Graph: Part 2 - Subgraph and Operations on Graphs

Adila A. Krisnadhi

Faculty of Computer Science, Universitas Indonesia

References and acknowledgements

- Materials of these slides are taken from:
  - Kenneth H. Rosen. *Discrete Mathematics and Its Applications*, 8ed. McGraw-Hill, 2019. Section 10.1.
  - Jean Gallier. *Discrete Mathematics Second Edition in Progress*, 2017 [Draft]. Section 4.1, 4.2, 4.4
  - Robin J. Wilson. *Introductio to Graph Theory*, 4ed, 1996. Chapter 2.
- Figures taken from the above books belong to their respective authors. I do not claim any rights whatsoever.

We can apply operations on one or more graphs to obtain another graph. Here, we discuss:

- underlying undirected graph of a digraph;
- subgraph operation;
- edge addition and removal;
- edge contraction;
- node addition and removal; and
- graph union.

Underlying undirected graph

FACULTY OF COMPUTER SCIENCE

UNIVERSITAS INDONESIA

Definition

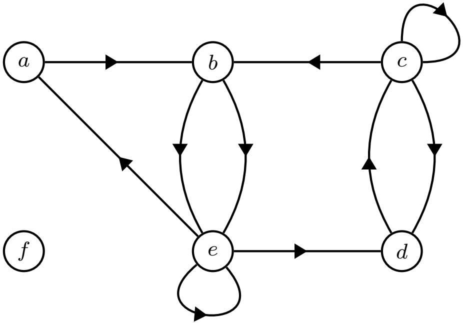
Given a directed graph $G = (V, E, s, t)$, the **underlying undirected graph** $G' = (V', E', st)$ of G is the graph obtained from G as follows:

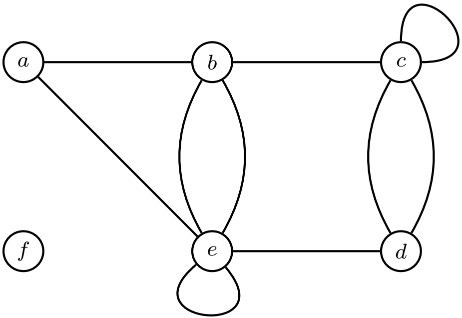
1. set $V' := V$
2. for each $e \in E$, construct an edge $e' \in E'$ such that $st(e') = \{s(e), t(e)\}$ – if $s(e) = t(e)$, then $st(e')$ contains a single node.

- That is, the underlying undirected graph of a directed graph is obtained by simply ignoring the direction of its edges.
- $|V'| = |V|$ and $|E'| = |E|$.
- If the directed graph has an edge from a to b and an edge from b to a, then its underlying undirected graph would contain parallel edges between a and b.

Example

A digraph (left figure) and its underlying undirected graph (right figure).





The following applies to both directed and undirected graphs.

Definition

Given a graph $G = (V, E)$, a **subgraph** of G is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If $G' \neq G$ (i.e,. $V' \subsetneq V$ or $E' \subsetneq E$), then G' is a **proper subgraph** of G.

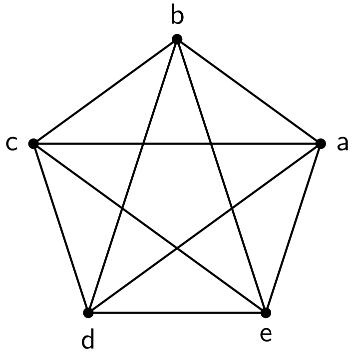
Definition

Given a graph $G = (V, E)$ and a set of nodes $W \subseteq V$, the **induced subgraph** of G **with respect to** W is a subgraph G whose nodes are in W and edges are those from E that connect only pairs of nodes in W.

Note: if G is a (proper) subgraph of H, then we also say that H is a **(proper) supergraph** of G.

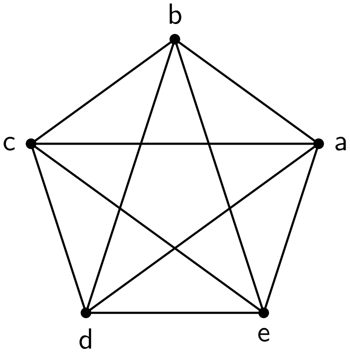
Example

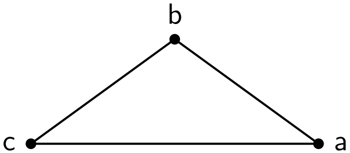
Given the following complete graph K_5, give its induced subgraph w.r.t. $\{a, b, c\}$.



Example

Given the following complete graph K_5, give its induced subgraph w.r.t. $\{a, b, c\}$.



