CANADIAN COLLEGE OF TECHNOLOGY AND BUSINESS

2023

Canadian College of Technology and Business

Invest in Yourself

Probability

CANADIAN COLLEGE
OF TECHNOLOGY
AND BUSINESS

Probability

Overview

- A mathematical concept that describes the **likelihood or chance** of an event occurring
- A measure of the **strength of belief in the occurrence** of an uncertain event
- It is expressed as **a number between 0 and 1**
  - 0 indicating that an event is impossible
  - 1 indicating that an event is certain to occur
  - A probability of 0.5 means there is an equal chance of an event occurring or not occurring

Basic Definitions

Experiment

- A procedure or process with uncertain outcomes
    - These outcomes are usually represented in a **sample space**
    - Examples
        - Flip a coin (heads, tails)
        - Roll a die (1, 2, 3, 4, 5, 6)
        - Draw a card from a deck (Jack of Spades (JS), Queen of Clubs (QC), King of Hearts (KH), and Ace of Diamonds (AD), etc.)

Basic Definitions

Sample Space

- The **set of all possible outcomes** of an experiment
    - Provide the basis for determining the probability of a particular event
- A fundamental concept representing the **entire set of possibilities for** a given situation
- The sample space is typically **denoted by the symbol Ω**
- An example of a sample space is the experiment of rolling a six-sided die

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Basic Definitions

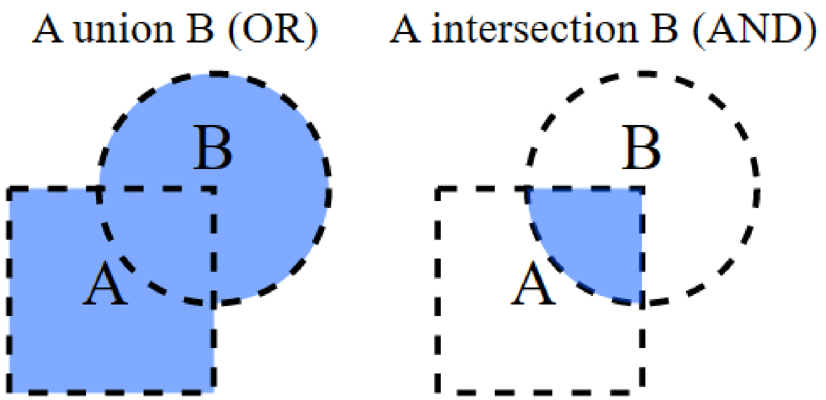
Event

- **Subset of a sample space** representing a specific outcome or set of outcomes of an experiment
- **Probability of an event** is a measure of the likelihood that the event occurs in each experiment
- For example, consider the experiment of flipping a coin
  - The sample space: **Ω = {Heads, Tails}**
  - An event of interest could be getting heads, which is represented by the set {Heads}
  - The probability of getting heads in this experiment is 1/2

Basic Definitions

Event

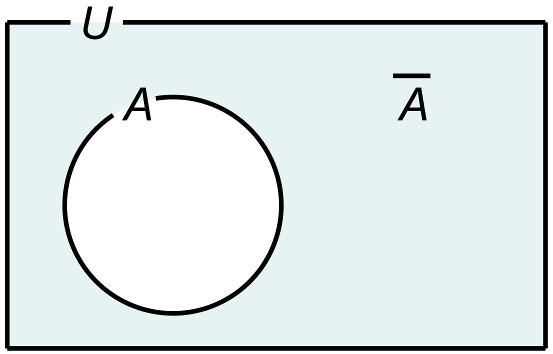
- **Union of events:** Consist of all outcomes that **belong to at least one of the events**
  - Union of events A and B can be expressed as **A ∪ B**, which represents event **A or B**
- **Intersection of events:** Consist of all outcomes that **belong to both events**
  - Intersection of events A and B can be expressed as **A ∩ B**, which represents event **A and B**



Basic Definitions

Event

- **Complement of an event:** Consist of all outcomes that are **not part of the original event**
  - The complement of an event A is denoted as **A'** or **Ā**
  - The complement event and the original event are mutually exclusive and together they form the sample space



