Relation: Part 3 - Properties of Binary Relation

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Properties of (binary) relation

Properties of binary relation:

- Function(ality)
- Reflexivity, irreflexivity
- Symmetry, asymmetry, antisymmetry
- Transitivity
- Equivalence (discussed in a separate video series)
- Partial ordering (discussed in a separate video series)

Note: property of a relation means "a condition/characteristic that can be true or false for a given relation".

Functions

- Given nonemptysets A, B, a **function** from A to B, written $f : A \to B$, is an assignment of **exactly one** element of B to **each** element of A.
  - $f(a) = b$ means $b \in B$ is the unique element of B assigned to $a \in A$.
- A function $f : A \to B$ can be viewed as a relation by writing it as

$$R_f = \{(a, f(a)) \mid a \in A\}$$

  For every $a \in A$, there exists exactly one $b \in B$ such that $(a, b) \in R_f$, namely $b = f(a)$.

Let $A = \{0, 1, 2\}$, $B = \{a, b\}$. Are the following relations a function? Give their graph and matrix representation.

- $R_1 = \{(0, a), (1, b), (2, a)\}$
  ☐ Yes ☐ No

- $R_2 = \{(0, a), (1, b), (2, a), (2, b)\}$
  ☐ Yes ☐ No

- $R_3 = \{(0, a), (2, b)\}$
  ☐ Yes ☐ No

Graph and matrix representation of functional relation

- Graph representation of functions/functional relation $R \subseteq A \times B$:
    - Each $a \in A$ has exactly one outgoing edge to some $b \in B$.
- Matrix representation of functions/functional relation $R \subseteq A \times B$:
    - Each row corresponding to an $a \in A$ contains exactly one value of 1, while the rest is 0.

Determine if the following relation is a function.

1. $R_1 = \{(a, b) \in A^2 \mid a \geqslant b\}$ with $A = \{5, 6, 7, 8\}$
2. $R_2 = \{(a, b) \in \mathbb{N}^2 \mid a = \log_2 b\}$
3. $R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{Z} \mid x \leqslant y \leqslant x + 1\}$.

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Is the following relation a function?

$$R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{Z} \mid x \leqslant y \leqslant x + 1\}$$

Reflexivity, irreflexivity

Definition

Let $R \subseteq A^2$ be a binary relation over a set A.

- R is **reflexive** iff $(a, a) \in R$ for every $a \in A$.
- R is **irreflexive** iff $(a, a) \notin R$ for every $a \in A$.

- Irreflexive $\neq$ not reflexive

Are $R_1, \ldots, R_5$ reflexive or irreflexive relations over $\{1, 2, 3, 4\}$?

- $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
  Reflexive (Yes / No). Irreflexive (Yes / No).

- $R_2 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
  Reflexive (Yes / No). Irreflexive (Yes / No).

- $R_3 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
  Reflexive (Yes / No). Irreflexive (Yes / No).

Graph and matrix representation of (ir)reflexive relations

Reflexive relation $R \subseteq A \times A$.

- Graph: There is a **loop** on **all** elements of A.
- Matrix: **All main diagonal** elements of the matrix is **1**.

Irreflexive relation $R \subseteq A \times A$:

- Graph: There is **no loop** on **all** elements of A.
- Matrix: **All main diagonal** elements of the matrix is **0**.

Exercises

Determine if the following relations are reflexive, irreflexive, or neither.

1. $R_1 = \emptyset$ on any nonempty domain.
2. $R_2 = \{(a, b) \in A^2 \mid ab \geqslant 1\}$ with $A = \{1, 2, 3, 4\}$
3. $R_3 = \{(a, b) \in A^2 \mid ab < 1\}$ with $A = \{1, 2, 3, 4\}$
4. $R_4 = \{(a, b) \in \mathbb{N}^2 \mid ab \geqslant 1\}$
5. $R_5 = \{(a, b) \in \mathbb{N}^2 \mid ab < 1\}$

Is the following relation reflexive, irreflexive, or neither?

$R_1 = \emptyset$ on any nonempty domain.

Is the following relation reflexive, irreflexive, or neither?

$R_2 = \{(a, b) \in A^2 \mid ab \geqslant 1\}$ with $A = \{1, 2, 3, 4\}$

Is the following relation reflexive, irreflexive, or neither?

$R_3 = \{(a, b) \in A^2 \mid ab < 1\}$ with $A = \{1, 2, 3, 4\}$

Is the following relation reflexive, irreflexive, or neither?

$$R_4 = \{(a, b) \in \mathbb{N}^2 \mid ab \geqslant 1\}$$

Is the following relation reflexive, irreflexive, or neither?

$$R_5 = \{(a, b) \in \mathbb{N}^2 \mid ab < 1\}$$

Symmetry, asymmetry, antisymmetry

Definition

Let $R \subseteq A^2$ be a binary relation over a set A.

- R is **symmetric** iff whenever $(a, b) \in R$, then $(b, a) \in R$.
- R is **asymmetric** iff whenever $(a, b) \in R$, then $(b, a) \notin R$.
- R is **antisymmetric** iff for every $(a, b) \in R$, whenever both $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

- Asymmetry and antisymmetry are different, and both are different from not symmetric.
- Antisymmetry vacuously holds when there are **no** elements $a, b \in A$ such that both $(a, b) \in R$ and $(b, a) \in R$.

Examples

Determine if the following relations over $\{1, 2, 3, 4\}$ symmetric, asymmetric, and/or antisymmetric.

- $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
- $R_2 = \{(1,1), (1,2), (2,1)\}$
- $R_3 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
- $R_4 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

Is the relation over $\{1, 2, 3, 4\}$ (a/anti)symmetric?