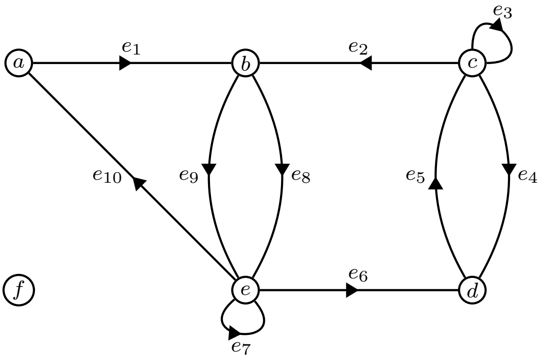


Node set $V' = \{a, b, c\}$

Edge set $E' = \{\{a, b\}, \{b, c\}, \{a, c\}\}$

Exercise

Find the induced subgraph w.r.t. $\{a, c, e\}$



Edge addition and removal

Definition

Let $G = (V, E)$ be a graph.

- If $e \in E$ is an edge in E, then $G - e = (V, E - \{e\})$ is the subgraph obtained from G by removing the edge e.
  - Removing multiple edges can be done analogously.
- If $e \in E$ is a new edge not in E, but already connects the nodes $u, v \in V$, then $G + e = (V, E \cup \{e\})$ is the graph obtained from G by adding the edge e to it.
  - Adding multiple edges can be done analogously.

Example

Let K_5 be the complete graph with nodes $\{a, b, c, d, e\}$. Give the graph obtained by removing edges $\{a, d\}, \{b, d\}, \{c, d\}, \{d, e\}, \{c, e\}$. Further, what graph do we obtain by adding $\{c, e\}$ to it?

Node addition and removal

Definition

Let $G = (V, E)$ be a graph and $v \in V$ is a node in it. The removal of v from G yields a subgraph $G' = (V', E')$ such that:

- $V' = V - \{v\}$
- if G is undirected: $E' = E - \{e \in E \mid v \in st(e)\}$
  if G is directed: $E' = E - \{e \in E \mid v = s(e) \text{ or } v = t(e)\}$.

Meanwhile, adding a node v to G is straightforward: simply add v to the node set V.

Example

Let K_5 be the complete graph with nodes $\{a, b, c, d, e\}$. What is the result of removing a from it? Is the resulting graph still complete?

Edge contraction

Definition

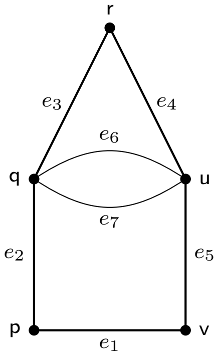
Let $G = (V, E)$ and $e \in E$ is an edge in G connecting u and v. **Contracting** the **edge** e from G yields a new graph $G' = (V', E')$ according to the following steps:

1. remove e from E;
2. create a new node w and add w to V;
3. if G is undirected graph: for each edge e' such that $st(e') = \{u, x\}$ or $st(e') = \{v, x\}$ for some node $x \in V$, set $st(e') := \{w, x\}$;
4. if G is directed graph:
    - for each edge e' such that $s(e') = u$ or $s(e') = v$, set $s(e') := w$;
    - for each edge e' such that $t(e') = u$ or $t(e') = v$, set $t(e') := w$;
5. remove u and v from V.

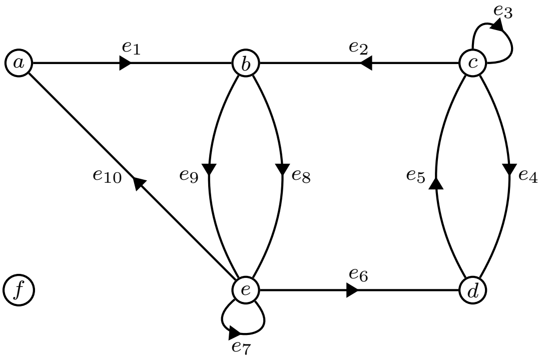
Edge contraction is like edge removal, but also accompanied with node merging operation. Hence, the result is not a subgraph of the original graph.

Example

Given the following graph, what is the result of contracting the edge e_7?



What is the result of contracting the edge e_8 in the following graph?



Graph union

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The **union** of G_1 and G_2, denoted $G_1 \cup G_2$, is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$.

Example

Compute $G_1 \cup G_2$ if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 = \{a, b, c, d, e\}$, $E_1 = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, e\}, \{c, e\}, \{d, e\}\}$, $V_2 = \{a, b, c, d, f\}$, and $E_2 = \{\{a, b\}, \{b, c\}, \{b, d\}, \{b, f\}, \{c, f\}\}$.