Basic Definitions

Simple Event

- A **single outcome of an experiment**, which is **distinct and indivisible**
- An example of a simple event can be drawing a single card from a deck of playing cards
    - Each card drawn is a distinct and indivisible outcome
    - So, drawing the Ace of Spades or the Queen of Hearts in a single draw is a simple event

Probability

Types of Probability

- Classical probability
- Empirical probability
- Subjective probability

Types of Probability

Classical Probability

- Used when **the number of possible outcomes for a particular event is known**
- Assume that each possible outcome of a random event is equally likely to occur

$$\boxed{P(A) = \frac{\text{The number of outcomes that are favorable to Event } A}{\text{The total number of possible outcomes in the sample space}}}$$

where:

$P(A)$ = The probability of event A happening

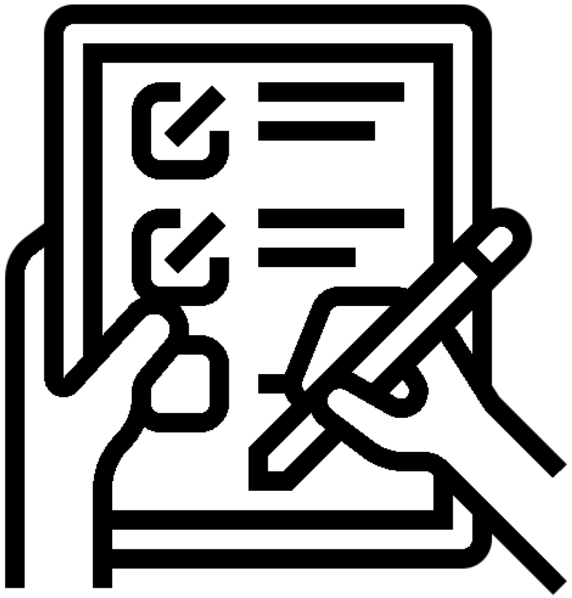
Types of Probability

Classical Probability

- For example, consider a simple coin flip
  - The classical probability of getting heads would be 1/2 because there are two possible outcomes (heads or tails)
  - Similarly, the classical probability of getting tails would also be 1/2

Types of Probability

Activity 1



Types of Probability

Empirical Probability

- Also known as experimental probability
- Based on **the frequency of occurrence of an event in many trials or observations**
- Useful when the theoretical probability of an event is unknown or difficult to determine
  - Provide an estimate of the probability based on actual data
  - The accuracy depends on the size of the sample and the representativeness of the data

$$\boxed{P(A) = \frac{\text{The number of times Event } A \text{ occurs}}{\text{Total number of observations}}}$$

Types of Probability

Empirical Probability

- For example, consider flipping a coin 100 times
  - Can calculate the empirical probability of getting heads by counting the number of heads that we observe and dividing that number by 100
  - If we observe 60 heads, the empirical probability of getting heads would be **60/100 = 0.6**

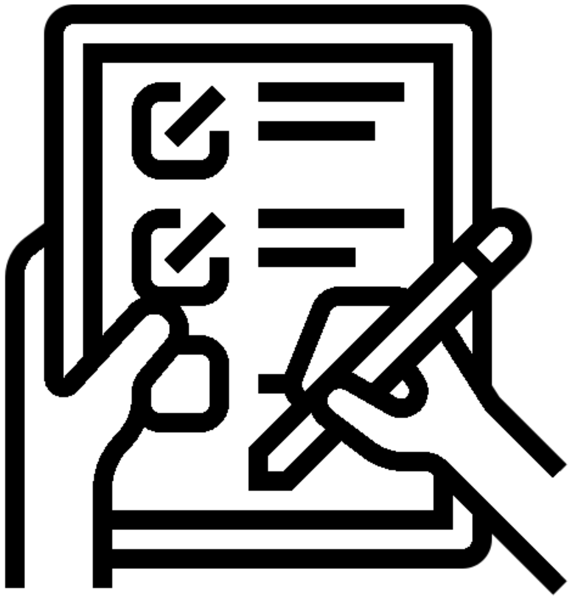
Types of Probability

Empirical Probability

- **Law of large numbers:** State that as the number of independent trials or events increases, the average of the results will tend to converge towards the expected value or mean
- For example, if we flip a fair coin many times, the proportion of heads/tails should converge towards **0.5 – the expected value**

