^{(k)}(\mathbf{x}) - \mathbf{p}_{0,\text{set}}^{(k)} \right) - E_p^{(k)} \mathbb{1}_3 \right) \\
& + \boldsymbol{\mu}^\top \left(s^{(k)} \left(\mathbf{p}_{0,\text{set}}^{(k)} - \mathbf{p}_0^{(k)}(\mathbf{x}) \right) - E_p^{(k)} \mathbb{1}_3 \right) \\
& + \boldsymbol{\eta}^\top \left(s^{(k)} \left(\mathbf{q}_0^{(k)}(\mathbf{x}) - \mathbf{q}_{0,\text{set}}^{(k)} \right) - E_q^{(k)} \mathbb{1}_3 \right) \\
& + \boldsymbol{\psi}^\top \left(s^{(k)} \left(\mathbf{q}_{0,\text{set}}^{(k)} - \mathbf{q}_0^{(k)}(\mathbf{x}) \right) - E_q^{(k)} \mathbb{1}_3 \right) \\
& + \boldsymbol{\gamma}^\top \left(|\mathbf{v}^{(k)}(\mathbf{x})| - \overline{\mathbf{v}} \right) \\
& + \boldsymbol{\nu}^\top \left(\underline{\mathbf{v}} - |\mathbf{v}^{(k)}(\mathbf{x})| \right) \\
& + \boldsymbol{\zeta}^\top \left(|\mathbf{i}_L^{(k)}(\mathbf{x})| - \bar{\mathbf{i}} \right)
\end{aligned}
\tag{3.26}
$$

Which can be re-written as

$$
\begin{aligned}
L^{(k)}(\mathbf{x}, \boldsymbol{d^{(k)}}) \coloneqq & \sum_{i\in\mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\
& + \sum_{i\in\mathcal{D}} \left(s^{(k)} \left[(\boldsymbol{\lambda} - \boldsymbol{\mu})^\top \mathbf{M}_i \mathbf{x}_i + (\boldsymbol{\eta} - \boldsymbol{\psi})^\top \mathbf{H}_i \mathbf{x}_i \right] + (\boldsymbol{\gamma} - \boldsymbol{\nu})^\top \mathbf{A}_i \mathbf{x}_i + \boldsymbol{\zeta}^\top \mathbf{B}_i \mathbf{x}_i \right) \\
& + s^{(k)} \left[(\boldsymbol{\lambda} - \boldsymbol{\mu})^\top (\mathbf{m}^{(k)} - \mathbf{p}_{0,\text{set}}) + (\boldsymbol{\eta} - \boldsymbol{\psi})(\mathbf{h}^{(k)} - \boldsymbol{q}_{0,\text{set}}) \right] \\
& - (\boldsymbol{\lambda} + \boldsymbol{\mu})^\top E_p^{(k)} \mathbb{1} - (\boldsymbol{\eta} + \boldsymbol{\psi})^\top E_q^{(k)} \mathbb{1} \\
& + \boldsymbol{\gamma}^\top (\mathbf{a}^{(k)} - \overline{\mathbf{v}}) + \boldsymbol{\nu}^\top (\underline{\mathbf{v}} - \mathbf{a}^{(k)}) + \boldsymbol{\zeta}^\top (\mathbf{b}^{(k)} - \bar{\mathbf{i}})
\end{aligned}
\tag{3.27}
$$

Algorithm 3 Real-time optimization algorithm

At each t_k

[Step 1]: Collect $|\mathbf{v}^{(k)}|$ at $\mathcal{M}_v$, $|\mathbf{i}^{(k)}|$ at $\mathcal{M}_i$, $\mathbf{p}_0^{(k)}$ and $\mathbf{q}_0^{(k)}$, and perform the following updates to the dual variables:

$$\boldsymbol{\nu}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}} \left\{\boldsymbol{\nu}^{(k)} + \alpha \left(\underline{\mathbf{v}}\mathbb{1} - |\mathbf{v}^{(k)}| - r_d\boldsymbol{\nu}^{(k)}\right)\right\} \tag{3.28}$$

$$\boldsymbol{\gamma}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}} \left\{\boldsymbol{\gamma}^{(k)} + \alpha \left(|\mathbf{v}^{(k)}| - \overline{\mathbf{v}}\mathbb{1} - r_d\boldsymbol{\gamma}^{(k)}\right)\right\} \tag{3.29}$$

$$\boldsymbol{\zeta}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}} \left\{\boldsymbol{\zeta}^{(k)} + \alpha \left(|\mathbf{i}^{(k)}| - \overline{\mathbf{v}}\mathbb{1} - r_d\boldsymbol{\zeta}^{(k)}\right)\right\} \tag{3.30}$$

$$\boldsymbol{\lambda}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^3} \left\{\boldsymbol{\lambda}^{(k)} + \alpha \left(\mathbf{p}_0^{(k)} - \mathbf{p}_{0,\mathsf{set}}^{(k)} - E_p^{(k)}\mathbb{1} - r_d\boldsymbol{\lambda}^{(k)}\right)\right\} \tag{3.31}$$

$$\boldsymbol{\mu}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^3} \left\{\boldsymbol{\mu}^{(k)} + \alpha \left(\mathbf{p}_{0,\mathsf{set}}^{(k)} - \mathbf{p}_0^{(k)} - E_p^{(k)}\mathbb{1} - r_d\boldsymbol{\mu}^{(k)}\right)\right\} \tag{3.32}$$

$$\boldsymbol{\eta}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^3} \left\{\boldsymbol{\eta}^{(k)} + \alpha \left(\mathbf{q}_0^{(k)} - \mathbf{q}_{0,\mathsf{set}}^{(k)} - E_q^{(k)}\mathbb{1} - r_d\boldsymbol{\eta}^{(k)}\right)\right\} \tag{3.33}$$

$$\boldsymbol{\psi}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^3} \left\{\boldsymbol{\psi}^{(k)} + \alpha \left(\mathbf{q}_{0,\mathsf{set}}^{(k)} - \mathbf{q}_0^{(k)} - E_q^{(k)}\mathbb{1} - r_d\boldsymbol{\psi}^{(k)}\right)\right\} \tag{3.34}$$

[Step 2]: Each DER device $i \in \mathcal{D}$ measures the output powers $\mathbf{x}_i^{(k)}$ and updates the set-point as follows

$$\begin{aligned}\mathbf{x}_i^{(k+1)} =& \mathsf{proj}_{\mathcal{X}^{(k)}}\Bigg\{\mathbf{x}_i^{(k)} - \alpha\Bigg[\nabla_{\mathbf{x}_i} f_i^{(k)}(\mathbf{x}_i^{(k)}) \\ &+ s^{(k)}(\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\mu}^{(k+1)})^\top \mathbf{M}_i + s^{(k)}(\boldsymbol{\eta}^{(k+1)} - \boldsymbol{\psi}^{(k+1)})^\top \mathbf{H}_i \\ &+ \boldsymbol{\zeta}^{(k+1)}\mathbf{B}_i + (\boldsymbol{\gamma}^{(k+1)} - \boldsymbol{\nu}^{(k+1)})^\top \mathbf{A}_i + r_d\mathbf{x}_i^{(k)}\Bigg]\Bigg\}\end{aligned} \tag{3.35}$$

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