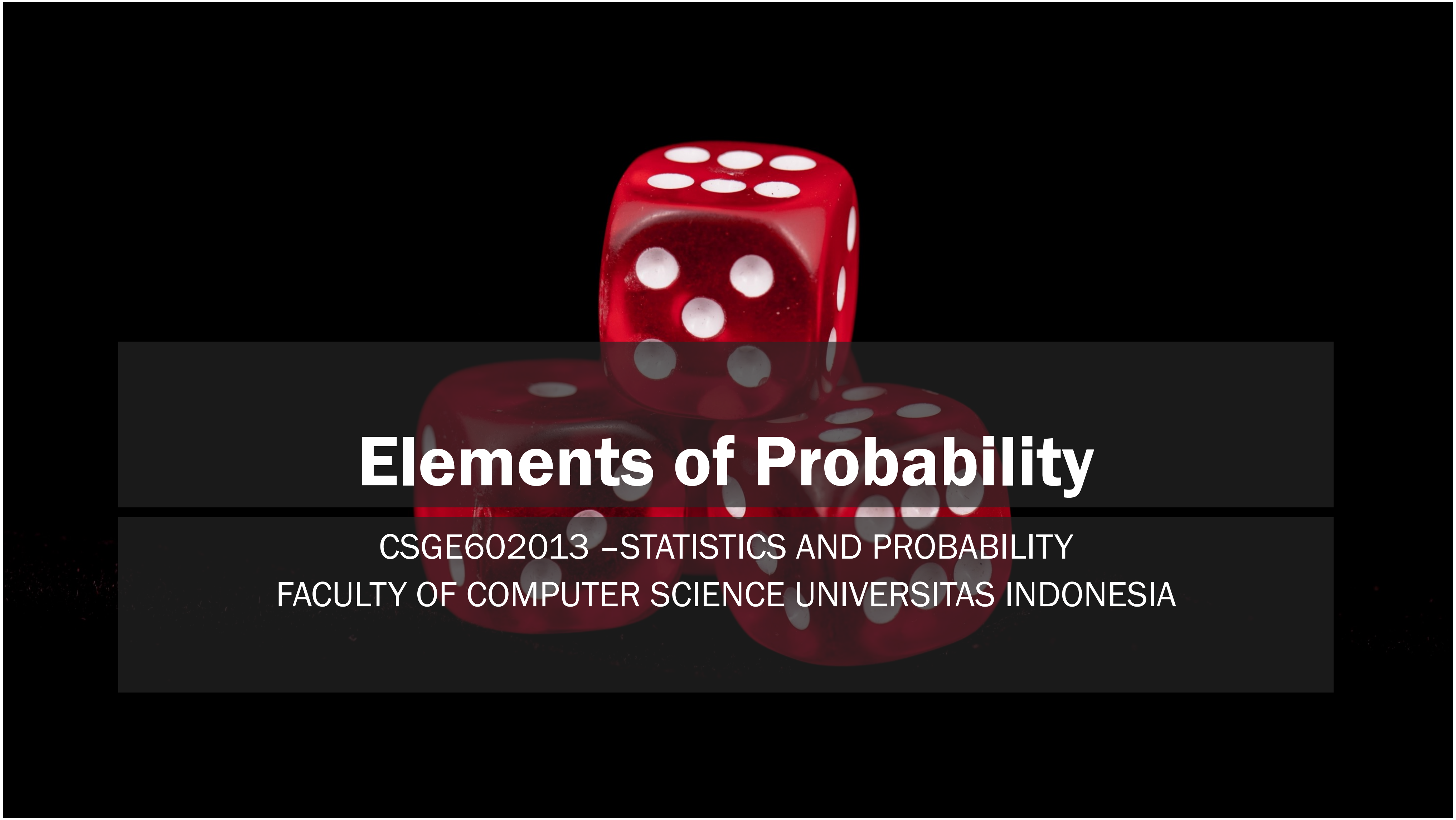


Elements of Probability

CSGE602013 –STATISTICS AND PROBABILITY

FACULTY OF COMPUTER SCIENCE UNIVERSITAS INDONESIA

References

- Introduction to Probability and Statistics for Engineers & Scientists, 4th ed., Sheldon M. Ross, Elsevier, 2009.
- Probability and Statistics for Engineers & Scientists, 4th Edition, Anthony J. Hayter, Thomson Higher Education
- Probability, Statistics, and Queueing Theory with Computer Science Applications, Arnold O. Allen
- Probability and Statistics for Engineers & Scientists, 4th Edition, Ronald E. Walpole, Raymond H. Myers

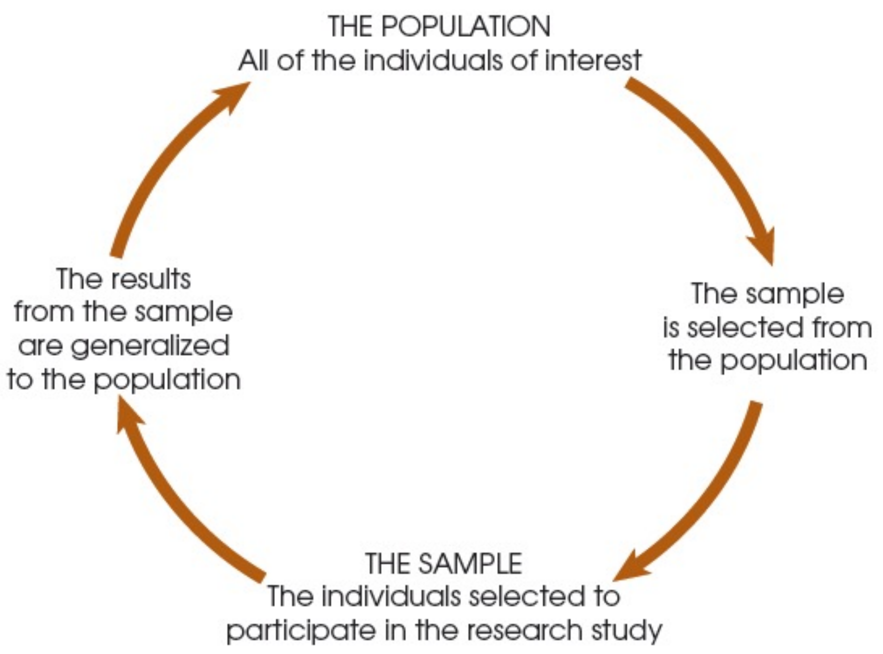
Introduction

- Probability theory provides a basis for the science of statistical inference from data.
- Probabilistic model is used to quantify random phenonema.

Inferential Statistics

- Probability theory provides a basis for the science of statistical inference from data.
- How?

FIGURE 1.1
The relationship between a population and a sample.



The 3-Step Process

1. Sampling

- A sample (of size n) is obtained from a population
- Assume the probability distribution of the population.

