T&D Coordination

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Chapter 1

Iteration #4

We build iteration #4 based on the multi-phase distribution model presented in [1, 2]. However, we consider a different problem setup in this iteration, and we follow a regularized Lagrangian optimization function, similar to what presented in [3].

1.1 Multi-level controller illustration

In this section, we describe the formulation of a multi-level controller for management of DERs in large distribution system, with a focus on T&D coordination. In particular, we will explain how the sub-controllers within the multi-level scheme communicate in order to meet the the power set-point provided by the TNO.

In Fig. 1.1, we illustrate a multi-feeder distribution network that is connected with the transmission network at the interface bus. The Level 1 controller receives the power set-point $\mathbf{P}^{\star}$ from the transmission network, and then generates the set-point $P_0^{f_i}$ for the i^{th} feeder such that:

$$\mathbf{P}^{\star} = \sum_i P_0^{f_i}. \tag{1.1}$$

It is important to note that $\mathbf{P}^{\star}$ and $P_0^{f_i}$ are scalar values. The controllers would track the set-points by changing the power flows in the phases such that the $\sum\limits_{\phi} P_0^{f_i,\phi} = P_0^{f_i}$.

As a simple example, the Level 1 controller could define $P_0^{f_i}$ via a set of participation factors

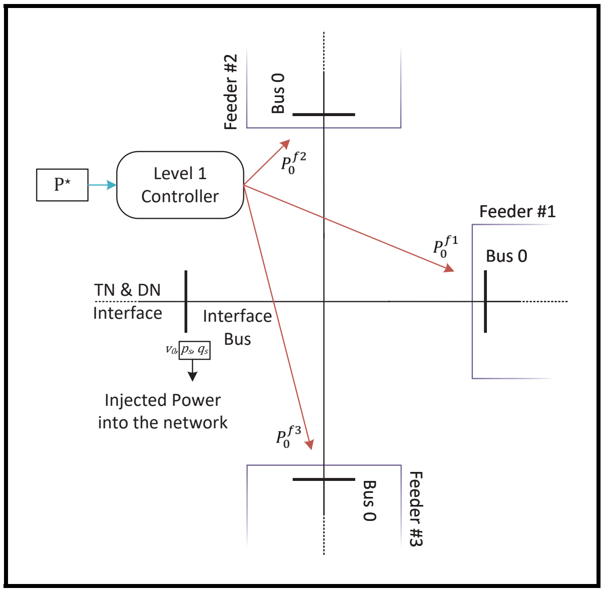


Figure 1.1: Level 1 controller in Distribution Network and feeders networks diagram.

for each distribution feeder, i.e.,

$$P_0^{f_i} = \alpha_i \mathbf{P}^\star$$

where $\sum_i \alpha_i = 1$ and $\alpha_i > 0$. The participation factors could be defined a priori, or could be computed as a result of an optimization problem. One approach could be based on the available DER-powers (reserves) in each feeder. For such implementation, a power estimation problem can be considered following the approach in [4]. With this approach, one can estimate the available power (flexibility) in a feeder, which can be a function running all the time. This function updates the flexibility estimation periodically (every 15 mins for example). Then, Level 1 controller can use this information, along with cost function for example (assuming different operators and different costs). The Level 1 controller would change the participation factors for each feeder based on the information it receives. The development of such controller is beyond the scope of this research problem considered here.

1.2 Feeder multi-level control structure

In each feeder, we consider a multi-level control structure that divides the feeder into smaller control areas. Fig. 1.2 illustrates the generic structure of this multi-level control scheme, consisting of two types of controllers: (i) Level 2 controllers and (ii) lower level controllers.

1.2.1 Level 2 Controller

The Level 2 controller is the highest level controller for each feeder. This controller is responsible for providing set-points to the lower level controllers within the feeder, and is ultimately responsible for tracking the set-point $\mathrm{p}_{0,\mathsf{set}} := P_0^{f_i}$ provided by Level 1 controller. The available inputs to the Level 2 controller are any available voltage and current measurements within its "control area", including current and voltage measurements at the feeder head. The controller generates updated "dual variables" and broadcasts them to the DERs and DER aggregators within its control area. Each DER and DER aggregator evaluates and updates its P and Q set-points. The definition of "control area" and "DER aggregator" will be discussed in Section 1.2.3.

1.2.2 Lower Level Controller

Lower level controllers are similar to Level 2 controllers in their structure and operation; however, they differ in the power set-point provided to them. Lower level controllers need to track two power set-points, P and Q. Like the Level 2 controller, the lower level controllers receive voltage and current measurements from within their control areas, and analogously update their dual variables. DERs and DER aggregators within the control area update their P and Q set-points accordingly.

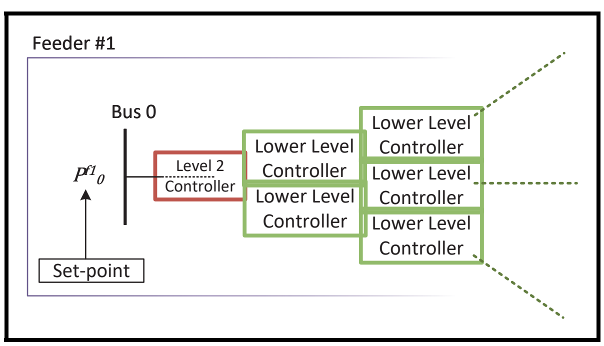


Figure 1.2: Feeder control structure.

1.2.3 Feeder Example

We now describe the meaning of “control areas”, DER aggregators, and how the feeder is divided into control areas. The following example is used to help clarify these concepts.

Consider the example feeder shown in Figure 1.3. This feeder consists of 19 buses. At each bus we assume that we have a non-controllable load, a DER device, or both. The feeder is partitioned into layered control areas, shown in red, green, and blue boxes. These control areas may represent contractual arrangements for the management of DER resources via aggregators, or may be defined based on other operational criteria.

The resources within the red box are managed by the Level 2 controller. This controller has visibility over the system within the red box, but does not have visibility within the green boxes or beyond into the blue boxes. The boundary of the red box is defined by the two interface buses, Bus 2 and Bus 3, which connect to new areas that are independently managed by DER aggregators. The Level 2 controller will provide P and Q set-points for each interface bus. The green boxes indicate control areas under the responsibility of DER aggregators. These DER aggregators are responsible for managing DER resources and maintaining operational constraints within the control area, and for tracking the set-points provided by the Level 2 controller. This hierarchical structure can then be further nested, with the blue boxes indicating an additional layer of DER aggregators; this structure can be repeated as required based on operational/information constraints, until all DER and DER aggregators are covered in the control-areas. The green box DER aggrgeators do not have visibility into the blue control areas, but provide set-points to the interface buses between the green and blue control areas.

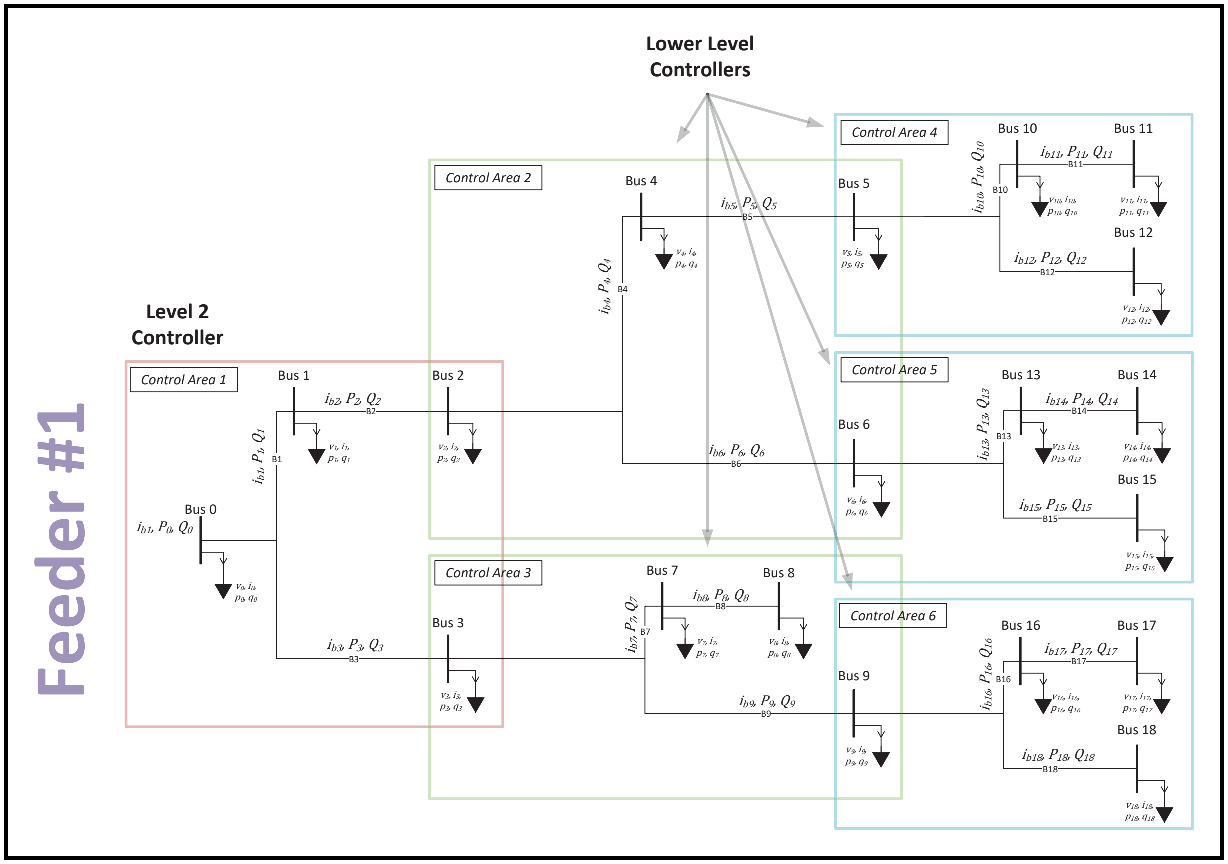
The DER aggregators in each control area would be represented by their Thevenin Equivalent circuit/components. This is necessary for DN modeling equation. The Thevenin equivalent will be provided by the DER aggregator to the controller. The DER aggregator shall update its Thevenin equivalent model with each iteration, or periodically. More information about this implementation can be found in [5, 6].