Types of Probability

Activity 2



Types of Probability

Subjective Probability

- Based on **individual's beliefs, opinions, and attitudes** towards a certain event or outcome
    - Can be influenced by prior experiences, available information, emotions, and personal biases
- Used when classical and empirical probabilities are not available
    - Data is limited
    - Information is subjective (e.g., likelihood of a person falling in love)
    - Decisions need to be made quickly
    - The event is unique

Types of Probability

Subjective Probability

- Some examples of subjective probability
  - **Weather forecast:** A person's subjective probability of rain tomorrow may be based on their past experiences, local weather patterns, and current weather conditions
  - **Investment decisions:** An individual's subjective probability of a stock performing well may be influenced by their personal biases and level of familiarity with the stock and industry

Probability

Basic Rules for Probability

- **Non-negativity:** Probability is always a non-negative value

$$\boldsymbol{P(A) \geq 0}$$

- **Range of Values:** The probability of any event must range from 0 to 1

$$\boldsymbol{0 \leq P(A) \leq 1}$$

- **Normalization:** Sum of the probabilities of all possible outcomes of an experiment is equal to 1

$$\boldsymbol{P(S) = 1}$$

- **Complements:** Probability of complement of event A is equal to 1 minus probability of the event

$$\boldsymbol{P(\bar{A}) = 1 - P(A)}$$

Probability

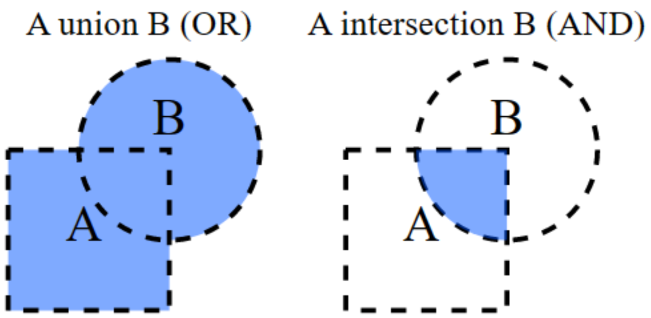
Basic Rules for Probability

- **Union:** The number of instances where either Event A **or** B occur or **both** events occur together

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **Intersection:** The number of instances in which Events A **and** B occur at the same time

$$P(A \cap B) = n(A \cap B)/n(S)$$



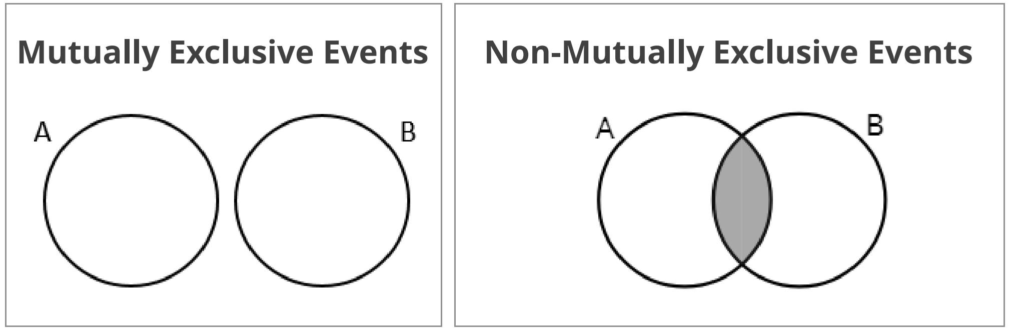
Probability

Basic Rules for Probability

- **The addition rule:** The probability of the union of two events is equal to the **sum of their individual probabilities**, provided that the events are disjoint or mutually exclusive
  - If events **A and B** are **mutually exclusive** (cannot occur at the same time)

$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$$



Probability

Contingency Tables

- A table used to **summarize and display the relationship between two variables**
  - Rows and columns each represent one of the variables
  - Cells contain the count or frequency of occurrences of each combination of values
    - Determine if there is any association or dependence between them

Probability

Contingency Tables

- Suppose we want to see if there is a relationship between the type of movie a person watches (Action, Drama, or Comedy) and their age group (18-30, 31-50, or 51+)