2. Descriptive Statistics

- Describe the sample

3. Inference Statistics

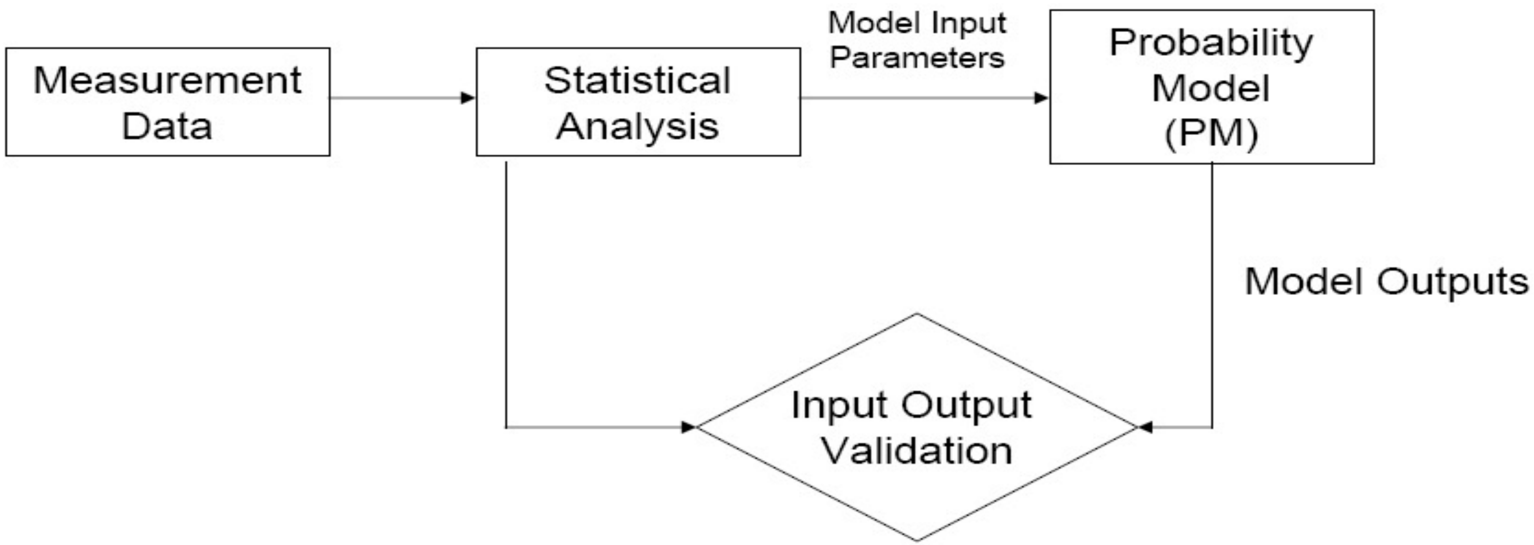
- Make a decision from a sample for our problem

Random Phenomena

- Random phenomena characteristic → the future behaviour is not predictable in deterministic ways.

  Example in computer systems
  - job/message/request arrival
  - job/message/request execution time
  - component or resource failure
- Study of random phenomena → to make it manageable and predictable
- How to quantify the randomness → using probabilistic model
- How to estimate the quantifiers → Using methods of statistics and data measurements.

Random Phenomena Modelling



- Random phenomena can be described mathematically by constructing a probabilistic model
- It consists of a list of all possible outcomes and an assignment of their probabilities
- It allows us to predict or deduce patterns of future outcomes
- Prediction based on the model must be validated against actual measurements collected from real phenomena
- The theory of statistics facilitates the validation process by drawing inferences about the model

Probability Models

- Sample Space (S): all of the possible outcomes or status of the random phenomena that can be observed.
- Events ($\mathfrak{I}$): a collection of certain sample points, that is, a subset of sample space.
- Event probability (P): consistent description/measurement of likely to occur of an event.
- Therefore, a Probability Model consists of triple (S, $\mathfrak{I}$, P)

Outline

- Sample Space and Events
- Complement, Combinations, & Algebra of Events
- Axioms of Probability
- Conditional Probability
- Bayes Rule & Law of Total Probability
- Independent Events

SAMPLE SPACE AND EVENTS

Random Experiment

- **Observation**: an activity that gives some possible outcomes
  - Example: measuring a student's height, observing the number that appears in a die rolling.
- **Random Experiment/Sampling**: an observation for which more than one outcome is possible
  - Cannot guarantee "representativeness" on all traits of interest.
  - A possible outcome from an observation is called **sample**.

Sample Space Definition

- **Sample Space *S***: a space that is formed from all possible outcomes of a random experiment.
- **Sample**: is a "point" in a sample space.
  - A sample in a sample space terminology is more related to the outcome not the object itself.

Sample Space Classification

- Finite vs Infinite Sample Space:
  - Finite example: observe the sum number resulted from rolling two dice.
  - Infinite example: observe how many times we need to roll a die to get number 6.
- Countable vs Uncountable Sample Space:
  - Countable example: how many iterations can be done in ‘while’ statement.
  - Uncountable example : how many real numbers between 0.0 and 1.0.
- Discrete vs Continuous Sample Space:
  - Discrete : finite or countable infinite sample space
  - Continuous : uncountable sample space

Sample Space Examples

- If the experiment consists of the tossing of a coin, then

$$S = \{Head, Tail\}$$

- If the experiment consists of the running of a race among the six horses having post positions 1, 2, 3, 4, 5, 6, then

$$S = \{all\ orderings\ of\ (1,2,3,4,5,6)\}$$

- Experiment consists of determining the amount of dosage that must be given to a patient until that patients reacts positively, then

$$S = \{1,2,3,4,...\}$$

Games of Chance

- Games of chance commonly involve the toss of a coin, the roll of a die, or the use of a pack of cards.
- The roll of a die:
  - A usual six-sided die has a sample space, $S = \{1, 2, 3, 4, 5, 6\}$
  - If two dice are rolled, the sample space is...

FIGURE 1.2
Sample space for rolling two dice

$\mathcal{S}$

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Events

- **An event ($\boldsymbol{E}$)** is a subset of the sample space S.
- $\therefore$ If the outcome of the experiment is contained in E, then we say that E has occurred.

Events (2)

- The event that an even score is recorded on the roll of die

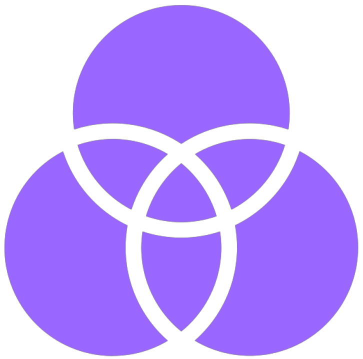
$$E = \{2,4,6\}$$

- The event that we get Head on the toss of coin =

$$E = \{Head\}$$

- Event that the number 3 horse wins the race

$$E = \{all\ outcomes\ in\ S\ starting\ with\ a\ 3\}$$



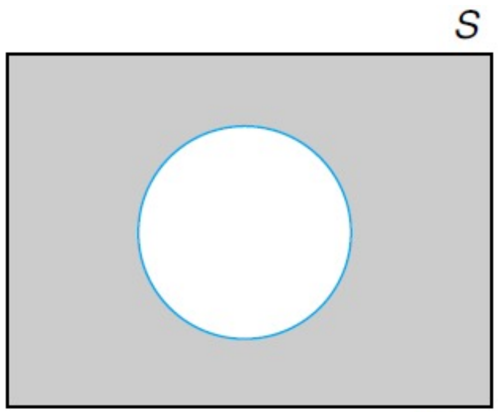
RECALL: SET THEORY

Complement, Combinations, & Algebra of Events

Complement of E

- For any event E, we define the event E^c, referred to as the complement of E, to consist of all outcomes in the sample space S that are not in E.