To summarize the role of each controller and the DER/DER aggregators, we list them below:

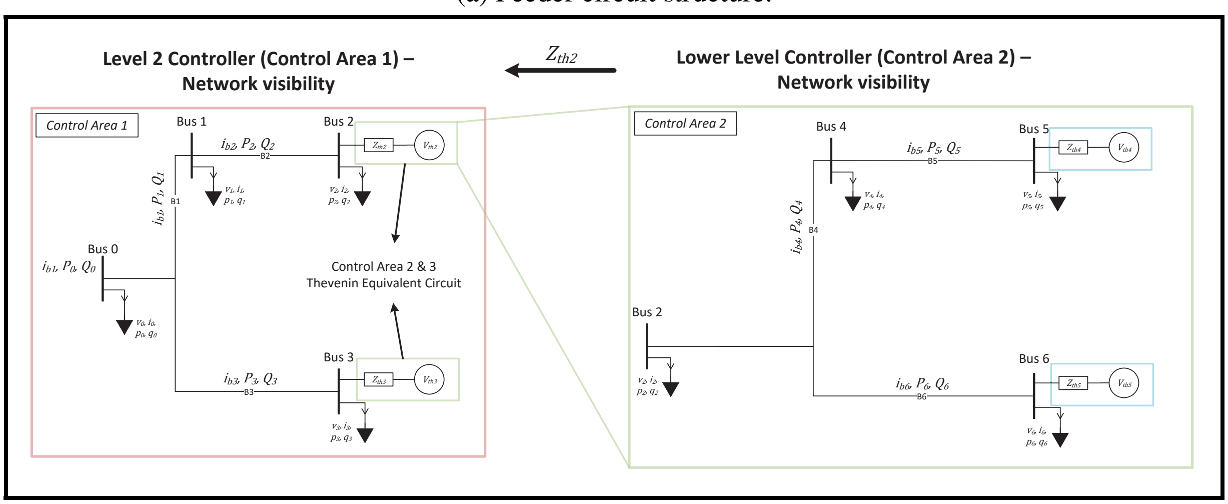
- Level 2 Controller
  - ◇ Receive $\mathrm{p}_{0,\mathsf{set}}$

◇ Acquire measurements from the visible block ($V_{\text{meas}}, i_{\text{meas}}, \mathbf{p}_0$)

◇ Generates updated dual variables $\mathbf{d}$ and broadcast them to connected DERs (DER devices and aggregators)



(a) Feeder circuit structure.



(b) Control area and controller network visibility. Control area 1 and control area 2 considered here.

Figure 1.3: Feeder Example.

- Lower Level Controllers
  - ◇ Receive $\mathrm{p}_{0,\mathsf{set}}$ and $\mathrm{q}_{0,\mathsf{set}}$
  - ◇ Acquire measurements from the visible block ($V_{\mathsf{meas}}, i_{\mathsf{meas}}, \mathbf{p}_0, \mathbf{q}_0$)
  - ◇ Generates updated dual variables $\mathbf{d}$ and broadcast them to connect DERs (DER devices and aggregators)
- DERs (devices and aggregators)
  - ◇ Receive updated dual variables $\mathbf{d}$
  - ◇ Acquire output power $\mathbf{x}_i$ measurements
  - ◇ Generates updated set points $\mathrm{p}_{0,\mathsf{set}}$ and $q_{0,\mathsf{set}}{}^*$
  - * if DER aggregator, the updated set-points will be fed to the lower level controller as the new/updated set-points

1.3 DER Parameters

- $\mathcal{P} := \{a, b, c\} \cup \{ab, bc, ca\}$ set of possible connections ($\{a, b, c\}$ representing wye-connection (line-to-ground) and $\{ab, bc, ca\}$ representing delta-connection (phase-to-phase))
- Let $\mathcal{D}$ represent the set of DERs that are controlled directly by the main controller (the set includes DERs and group of DERs that controlled by one controller)
- Let $\boxed{\mathbf{x}_i = [P_i, Q_i]^\top \in \mathbb{R}^2}$ be the active and reactive power set-points of the i^{th} DER. Note that the units of P_i and Q_i are W and Var.
- We define $\mathcal{P}_i \subset \mathcal{P}$ as the set that collects the phases where DER i is connected.
- For any DER $i \in \mathcal{D}$, we define $\boxed{\mathcal{X}_i \subset \mathbb{R}^2 \text{ as the set of possible power set-points } \mathbf{x}_i}$.
- We define a $\boxed{\text{cost function } f_i(\mathbf{x}_i)}$ for each DER, for which the gradient of $f_i(x_i)$ has the units of W and Var. (Hence, the cost function is a quadratic function of p and q)

examples of $f(\mathrm{x})$ [3]

$$f_i(\mathbf{x_i}) = \begin{cases} (p_{i,av} - p_i)^2 + (q_i)^2 & \text{, for three-phase PV systems, where } p_{i,av} \text{ is} \\ & \text{max. real power available} \\ 100\,(p_{i,av} - p_i)^2 + 10\,(q_i)^2 & \text{, for single-phase PV systems} \\ (p_{i,\phi})^2 + (q_{i,\phi})^2 & \text{, for batteries} \\ 100\,(p_i - p_{i,max})^2 & \text{, for EVs, where } p_{i,max} \text{ is the max. charging rate} \end{cases} \quad (1.2)$$

1.4 Distribution Network Model

We now describe the distribution network model which will be integrated into the controller within each control area.

1.4.1 Basic Notation

- $\boxed{\mathcal{N} = \{1, 2, \ldots, N\}\text{: the set of buses (all considered as PQ buses)}}$. We consider node "0" as the point of connection with the rest of the electrical system (for example $\mathbf{p}_0$ represents the power injection at the interface bus/node)
- $\mathbf{s}_j^Y = \left[s_j^a, s_j^b, s_j^c\right]^\top \in \mathbb{C}^3$: the complex power injections at each phase at bus j for Y grounded sources
- $\mathbf{s}_j^\Delta = \left[s_j^{ab}, s_j^{bc}, s_j^{ca}\right]^\top \in \mathbb{C}^3$: the complex power injections for delta-connected sources.
- $\mathbf{v}_j = \left[v_j^a, v_j^b, v_j^c\right]^\top \in \mathbb{C}^3$, $\mathbf{i}_j = \left[i_j^a, i_j^b, i_j^c\right]^\top \in \mathbb{C}^3$ and $\mathbf{i}_j^\Delta = \left[i_j^{ab}, i_j^{bc}, i_j^{ca}\right]^\top \in \mathbb{C}^3$: phase-to-ground voltages, phase net current injections and phase-to-phase currents at bus j.
- $\mathbf{v}_0 = \left[v_0^a, v_0^b, v_0^c\right]^\top \in \mathbb{C}^3$: voltages at the slack bus (interface bus)
- $\mathbf{v} = \left[\mathbf{v}_1^\top, \ldots, \mathbf{v}_N^\top\right]^\top$: voltages at PQ buses (network buses)
- $\mathbf{i} = \left[\mathbf{i}_1^\top, \ldots, \mathbf{i}_N^\top\right]^\top$: current injections PQ buses
- $\mathbf{i}^\Delta = \left[\mathbf{i}_1^{\Delta\top}, \ldots, \mathbf{i}_N^{\Delta\top}\right]^\top$: phase-to-phase currents at PQ buses

- $\mathbf{s}^Y = \left[\mathbf{s}_1^{Y\top}, \ldots, \mathbf{s}_N^{Y\top}\right]^\top$: wye sources at PQ buses

- $\mathbf{s}^\Delta = \left[\mathbf{s}_1^{\Delta\top}, \ldots, \mathbf{s}_N^{\Delta\top}\right]^\top$: delta sources at PQ buses

- $\mathbf{Y}$ is three-phase admittance matrix

$$\boxed{\mathbf{Y} := \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{0L} \\ \mathbf{Y}_{L0} & \mathbf{Y}_{LL} \end{bmatrix} \in \mathbb{C}^{3(N+1)\times 3(N+1)}} \tag{1.3}$$