	Age Group: 18-30	Age Group: 31-50	Age Group: 51+	Total
Action	20	15	5	40
Drama	10	25	20	55
Comedy	15	10	15	40
Total	45	50	40	135

$$P(Drama) = \frac{55}{135}$$

$$P(18-30) = \frac{45}{135}$$

$$P(31-50 \cap Drama) = \frac{25}{135}$$

Probability

Contingency Tables

- Suppose we want to see if there is a relationship between the type of movie a person watches (Action, Drama, or Comedy) and their age group (18-30, 31-50, or 51+)

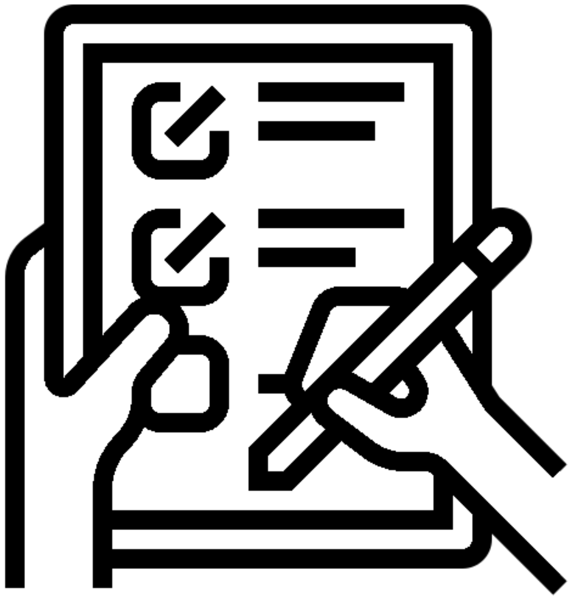
$$P(18-30 \cup Action) = \frac{20+15+5+10+15}{135} = \frac{65}{135}$$

	Age Group: 18-30	Age Group: 31-50	Age Group: 51+	Total
Action	20	15	5	40
Drama	10	25	20	55
Comedy	15	10	15	40
Total	45	50	40	135

$$P(18-30 \cup Action) = P(18\text{–}30) + P(Action) - P(18\text{–}30 \cap Action) = \frac{45+40-20}{135} = \frac{65}{135}$$

Probability

Activity 3



Probability

Conditional Probability

- The probability of an event **occurring given that another event has already occurred**
- It is expressed as $\boldsymbol{P(A|B)}$
    - where A and B are two events, and B is the event that has already occurred
- For example, the probability of rain tomorrow given the current temperature and humidity can be calculated using conditional probability

Probability

Conditional Probability

- The conditional probability of **event A given event B** is calculated as follows

$$\boxed{P(A|B) = \frac{P(A \text{ and } B)}{P(B)}}$$

where:

$P(A \text{ and } B)$ represents the probability of both events A and B occurring together

$P(B)$ is the probability of event B occurring

Probability

Conditional Probability

- It's important to note that the conditional probability of event A given event B is **not equal** to the probability of event B given event A

$$\boxed{P(B|A) = \frac{P(B \text{ and } A)}{P(A)}}$$

where:

$P(A \text{ and } B)$ represents the probability of both events A and B occurring together

$P(A)$ is the probability of event A occurring

Probability

Conditional Probability

- Suppose we have a sample of 200 people and want to find the probability that a person has a certain type of vehicle given that they live in a certain location
  - Contingency tables can be used to represent the probabilities of two or more events
  - Probabilities are calculated based on the number of occurrences of each event and the total number of observations

Location	Car	Bike	Total
City	80	20	100
Suburbs	50	50	100
Total	130	70	200

Location	Car	Bike	Total
City	0.4	0.1	0.5
Suburbs	0.25	0.25	0.5
Total	0.65	0.35	1

Probability

Conditional Probability

- For example, the conditional probability of owning a car given a person lives in city

$$\boldsymbol{P(Car|City) = \frac{P(Car\ and\ City)}{P(City)} = \frac{0.4}{0.5} = 0.8}$$

- This means that if a person lives in the city, there is an 80% chance that they own a car
- This information can be used to understand relationship between location and vehicle ownership

