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Divide and Conquer (2)

Convolution and Fast Fourier Transform

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Introduction

- **Fast Fourier Transform (FFT)** is one of many algorithms that implements the Divide and Conquer paradigm.
  - It is commonly used in signal processing, i.e. to convert between time domain and frequency domain.
  - It is also can be viewed as a fast way to multiply and evaluate polynomials.
    - Suppose we have two polynomials of degree n. Addition of these two polynomials takes $\Theta(n)$ but the multiplication of them takes $\Theta(n^2)$
    - Fast Fourier Transform (FFT) can reduce the time to multiply two polynomials to $\Theta(n \lg n)$.

Polynomials

A polynomial in the variable x over an algebraic field F represents a function $A(x)$ as a formal sum:

$$A(x) = \Sigma_{i=0}^{n-1} a_i x^i$$

- a_i is the coefficient with $i = 0,1, \dots, n-1$
- degree of the polynomial is the largest power of x whose coefficient is not equals to zero.

Any integer strictly greater than the degree of a polynomial is a ***degree-bound*** of that polynomial. Therefore, the degree of a polynomial of degree-bound n may be any integer between 0 and $n-1$, inclusive.

Example: $x^3 + 2x^2 - x + 1$

- The degree of this polynomial is 3 and this is a polynomial of degree-bound 4

Polynomials

Suppose we have two polynomials of degree-bound n:

$A(x) = \sum_{i=0}^{n-1} a_i x^i$ and $B(x) = \sum_{i=0}^{n-1} b_i x^i$

- The **sum** of these polynomials is also a polynomial of **degree-bound n** $C(x)$ such that $C(x) = A(x) + B(x) = \sum_{i=0}^{n-1} c_i x^i$ where $c_i = a_i + b_i$.

- Example: $A(x) = x^2 + 2x - 1$ and $B(x) = x^2 - x + 1$ are polynomials of degree-bound 3.
  - The sum result: $C(x) = 2x^2 + x$ (a polynomial of degree-bound 3)

Polynomials

- The **product** of $A(x)$ and $B(x)$ is a polynomial of **degree bound $\boldsymbol{2n-1}$** $D(x)$ such that $D(x) = A(x)B(x)$.

- Example: $A(x) = x^2 + 2x - 1$ and $B(x) = x^2 - x + 1$
- The product result: $D(x) = x^4 + x^3 - 2x^2 + 3x - 1$ (a polynomial of degree-bound 5)
  - To obtain $D(x)$, we can use the standard multiplication by multiplying each term in $A(x)$ by each term in $B(x)$ and then combining terms with equal powers.
  - The other way, $D(x)$ can also be expressed as $D(x) = A(x)B(x) = \Sigma_{i=0}^{2n-2} d_i x^i$ where $\boldsymbol{d_i = \Sigma_{k=0}^{i} a_k b_{i-k}}$.

This formula represents the **convolution** of input vectors a and b, denoted $d = a \otimes b$

Polynomials Representation

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Coefficient Representation

- Given a polynomial $A(x) = \sum_{i=0}^{n-1} a_i x^i$. Its **coefficient representation** is a vector of coefficients $a = (a_0, a_1, \ldots, a_{n-1})$.

- Example:
  - $A(x) = 6x^3 + 7x^2 - 10x + 9$ can be represented as a vector $a = (9, -10, 7, 6)$
  - $B(x) = -2x^3 + 4x - 5$ can be represented as a vector $b = (5, 4, 0, -2)$

Polynomials Representation

Coefficient Representation (2)

- Evaluating a polynomial $A(x) = \Sigma_{i=0}^{n-1} a_i x^i$ at a point x_o takes $\Theta(n)$ time using ***Horner's Rule***:

$$A(x_0) = a_0 + a_1(x_0)^2 + a_2(x_0)^3 + \cdots + a_{n-1}(x_0)^{n-1}$$
$$= a_0 + x_0\left(a1 + x_0\left(a_2 + \cdots + x_0(a_{n-2} + x_0(a_{n-1}))\right)\right)$$