- $\mathbf{Y}_{00} \in \mathbb{C}^{3\times 3}$, $\mathbf{Y}_{L0} \in \mathbb{C}^{3N\times 3}$, $\mathbf{Y}_{0L} \in \mathbb{C}^{3\times 3N}$ and $\mathbf{Y}_{LL} \in \mathbb{C}^{3N\times 3N}$

- $\mathbf{G}$ is $3N \times 3N$ block diagonal matrix (phase-to-ground to phase-to-phase voltage 3-phase conversion matrix).

$$\mathbf{G} := \begin{bmatrix} \mathbf{\Gamma} & & \\ & \ddots & \\ & & \mathbf{\Gamma} \end{bmatrix}, \qquad \mathbf{\Gamma} := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \tag{1.4}$$

- $\mathbf{w}$ is zero load voltage profile

- Let $\mathcal{M}_v$ denote the locations/buses where voltage measurements (phase-to-ground) are available, and let $\boxed{\mathbf{v}_{\mathcal{M}_v} \text{ be the vector of measurements}}$. $\boxed{\text{We denote } |\mathcal{M}_v| = r_v}$

- Let $\boxed{\mathbf{i}_{L,\mathcal{M}_i} \text{ denotes the vector of line currents for a subset of monitored distribution lines } \mathcal{M}_i}$ (this can be measured or given by pseudo-measurements, possibly relying on network model and voltage measurements, or using previous measurements). $\boxed{\text{We denote } |\mathcal{M}_i| = r_i}$

- $\mathbf{p}^Y, \mathbf{q}^Y, \mathbf{p}^\Delta, \mathbf{q}^\Delta$: active and reactive power injections (from $\mathbf{s}^Y$ & $\mathbf{s}^\Delta$)

- $\mathbf{x}^Y = \left[\mathbf{p}^{Y\top}, \mathbf{q}^{Y\top}\right]^\top$

- $\mathbf{x}^\Delta = \left[\mathbf{p}^{\Delta\top}, \mathbf{q}^{\Delta\top}\right]^\top$

- Let $\boxed{\mathbf{p}_0 \in \mathbb{R}^3 \text{ denotes the real-power entering interface bus}}$ at phases $\{a, b, c\}$.

- Let $\boxed{\mathbf{q}_0 \in \mathbb{R}^3 \text{ denotes the reactive-power entering interface bus}}$ at phases $\{a, b, c\}$. Note that this would be utilized in the lower level controllers.

1.4.2 Voltage Phasors

$$\tilde{\mathbf{v}} = \mathbf{M}^Y \mathbf{x}^Y + \mathbf{M}^\Delta \mathbf{x}^\Delta + \mathbf{a} \tag{1.5a}$$

where

$$\mathbf{M}^Y = \left[\mathbf{Y}_{LL}^{-1} \mathsf{diag}(\overline{\widehat{\mathbf{v}}})^{-1}, -\jmath \mathbf{Y}_{LL}^{-1} \mathsf{diag}(\widehat{\mathbf{v}})^{-1}\right], \qquad \text{in} \quad \frac{1}{SV} = \frac{\Omega}{V} = \frac{1}{A} = \frac{V}{W} \tag{1.5b}$$

$$\mathbf{M}^\Delta = \left[\mathbf{Y}_{LL}^{-1} \mathbf{G}^\top \mathsf{diag}\left(\mathbf{G}\overline{\widehat{\mathbf{v}}}\right)^{-1}, -\jmath \mathbf{Y}_{LL}^{-1} \mathbf{G}^\top \mathsf{diag}\left(\mathbf{G}\widehat{\mathbf{v}}\right)^{-1}\right], \quad \text{in} \quad \frac{1}{SV} = \frac{\Omega}{V} = \frac{1}{A} = \frac{V}{W} \tag{1.5c}$$

$$\mathbf{a} = \mathbf{w}, \qquad \text{in} \quad V \tag{1.5d}$$

1.4.3 Voltage magnitudes

$$|\tilde{\mathbf{v}}| = \mathbf{K}^Y \mathbf{x}^Y + \mathbf{K}^\Delta \mathbf{x}^\Delta + \mathbf{b} \tag{1.6a}$$

where

$$\mathbf{W} = \mathsf{diag}(\mathbf{w}), \qquad \text{in} \quad V \tag{1.6b}$$

$$\mathbf{K}^Y = |\mathbf{W}| \, \mathsf{Re}\{\mathbf{W}^{-1} \mathbf{M}^Y\}, \qquad \text{in} \quad \frac{1}{A} = \frac{V}{W} \tag{1.6c}$$

$$\mathbf{K}^\Delta = |\mathbf{W}| \, \mathsf{Re}\{\mathbf{W}^{-1} \mathbf{M}^\Delta\}, \qquad \text{in} \quad \frac{1}{A} = \frac{V}{W} \tag{1.6d}$$

$$\mathbf{b} = |\mathbf{w}|, \qquad \text{in} \quad V \tag{1.6e}$$

1.4.4 Power flow at the interface bus/substation

$$\tilde{\mathbf{s}}_0 = \mathbf{G}^Y \mathbf{x}^Y + \mathbf{G}^\Delta \mathbf{x}^\Delta + \mathbf{c} \tag{1.7a}$$

where

$$\mathbf{G}^{Y} = \mathsf{diag}(\mathbf{v}_0)\overline{\mathbf{Y}}_{0L}\overline{\mathbf{M}}^{Y}, \qquad \text{in} \quad \frac{VS}{SV} = \frac{1}{1} = \text{unitless} \tag{1.7b}$$

$$\mathbf{G}^{\Delta} = \mathsf{diag}(\mathbf{v}_0)\overline{\mathbf{Y}}_{0L}\overline{\mathbf{M}}^{\Delta}, \qquad \text{in} \quad \frac{VS}{SV} = \frac{1}{1} = \text{unitless} \tag{1.7c}$$

$$\mathbf{c} = \mathsf{diag}(\mathbf{v}_0)\left(\overline{\mathbf{Y}}_{00}\overline{\mathbf{v}}_0 + \overline{\mathbf{Y}}_{0L}\overline{\mathbf{a}}\right), \qquad \text{in} \quad V(SV + SV) = V(\frac{W}{V}) = W \tag{1.7d}$$

1.4.5 Branch currents

$$\tilde{\mathbf{i}}_{ij} = \mathbf{J}_{ij}^{Y}\mathbf{x}^{Y} + \mathbf{J}_{ij}^{\Delta}\mathbf{x}^{\Delta} + \mathbf{c}_{ij} \tag{1.8a}$$

where

$$\mathbf{Z}_{ij} \in \mathbb{C}^{3\times 3} \text{ phase impedance matrix of line } (i,j), \qquad \text{in} \quad \Omega \tag{1.8b}$$

$$\mathbf{Y}_{ij}^{(s)} \in \mathbb{C}^{3\times 3} \text{ shunt admittance matrix of line } (i,j), \qquad \text{in} \quad S \tag{1.8c}$$

$$\mathbf{E}_i = \left[\mathbf{0}_{3\times 3(i-1)}, \mathbf{I}_3, \mathbf{0}_{3\times 3(N-i)}\right], \qquad \text{unitless} \tag{1.8d}$$

$$\mathbf{J}_{ij}^{Y} = \left[\left(\mathbf{Y}_{ij}^{(s)} + \mathbf{Z}_{ij}^{-1}\right)\mathbf{E}_i - \mathbf{Z}_{ij}^{-1}\mathbf{E}_j\right]\mathbf{M}^{Y}, \qquad \text{in} \quad [S]\frac{V}{W} = \frac{SV}{W} = \frac{A}{W} \tag{1.8e}$$

$$\mathbf{J}_{ij}^{\Delta} = \left[\left(\mathbf{Y}_{ij}^{(s)} + \mathbf{Z}_{ij}^{-1}\right)\mathbf{E}_i - \mathbf{Z}_{ij}^{-1}\mathbf{E}_j\right]\mathbf{M}^{\Delta}, \qquad \text{in} \quad [S]\frac{V}{W} = \frac{SV}{W} = \frac{A}{W} \tag{1.8f}$$

$$\mathbf{c}_{ij} = \left[\left(\mathbf{Y}_{ij}^{(s)} + \mathbf{Z}_{ij}^{-1}\right)\mathbf{E}_i - \mathbf{Z}_{ij}^{-1}\mathbf{E}_j\right]\mathbf{w}, \qquad \text{in} \quad [S]V = A \tag{1.8g}$$

1.5 Modified Equations of distribution network model

Our goal is to obtain a linearized model describing the distribution feeder of the following form

$$|\tilde{\mathbf{v}}_{\mathcal{M}_v}(\mathbf{x})| = \sum_{i\in\mathcal{D}} \mathbf{A}_i\mathbf{x}_i + \mathbf{a} \qquad = \mathbf{A}\mathbf{x} + \mathbf{a} \tag{1.9a}$$

$$|\tilde{\mathbf{i}}_{L,\mathcal{M}_i}(\mathbf{x})| = \sum_{i\in\mathcal{D}} \mathbf{B}_i\mathbf{x}_i + \mathbf{b} \qquad = \mathbf{B}\mathbf{x} + \mathbf{b} \tag{1.9b}$$

$$\tilde{\mathbf{p}}_0(\mathbf{x}) = \sum_{i\in\mathcal{D}} \mathbf{M}_i\mathbf{x}_i + \mathbf{m} \qquad = \mathbf{M}\mathbf{x} + \mathbf{m} \tag{1.9c}$$

$$\tilde{\mathbf{q}}_0(\mathbf{x}) = \sum_{i\in\mathcal{D}} \mathbf{H}_i\mathbf{x}_i + \mathbf{h} \qquad = \mathbf{H}\mathbf{x} + \mathbf{h} \tag{1.9d}$$

where $\mathbf{x} = \mathrm{col}(\mathbf{x}_1, \ldots, \mathbf{x}_{|\mathcal{D}|})$ is the vector of all DER/DER-aggregator set-points and all other symbols are constants defined based on the network information. The "$\sim$" over the variables indicates that these are not the true values, but the values obtained via a linearized model.