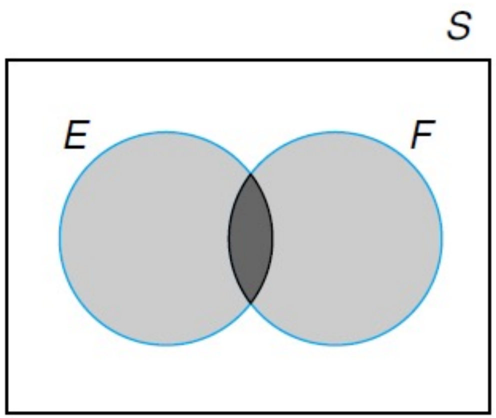
- Example:
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - $E = \{1, 3, 5\}$
  - $E^c = \{2, 4, 6\}$



(c) Shaded region: E^c

Union of E and F

- For any event E and F, we define the new event E ∪ F, called the union of the events E and F, to consists of all outcomes that are either in E or in F or both E and F.



(a) Shaded region: $E \cup F$

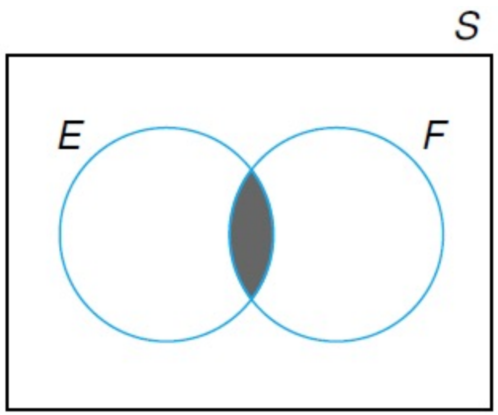
- Example:
  - E = {1, 3, 5}
  - F = {2, 4, 6}
  - E ∪ F = {1, 2, 3, 4, 5, 6} = S

$$\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup \ldots \cup E_n$$

Intersection of E and F

- For any event E and F, we define the new event EF, called the intersection of the events E and F, to consists of all outcomes that are both in E and F.

- If event $A = \emptyset$, A is a **null event.**
- If $EF = \emptyset$, E and F are **mutually exclusive**



(b) Shaded region: *EF*

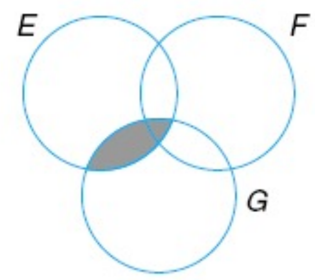
Subset & Proper Subset

- Subset ($\subseteq$) and proper subset ($\subset$)
  - $\{a, b\} \subseteq \{a, b, c\}$
  - $\{a, b\} \subset \{a, b, c\}$
  - $\{a, b, c\} \subseteq \{a, b, c\}$
  - $\{a, b, c\} \subset \{a, b, c\}$ → wrong

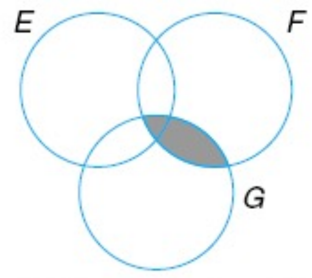
- If $\boldsymbol{E} \subseteq \boldsymbol{F}$ and $\boldsymbol{F} \subseteq \boldsymbol{E}$, we say E and F are equal, or $\boldsymbol{E} = \boldsymbol{F}$

Algebra of Events

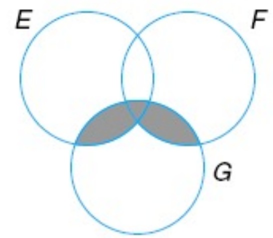
- Commutative Law
  - $E \cup F = F \cup E$
  - $EF = FE$
- Associative law
  - $(E \cup F) \cup G = E \cup (F \cup G)$
  - $(EF)G = E(FG)$
- Distributive law
  - $(E \cup F)G = EG \cup FG$
  - $EF \cup G = (E \cup G)(F \cup G)$
- DeMorgan's laws
  - $(E \cup F)^c = E^c F^c$
  - $(EF)^c = E^c \cup F^c$



(a) Shaded region: EG



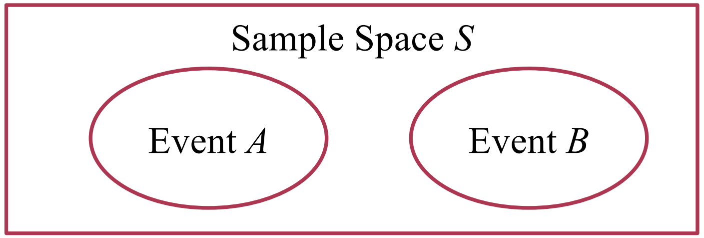
(b) Shaded region: FG



(c) Shaded region: $(E \cup F)G$
$(E \cup F)G = EG \cup FG$

Mutually Exclusive Events

- Two events A and B are disjoint (mutually exclusive) **if and only if** $A \cap B = \varnothing$



- In general, events $A_1, A_2, \ldots, A_n$, are mutually exclusive **if and only if**

$$A_i \cap A_j = \begin{cases} A_i, & i = j \\ \varnothing & \text{, other} \end{cases}$$

Collectively Exhaustive Events

- Two events A and B are Collectively Exhaustive **if and only if**
  - $A \cup B = S$
- In general, events $A_1, A_2, \ldots, A_n$, are collectively exhaustive **if and only if**
  - $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$

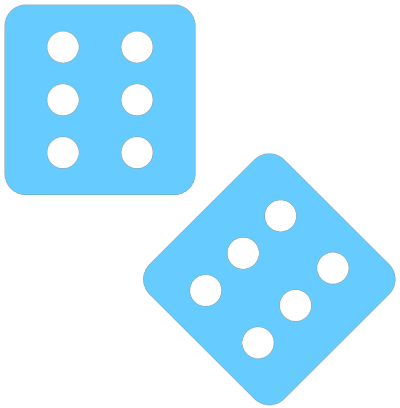
Events in a Sample Space

- A sample space is **partitioned** into events A_1, A_2, …, A_n, **if and only if** :
  - events A_1, A_2, …, A_n, are mutually exclusive

    $A_i \cap B_j = \varnothing$, for all $1 \leq i,j \leq n$ and $i \neq j$

  - events A_1, A_2, …, A_n, are collectively exhaustive

    $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$



AXIOMS OF PROBABILITY

Will it rain this afternoon?

Do I have enough gas to drive to PIM ?

Is Transmart crowded today?

What is the chance Fasilkom students get married exactly 3 years after graduating?

Probability

- The probability is a consistent description of a possibility/chance as a number between 0 (impossible) and 1 (certain).
  - Lower number indicates it is *less likely* to occur, 0 indicates it will *never* occur.
  - Higher number indicates it is *more likely* to occur, 1 indicates it will *definitely* occur.
  - 0.5: the possibility is the same between it occurs or not.

Assigning Probabilities

- **Classic Methods**: based on a certain assumption, for example, the same likelihood to every possible outcome occurence, or careful analisys of conditions underlying the random experiment.
- **Frequency Relative Methods**: based on the estimation of past experiment using inferential statistics.
- **Subjective Methods**: based on the judgement from an expert.