- Adding two polynomials also takes $\Theta(n)$ time but multiplying them takes $\mathbf{\Theta}\left(\boldsymbol{n}^{\mathbf{2}}\right)$ time by using standard multiplication or convolution.
  - Since multiplying polynomials and computing convolutions are fundamental computational problems of considerable practical importance, this chapter concentrates on efficient algorithms for them

Polynomials Representation

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Coefficient Representation (3)

- Multiplication Example:

$a = (9, -10, 7, 6)$

$b = (-5, 4, 0, -2)$

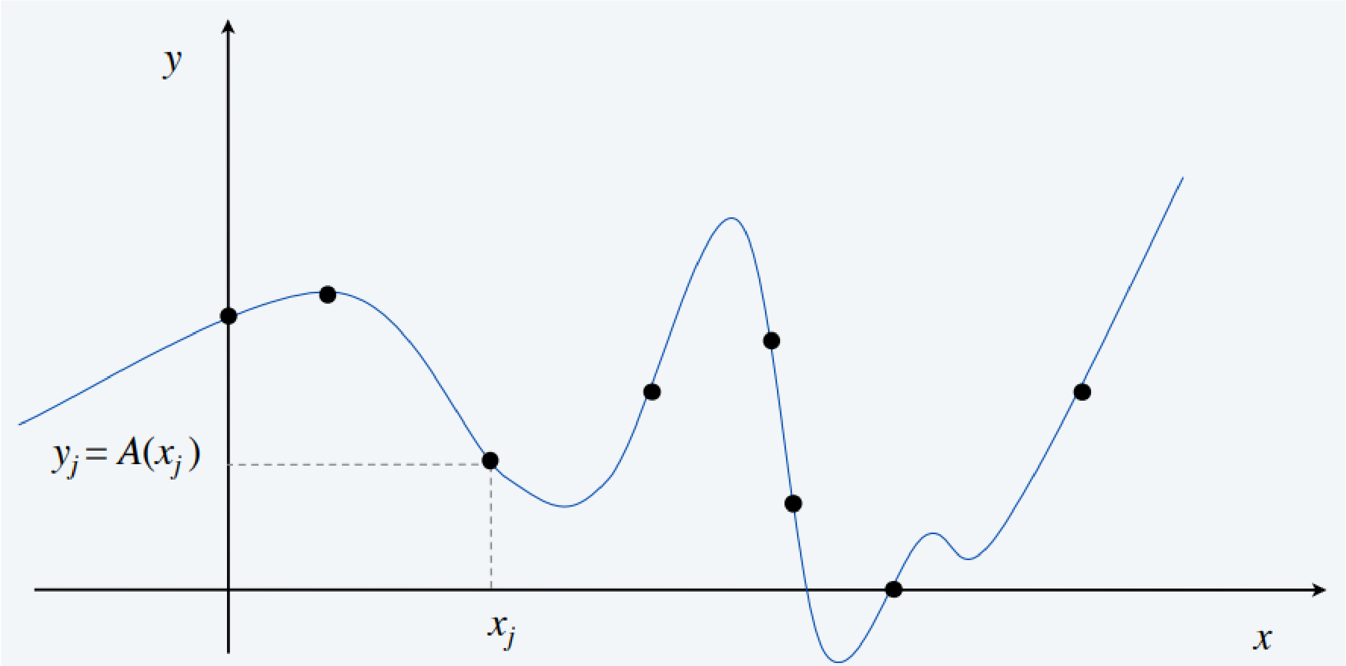
$$
\begin{array}{rrrrrrrr}
 & & & 6x^3 & + 7x^2 & - 10x & + 9 \\
 & & & - 2x^3 & & + 4x & - 5 \\
\hline
 & & & - 30x^3 & - 35x^2 & + 50x & - 45 \\
 & & 24x^4 & + 28x^3 & - 40x^2 & + 36x & \\
- 12x^6 & - 14x^5 & + 20x^4 & - 18x^3 & & & \\
\hline
- 12x^6 & - 14x^5 & + 44x^4 & - 20x^3 & - 75x^2 & + 86x & - 45
\end{array}
$$

convolution $a \otimes b = (-45, 86, -75, -20, 44, -14, -12)$

Polynomials Representation

Point-value Representation

- Given a polynomial $A(x) = \Sigma_{i=0}^{n-1} a_i x^i$. Its **point-value representation** is a set of n point-value pairs $\{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$ such that all the x_k are distinct and $y_k = A(k)$ for $k = 0,1, \dots, n-1$. It has many point-value representations.