1.5.1 Mapping $\mathrm{A}, \mathrm{B}, \mathrm{M}, \mathrm{H}, \mathrm{a}, \mathrm{b}, \mathrm{m}$ & h to multi-phase distribution network model matrices (Eqs. 1.6a, 1.7a & 1.8a

Important note: x^Y & x^Δ in Section 1.4 model are defined as follows:

- $\mathbf{p}^Y, \mathbf{q}^Y, \mathbf{p}^\Delta, \mathbf{q}^\Delta$: active and reactive power injections (from s^Y & s^Δ)
- $\mathbf{x}^Y = \left[\mathbf{p}^{Y\top}, \mathbf{q}^{Y\top}\right]^\top$
- $\mathbf{x}^\Delta = \left[\mathbf{p}^{\Delta\top}, \mathbf{q}^{\Delta\top}\right]^\top$

(i) A matrix

From Eq. 1.6a, we have that

$$\begin{aligned}
|\tilde{\mathbf{v}}| &= \mathbf{K}^Y \mathbf{x}^Y + \mathbf{K}^\Delta \mathbf{x}^\Delta + \mathbf{b} \\
&= \begin{bmatrix} \mathbf{K}^Y & \mathbf{K}^\Delta \end{bmatrix} \begin{bmatrix} \mathbf{x}^Y \\ \mathbf{x}^\Delta \end{bmatrix} + \mathbf{b} \\
&= \underbrace{\begin{bmatrix} \mathbf{K}^Y & \mathbf{K}^\Delta \end{bmatrix}}_{3N\times 12N} \underbrace{\begin{bmatrix} \mathbf{p}^Y \\ \mathbf{q}^Y \\ \mathbf{p}^\Delta \\ \mathbf{q}^\Delta \end{bmatrix}}_{12N\times 1} + \mathbf{b}
\end{aligned} \tag{1.10}$$

To extract only the measured values $|\mathbf{v}_{\mathcal{M}_\mathbf{v}}|$ out of $|\tilde{\mathbf{v}}|$, and to express the right-hand side as a function only of the DER point setpoints $\mathbf{x}$, we do the following manipulations. First, we split the powers $\mathbf{p}$ and $\mathbf{q}$ to DER and non-DER components, and then second, we extract the required components. To do this, let $\mathbf{Q}_L^v \in \mathbb{R}^{3r_v \times 3N}$ select the desired measured voltages from the vector of all voltages, with ith row defined by

$$(Q_L^v)_{ij} = \begin{cases} 1 & \text{if the } i\text{th voltage measurement is from bus } j \\ 0 & \text{otherwise} \end{cases}$$

and then set $\mathbf{Q}_L^v = Q_L^v \otimes I_3$. With this notation, we have that

$$|\tilde{\mathbf{v}}_{\mathcal{M}_v}| = \mathbf{Q}_L^v |\tilde{\mathbf{v}}|.$$

Next, we wish to express $|\tilde{\mathbf{v}}|$ in terms of $\mathbf{x}$, and we will define a matrix $\mathbf{Q}_R$ such that

$$\begin{bmatrix} \mathbf{p}^Y \\ \mathbf{q}^Y \\ \mathbf{p}^\Delta \\ \mathbf{q}^\Delta \end{bmatrix} = \mathbf{Q}_R \mathbf{x} + \mathbf{c}$$

Effectively, $\mathbf{Q}_R$ places DER powers at the correct phases, accounting for their method of interconnection. The elements are defined as

$$
\underbrace{|\mathbf{v}_{\mathcal{M}_v}|}_{3r_v\times 1} = \overbrace{\underbrace{\mathbf{Q}_L^v}_{3r_v\times 3N}\underbrace{\begin{bmatrix}\mathbf{K}^Y & \mathbf{K}^\Delta\end{bmatrix}}_{3N\times 12N}\underbrace{\mathbf{Q}_R}_{12N\times 2D}}^{\mathbf{A}}\underbrace{\overbrace{\begin{bmatrix}p_1\\ q_1\\ \vdots\\ p_D\\ q_D\end{bmatrix}}^{\mathbf{x}}}_{2D\times 1} + \overbrace{\mathbf{Q}_L^v\left(\begin{bmatrix}\mathbf{K}^Y & \mathbf{K}^\Delta\end{bmatrix}\begin{bmatrix}\mathbf{p}^Y_{\mathsf{non-DER}}\\ \mathbf{q}^Y_{\mathsf{non-DER}}\\ \mathbf{p}^\Delta_{\mathsf{non-DER}}\\ \mathbf{q}^\Delta_{\mathsf{non-DER}}\end{bmatrix}+\mathbf{b}\right)}^{\mathbf{a}} \tag{1.11a}
$$

where

- $\mathbf{Q}_R \in \mathbb{R}^{12N\times 2D}$, for DER j connect at bus i (here $e_j \in \mathbb{R}^{2D}$)

  ◇ Y-connected balanced DER

$$
\begin{bmatrix}\mathbf{Q}_{R,\mathsf{rows}(i:i+2)}\\ \mathbf{Q}_{R,\mathsf{rows}(3N+i:3N+i+2)}\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\mathbb{1}_3 e_{2j-1}{}^\top\\ \frac{1}{3}\mathbb{1}_3 e_{2j}{}^\top\end{bmatrix} \tag{1.11b}
$$

  ◇ Single-phase connected DER

$$
\begin{bmatrix}\mathbf{Q}_{R,\mathsf{rows}(i:i+2)}\\ \mathbf{Q}_{R,\mathsf{rows}(3N+i:3N+i+2)}\end{bmatrix} = \begin{cases}\begin{bmatrix}e_{2j-1} & \mathbb{0} & \mathbb{0} & e_{2j} & \mathbb{0} & \mathbb{0}\end{bmatrix}^\top & \text{, if connected at phase } a\\ \begin{bmatrix}\mathbb{0} & e_{2j-1} & \mathbb{0} & \mathbb{0} & e_{2j} & \mathbb{0}\end{bmatrix}^\top & \text{, if connected at phase } b\\ \begin{bmatrix}\mathbb{0} & \mathbb{0} & e_{2j-1} & \mathbb{0} & \mathbb{0} & e_{2j}\end{bmatrix}^\top & \text{, if connected at phase } c\end{cases} \tag{1.11c}
$$

  ◇ Δ-connected balanced DER

$$
\begin{bmatrix}\mathbf{Q}_{R,\mathsf{rows}(6N+i:6N+i+2)}\\ \mathbf{Q}_{R,\mathsf{rows}(9N+i:9N+i+2)}\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\mathbb{1}_3 e_{2j-1}{}^\top\\ \frac{1}{3}\mathbb{1}_3 e_{2j}{}^\top\end{bmatrix} \tag{1.11d}
$$

  ◇ phase-to-phase connected DER

$$
\begin{bmatrix}\mathbf{Q}_{R,\mathsf{rows}(6N+i:6N+i+2)}\\ \mathbf{Q}_{R,\mathsf{rows}(9N+i:9N+i+2)}\end{bmatrix} = \begin{cases}\begin{bmatrix}e_{2j-1} & \mathbb{0} & \mathbb{0} & e_{2j} & \mathbb{0} & \mathbb{0}\end{bmatrix}^\top & \text{, if } ab\text{-phase connected}\\ \begin{bmatrix}\mathbb{0} & e_{2j-1} & \mathbb{0} & \mathbb{0} & e_{2j} & \mathbb{0}\end{bmatrix}^\top & \text{, if } bc\text{-phase connected}\\ \begin{bmatrix}\mathbb{0} & \mathbb{0} & e_{2j-1} & \mathbb{0} & \mathbb{0} & e_{2j}\end{bmatrix}^\top & \text{, if } ca\text{-phase connected}\end{cases} \tag{1.11e}
$$

- ◇ Other connections - treat similarly to what presented here. We can have two phase-to-phase, or phase-to-ground connected DERs ("ab" and "bc", or "a" and "b" for example). In both cases, we consider their corresponding e_j vectors, and we scale them by $1/2$.

- Lastly, we define the matrix $\mathbf{A}$ as follows

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & \cdots & \mathbf{A}_D\end{bmatrix} \tag{1.11f}$$

where $\mathbf{A}_i \in \mathbb{R}^{3r_v \times 2}, \forall i \in \{1, \dots, D\}$

Note that the both, $\mathbf{Q}_L^v$ and $\mathbf{Q}_R$ are "unitless". Therefore, the unit of $\mathbf{A}$ is the same as $\mathbf{K}^Y$ and $\mathbf{K}^\Delta = \frac{V}{W}$.