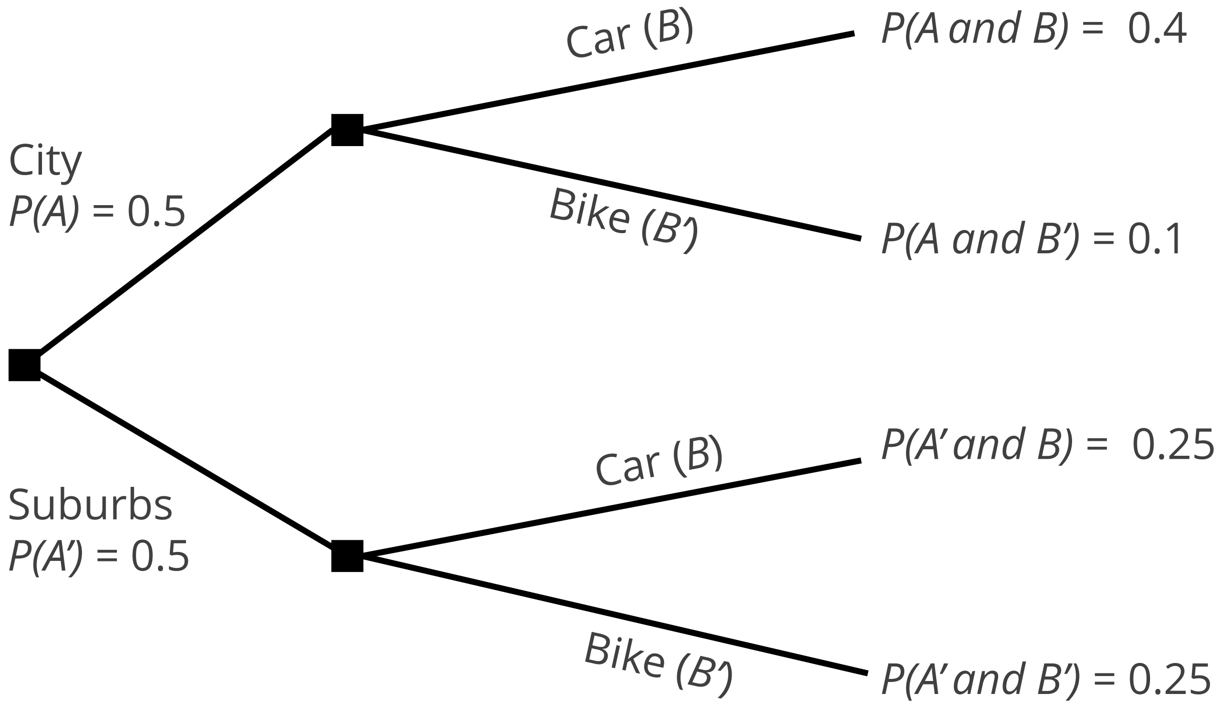
Location	Car	Bike	Total
City	0.4	0.1	0.5
Suburbs	0.25	0.25	0.5
Total	0.65	0.35	1

Probability

Decision Trees

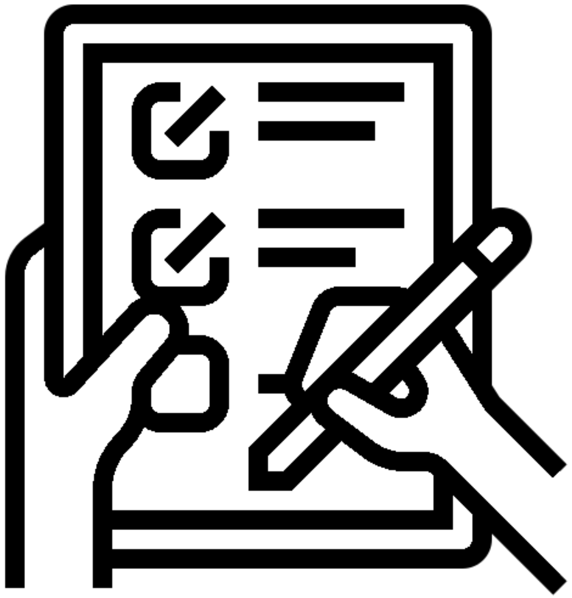
- Used to represent marginal and joint probabilities from a contingency table

Location	Car	Bike	Total
City	0.4	0.1	0.5
Suburbs	0.25	0.25	0.5
Total	0.65	0.35	1



Probability

Activity 4



Probability

Independent and Dependent Events

- **Dependent events:** Events that are **related to each other**
  - If two events are dependent, the occurrence of one event can **affect the occurrence** of the other event
  - For example, the outcome of picking a card from a deck and then picking another card from the remaining cards are dependent events