Event Probability

- The probability of an event is meant to represent the “relative likelihood” that a performance of the experiment will result in the occurence of that event.
- P(***E***) will denote the probability of the event ***E*** in the sample space ***S***.
- Any events has the probability to accur as the total probabilities of all sample point included in that event.
    - If we can identify all sample points in an experiment and assign a probability in each sample point, we can calculate the event probability for any event we define.

Axioms of Probability

For each event E of an experiment having a sample space S, there is a number P(E), where P(E) follows three axioms:

1. AXIOM 1

$$0 \leq P(E) \leq 1$$

2. AXIOM 2

$$P(S) = 1$$

3. AXIOM 3

For any sequence of mutually exclusive events $E1, E2,$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1,2,\ldots,\infty$$

we call $P(E)$ the probability of the event E

Axioms of Probability (2)

- Proposition 1

$$1 = P(S) = P(E \cup E^C) = P(E) + P(E^C)$$

  - Then, we obtain

$$P(E^C) = 1 - P(E)$$

- Proposition 2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Axioms of Probability (3)

- An experiment with sample space $S = \{O_1, O_2, \dots, On\}$
- Set of probability of outcome O_i, denoted by $P(O_i)$, satisfies

$$0 \le P(O_1) \le 1, \quad 0 \le P(O_2) \le 1, \quad \dots, \quad 0 \le P(O_n) \le 1$$

and

$$P(O_1) + P(O_2) + \dots + P(O_n) = 1$$

Axioms of Probability (4)

- Some properties of probability
  - $P(\emptyset) = 0$
  - $P(E^C) = 1 - P(E)$
  - $P(E \cup F) = P(E) + P(F) - P(EF)$
  - If $A \subset B$, then $P(A) \leq P(B)$

Example 1

- The probability that a student passes math is 2/3 and the probability he/she passes biology is 4/9. The probability he/she passes both courses is 1/4.

- How big is the probability he/she passes at least 1 course?
  - M: the event the student passes math
  - B: the event the student passes biology

$$\begin{aligned} P(M \cup B) &= P(M) + P(B) - P(MB) \\ &= 2/3 + 4/9 - 1/4 \\ &= 31/36 \end{aligned}$$

Example 2

- A total of 28% of American males smoke cigarettes, 7% smoke cigars, and 5% smoke both cigars and cigarettes.

- What percentage of males smoke neither cigars nor cigarettes?
  - E: event that a randomly chosen male is a cigarette smoker
  - F: event that a randomly chosen male is a cigar smoker

$$P(E \cup F) = P(E) + P(F) - P(EF) = 0.28 + 0.07 - 0.05 = 0.3$$
$$P(E \cup F)^C = 1 - P(E \cup F) = 1 - 0.3 = 0.7$$

Exercise

- A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities $0.1, 0.3, 0.5$ and 0.1 respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$ and C the event $\{d\}$.
- Find :
  - $P(A), P(B), \quad and\ P(C)$
  - $P(AC), P(BC), and\ P(CC)$
  - $P(A\ \cap\ B), P(A\ \cup\ B), and\ P(A\ \cap\ C)$

Sample Spaces Having Equally Likely Outcomes

- If a die is rolled, what is the probability that its face will equal 6 ?

$$1/6$$

- Are you sure ?

Actually, we cannot directly answer that question.

∴ But, if we **assume that all possible outcomes are equally likely** to occur, then 1/6 is a correct answer

Sample Spaces Having Equally Likely Outcomes (2)

- In many experiments, it is natural to assume that each point in the sample space is equally likely to occur.
- For $S = \{1, 2, 3, \dots, N\}$, it is natural to assume $P(\{1\}) = P(\{2\}) = P(\{3\}) \dots .. P(\{N\}) = p$
- Given $P(\{1\}) = P(\{2\}) = P(\{3\}) \dots .. P(\{N\}) = p$, and using axiom 2 & 3, we have:

$$P(S) = 1$$

$$P(S) = P(\{1\}) + P(\{2\}) + P(\{3\}) \dots .. P(\{N\})$$

$$P(S) = n.p$$

$$P(\{i\}) = p = \frac{1}{N}$$

$$\therefore P(E) = \frac{no\ of\ members\ in\ E}{N}$$

Rolling a Fair Dice

- The event in which an even score is recorded on the roll of a die, $even = \{ 2,4,6 \}$
  - For a fair die, the probability is

$$P(even) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Rolling Two Fair Die

- The event that the sum of the scores of two dice is equal to 6, $A = \{ (1,5), (2,4), (3,3), (4,2), (5,1) \}$
  - For two fair die the probability is

$$P(A) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$$

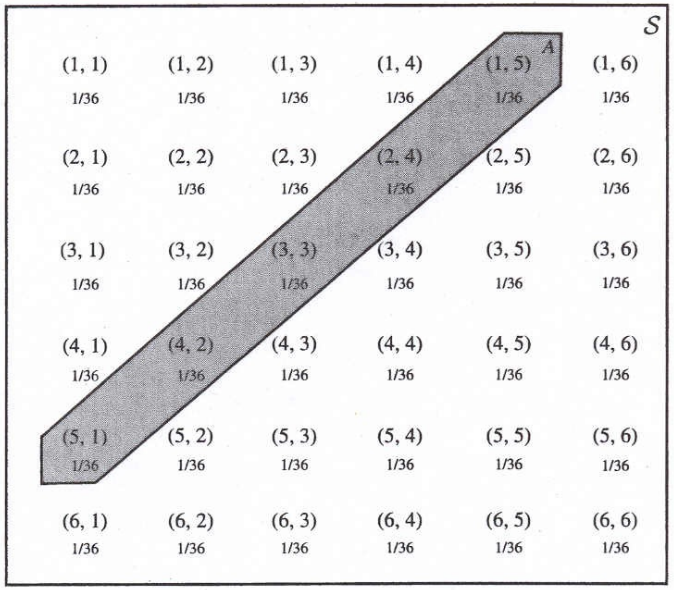


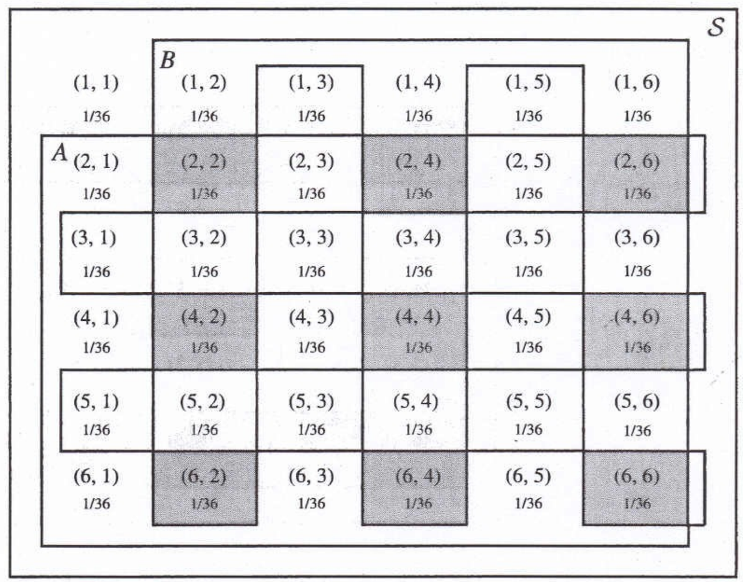
FIGURE 1.18 ●
Event A: sum equal to 6

Rolling Two Fair Die

- If we assume that all outcomes are considered equally likely,
- What is the probability that both dice have even scores ?
  - *A* : event that even score is obtained on the first die
  - *B* : event that even score is obtained on the second die