(ii) B matrix

Following a similar approach, we can achieve the following. We start with Eq. 1.8a.

$$\tilde{\mathbf{i}}_{ij} = \mathbf{J}_{ij}^Y \mathbf{x}^Y + \mathbf{J}_{ij}^\Delta \mathbf{x}^\Delta + \mathbf{c}_{ij} \tag{1.12}$$

Let the line ij map to the branch $k \in \mathcal{M}_i$. Then, we combine the J matrices, and collect all constants in new $\mathbf{b}$, to get the updated equation of $\tilde{\mathbf{i}}_k$

$$\tilde{\mathbf{i}}_k = \tilde{\mathbf{i}}_{ij} = \begin{bmatrix}\mathbf{J}_{ij}^Y & \mathbf{J}_{ij}^\Delta\end{bmatrix} \begin{bmatrix}\mathbf{x}^Y \\ \mathbf{x}^\Delta\end{bmatrix} + \mathbf{c}_{ij} \tag{1.13a}$$

$$\tilde{\mathbf{i}}_k = \overbrace{\begin{bmatrix}\mathbf{J}_{ij}^Y & \mathbf{J}_{ij}^\Delta\end{bmatrix} \mathbf{Q}_R}^{\overline{\mathbf{B}}_k} \overbrace{\begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_D \\ q_D \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix}\mathbf{J}_{ij}^Y & \mathbf{J}_{ij}^\Delta\end{bmatrix} \begin{bmatrix}\mathbf{p}_{\mathsf{non-DER}}^Y \\ \mathbf{q}_{\mathsf{non-DER}}^Y \\ \mathbf{p}_{\mathsf{non-DER}}^\Delta \\ \mathbf{q}_{\mathsf{non-DER}}^\Delta\end{bmatrix} + \mathbf{c}_{ij}}^{\overline{\mathbf{b}}_k} \tag{1.13b}$$

where $\overline{\mathbf{B}}_k \in \mathbb{R}^{3\times 2D}$. To calculate the current in all branches (similar to how we calculate the voltage), we can stack $\overline{\mathbf{B}}_k$ matrices next to each other to form a new $\mathbf{B} = \begin{bmatrix}\overline{\mathbf{B}}_1^\top & \cdots & \overline{\mathbf{B}}_{r_i}^\top\end{bmatrix}^\top$. We then can re-partition $\mathbf{B}$ per DER device (rather than per branch) as follows: $\mathbf{B} = \begin{bmatrix}\mathbf{B}_1 & \cdots & \mathbf{B}_{|\mathcal{D}|}\end{bmatrix}$, where $\mathbf{B}_i \in \mathbb{R}^{3r_i\times 2}$.

The unit of $\mathbf{B}$ is the same as $\mathbf{J}^Y$ and $\mathbf{J}^\Delta = \frac{A}{W}$.

(iii & iv) $\mathbf{M}$ & $\mathbf{H}$ matrices

We start with Eq. 1.7a.

$$\tilde{\mathbf{s}}_0 = \mathbf{G}^Y\mathbf{x}^Y + \mathbf{G}^\Delta\mathbf{x}^\Delta + \mathbf{c} \tag{1.14}$$

We then combine G matrices and collect all constants in g as follows:

$$\tilde{\mathbf{s}}_0 = \begin{bmatrix}\mathbf{G}^Y & \mathbf{G}^\Delta\end{bmatrix}\begin{bmatrix}\mathbf{x}^Y \\ \mathbf{x}^\Delta\end{bmatrix} + \mathbf{c} \tag{1.15a}$$

$$\tilde{\mathbf{s}}_0 = \overbrace{\begin{bmatrix}\mathbf{G}^Y & \mathbf{G}^\Delta\end{bmatrix}\mathbf{Q}_R}^{\overline{\mathbf{G}}}\overbrace{\begin{bmatrix}p_1 \\ q_1 \\ \vdots \\ p_D \\ q_D\end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix}\mathbf{G}^Y & \mathbf{G}^\Delta\end{bmatrix}\begin{bmatrix}\mathbf{p}^Y_{\text{non}-\text{DER}} \\ \mathbf{q}^Y_{\text{non}-\text{DER}} \\ \mathbf{p}^\Delta_{\text{non}-\text{DER}} \\ \mathbf{q}^\Delta_{\text{non}-\text{DER}}\end{bmatrix} + \mathbf{c}}^{\mathbf{g}} \tag{1.15b}$$

$$\tag{1.15c}$$

We then define $\mathbf{M}, \mathbf{H}, \mathbf{m}$ & $\mathbf{h}$ based on $\overline{\mathbf{G}}$ and g as follows:

$$\mathbf{M} = \mathsf{Re}(\overline{\mathbf{G}}) \tag{1.15d}$$

$$\mathbf{H} = \mathsf{Im}(\overline{\mathbf{G}}) \tag{1.15e}$$

$$\mathbf{m} = \mathsf{Re}(\mathbf{g}) \tag{1.15f}$$

$$\mathbf{h} = \mathsf{Im}(\mathbf{g}) \tag{1.15g}$$

The units of $\mathbf{M}$ & $\mathbf{H}$ are the same of $\overline{\mathbf{G}}$. $\overline{\mathbf{G}}$ has the same unit of $\mathbf{G}^{Y}$ and $\mathbf{G}^{\Delta}$, which are "unitless".

1.6 Controller and optimization problem setup

1.6.1 Level 2 Controller

$$\min_{\mathbf{x}} \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \tag{1.16}$$

subject to

$$\mathbf{x}_i \in \mathcal{X}_i, \qquad \forall i \in \mathcal{D} \tag{1.17a}$$

$$\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right)\ \mathbf{p}_0^{(k)}(\mathbf{x}) \leq 0 \tag{1.17b}$$

$$s^{(k)} \left(|\underbrace{\mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x})}_{\mathbf{p}_0^a + \mathbf{p}_0^b + \mathbf{p}_0^c} - \mathrm{p}_{0,\mathsf{set}}^{(k)}| \right) \leq E^{(k)} \tag{1.17c}$$

$$\underline{\mathbf{v}} \leq |\mathbf{v}^{(k)}(\mathbf{x})| \leq \overline{\mathbf{v}} \tag{1.17d}$$

$$|\mathbf{i}_L^{(k)}(\mathbf{x})| \leq \bar{\mathbf{i}}, mko \tag{1.17e}$$

where:

- k is the time-step (discrete)
- $f_i^{(k)}$ is a time-varying convex function associated with the i^{th} DER (or DER agg.)
- $s^{(k)} \in \{0, 1\}$ enable/disable the controller from tracking the power set-point
- $\mathrm{p}_{0,\mathsf{set}}^{(k)}$ is the power set-point at the interface bus
- $E^{(k)}$ is the tolerance or accuracy

To solve this optimization problem, we follow the approach in [3]. We form the Lagrangian function. To build this, we re-write the constraints as follows:

$$\mathbf{x}_i \in \mathcal{X}_i, \qquad \forall i \in \mathcal{D} \tag{1.18a}$$

$$\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right)\ \mathbf{p}_0^{(k)}(\mathbf{x}) \leq \mathbb{0}_3 \tag{1.18b}$$

$$s^{(k)}\left(\mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x}) - \mathrm{p}_{0,\mathsf{set}}^{(k)}\right) \leq E^{(k)} \tag{1.18c}$$

$$s^{(k)}\left(\mathrm{p}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x})\right) \leq E^{(k)} \tag{1.18d}$$

$$|\mathbf{v}^{(k)}(\mathbf{x})| \leq \overline{\mathbf{v}} \tag{1.18e}$$

$$\underline{\mathbf{v}} \leq |\mathbf{v}^{(k)}(\mathbf{x})| \tag{1.18f}$$

$$|\mathbf{i}_L^{(k)}(\mathbf{x})| \leq \bar{\mathbf{i}} \tag{1.18g}$$

DERs Assumptions

- $\mathcal{X}_i^{(k)}$ is convex and compact for all t_k
- $f_i^{(k)}(\mathbf{x}_i)$ is convex and continuously differentiable, and its gradient is Lipschitz continuous for all t_k

To build the Lagrangian function of our problem, we associate the dual variables $\boldsymbol{\rho}^{(k)}$, $\lambda^{(k)}$, $\mu^{(k)}$, $\boldsymbol{\gamma}^{(k)}$, $\boldsymbol{\nu}^{(k)}$ and $\boldsymbol{\zeta}^{(k)}$ with constraints 1.18b, 1.18c, 1.18d, 1.18e , 1.18f and 1.18g, respectively. We will use $\boldsymbol{d}^{(k)}$ to represent the dual variables. At time t_k, the Lagrangian function associated with the optimization problem is given as

$$\begin{aligned}
L^{(k)}(\mathbf{x}, \boldsymbol{d}) := & \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\
& + \boldsymbol{\rho}^\top \left(\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right) \mathbf{p}_0^{(k)}(\mathbf{x})\right) \\
& + \lambda \left(s^{(k)}\left(\mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x}) - \mathrm{p}_{0,\mathsf{set}}^{(k)}\right) - E^{(k)}\right) \\
& + \mu \left(s^{(k)}\left(\mathrm{p}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x})\right) - E^{(k)}\right) \\
& + \boldsymbol{\gamma}^\top \left(|\mathbf{v}^{(k)}(\mathbf{x})| - \overline{\mathbf{v}}\right) \\
& + \boldsymbol{\nu}^\top \left(\underline{\mathbf{v}} - |\mathbf{v}^{(k)}(\mathbf{x})|\right) \\
& + \boldsymbol{\zeta}^\top \left(|\mathbf{i}_L^{(k)}(\mathbf{x})| - \bar{\mathbf{i}}\right)
\end{aligned} \tag{1.19}$$