Probability

Independent and Dependent Events

- **Independent events:** Events that are not affected by each other
  - If two events are independent, the occurrence of one event **does not affect** the occurrence of the other event
  - For example, the outcome of rolling a dice and flipping a coin are independent events because the outcome of one does not affect the outcome of the other

Probability

Independent and Dependent Events

- Independent events and mutually exclusive events are **related but distinct concepts** in probability theory
  - **Independent events:** Have two events derived from **two different trials** (two different events like rolling a dice and flipping a coin)
    - Probability of occurrence of one does not affect the probability of occurrence of other
  - **Mutually exclusive events:** have also two events (may be more than two) but the events derived from **the same trial** (facing even numbers and odd numbers in rolling a dice)

Probability

Independent and Dependent Events

- If events A and B are independent

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

Probability

Independent and Dependent Events

- **The multiplication rule for independent events:** The probability of two independent events A and B **both occurring** is equal to the product of their individual probabilities

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \Rightarrow P(A \text{ and } B) = P(A|B)P(B)$$

$$\Rightarrow P(A \text{ and } B) = P(A)P(B)$$

Probability

Independent and Dependent Events

- The probability of the **intersection of several independent events** is the product of their separate individual probabilities

$$P(A_1 \text{ and } A_2 \text{ and } \dots \text{ and } A_n) = P(A_1)P(A_2)\dots P(A_n)$$

- The probability of the **union of several independent events** is 1 minus the product of probabilities of their complements

$$P(A_1 \text{ or } A_2 \text{ or } \dots \text{ or } A_n) = 1 - P(\overline{A_1})P(\overline{A_2})\dots P(\overline{A_n})$$

Probability

Bayes' Theorem

- Provide a way to describe the probability of an event **based on prior knowledge of conditions**
  - Calculate $P(A|B)$ from information about $P(B|A)$
- The formula for Bayes' Theorem is given by

$$\boldsymbol{P(A|B) = \frac{P(A)P(B|A)}{P(B)}}$$

where:

$P(A|B)$ is the conditional probability of event A given that event B has occurred

$P(B|A)$ is the conditional probability of event B given that event A has occurred

Probability

Bayes' Theorem

- With two (or more) cases, apply the law of total probability to the **denominator**

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

where:

A_i = The i^{th} event of interest from a choice of n events

B = An event that has already occurred

- The probabilities $P(A_1)$ and $P(A_2)$ are known as prior probabilities because they are determined without any other information

Probability

Bayes' Theorem

- Weather forecaster claims **60% chance of rain (A_1) and 40% chance of sunny (A_2) tomorrow**
  - On **rainy days**, the forecaster **correctly** predicts **rain 90% of the time**
  - On **sunny days** they **correctly** predict **no rain 80% of the time**
- What's the probability of **raining tomorrow given the forecaster's prediction?**

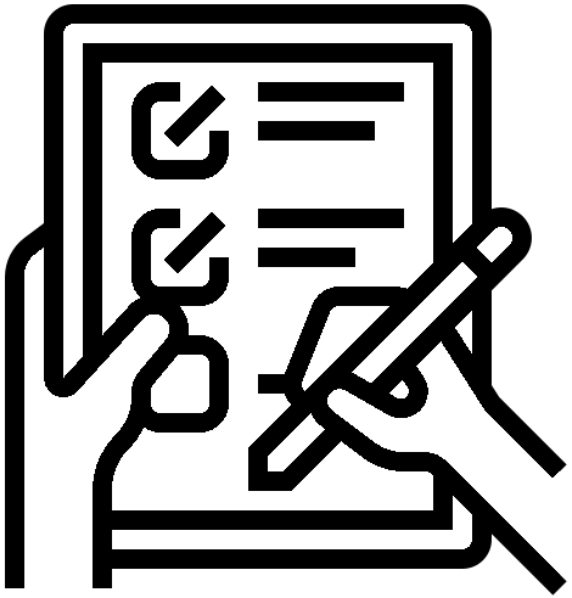
$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$P(Rain|Prediction(=Rain)) = \frac{P(Rain)P(Prediction|Rain)}{P(Rain)P(Prediction|Rain) + P(Sunny)P(Prediction|Sunny)}$$

$$= \frac{0.6 \times 0.9}{0.6 \times 0.9 + 0.4 \times 0.2} = 0.87$$

Probability

Activity 5



Probability

Counting Principles

- The methods used in combinatorics **to determine the number of ways** to arrange or select items from a given set