$$P(AB) = \frac{9}{36} = \frac{1}{4}$$

FIGURE 1.46 ●
Event $A \cap B$

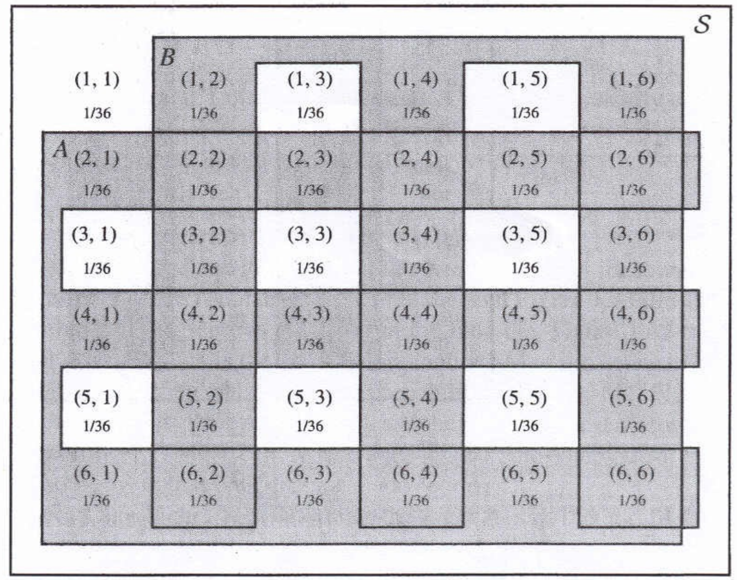


Rolling Two Fair Die (2)

- If we assume that all outcomes are considered equally likely,
- What is the probability that at least one die has even score?
  - *A* : event that even score is obtained on the first die
  - *B* : event that even score is obtained on the second die

$$P(A \cup B) = \frac{27}{36} = \frac{3}{4}$$

FIGURE 1.47 ●
Event $A \cup B$



Example

- If a family has three children, find the probability that two of the three children are girls!
- Suppose all outcomes are considered equally likely.
  - $S = \{BBB, BBG, \dots, GGG\}$
- There are 8 outcomes! Each has probability of 1/8.
  - E = event that we found that 2 of 3 are girls

    $E = \{GGB, BGG, GBG\}$
- There are 3 outcomes in the event.
  - $P(E) = 3 \times 0.125 = 3/8$

Basic Principles of Counting

- Product Rule
  - In a sequence of r **experiments** in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, then there are a total of

$$k_1 . k_2 . k_3 \dots k_r$$

possible outcomes of the r experiments.

- Example:
  - How many possible outcomes if we toss a coin, and subsequently roll a die?

Recall: Permutation and Combination

Permutation

- The number of different groups of size k that can be selected from a set of size n **in a specific order** at a time

$$P_k^n = \frac{n!}{(n-k)!}$$

- P_k^n: number of permutations of k objects taken from n **in a specific order** at a time.
- Also P_k^n, ${}_nP_k$, ${}^nP_k$, $P_{n,k}$, or $P(n,k)$

Combination

- The number of different groups of size k that can be selected from a set of size n **when the order of selection is not considered.**

$$\mathrm{C}(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- $\binom{n}{k}$: number of combinations of k objects taken from n at a time.

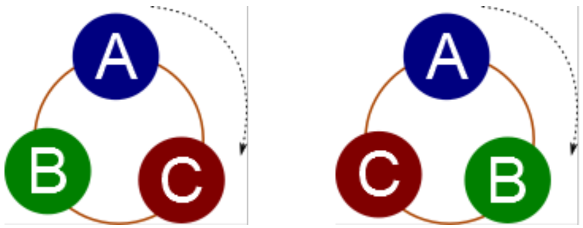
Circular & Ring Permutation

Circular Permutation

- The arrangement of n objects in a circular order.
- Formula:

$$(n-1)!$$

- Considers the clockwise / counter clockwise as different



Ring Permutation

- Considers the clockwise/ counter clockwise as the same
- Formula:

$$\begin{cases} \frac{1}{2}(n-1)! & for\ \ n \geq 3 \\ \quad 1 \quad for\ n = 1\,,2 \end{cases}$$

- Example: A. B and C → there is a way.

Example 1

- A committee of size 5 is to be selected from a group of 6 men and 9 women. The selection is made randomly.
- What is the probability that the committee consists of 3 men and 2 women ?

- Assume that "randomly selected" means that each of the $C(15, 5)$ possible combinations is equally likely to be selected.
- The probability is

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

Example 2

- A class consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance (no two students obtain the same score).
- If all rankings are considered equally likely, what is the probability that women receive the top 4 scores?

- In total, there **10!** possible rankings. There are **4!** possible rankings of the women among themselves, and **6!** for men.
- The probability is

$$\frac{4!\,6!}{10!} = \frac{1}{210}$$

CONDITIONAL PROBABILITY

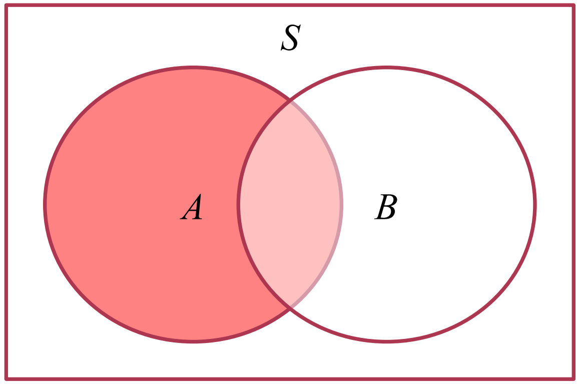
Conditional Probability Definition

- Conditional probability is the probability of an event with condition that another event is known occuring.
- Notation → P(A|B)
  - The probability of event *A*, given that event *B* occurs.
  - Event A: event that we want to compute its probability.
  - Event B: as a condition (it is known the even occurs).
- Do not confused with:
  - P(A): the probability of event A (unconditional).
  - P(AB): the probability of both event A and event B occur.
- Example: random phenomenom "chosing a person randomly".
  - Event *A*: the person has a lung cancer
  - Event *B*: the person is a heavy smoker.
  - $P(A \mid B)$: the probability that the person has a lung cancer given the person is a heavy smoker.
- The probability of an event can be changed (or not), given another event occurs.

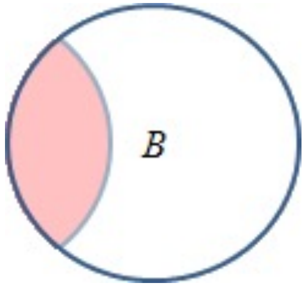
About Both Events

- In some experiments, a relevant prior information may be available, for example:
  - The probability of X getting grade A in math subject, given his GPA in previous semester is > 3.6.
  - The probability it rains this afternoon, given the claudy sky since this morning.
- Note:
  - both events are not always in chronological order.
  - Sometimes, there is no relation between both events (*mutual indenpendence*), for example:
    - The probability of the second rolling a die has a value of 6, given is the first rolling has a value of 6.