Let $[\dots]$ denote the units of the quantity. Looking at the units of the Lagrangian function, we have the following

$$W^2 = W^2 + [\boldsymbol{\rho}](W) + [\lambda](W) + [\mu](W) + [\boldsymbol{\gamma}](V) + [\boldsymbol{\nu}](V) + [\boldsymbol{\zeta}](A) \tag{1.20}$$

Therefore, we can extract the units of the dual variables as follows:

- $[\boldsymbol{\rho}] = W$
- $[\lambda] = W$
- $[\mu] = W$
- $[\boldsymbol{\gamma}] = \frac{W^2}{V}$
- $[\boldsymbol{\nu}] = \frac{W^2}{V}$
- $[\boldsymbol{\zeta}] = \frac{W^2}{A}$

Using Eq. 1.9, we can re-write Eq. 1.19 as follows:

$$\begin{aligned} L^{(k)}(\mathbf{x}, \boldsymbol{d}) \coloneqq & \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\ & + \sum_{i \in \mathcal{D}} \left[\left(s^{(k)}(\lambda - \mu)\mathbb{1}_3^\top + \boldsymbol{\rho}^\top \,\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right)\right) \mathbf{M}_i \mathbf{x}_i + (\boldsymbol{\gamma} - \boldsymbol{\nu})^\top \mathbf{A}_i \mathbf{x}_i + \boldsymbol{\zeta}^\top \mathbf{B}_i \mathbf{x}_i \right] \\ & + s^{(k)}(\lambda - \mu)(\mathbb{1}_3^\top \mathbf{m}^{(k)} - \mathrm{p}_{0,\mathsf{set}}) - (\lambda + \mu) E^{(k)} + \boldsymbol{\rho}^\top \,\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right) \mathbf{m} \\ & + \boldsymbol{\gamma}^\top (\mathbf{a}^{(k)} - \overline{\mathbf{v}}) + \boldsymbol{\nu}^\top (\underline{\mathbf{v}} - \mathbf{a}^{(k)}) + \boldsymbol{\zeta}^\top (\mathbf{b}^{(k)} - \bar{\mathbf{i}}) \end{aligned} \tag{1.21}$$

We then define the regularized Lagrangian function, with r_p and $\boldsymbol{r_d} > 0$ (regularization factors):

$$\begin{aligned} L_r^{(k)}(\mathbf{x}, \boldsymbol{d}) = & L^{(k)}(\mathbf{x}, \boldsymbol{d}) \\ & + \frac{r_p}{2}\|\mathbf{x}\|_2^2 - \frac{\boldsymbol{r_d}}{2}\|\boldsymbol{d}\|_2^2 \\ = & L^{(k)}(\mathbf{x}, \boldsymbol{d}) \\ & + \frac{r_p}{2}\|\mathbf{x}\|_2^2 - \left(\frac{r_{d,\rho}}{2}\|\boldsymbol{\rho}\|_2^2 + \frac{r_{d,\lambda}}{2}\|\lambda\|_2^2 + \frac{r_{d,\mu}}{2}\|\mu\|_2^2 + \frac{r_{d,\gamma}}{2}\|\boldsymbol{\gamma}\|_2^2 + \frac{r_{d,\nu}}{2}\|\boldsymbol{\nu}\|_2^2 + \frac{r_{d,\zeta}}{2}\|\boldsymbol{\zeta}\|_2^2 \right) \end{aligned} \tag{1.22}$$

We define a parameterized step sizes and regularization parameters for dual variables as follows. First, we define the general step size $\alpha > 0$ and $\tilde{r}_d > 0$, both being "unitless". Then, we define the following:

- $\boldsymbol{\rho} \rightarrow \alpha_\rho = a_\rho \alpha \quad \& \quad r_{d,\rho} = c_\rho \tilde{r}_d$
- $\lambda \rightarrow \alpha_\lambda = a_\lambda \alpha \quad \& \quad r_{d,\lambda} = c_\lambda \tilde{r}_d$
- $\mu \rightarrow \alpha_\mu = a_\mu \alpha \quad \& \quad r_{d,\mu} = c_\mu \tilde{r}_d$
- $\boldsymbol{\gamma} \rightarrow \alpha_\gamma = a_\gamma \alpha \quad \& \quad r_{d,\gamma} = c_\gamma \tilde{r}_d$
- $\boldsymbol{\nu} \rightarrow \alpha_\nu = a_\nu \alpha \quad \& \quad r_{d,\nu} = c_\nu \tilde{r}_d$
- $\boldsymbol{\zeta} \rightarrow \alpha_\zeta = a_\zeta \alpha \quad \& \quad r_{d,\zeta} = c_\zeta \tilde{r}_d$
- $\boldsymbol{x}_i \rightarrow \alpha_{x_i} = a_{x_i} \alpha \quad \& \quad r_{p,x_i} = c_{x_i} r_p$

Note that a_ρ, a_λ, a_μ, a_γ, a_ν, a_ζ, a_{x_i}, c_ρ, c_λ, c_μ, c_γ, c_ν, c_ζ and c_{x_i} are all > 0.

The units of the coefficients of α and $\overline{r}_d$ parameters are:

- $[a_\rho] =$ "unitless" $\quad \& \quad [c_\rho] =$ "unitless"
- $[a_\lambda] =$ "unitless" $\quad \& \quad [c_\lambda] =$ "unitless"
- $[a_\mu] =$ "unitless" $\quad \& \quad [c_\mu] =$ "unitless"
- $[a_\gamma] = \frac{W^2}{V^2} \quad \& \quad [c_\gamma] = \frac{V^2}{W^2}$
- $[a_\nu] = \frac{W^2}{V^2} \quad \& \quad [c_\nu] = \frac{V^2}{W^2}$
- $[a_\zeta] = \frac{W^2}{A^2} \quad \& \quad [c_\zeta] = \frac{A^2}{W^2}$
- $[a_{x_i}] =$ "unitless" $\quad \& \quad [c_{x_i}] =$ "unitless"

Algorithm 1 Real-time optimization algorithm

At each t_k

[Step 1]: Collect $|\mathbf{v}^{(k)}|$ at $\mathcal{M}_v$, $|\mathbf{i}^{(k)}|$ at $\mathcal{M}_i$ and $\mathbf{p}_0^{(k)}$, and perform the following updates to the dual variables:

$$\boldsymbol{\rho}^{(k+1)} = \mathsf{proj}_{\mathbb{R}^3_+}\left\{\boldsymbol{\rho}^{(k)} + \alpha_\rho\left[\mathsf{sign}\left(-\mathrm{p}_{0,set}^{(k)}\right)\mathbf{p}_0^{(k)} - r_{d,\rho}\boldsymbol{\rho}^{(k)}\right]\right\} \tag{1.23}$$

$$\lambda^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+}\left\{\lambda^{(k)} + \alpha_\lambda\left(\mathbb{1}_3^\top\mathbf{p}_0^{(k)} - \mathrm{p}_{0,\mathsf{set}}^{(k)} - E^{(k)} - r_{d,\lambda}\lambda^{(k)}\right)\right\} \tag{1.24}$$

$$\mu^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+}\left\{\mu^{(k)} + \alpha_\mu\left(\mathrm{p}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top\mathbf{p}_0^{(k)} - E^{(k)} - r_{d,\mu}\mu^{(k)}\right)\right\} \tag{1.25}$$

$$\boldsymbol{\gamma}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}}\left\{\boldsymbol{\gamma}^{(k)} + \alpha_\gamma\left(|\mathbf{v}^{(k)}| - \overline{\mathbf{v}}\mathbb{1} - r_{d,\gamma}\boldsymbol{\gamma}^{(k)}\right)\right\} \tag{1.26}$$

$$\boldsymbol{\nu}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}}\left\{\boldsymbol{\nu}^{(k)} + \alpha_\nu\left(\underline{\mathbf{v}}\mathbb{1} - |\mathbf{v}^{(k)}| - r_{d,\nu}\boldsymbol{\nu}^{(k)}\right)\right\} \tag{1.27}$$

$$\boldsymbol{\zeta}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_i|}}\left\{\boldsymbol{\zeta}^{(k)} + \alpha_\zeta\left(|\mathbf{i}^{(k)}| - \bar{\mathbf{i}}\mathbb{1} - r_{d,\zeta}\boldsymbol{\zeta}^{(k)}\right)\right\} \tag{1.28}$$

