Graph: Part 3 - Bipartite Graph

Adila A. Krisnadhi

Faculty of Computer Science, Universitas Indonesia

References and acknowledgements

- Materials of these slides are taken from:
  - Kenneth H. Rosen. *Discrete Mathematics and Its Applications*, 8ed. McGraw-Hill, 2019. Section 10.1.
  - Jean Gallier. *Discrete Mathematics Second Edition in Progress*, 2017 [Draft]. Section 8.5
  - Robin J. Wilson. *Introductio to Graph Theory*, 4ed, 1996. Chapter 2 and 8.
- Figures taken from the above books belong to their respective authors. I do not claim any rights whatsoever.

Motivation (1)

Consider the following table of employees and their expertise.

Employee	Expertise
Andi	Requirement analysis (RA), Testing (T)
Beni	Architecture design (AD), Implementation (I), Testing (T)
Cita	Requirement analysis (RA), Architecture design (AD), Implementation (I)
Desi	Requirement analysis (RA)

To finish the project, those four employees need to complete four tasks: requirement analysis, architecture design, implementation, and testing. How can we assign tasks to each of the employee to finish the project such that each task is assigned to exactly one employee and no employee is given more than one task?

Motivation (2)

- The previous problem is an example of a **matching** problem: finding an assignment of workers to tasks so that no two workers share the same task and no worker is assigned more than one task.
  - Matching is thus a set of worker-task pairs.
- In undirected graph, matching is assigning nodes to their proper neighbors so that no two nodes are matched to the same node.
- Matching problem is usually easier to solve in bipartite graphs.
- Two variants of matching problem considered here:
  - Maximum matching: the number of worker-task pairs is maximum.
  - Complete matching: all tasks appear in some worker-task pair of the matching.

Bipartite graphs

Definition

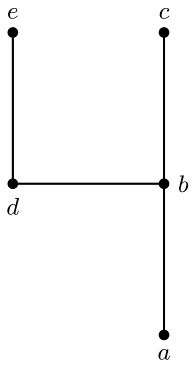
Let $G = (V, E, st)$ be an undirected graph.

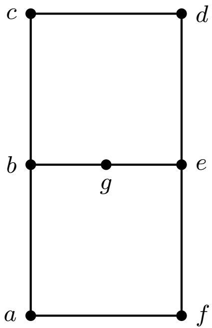
- G is called **bipartite** iff V can be partitioned into two nonempty disjoint sets V_1 and V_2 such that for every edge $e \in E$, one endpoint of e is in V_1 and the other endpoint is in V_2.
- Furthermore, such a G is **complete bipartite** if G is bipartite and every two nodes $u_1 \in V_1$ and $u_2 \in V_2$ are adjacent.

- Notice that V_1 and V_2 form a partition of V. Hence, both cannot be empty.
- Since V_1 and V_2 are disjoint, the bipartite graph G contains no edge that connects two nodes in V_1 or two nodes in V_2. In particular, G cannot contain loops.
- The pair (V_1, V_2) of the two disjoint sets above is called a **bipartition**.
- A complete bipartite graph whose bipartition is (V_1, V_2) with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

Example

Which of the following graphs is a bipartite graph?





Example

Draw $K_{2,3}$ and $K_{3,5}$.

Is the 3-dimensional hypercube Q_3 bipartite? What about n-dimensional hypercubes generally?

How to determine if a simple graph is bipartite

Theorem

A simple graph is bipartite iff if each of its nodes can be assigned one of two colors so that no two adjacent nodes share the same color.

Matching

Definition

Let $G = (V, E, st)$ be an undirected graph. A **matching** is a subset $M \subseteq E$ of edges such that if no two edges in M are incident at the same node.
A node $u \in V$ is **matched** iff there is an edge $e \in M$ in the matching with $u \in st(e)$ (otherwise, u is **unmatched**). If for such an edge $e \in M$, $st(e) = \{u, v\}$, then specifically u is matched with v and vice versa.

- For a matching M, if $e_1, e_2 \in M$, then e_1 and e_2 are not loop, and $st(e_1) \cap st(e_1) = \emptyset$, i.e., their endpoints are all different.
- Matching can exist in non-bipartite graphs, but finding matching in a bipartite graph is usually easier.
- A graph can contain more than one matching. See next.

Maximal, maximum, complete, perfect matching

Definition

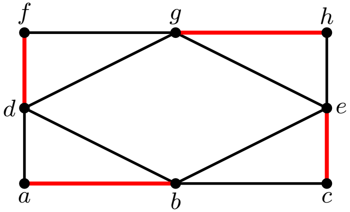
Let $G = (V, E)$ be a graph.