Conditional Probability Model



A is red area, *B is white*, *A* and *B are overlapping in pink* area, that is $A \cap B$.
P(A): area *A* relatively to *S*.
P(B): area *B* relatively to *S*.
P(*AB*): pink area relatively to *S*



P(*A* | *B*): are *P*(*AB*) relatively to *P*(*B*), then

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

for $P(B) \neq 0$.

If $AB = \phi$, then

$$P(A \mid A) = \frac{P(AB)}{P(B)} = 0$$

Using Tabulation

- If the number of events is two, then the combination of both events can be described using table of 2x2 plus one row for total per column and one column for total per row.

	B	B^C	Total per row
A	P(AB)	$P(AB^C)$	P(A)
A^C	$P(A^CB)$	$P(A^CB^C)$	$P(A^C)$
Total per column	P(B)	$P(B^C)$	P(S) = 1

- Conditional probability: the probability of a cell relatively to its total.
  - $P(A|B)$ = cell (A,B) / total column B = P(AB) / P(B)
  - $P(B|A)$ = cell (A,B) / total row A = P(AB) / P(A)

Conditional Probability

- The probability of event ***A*** given that the event ***B*** has occurred is called the conditional probability, is denoted by

$$P(A \mid B)$$

- In this case, ***F*** becomes our new sample space, so

$$P(A \mid B) = \frac{P(AB)}{P(B)} \qquad for\ P(B) > 0$$

F is also called the ***conditioning event***

$\bullet\ AB = \phi$

$$P(A \mid B) = \frac{P(AB)}{P(B)} = 0$$

$\bullet\ B \subset A$

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Example 1

- A red die and blue die are thrown:

A = {The red die = 5}

= {**(5,1)**, (5,2), (5,3), (5,4), (5,5), (5,6)}

B = {Sum of scores of two dices is 6}

= {(1,5), (2,4), (3, 3), (4, 2), **(5, 1)**}

$$P(A) = \frac{6}{36} = \frac{1}{6} \qquad P(B) = \frac{5}{36} \qquad P(AB) = \frac{1}{36}$$

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{1/36}{5/36} = 0.2$$

- Given *B* has occurred, what is the probability of *A* ?

Example 2

- Each employee is invited to attend the party along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner ?
  - b = boy, g = girl
  - *S* = {(b,b), (b,g), (g,b), (g,g)}
- Assume that all outcomes are **equally likely.**
  - *B* = event that both children are boys
  - *A* = event that at least one of them is a boy

$$P(B \mid A) = \frac{P(BA)}{P(A)} = \frac{P(\{(b,b)\})}{P(\{(b,b),(b,g),(g,b)\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

General Multiplication Rule

- For conditional events $P(A \mid B) = \dfrac{P(AB)}{P(B)}$

$$P(AB) = P(B)P(A \mid B) = P(A)P(B \mid A)$$

$$P(C \mid AB) = \frac{P(ABC)}{P(AB)}$$

$$P(ABC) = P(AB)P(C \mid AB) = P(A)P(B \mid A)P(C \mid AB)$$

- Then, probability of the intersection of a series of events:

Chain Rule $P(A_1 A_2 \ldots A_n) = P(A_1)P(A_2 \mid A_1)\ldots P(A_n \mid A_1 A_2 \ldots A_{n-1})$

Exercise

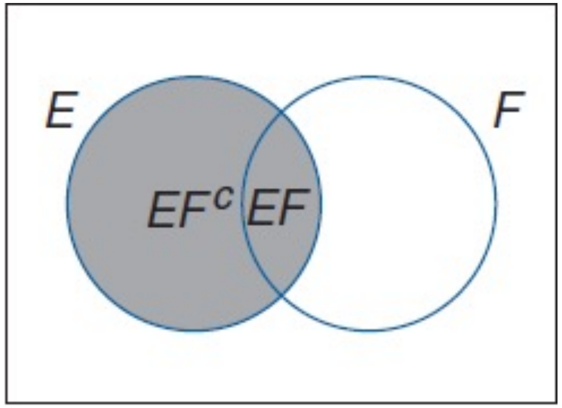
- A box contains 10 red balls and 10 blue balls. If 3 balls are selected randomly, without being returned each time, what is the probability that all three balls are red? Assume that each ball gas the same probability to be chosen (equally likely)
  - A1 : taking the first red ball
  - A2: taking the second red ball
  - A3: taking the third red ball

$$P(A1A2A3) = \frac{10}{20} \times \frac{9}{19} \times \frac{8}{18}$$

Multiplication Rule

BAYES RULE

Marginal Probability



- Let E and F be events. We can express E as

$$E = EF \cup EF^C$$

- Since EF and EF^C are mutually exclusive, the marginal probability (or total probability) of E, P(E) is:

$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E \mid F)P(F) + P(E \mid F^C)P(F^C) \\ &= P(E \mid F)P(F) + P(E \mid F^C)[1 - P(F)] \end{aligned}$$

- It enables us to determine the probability of an event by first “conditioning” on whether or not some second event has occurred.

Example

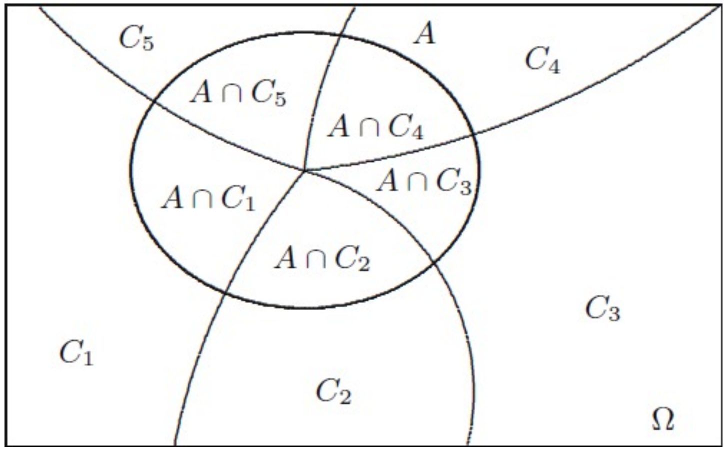
- An insurance company believes that people can be divided into two classes:

  Accident-prone person & Non-accident-prone person

- Their statistics show that an accident-prone person will have an accident with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person. If we assume 30% of the population is accident prone.
- What is the probability that new holder will have an accident ?
  - *A* : event that accident will happen
  - *F* : event that a holder is accident prone

$$
\begin{aligned}
P(A) &= P(A \mid F)P(F) + P(A \mid F^C)P(F^C) \\
&= (0.4)(0.3) + (0.2)(0.7) \\
&= 0.26
\end{aligned}
$$

Law of Total Probability



- Generalization of the previous notion

$$S = C_1 \cup C_2 \cup \ldots \cup C_n$$

C_i= mutually exclusive

$$P(C_i) > 0$$

- Then, the following equations hold:

$$A = (A \cap C_1) \cup (A \cap C_2) \cup \ldots \cup (A \cap C_n)$$

$$P(A) = P(A \cap C_1) + P(A \cap C_2) + \ldots + P(A \cap C_n)$$

$$= P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \ldots + P(A \mid C_n)P(C_n)$$

Law of total probability

Bayes' Theorem

- Suppose we know ...

$$P(C_1), P(C_2), \ldots, P(C_n)$$ → Prior probabilities

$$P(A \mid C_1), P(A \mid C_2), \ldots, P(A \mid C_n)$$ → Likelihoods

- We want to compute ...