[Step 2]: Each DER device $i \in \mathcal{D}$ measures the output powers $\mathbf{x}_i^{(k)}$ and updates the set-point as follows

$$\begin{aligned}\mathbf{x}_i^{(k+1)} =& \mathsf{proj}_{\mathcal{X}^{(k)}}\Bigg\{\mathbf{x}_i^{(k)} - \alpha_{x_i}\Bigg[\nabla_{\mathbf{x}_i} f_i^{(k)}(\mathbf{x}_i^{(k)}) \\ &+\mathbf{M}_i^\top\left[s^{(k)}(\lambda^{(k+1)} - \mu^{(k+1)})\mathbb{1}_3 + \boldsymbol{\rho}^{(k+1)}\mathsf{sign}\left(-\mathrm{p}_{0,set}^{(k)}\right)\right] \\ &+\mathbf{B}_i^\top\boldsymbol{\zeta}^{(k+1)} + \mathbf{A}_i^\top(\boldsymbol{\gamma}^{(k+1)} - \boldsymbol{\nu}^{(k+1)}) + r_{p,x_i}\mathbf{x}_i^{(k)}\Bigg]\Bigg\}\end{aligned} \tag{1.29}$$

1.6.2 Lower-level controllers

Each controller at the lower-levels will receive two power set-points, $\mathrm{p}_{0,\mathsf{set}}$ and $\mathrm{q}_{0,\mathsf{set}}$. Therefore, we update the problem formulation for these controllers as follows:

$$\min_{\mathbf{x}} \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \tag{1.30}$$

subject to

$$
\begin{align}
& \mathbf{x}_i \in \mathcal{X}_i, \qquad \forall i \in \mathcal{D} \tag{1.31a}\\
& \mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right)\ \mathbf{p}_0^{(k)}(\mathbf{x}) \leq \mathbb{0}_3 \tag{1.31b}\\
& \mathsf{sign}\left(-\mathrm{q}_{0,\mathsf{set}}^{(k)}\right)\ \mathbf{q}_0^{(k)}(\mathbf{x}) \leq \mathbb{0}_3 \tag{1.31c}\\
& s^{(k)}\left(\mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x}) - \mathrm{p}_{0,\mathsf{set}}^{(k)}\right) \leq E_p^{(k)} \tag{1.31d}\\
& s^{(k)}\left(\mathrm{p}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x})\right) \leq E_p^{(k)} \tag{1.31e}\\
& s^{(k)}\left(\mathbb{1}_3^\top \mathbf{q}_0^{(k)}(\mathbf{x}) - \mathrm{q}_{0,\mathsf{set}}^{(k)}\right) \leq E_q^{(k)} \tag{1.31f}\\
& s^{(k)}\left(\mathrm{q}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top \mathbf{q}_0^{(k)}(\mathbf{x})\right) \leq E_q^{(k)} \tag{1.31g}\\
& |\mathbf{v}^{(k)}(\mathbf{x})| \leq \overline{\mathbf{v}} \tag{1.31h}\\
& \underline{\mathbf{v}} \leq |\mathbf{v}^{(k)}(\mathbf{x})| \tag{1.31i}\\
& |\mathbf{i}_L^{(k)}(\mathbf{x})| \leq \bar{\mathbf{i}} \tag{1.31j}
\end{align}
$$

Similarly, we update the Lagrangian function. We associate three additional dual variables, $\boldsymbol{\sigma}$, η and ψ with the new constraints Eqs. 1.31c,1.31f & 1.31g.

$$
\begin{aligned}
L^{(k)}(\mathbf{x}, \boldsymbol{d^{(k)}}) \coloneqq & \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\
& + \boldsymbol{\rho}^\top \left(\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right) \mathbf{p}_0^{(k)}(\mathbf{x}) \right) \\
& + \boldsymbol{\sigma}^\top \left(\mathsf{sign}\left(-\mathrm{q}_{0,\mathsf{set}}^{(k)}\right) \mathbf{q}_0^{(k)}(\mathbf{x}) \right) \\
& + \lambda \left(s^{(k)} \left(\mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x}) - \mathrm{p}_{0,\mathsf{set}}^{(k)} \right) - E_p^{(k)} \right) \\
& + \mu \left(s^{(k)} \left(\mathrm{p}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top \mathbf{p}_0^{(k)}(\mathbf{x}) \right) - E_p^{(k)} \right) \\
& + \eta \left(s^{(k)} \left(\mathbb{1}_3^\top \mathbf{q}_0^{(k)}(\mathbf{x}) - \mathrm{q}_{0,\mathsf{set}}^{(k)} \right) - E_q^{(k)} \right) \\
& + \psi \left(s^{(k)} \left(\mathrm{q}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top \mathbf{q}_0^{(k)}(\mathbf{x}) \right) - E_q^{(k)} \right) \\
& + \boldsymbol{\gamma}^\top \left(|\mathbf{v}^{(k)}(\mathbf{x})| - \overline{\mathbf{v}} \right) \\
& + \boldsymbol{\nu}^\top \left(\underline{\mathbf{v}} - |\mathbf{v}^{(k)}(\mathbf{x})| \right) \\
& + \boldsymbol{\zeta}^\top \left(|\mathbf{i}_L^{(k)}(\mathbf{x})| - \bar{\mathbf{i}} \right)
\end{aligned} \tag{1.32}
$$

Similar to the work shown in Level 2 Controller, we conduct the same analysis for the units of the dual variables here. Let $[\dots]$ denote the units of the quantity. Looking at the units of the Lagrangian function, we have the following

$$
W^2 = W^2 + [\boldsymbol{\rho}](W) + [\boldsymbol{\sigma}](Var) + [\lambda](W) + [\mu](W) + [\eta](Var) + [\psi](Var) + [\boldsymbol{\gamma}](V) + [\boldsymbol{\nu}](V) + [\boldsymbol{\zeta}](A) \tag{1.33}
$$

Therefore, we can extract the units of the dual variables as follows:

- $[\boldsymbol{\rho}] = W$
- $[\boldsymbol{\sigma} = Var$
- $[\lambda] = W$
- $[\mu] = W$
- $[\eta] = Var$
- $[\psi] = Var$

- $[\boldsymbol{\gamma}] = \frac{W^2}{V}$
- $[\boldsymbol{\nu}] = \frac{W^2}{V}$
- $[\boldsymbol{\zeta}] = \frac{W^2}{A}$

Which can be re-written as

$$
\begin{aligned}
L^{(k)}(\mathbf{x}, \boldsymbol{d}) \coloneqq & \sum_{i \in \mathcal{D}} f_i^{(k)}(\mathbf{x}_i) \\
& + \sum_{i \in \mathcal{D}} \Big[\quad \left(s^{(k)}(\lambda - \mu)\mathbb{1}_3^\top + \boldsymbol{\rho}^\top \,\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right)\right) \mathbf{M}_i \mathbf{x}_i \\
& \qquad + \left(s^{(k)}(\eta - \psi)\mathbb{1}_3^\top + \boldsymbol{\sigma}^\top \,\mathsf{sign}\left(-\mathrm{q}_{0,\mathsf{set}}^{(k)}\right)\right) \mathbf{H}_i \mathbf{x}_i \\
& \qquad + (\boldsymbol{\gamma} - \boldsymbol{\nu})^\top \mathbf{A}_i \mathbf{x}_i + \boldsymbol{\zeta}^\top \mathbf{B}_i \mathbf{x}_i \Big] \\
& + s^{(k)} \left[(\lambda - \mu)(\mathbb{1}_3^\top \mathbf{m}^{(k)} - \mathrm{p}_{0,\mathsf{set}}) + (\eta - \psi)(\mathbb{1}_3^\top \mathbf{h}^{(k)} - \mathrm{q}_{0,set}) \right] \\
& - (\lambda + \mu) E_p^{(k)} - (\eta + \psi) E_q^{(k)} \\
& + \boldsymbol{\rho}^\top \,\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right) \mathbf{m} + \boldsymbol{\sigma}^\top \,\mathsf{sign}\left(-\mathrm{q}_{0,\mathsf{set}}^{(k)}\right) \mathbf{h} \\
& + \boldsymbol{\gamma}^\top (\mathbf{a}^{(k)} - \overline{\mathbf{v}}) + \boldsymbol{\nu}^\top (\underline{\mathbf{v}} - \mathbf{a}^{(k)}) + \boldsymbol{\zeta}^\top (\mathbf{b}^{(k)} - \bar{\mathbf{i}})
\end{aligned}
\tag{1.34}
$$

Similar to Level 2 Controller, we define a parameterized step sizes and regularization parameters for dual variables as follows. First, we define the general step size $\alpha > 0$ and $\tilde{r}_d > 0$, both being "unitless". Then, we define the following:

- $\boldsymbol{\rho} \to \alpha_\rho = a_\rho \alpha \quad \& \quad r_{d,\rho} = c_\rho \tilde{r}_d$
- $\boldsymbol{\sigma} \to \alpha_\sigma = a_\sigma \alpha \quad \& \quad r_{d,\sigma} = c_\sigma \tilde{r}_d$
- $\lambda \to \alpha_\lambda = a_\lambda \alpha \quad \& \quad r_{d,\lambda} = c_\lambda \tilde{r}_d$
- $\mu \to \alpha_\mu = a_\mu \alpha \quad \& \quad r_{d,\mu} = c_\mu \tilde{r}_d$
- $\eta \to \alpha_\eta = a_\eta \alpha \quad \& \quad r_{d,\eta} = c_\eta \tilde{r}_d$
- $\psi \to \alpha_\psi = a_\psi \alpha \quad \& \quad r_{d,\psi} = c_\psi \tilde{r}_d$
- $\boldsymbol{\gamma} \to \alpha_\gamma = a_\gamma \alpha \quad \& \quad r_{d,\gamma} = c_\gamma \tilde{r}_d$