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$$

Is the relation over $\{1, 2, 3, 4\}$ (a/anti)symmetric?

$$R_2 = \{(1,1),(1,2),(2,1)\}$$

Is the relation over $\{1, 2, 3, 4\}$ (a/anti)symmetric?

$R_3 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

Is the relation over $\{1, 2, 3, 4\}$ (a/anti)symmetric?

$$R_4 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$

Symmetric relations

- Graph: each edge **always has** a counterpart edge going to the opposite direction; loops are considered going in both directions.
- Matrix: matrix is symmetric with respect to the main diagonal, i.e., if the matrix of the relation is $\mathbf{M}_R = [m_{ij}]$, then $m_{ij} = m_{ji}$

Asymmetric relations

- Graph: each edge **never has** a counterpart edge going to the opposite direction; **loops** are **disallowed**.
- Matrix: each entry at **non-diagonal position** can **never** be **both 1**, and the **main diagonal** entries can **never** be **1**. That is, if the matrix of the relation is $\mathbf{M}_R = [m_{ij}]$, then $m_{ii} = 0$ and when $i \neq j$, either $m_{ij} = 0$ or $m_{ji} = 0$.

Graph and matrix representation of (a/anti)symmetric relations (2)

Antisymmetric relations

- Graph: similar to graph for asymmetric relation, except that **loops** are **allowed**.
- Matrix: similar to matrix for asymmetric relation, except that the **main diagonal may contain 1** as entry, i.e., if the matrix of the relation is $\mathbf{M}_R = [m_{ij}]$, then either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$, while m_{ii} can either be 0 or 1.

Exercises

Determine if the following relations symmetric, asymmetric or antisymmetric.

1. $R_1 = \emptyset$ over any nonempty domain.
2. $R_2 = \{(a, b) \in A^2 \mid a = 2b\}$ with $A = \{0, 1, 2, 3, 4\}$
3. R_3 is a binary relation defined on the set of all Youtube videos where $(a, b) \in R_3$ iff everyone who has watched the video a has also watched the video b.
4. $R_4 = \{(x, y) \in \mathbb{Z}^2 \mid |x - y| = 1\}$ with $|z|$ the absolute value of z.
5. $R_5 = \{(x, y) \in A^2 \mid x \geqslant y^2\}$ with $A = \{z \in \mathbb{Z} \mid z > 1\}$

Is the following relation symmetric? Asymmetric? Antisymmetric?

$R_1 = \emptyset$ over any nonempty domain.

Is the following relation symmetric? Asymmetric? Antisymmetric?

$R_2 = \{(a, b) \in A^2 \mid a = 2b\}$ with $A = \{0, 1, 2, 3, 4\}$

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R_3 is a binary relation defined on the set of all Youtube videos where $(a, b) \in R$ iff everyone who has watched the video a has also watched the video b.

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$R_5 = \{(x, y) \in A^2 \mid x \geqslant y^2\}$ with $A = \{z \in \mathbb{Z} \mid z > 1\}$

Transitivity

Definition

Let $R \subseteq A^2$ be a binary relation over a set A. We say that R is **transitive** iff whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ also holds.

When checking transitivity, we must check the condition on every two pairs in R that are of the form $(a, b), (b, c)$.

- If no such two pairs are found, the transitivity condition vacuously holds.

Example

Are the following relations over $\{1, 2, 3, 4\}$ transitive?

- $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
- $R_2 = \{(1,1), (1,2), (2,1)\}$
- $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
- $R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

Is the following relation over $\{1, 2, 3, 4\}$ transitive?

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$$

Is the following relation over $\{1, 2, 3, 4\}$ transitive?

$$R_2 = \{(1,1), (1,2), (2,1)\}$$

Is the following relation over $\{1, 2, 3, 4\}$ transitive?

$$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$

Is the following relation over $\{1, 2, 3, 4\}$ transitive?

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

Graph representation of transitive relations

Graph representation for transitive relations satisfies:

- Every pair of vertices connected through two hops of edges have a shortcut edge between them.

There is no simple characteristic for matrix representation of transitive relations.

Exercises

Determine if the following relations are transitive.

1. $R_1 = \emptyset$ on any nonempty domain.
2. $R_2 = \{(x, y) \in A^2 \mid xy = 0\}$ with $A = \{0, 1, 2, 3\}$.
3. R_3 is a binary relation defined over a set of Youtube videos where $(a, b) \in R_3$ iff everyone who has watched video a has also watched video b.
4. $R_4 = \{(x, y) \in \mathbb{Z}^2 \mid |x - y| = 1\}$ with $|z|$ the absolute value of z.
5. $R_5 = \{(x, y) \in A^2 \mid x \geqslant y^2\}$ with $A = \{z \in \mathbb{Z} \mid z > 1\}$

Is the following relation transitive?

$R_1 = \emptyset$ on any nonempty domain.

Is the following relation transitive?

$R_2 = \{(x, y) \in A^2 \mid xy = 0\}$ with $A = \{0, 1, 2, 3\}$.

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R_3 is a binary relation defined over a set of Youtube videos where $(a, b) \in R_3$ iff everyone who has watched video a has also watched video b.

Is the following relation transitive?

$R_4 = \{(x, y) \in \mathbb{Z}^2 \mid |x - y| = 1\}$ with $|z|$ the absolute value of z.

Is the following relation transitive?

$R_5 = \{(x, y) \in A^2 \mid x \geqslant y^2\}$ with $A = \{z \in \mathbb{Z} \mid z > 1\}$