$$P(C_1 \mid A), P(C_2 \mid A), \ldots, P(C_n \mid A)$$ → Posterior probabilities

- Then

$$P(C_i \mid A) = \frac{P(C_i A)}{P(A)}$$

$$= \frac{P(A \mid C_i)P(C_i)}{P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \ldots + P(A \mid C_n)P(C_n)}$$

Example 1

- On a multiple-choice test, the probability that a student knows the answer is 0.4. Assume that a student who guesses at the answer will be correct with probability 0.2. What is the conditional probability that a student knew the answer to a question given that he answered it correctly ?
  - *C*: events that the student answers correctly
  - *K*: events that the student knows the answer

$$P(K \mid C) = \frac{P(C \mid K)P(K)}{P(C \mid K)P(K) + P(C \mid K^C)P(K^C)}$$

$$= \frac{(1)(0.4)}{(1)(0.4) + (0.2)(0.6)} = 0.71$$

Exercise

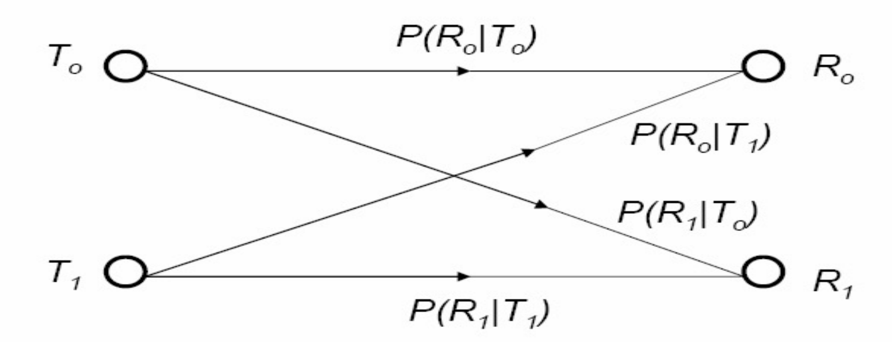
- The chances of A, B and C becoming manager of a certain company are 5 : 3 : 2. The probabilities that the office canteen will be improved if A, B, and C become managers are 0.4, 0.5 and 0.3 respectively. If the office canteen has been improved, what is the probability that B was appointed as the manager?

Bayes' Applications

- Bayes' rule is one of the important rules in statistics. Statistics methods that is based on Bayes' Rule is called Bayesian Statistics.
- Bayesian Statistics is widely used in many applications, such as:
  - Decision Theory
  - In the internet wolrd : **Spam Filter**!
  - Speech Recognition
  - Machine Translation
  - etc

Application: Data Transmission

- A data transmitter carries data as one of two types of signals denoted by "0" and "1". As a result of noise, a transmitted "0" is sometimes received as "1", and a transmitted "1" is sometimes recieved as "0".



  - Event T_0 = a "0" is transmitted and event T_1 = a "1" is transmitted
    $T_1 = T_0$', $P(T_0) = 1 - P(T_1)$
  - Event R_0 = a "0" is received and event R_1 = a "1" is recieved,
    $R_1 = R_0$', $P(R_0) = 1 - P(R_1)$

Example

- Given,
  - The probability a signal "0" is transmitted from all data trasmitted, $P(T_0) = 0.45$
  - The probability a signal "0" is recieved given a signal "0" is transmitted, $P(R_0|T_0) = 0.92$
  - The probability a signal "1" is recieved given a signal "1" is transmitted, $P(R_1|T_1) = 0.95$
- Question:
  - How much $P(R_0)$ and $P(R_1)$?
  - What is the probability of a "0" is transmitted, if the signal recieved is "0"?
  - What is the probability of a "1" is transmitted, if the signal recieved is "1"?
  - What is P(*error*)? (*error*: the signal received is different from which transmitted)

- How much $P(R_0)$ and $P(R_1)$?

  $P(R_0) = P(R_0|T_0)P(T_0) + P(R_0|T_1)P(T_1) = 0.92 \text{ x } 0.45 + 0.05\text{x}0.55 = 0.4415$

  $P(R_1) = 1 - P(R_0) = 0.5585$

- What is the probability of a "0" is transmitted, if the signal recieved is "0"?

  $P(T_0 \mid R_0) = (P(R_0|T_0).P(T_0))/P(R_0)$

  $= 0.92 \text{ x } 0.45 / 0.4415 = 0.9377$

- What is the probability of a "1" is transmitted, if the signal recieved is "1"?

  $P(T_1 \mid R_1) = (P(R_1|T_1).P(T_1))/P(R_1)$

  $= 0.95 \text{ x } 0.55 / 0.5585 = 0.9355$

- What is P(*error*)? (*error*: the signal received is different from which transmitted)

  $P(T_0 \cap R_1) + P(T_1 \cap R_0) = P(R_1|T_0).P(T_0) + P(R_0|T_1).P(T_1)$

Application: “False Postive” vs “False Negative”

- A medical tester is used to detect whether a person has a certain desease or not.
  - **“False positive”** results when a test falsely or incorrectly reports a positive result. For example, this medical tester may return a positive result indicating that patient has a disease even if the patient does not have the disease.
  - **“False negative”** results when a test falsely or incorrectly reports a negative result. For example, this medical tester may return a negative result indicating that patient does not have a disease even if the patient has the disease.

Example

- A laboratory blood test is 95% effective in detecting a certain desease when it is, in fact, present. However, the test also yields a “false positive” result for 1% of the healthy person tested. If 5% of the population actually has the dease, what is the probability a person has the desease given that the test result is positive?
  - Let D be the event that the tested person has the dease and E that the test result is positive.
  - Question: $P(D|E)$

- From the passage:
  - P(E|D) = 0.95
  - P(E|D’) = 0.01
  - P(D) = 0.05
- Question: P(D|E)

$$
\begin{aligned}
P(D \mid E) &= \frac{P(E \mid D)P(D)}{P(E)} \\
&= \frac{P(E \mid D)P(D)}{P(E \mid D)P(D) + P(E \mid D')P(D')} \\
&= \frac{0.95x0.05}{0.95x0.05 + 0.01x0.95} = \frac{5}{6} = 0.833
\end{aligned}
$$

INDEPENDENT EVENTS

Independent Events

- Event E and F are said to be independent if and only if

$$P(EF) = P(E)P(F)$$

- Meaning:

The fact whether the event E occurred or not does not change the probability of the event F occurring (and vice versa).

Example 1

- Tossing a coin & subsequently rolling a die
  - E: even outcome on rolling a die
  - H: head outcome
  - T: tail outcome
- Compute $P(E), P(T)$, and $P(E|T)$

$$P(E) = 0.5$$

$$P(T) = 0.5$$

$$P(E|T) = 0.5$$

$$P(ET) = P(E|T).P(T) = 0.25$$

$$P(E)P(T) = 0.25$$

$$\therefore P(ET) = P(E)P(T)$$

- So, based on the definition, E & T are independent.
- No surprise since the two events are "physically" independent!