- $\boldsymbol{\nu} \to \alpha_\nu = a_\nu \alpha \quad \& \quad r_{d,\nu} = c_\nu \tilde{r}_d$

- $\boldsymbol{\zeta} \to \alpha_\zeta = a_\zeta \alpha \quad \& \quad r_{d,\zeta} = c_\zeta \tilde{r}_d$

- $\boldsymbol{x}_i \to \alpha_{x_i} = a_{x_i} \alpha \quad \& \quad r_{p,x_i} = c_{x_i} r_p$

Note that a_ρ, a_σ, a_λ, a_μ, a_η, a_ψ, a_γ, a_ν, a_ζ, a_{x_i}, c_ρ, c_σ, c_λ, c_μ, c_η, c_ψ, c_γ, c_ν, c_ζ and c_{x_i} are all > 0. The units of the coefficients of α and $\overline{r}_d$ parameters are:

- $[a_\rho] =$ "unitless" $\quad \& \quad$ $[c_\rho] =$ "unitless"

- $[a_\sigma] =$ "unitless" $\quad \& \quad$ $[c_\sigma] =$ "unitless"

- $[a_\lambda] =$ "unitless" $\quad \& \quad$ $[c_\lambda] =$ "unitless"

- $[a_\mu] =$ "unitless" $\quad \& \quad$ $[c_\mu] =$ "unitless"

- $[a_\eta] =$ "unitless" $\quad \& \quad$ $[c_\eta] =$ "unitless"

- $[a_\psi] =$ "unitless" $\quad \& \quad$ $[c_\psi] =$ "unitless"

- $[a_\gamma] = \frac{W^2}{V^2} \quad \& \quad [c_\gamma] = \frac{V^2}{W^2}$

- $[a_\nu] = \frac{W^2}{V^2} \quad \& \quad [c_\nu] = \frac{V^2}{W^2}$

- $[a_\zeta] = \frac{W^2}{A^2} \quad \& \quad [c_\zeta] = \frac{A^2}{W^2}$

- $[a_{x_i}] =$ "unitless" $\quad \& \quad$ $[c_{x_i}] =$ "unitless"

We then define the regularized Lagrangian function, with r_p and $r_d > 0$ (regularization factors):

$$
\begin{aligned}
L_r^{(k)}(\mathbf{x}, \boldsymbol{d}) =& L^{(k)}(\mathbf{x}, \boldsymbol{d}) \\
&+ \frac{r_p}{2}\|\mathbf{x}\|_2^2 - \frac{\boldsymbol{r_d}}{2}\|\boldsymbol{d}\|_2^2 \\
=& L^{(k)}(\mathbf{x}, \boldsymbol{d}) \\
&+ \frac{r_p}{2}\|\mathbf{x}\|_2^2 - \Big(\frac{r_{d,\rho}}{2}\|\boldsymbol{\rho}\|_2^2 + \frac{r_{d,\sigma}}{2}\|\boldsymbol{\sigma}\|_2^2 \\
&+ \frac{r_{d,\lambda}}{2}\|\lambda\|_2^2 + \frac{r_{d,\mu}}{2}\|\mu\|_2^2 + \frac{r_{d,\eta}}{2}\|\eta\|_2^2 + \frac{r_{d,\psi}}{2}\|\psi\|_2^2 \\
&+ \frac{r_{d,\gamma}}{2}\|\boldsymbol{\gamma}\|_2^2 + \frac{r_{d,\nu}}{2}\|\boldsymbol{\nu}\|_2^2 + \frac{r_{d,\zeta}}{2}\|\boldsymbol{\zeta}\|_2^2\Big)
\end{aligned}
\tag{1.35}
$$

Algorithm 2 Real-time optimization algorithm

At each t_k

[Step 1]: Collect $|\mathbf{v}^{(k)}|$ at $\mathcal{M}_v$, $|\mathbf{i}^{(k)}|$ at $\mathcal{M}_i$, $\mathbf{p}_0^{(k)}$ and $\mathbf{q}_0^{(k)}$, and perform the following updates to the dual variables:

$$\boldsymbol{\rho}^{(k+1)} = \mathsf{proj}_{\mathbb{R}^3_+}\left\{\boldsymbol{\rho}^{(k)} + \alpha_\rho\left[\mathsf{sign}\left(-\mathrm{p}_{0,set}^{(k)}\right)\mathbf{p}_0^{(k)} - r_{d,\rho}\boldsymbol{\rho}^{(k)}\right]\right\} \tag{1.36}$$

$$\boldsymbol{\sigma}^{(k+1)} = \mathsf{proj}_{\mathbb{R}^3_+}\left\{\boldsymbol{\sigma}^{(k)} + \alpha_\sigma\left[\mathsf{sign}\left(-\mathrm{q}_{0,set}^{(k)}\right)\mathbf{q}_0^{(k)} - r_{d,\sigma}\boldsymbol{\sigma}^{(k)}\right]\right\} \tag{1.37}$$

$$\lambda^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+}\left\{\lambda^{(k)} + \alpha_\lambda\left(\mathbb{1}_3^\top\mathbf{p}_0^{(k)} - \mathrm{p}_{0,\mathsf{set}}^{(k)} - E_p^{(k)} - r_{d,\lambda}\lambda^{(k)}\right)\right\} \tag{1.38}$$

$$\mu^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+}\left\{\mu^{(k)} + \alpha_\mu\left(\mathrm{p}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top\mathbf{p}_0^{(k)} - E_p^{(k)} - r_{d,\mu}\mu^{(k)}\right)\right\} \tag{1.39}$$

$$\eta^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+}\left\{\eta^{(k)} + \alpha_\eta\left(\mathbb{1}_3^\top\mathbf{q}_0^{(k)} - \mathrm{q}_{0,\mathsf{set}}^{(k)} - E_q^{(k)} - r_{d,\eta}\eta^{(k)}\right)\right\} \tag{1.40}$$

$$\psi^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+}\left\{\psi^{(k)} + \alpha_\psi\left(\mathrm{q}_{0,\mathsf{set}}^{(k)} - \mathbb{1}_3^\top\mathbf{q}_0^{(k)} - E_q^{(k)} - r_{d,\psi}\psi^{(k)}\right)\right\} \tag{1.41}$$

$$\boldsymbol{\gamma}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}}\left\{\boldsymbol{\gamma}^{(k)} + \alpha_\gamma\left(|\mathbf{v}^{(k)}| - \overline{\mathbf{v}}\mathbb{1} - r_{d,\gamma}\boldsymbol{\gamma}^{(k)}\right)\right\} \tag{1.42}$$

$$\boldsymbol{\nu}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_v|}}\left\{\boldsymbol{\nu}^{(k)} + \alpha_\nu\left(\underline{\mathbf{v}}\mathbb{1} - |\mathbf{v}^{(k)}| - r_{d,\nu}\boldsymbol{\nu}^{(k)}\right)\right\} \tag{1.43}$$

$$\boldsymbol{\zeta}^{(k+1)} = \mathsf{proj}_{\mathbb{R}_+^{|\mathcal{M}_i|}}\left\{\boldsymbol{\zeta}^{(k)} + \alpha_\zeta\left(|\mathbf{i}^{(k)}| - \bar{\mathbf{i}}\mathbb{1} - r_{d,\zeta}\boldsymbol{\zeta}^{(k)}\right)\right\} \tag{1.44}$$

[Step 2]: Each DER device $i \in \mathcal{D}$ measures the output powers $\mathbf{x}_i^{(k)}$ and updates the set-point as follows

$$\begin{aligned}\mathbf{x}_i^{(k+1)} =& \mathsf{proj}_{\mathcal{X}^{(k)}}\Big\{\mathbf{x}_i^{(k)} - \alpha_{x_i}\Big[\nabla_{\mathbf{x}_i} f_i^{(k)}(\mathbf{x}_i^{(k)}) \\ &+\mathbf{M}_i^\top\left[s^{(k)}(\lambda^{(k+1)} - \mu^{(k+1)})\mathbb{1}_3 + \boldsymbol{\rho}^{(k+1)}\mathsf{sign}\left(-\mathrm{p}_{0,\mathsf{set}}^{(k)}\right)\right] \\ &+\mathbf{H}_i^\top\left[s^{(k)}(\eta^{(k+1)} - \psi^{(k+1)})\mathbb{1}_3 + \boldsymbol{\sigma}^{(k+1)}\mathsf{sign}\left(-\mathrm{q}_{0,\mathsf{set}}^{(k)}\right)\right] \\ &+\mathbf{B}_i^\top\boldsymbol{\zeta}^{(k+1)} + \mathbf{A}_i^\top(\boldsymbol{\gamma}^{(k+1)} - \boldsymbol{\nu}^{(k+1)}) + r_d\mathbf{x}_i^{(k)}\Big]\Big\}\end{aligned} \tag{1.45}$$

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