Relation: Part 6 - Equivalence Relations

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Reference: Rosen, *Discrete Mathematics and Its Applications*, 8ed, 2019, Sec. 9.5

Equivalence relation

Equivalence is a property of binary relations, like reflexive, symmetric, etc.

Definition

Let $R \subseteq A \times A$ be a binary relation. We say that R is an **equivalence relation** iff R is (simultaneously) reflexive, symmetric, and transitive.

If $(a, b) \in R$ and R is an equivalence relation, then we call a and b **equivalent**, and we often write $a \sim_R b$. If the relation R is clear from the context, we simply write $a \sim b$.

Let R_1, R_2 be a binary relation on $A = \{1, 2, 3, 4\}$ with $R_1 = \{(1,1), (1,3), (1,4), (2,2), (3,3), (3,1), (3,4), (4,1), (4,3), (4,4)\}$ and $R_2 = R_1 \cup \{(2,4), (4,2)\}$. Determine if R_1 and R_2 are equivalence relations.

Exercises

Are the following relations equivalence relations?

- $R_1 = \{(a, b) \in \mathbb{N}^2 \mid 7 \text{ divides } a - b\}$
- $R_2 = \{((a, b), (c, d)) \in (\mathbb{Z}^+ \times \mathbb{Z}^+)^2 \mid a + d = b + c\}$
- $R_3 = \{(a, b) \in \mathbb{Z}^2 \mid |a - b| = 1\}$

Is $R_1 = \{(a,b) \in \mathbb{N}^2 \mid 7 \text{ divides } a - b\}$ an equivalence relation?

Is $R_2 = \{((a, b), (c, d)) \in (\mathbb{Z}^+ \times \mathbb{Z}^+)^2 \mid a + d = b + c\}$ an equivalence relation?

Is $R_3 = \{(a, b) \in \mathbb{Z}^2 \mid |a - b| = 1\}$ an equivalence relation?

Equivalence class

Definition

Let $R \subseteq A \times A$ be an equivalence relation on A and $c \in A$ is an element of A. Then, the **equivalence class** of c with respect to R, denoted $[c]_R$, is the set of all elements $d \in A$ such that $(c, d) \in R$:

$$[c]_R = \{d \mid (c, d) \in R\}$$

If $b \in [c]_R$, we say that b is a **representative** of the equivalence class $[c]_R$.

If R is clear from the context, we simply write $[c]$ instead of $[c]_R$.

Can $[a]_R$ be an empty set?

Let $R \subseteq A \times A$ with $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,3), (1,4), (2,2), (3,3), (3,1), (3,4), (4,1), (4,3), (4,4)\}$. We've seen that R is an equivalence relation. Give all of the corresponding equivalence classes.

Exercises

Give all of the equivalence classes that correspond to the following equivalence relations:

- $R_1 = \{(a, b) \in \mathbb{N}^2 \mid 7 \text{ divides } a - b\}$
- $R_2 = \{((a, b), (c, d)) \in (\mathbb{Z}^+ \times \mathbb{Z}^+)^2 \mid a + d = b + c\}$

How many equivalence classes are there for each of the above relation?

Give all equivalence classes of $R_1 = \{(a, b) \in \mathbb{N}^2 \mid 7 \text{ divides } a - b\}$. How many equivalence classes are there?

Give all equivalence classes of $R_2 = \{((a,b),(c,d)) \in (\mathbb{Z}^+ \times \mathbb{Z}^+)^2 \mid a+d = b+c\}$. How many equivalence classes are there?

Theorem

Let $R \subseteq A \times A$ be an equivalence relation on a set A. Suppose a and b are an arbitrary pair of elements of A. Then the following three statements are equivalent:

(i) $(a, b) \in R$

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$.

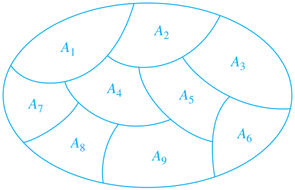
Partition of a set

Definition

A **partition** of a set S is a collection of sets $A_1, A_2, \ldots$ such that:

- $A_i \neq \emptyset$ for all i;
- $A_i \cap A_j = \emptyset$ for all i, j with $i \neq j$; and
- $\bigcup_{i \geqslant 1} A_i = A_1 \cup A_2 \cup \cdots = S$.

Note that from the above definition, $A_i \subseteq S$ for all i.



Partition of a set into 9 sets.
Source: Fig. 1, Rosen, page 644.

Connection between equivalence classes and partition of a set

Theorem

1. Let R be an equivalence relation on a set S. Then, all equivalence classes of R form a partition of S.
2. Let $A_1, A_2, \ldots,$ be a partition of a set S. Then, there exists an equivalence relation R on S such that each A_i, $i = 1, 2, \ldots,$ is an equivalence class with respect to R.

Proof of Theorem 5 part (1)

Let R be an equivalence relation on a set S. Then, all its equivalence classes form a partition of S.

Proof of Theorem 5 part (2)

Let $A_1, A_2, \ldots,$ be a partition of a set S. Then, there exists an equivalence relation R on S such that each A_i, $i = 1, 2, \ldots,$ is an equivalence class with respect to R.

Example

Verify that the equivalence classes with respect to $R_1 = \{(a, b) \in \mathbb{N}^2 \mid 7 \text{ divides } a - b\}$ form a partition of $\mathbb{N}$.

Example

The set $S = \{1, 2, 3, 4, 5, 6\}$ is partitioned into $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$. Find the equivalence relation R that corresponds to the partition.

Exercises

1. Let S be the set of all possible bit strings. Given a bit string x, $\ell(x) \geqslant 0$ is the length of x and $h_2(x)$ be the string formed of the first two bits of x. We assume $h_2(x) = \epsilon$ (the empty string) if $\ell(x) < 2$. Let R be the following:

$$R = \{(x, y) \in S^2 \mid \ell(x) = \ell(y) \text{ and } h_2(x) = h_2(y)\}$$

   Is R an equivalence relation on S? If so, give the partition of S due to R.

2. Let
   - $A_1 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \text{ and } y \text{ are both odd}\}$,
   - $A_2 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid \text{ exactly one of } x \text{ and } y \text{ is odd}\}$, and
   - $A_3 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \text{ and } y \text{ are both even}\}$.

   Does the collection $\{A_1, A_2, A_3\}$ form a partition of $(\mathbb{Z} \times \mathbb{Z})^2$? If so, give the equivalence relation R that yields the partition.

Let S be the set of all possible bit strings. Given a bit string x, $\ell(x) \geqslant 0$ is the length of x and $h_2(x)$ be the string formed of the first two bits of x. We assume $h_2(x) = \epsilon$ (the empty string) if $\ell(x) < 2$. Let R be the following:

$$R = \{(x, y) \in S^2 \mid \ell(x) = \ell(y) \text{ and } h_2(x) = h_2(y)\}$$

Is R an equivalence relation on S? If so, give the partition of S due to R.

Let

- $A_1 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x$ and y are both odd$\}$,
- $A_2 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid$ exactly one of x and y is odd$\}$, and
- $A_3 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x$ and y are both even$\}$.

Does the collection $\{A_1, A_2, A_3\}$ form a partition of $(\mathbb{Z} \times \mathbb{Z})^2$? If so, give the equivalence relation R that yields the partition.