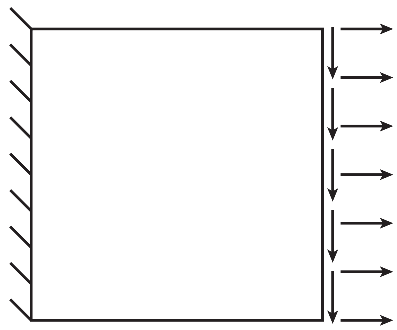

Figure 1: Problem setup for patch tests.

# FEA of Linear Elastic Problems

In this assignment you will build a finite element code to solve two-dimensional linear elasticity problems. You will then solve two problems with your code.

# Problem 1

The problem definition is shown in Figure 1. In this case, the elastic block is square with $M, N = 1$. The left side of the block is restrained from moving in the $x$ and $y$ directions and the right side is subject to prescribed displacements and tractions in the $x$ and $y$ directions as specified below. The material parameters $E$ and $\nu$ will also be specified below. The top and bottom of the block are traction free and there is no body force.

1. In this problem we will solve a simple patch test. Patch tests are commonly used to assess the correctness of a finite element code. We set our displacement boundary conditions on the right end to be $u_x = 1$. We take $E = 1$ and $\nu = 0$. Solve the problem for $p, q = 1, 2$ and $m, n = 1, 4, 32$. Turn in contour plots of each component of the computed displacement field $u_x$ and $u_y$ and each component of the stress $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{12}$.

2. Solve the same problem but now set a traction boundary condition $t_x = 1000$, $t_y = 0$, $E = 1e7$, and $\nu = 0.3$. The material parameters are an accurate model for steel. Solve the problem for $p, q = 1, 2$ and generate a mesh of sufficient resolution to resolve the physics of the problem. Turn in contour plots of each component of the computed displacement field $u_x$ and $u_y$ and each component of the stress $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{12}$ for your converged solution.

# Problem 2

The problem of a solid circular cylinder subjected to an internal pressure loading is shown in Figure 2. The exact solution, in terms of displacement and stresses in polar

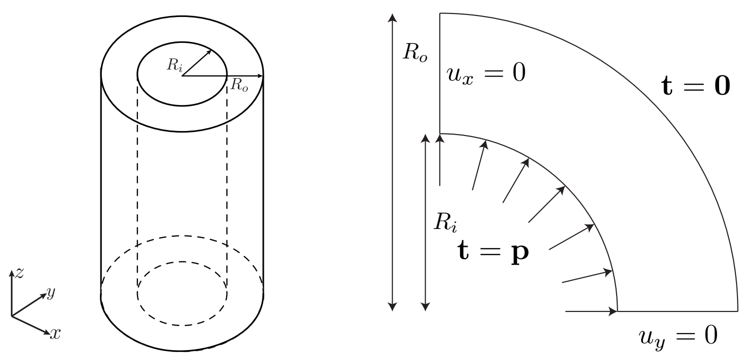


Figure 2: Problem setup for a solid circular cylinder subject to internal pressure loading.

coordinates $(R, \theta)$, for the case with constant pressure, is

$$u_R(R) = \frac{1}{E}\frac{PR_i^2}{R_o^2 - R_i^2}\left((1-\nu)R + \frac{R_o^2(1+\nu)}{R}\right) \tag{1}$$

$$\sigma_{RR}(R) = \frac{PR_i^2}{R_o^2 - R_i^2} - \frac{PR_i^2R_o^2}{R^2(R_o^2 - R_i^2)} \tag{2}$$

$$\sigma_{\theta\theta}(R) = \frac{PR_i^2}{R_o^2 - R_i^2} + \frac{PR_i^2R_o^2}{R^2(R_o^2 - R_i^2)}, \tag{3}$$

where $R_i$ is the radius of the inner cylinder, $R_o$ is the radius of the outer cylinder, $P$ is the pressure applied to the inner cylinder, $E$ is Young's modulus, and $\nu$ is Poisson's ratio. To solve this problem we take a slice of the cylinder and, employing symmetry, only model one quarter of the resulting annulus as shown in Figure 2 on the right. We will assume the cylinder is infinitely long and employ the plane strain hypothesis which says that

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \tag{4}$$

Confirm that the exact solution is obtained under the limit of mesh refinement (i.e., uniform subdivision of elements) for both polynomial degrees $p, q = 1$ and $p, q = 2$. To do so, it is sufficient to show that the $L^2$-norm of the error

$$||u - u^h||_{L^2(\Omega)} = \left(\int_\Omega \left(u_R - u_R^h\right)^2 dR\right)^{\frac{1}{2}} \tag{5}$$

goes to zero in the limit of mesh refinement. What is the rate of convergence in terms of the number of degrees of freedom? Generate contour plots of the computed displacements $u_R^h$ and stresses $\sigma_{RR}^h$ and $\sigma_{\theta\theta}^h$ for $p, q = 1, 2$.