Relation: Part 5 - Closures of Relation

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Closures of Relation

- Sometimes we are given a relation R that, unfortunately, does not satisfy some desired property.
- Example: not every pair of cities in Indonesia are connected by a direct flight (even when both possess an airport).
  - If there is no direct flight from two given cities, can the two cities be still be connected by a flight route, possibly with a layover?
  - The answer can be obtained by computing the **transitive closure** of the flight relation.

General definition

Definition

Let R be a binary relation on A and P be a property of a relation. A **$\mathbf{P}$-closure** of R, if it exists, is a binary relation S on A such that

1. S satisfies the property $\mathbf{P}$;
2. $R \subseteq S$, i.e., S contains R.
3. S is the smallest relation satisfying both (1) and (2) above. That is, if there is a binary relation R' on A such that R' satisfies $\mathbf{P}$ and $R \subseteq R'$, then it must be the case that $S \subseteq R'$.

- $\mathbf{P}$ is a property of relation, e.g., reflexive, symmetric, transitive, etc.
- If $\mathbf{P}$-closure of R exists, then it must be unique.

How to obtain $\mathbf{P}$-closure of a relation

Let $R \subseteq A \times A$ be a binary relation on A. The $\mathbf{P}$-closure of R can be obtained in principle by the following steps:

1. Check if R already satisfies the property $\mathbf{P}$. If so, then R is the $\mathbf{P}$-closure to itself, and we're done.
2. Otherwise, start with $R' := R$, and keep adding some pairs $(x, y) \in A \times A$ to R' until R' satisfies the property $\mathbf{P}$
   - The added pairs must bring R' closer to satisfying $\mathbf{P}$, hence they depend on the property $\mathbf{P}$.

Reflexive closure

Definition

Let $R \subseteq A \times A$. The **reflexive closure** of R is the **smallest reflexive** binary relation $R' \subseteq A \times A$ such that $R \subseteq R'$.

Give the reflexive closure of $R \subseteq A \times A$ on $A = \{1, 2, 3, 4\}$ if $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$.

Give the reflexive closure of $R_1 = \{(a,b) \in \mathbb{Z}^2 \mid a < b\}$ and $R_2 = \{(a,b) \in \mathbb{Z}^2 \mid a \leqslant b\}$.

Symmetric closure

Definition

Let $R \subseteq A \times A$. The **symmetric closure** of R adalah is the **smallest symmetric** binary relation $R' \subseteq A \times A$ such that $R \subseteq R'$.

Give the symmetric closure of $R \subseteq A \times A$ on $A = \{1, 2, 3, 4\}$ if
$R = \{(1, 1), (1, 2), (2, 1), (3, 2), (3, 4)\}$.

Give the symmetric closure of $R = \{(a, b) \in \mathbb{Z}^2 \mid a > b\}$.

Transitive closure

Definition

Let $R \subseteq A \times A$. The **transitive closure** of R is the **smallest transitive** binary relation $R' \subseteq A \times A$ such that $R \subseteq R'$.

Let R be a binary relation on $A = \{1, 2, 3, 4\}$ with $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$. Find the transitive closure of R.

Find the transitive closure of $R = \{(a, b) \in \mathbb{N}^2 \mid b = a + 1\}$