Counting Principles

The Multiplication Principle

- The multiplication principle states that if there are

  $\boldsymbol{k_1}$ ways to choose for the 1st event

  $\boldsymbol{k_2}$ ways to choose for the 2nd event

  …

  $\boldsymbol{k_n}$ ways to choose for the nth event

- Then the total number of possible outcomes are:

  $$(\boldsymbol{k_1})(\boldsymbol{k_2}) \dots (\boldsymbol{k_n})$$

Counting Principles

The Multiplication Principle

- Suppose we want to choose two items from a collection of 4 books (A, B, C, and D) and 3 movies (W, X, and Y)
- Using the multiplication principle to calculate the number of ways to select two items as follows
  - There are 4 ways to choose the first book
  - Once the first book is selected, there are 3 ways to choose the movie

  $\Rightarrow$ There are 4 x 3 = 12 ways to choose two items from this collection

Probability

Permutations

- Permutations are the possible **ordered** selections of r objects out of a total of n objects
- The number of permutations of **n objects taken r** at a time is given by the formula

$$ {}_nP_r = \frac{n!}{(n - r)!} $$

where:

n = The total number of objects

r = The number of objects to be selected

Probability

Permutations

- Suppose we have a set of 3 different balls: red, green, and blue and want to find the number of ways to choose 2 balls and arrange them in a row
- In this case, n = 3 and r = 2

$$_{3}P_{2} = \frac{3!}{(3\ -\ 2)!} = 6$$

- So, there are 6 different ways to choose 2 balls from a set of 3 balls to arrange them in a row
- These permutations can be listed as: **RG, RB, GR, GB, BR, BG**

Probability

Combinations

- Combinations are arrangements of objects where **the order does not matter**
- The number of combinations of **n objects taken r** at a time is given by the formula

$$ {}_{n}C_{r} = \frac{n!}{(n - r)!\,r!} $$

where:

n = The total number of objects

r = The number of objects to be selected

Probability

Combinations

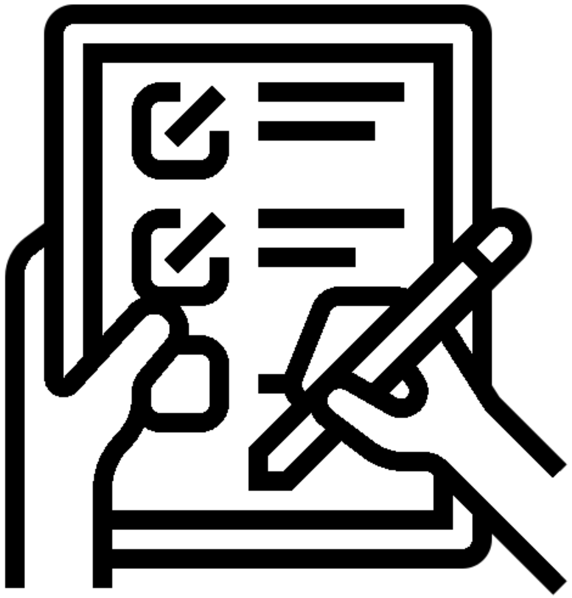
- Suppose we have a set of 3 different balls: red, green, and blue and want to find the number of ways to choose 2 balls **without regard to the order**
- In this case, n = 3 and r = 2

$$ {}_{3}C_{2} = \frac{3!}{(3\ -\ 2)!\,2!} = 3 $$

- So, there are 3 different ways to choose 2 balls from a set of 3 balls without regard to the order
- These permutations can be listed as: **RG, RB, GB**

Probability

Activity 6



Any Questions?

CANADIAN COLLEGE
OF TECHNOLOGY
AND BUSINESS

Thank You

CANADIAN COLLEGE OF TECHNOLOGY AND BUSINESS

Website www.canadianctb.ca

Email info@canadianctb.ca

Telephone +1 604-515-7880

Address 626 West Pender Street - Suite 600
Vancouver, British Columbia, V6B 1V9, Canada

Connect with CCTB

@CanadianCTB

DLI O134304821852