Example 2

- Rolling a die
  - E: The outcome of a die is even
  - F: The outcome is ≤ 4
  - EF: an even outcome is ≤ 4

$$P(E) = \frac{3}{6} \qquad P(F) = \frac{4}{6} \qquad P(EF) = \frac{2}{6}$$

$$P(EF) = P(E)P(F) = \frac{3}{6} \cdot \frac{4}{6} = \frac{12}{36} = \frac{2}{6}$$

$$\therefore P(ET) = P(E)P(T)$$

- So, based on the definition, E & T are independent.
- “independent events” do not have to be “independent physical processes”

Independent Events

- Using the previous definition, the following propositions hold

$$\text{if } P(F) > 0 \text{ then}$$

$$E \text{ and } F \text{ are independent} \quad \Leftrightarrow \quad P(E \mid F) = P(E)$$

$$\text{if } P(E) > 0 \text{ then}$$

$$E \text{ and } F \text{ are independent} \quad \Leftrightarrow \quad P(F \mid E) = P(F)$$

$$\text{if } E \text{ and } F \text{ are independent then}$$

$$\bullet\, E \text{ and } F \text{ are independent}$$

$$\bullet\, E \text{ and } F^C \text{ are independent}$$

$$\bullet\, E^C \text{ and } F \text{ are independent}$$

$$\bullet\, E^C \text{ and } F^C \text{ are independent}$$

Example

- Two fair dice are thrown.
  - E_7: event that the sum of the dice is 7
  - F : event that the first die equals 4
  - T : event that the second die equals 3

- E_7 and F are independent

$$P(E_7) = \frac{6}{36} \qquad P(E_7 \mid F) = \frac{1}{6} \qquad P(E_7 \mid F) = P(E_7)$$

$$P(E_7 F) = \frac{1}{36} \qquad P(E_7 F) = P(E_7) P(F)$$

Exercise

- Two fair dice are thrown.
  - E_7: event that the sum of the dice is 7
  - F : event that the first die equals 4
  - T : event that the second die equals 3

- What about E_7 and T?

Independent Events

- The three events E, F, and G are said to be independent if all of the following conditions hold:

$$P(EFG) = P(E)P(F)P(G)$$
$$P(EF) = P(E)P(F)$$
$$P(EG) = P(E)P(G)$$
$$P(FG) = P(F)P(G)$$

Independent Events (2)

- If the events E, F, and G are independent, then E will be independent of any event formed from F and G.
- For example, E is independent of $F \cup G$

$$
\begin{aligned}
P(E(F \cup G)) &= P(EF \cup EG) \\
&= P(EF) + P(EG) - P(EFG) \\
&= P(E)P(F) + P(E)P(G) - P(E)P(F)P(G) \\
&= P(E)[P(F) + P(G) - P(F)P(G)] \\
&= P(E)P(F \cup G)
\end{aligned}
$$

Independence in More than 3 Events

- The events $E_1, E_2, E_3, \dots E_n$ are said to be independent if and only if for every subset $E_1', E_2', E_3', \dots E_n'$, $r \leq n$, of these events:

$$P(E_{1'}E_{2'}\dots E_{r'}) = P(E_{1'})P(E_{2'})\dots P(E_{r'})$$

- It should be noted, **pairwise independent** does **NOT** imply **mutually independent.**
- Example: Perform two independent tosses of a coin.
  - *A* = head on toss 1
  - *B*= head on toss 2
  - *C*=both tosses are equal

It's easily seen that the three events are pairwise independent.
But they are not independent!
$P(ABC) \neq P(A)P(B)P(C)$.

Example

- Sample space:

	D2 = 1	D2 = 2	D2 = 3	D2=4	D2=5	D2=6
D1=1	(1, 1)	(1, 2)	(1, 3)	(1,4)	(1,5)	(1,6)
D1=2	(2, 1)	(2, 2)	(2, 3)	(2,4)	(2,5)	(2,6)
D1=3	(3, 1)	(3, 2)	(3, 3)	(3,4)	(3,5)	(3,6)
D1=4	(4, 1)	(4, 2)	(4, 3)	(4,4)	(4,5)	(4,6)
D1=5	(5, 1)	(5, 2)	(5, 3)	(5,4)	(5,5)	(5,6)
D1=6	(6, 1)	(6, 2)	(6, 3)	(6,4)	(6,5)	(6,6)

- Events A, B, & C are defined as follow:

  A: the first die has a value of 1,2, or 3 → 18 sample points.

  B: the first die has a value of 3, 4, or 5 → 18 sample points.

  C: the sum of both dice is 9 →4 sample points {(3, 6), (4, 5), (5, 4), 6, 3)}

- Intersection of events A, B, & C

  $A \cap B$: the first die has a value of 3 → 6 sample points

  $A \cap C$: there is only 1 sample point {(3, 6)}

  $B \cap C$: there are 3 sample points {(3, 6), (4, 5), (5, 4)}

  $A \cap B \cap C$: there is 1 sample point {(3, 6)}

- $P(A) = 18/36 = 1/2$
- $P(B) = 18/36 = 1/2$
- $P(C) = 4/36 = 1/9$
- $P(A \cap B) = 6/36$ while $P(A)\, P(B) = 9/36$

  so that $P(A \cap B) \neq P(A)\, P(B)$
- $P(A \cap C) = 1/36$ while $P(A)P(C) = 2/36$

  so that $P(A \cap C) \neq P(A)\, P(C)$
- $P(B \cap C) = 3/36$ while $P(B)P(C) = 2/36$

  so that $P(B \cap C) \neq P(B)\, P(C)$
- Thus $\{A, B, C\}$ are **NOT** pairwise independent set **NOR** mutually independent set eventhough

  $P(A \cap B \cap C) = 1/36 = P(A)\, P(B)\, P(C)$

Example (2)

- Events A, B and C are defined as follow:

  A: the first die has a value of 1, 2, or 3 → 18 sample points

  B: the second die has a value of 4, 5, or 6 → 18 sample points

  C: the sum of values from both dice is 7 → 6 sample points {(1, 6),(2, 5), (3, 4), (4, 3), (5, 2), (6,1)}

- The intersection of events A, B, & C

  $A \cap B$: 9 sample points {(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}.

  $A \cap C$: 3 sample points {(1, 6), (2, 5), (3, 4)}

  $B \cap C$: 3 sample points {(1, 6), (2, 5), (3, 4)}

  $A \cap B \cap C$: 3 sample points {(1, 6), (2, 5), (3, 4)}

- So,

  $P(A)\ P(B) = (18/36) \text{ x } (18/36) = 9/36 = P(A \cap B)$

  $P(A)\ P(C) = (18/36) \text{ x } (6/36) = 3/36 = P(A \cap C)$

  $P(B)\ P(C) = (18/36) \text{ x } (6/36) = 3/36 = P(B \cap C)$

- This means events $\{A, B, C\}$ are pairwise independent.
- But, since

  $P(A \cap B \cap C) = 3/36$

  $P(A)\ P(B)\ P(C) = (18/36).(18/36).(6/36) = 1/24$

  Then events $\{A, B, C\}$ are **NOT** mutually independent.

- Note: A pairwise independent set does not necessarily it is a mutually independent set!!
  $P(A \cap B \cap C) \neq P(A)\ P(B)\ P(C)$, but, it can happen
  $P(A \cap B) = P(A)\ P(B)$.

Exercise

Let say, **A**, **B**, **C** are events that have probabilities as follow:

P(A) = 0.2, **P(B) = 0.3**, dan **P(C) = 0.4.**

Find the probability **at least one of A or B** occur if

(1) A dan B *mutually exclusive*

(2) A dan B *independent*

Find the probability **all A, B, and C occur** if

(1) A, B, dan C *independent*

(2) A, B, dan C *mutually exclusive*