- A matching M is **maximal** in G iff M is not a proper subset of any other matching in G.
- A matching M is **maximum** in G iff $|M| \geqslant |M'|$ for every matching M' in G.
- A matching M is **perfect** in G iff all nodes in G are matched.
- If G is a bipartite graph with V_1, V_2 as the bipartition, then a matching M is **complete from** V_1 **to** V_2 iff every node in V_1 is matched with some node in V_2, i.e., $|M| = |V_1|$.

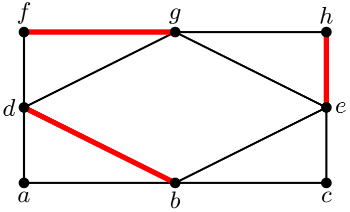
- If $G = (V, E)$ has a perfect matching M, then $|V|$ is even and $|M| = |V|/2$.

Example (red lines give the matching)

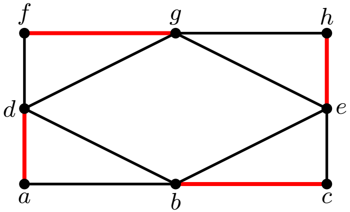
Perfect (and maximum) matching of size 4



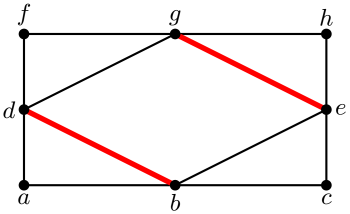
Maximal matching of size 3



Perfect (and maximum) matching of size 4

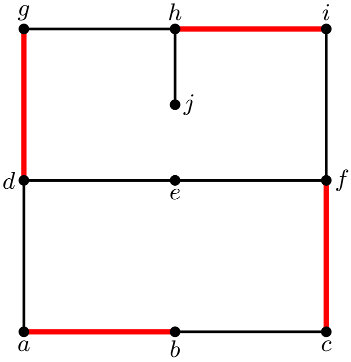


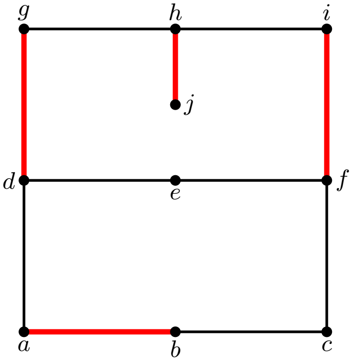
Maximal matching of size 2



Example

Two maximum (and maximal) matchings of size 4, but no perfect matching (should be of size 5 if one exists).





Example

Recall the following table of employees and their expertise.

Employee	Expertise
Andi	Requirement analysis (RA), Testing (T)
Beni	Architecture design (AD), Implementation (I), Testing (T)
Citra	Requirement analysis (RA), Architecture design (AD), Implementation (I)
Desi	Requirement analysis (RA)

To finish the project, those four employees need to complete four tasks: requirement analysis, architecture design, implementation, and testing. How can we assign tasks to each of the employee to finish the project such that each task is assigned to exactly one employee and no employee is given more than one task?

Answer to the previous question amounts to finding a complete matching from the set of employees to the set of expertise.

Example

Suppose there are m men and n women in an island. Everyone has a list of names of their opposite gender who (s)he can accept as husband/wife. This can be modeled as a bipartite graph with bipartition (V_1, V_2) where V_1 is the set of those m men, V_2 is the set of those n women, and there is an edge between a man and a woman iff they are both willing to be paired as husband and wife.

- A matching is a graph whose edges are between men and women who are actually paired as husband and wife.
- A maximum matching corresponds to the largest possible set of husband-wife pairs that can be obtained.
- A complete matching from V_1 to V_2 is the set of husband-wife pairs such that every man is married to some woman (but possibly not every woman is married to a man).

Exercise

In a company, four employees: Soni, Toni, Weni, and Yeni are tasked to finish a project, which requires them to complete four tasks: requirement analysis (RA), architecture design (AD), implementation (I), and testing (T). Soni's expertise is on AD. Toni's expertise is on RA, I, and T. Weni's expertise is on AD. Yeni's expertise is on RA, AD, and T. How can we assign tasks to each of the employee to finish the project such that each task is assigned to exactly one employee and no employee is given more than one task?

Sufficient and necessary conditions for a complete matching

Theorem (Hall's marriage theorem)

A bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) contains a complete matching from V_1 to V_2 if and only if $|N(A)| \geqslant |A|$ for every subset $A \subseteq V_1$. (Recall: $N(A)$ is the set of nodes neighbor to some node in A).

Exercise

Does the following graph have a complete matching from $V_1 = \{a, b, c, d\}$ to $V_2 = \{p, q, r, s, t\}$? If not, when does the Hall's marriage condition fail?