Polynomials Representation

Point-value Representation (2)

- Adding two polynomials takes $\Theta(n)$.
- If $C(x) = A(x) + B(x)$, then $C(x_k) = A(x_k) + B(x_k)$

- Suppose $A(x)$ is represented as $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$

and for $B(x)$ is $\{(x_0, y'_0), (x_1, y'_1), \ldots, (x_{n-1}, y'_{n-1})\}$.

The point-value representation for $C(x)$ is

$$\{(x_0, y_0 + y'_0), (x_1, y_1 + y'_1), \ldots, (x_{n-1}, y_{n-1} + y'_{n-1})\}$$

Polynomials Representation

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Point-value Representation (3)

- Multiplying two polynomials also takes $\Theta(n)$.
    - It is much less than the operation in coefficient representation.
- If $C(x) = A(x)B(x)$, then $C(x_k) = A(x_k)B(x_k)$ for any point x_k.

- The multiplication use the **extended point-value representations** consisting of $2n$ point-value pairs each
    - Suppose $A(x)$ and $B(x)$ are expressed in $\{(x_0, y_0), (x_1, y_1), \dots, (x_{2n-1}, y_{2n-1})\}$ and $\{(x_0, y'_0), (x_1, y'_1), \dots, (x_{2n-1}, y'_{2n-1})\}$ respectively.
    - The point-value representation for $C(x)$ is: $\{(x_0, y_0 y'_0), (x_1, y_1 y'_1), \dots, (x_{2n-1}, y_{2n-1}\, y'_{2n-1})\}$

Polynomials Representation

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Converting coefficient to point-value representation

- Computing a point-value representation for a polynomial given in coefficient form is in principle straightforward, since all we have to do is select n distinct points $x_0, x_1, \ldots, x_{n-1}$ and then evaluate $A(x_k)$ for $k = 0, 1, \ldots, n-1$. With Horner's method, evaluating a polynomial at n points takes time $\Theta(n^2)$. We shall see later that if we choose the points x_k cleverly, we can accelerate this computation to run in time $\Theta(n \lg n)$.

Polynomials Representation

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Converting coefficient to point-value representation (2)

Coefficient $\Rightarrow$ point-value. Given a polynomial $A(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Polynomials Representation

Converting point-value to coefficient representation

- Determining the coefficient-representation of a polynomial from a point-value representation (inverse of evaluation) is called **interpolation.**
- **Uniqueness of an interpolating polynomial theorem:**

  For any set $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$ of n point-value pairs such that all the x_k values are distinct, there is a unique polynomial $A(x)$ of degree-bound n such that $y_k = A(x_k)$ for $k = 0, 1, \ldots, n-1$.

- An n-point interpolation based on

  Lagrange's Formula (takes $\Theta(n^2)$) :

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Polynomials Representation

Converting point-value to coefficient representation (2)

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $A(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff x_i distinct

The vector a can be obtained from:

$$a = V(x_0, x_1, \ldots, x_{n-1})^{-1} y$$

where V is the Vandermonde matrix

Polynomials Representation

Converting point-value to coefficient representation (3)

- Example: For a polynomial of degree-bound 3, we have:

$$A(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

- Suppose $A(x)$ is represented as $\{(0,1),(1,4),(2,9)\}$, we could identify the corresponding polynomial by using Lagrange's Formula:

$$\begin{aligned} A(x) &= \frac{(x-1)(x-2)}{(0-1)(0-2)} + 4\frac{(x-0)(x-2)}{(1-0)(1-2)} + 9\frac{(x-0)(x-1)}{(2-0)(2-1)} \\ &= \frac{(x-1)(x-2)}{2} - 4x(x-2) + \frac{9}{2}x(x-1) \\ &= \frac{(x^2-3x+2)}{2} - (4x^2-8x) + \frac{9}{2}x^2 - \frac{9}{2}x \\ &= \boldsymbol{x^2+2x+1} \end{aligned}$$

Fast Multiplication of Polynomials

- Coefficient vs point-value representation:

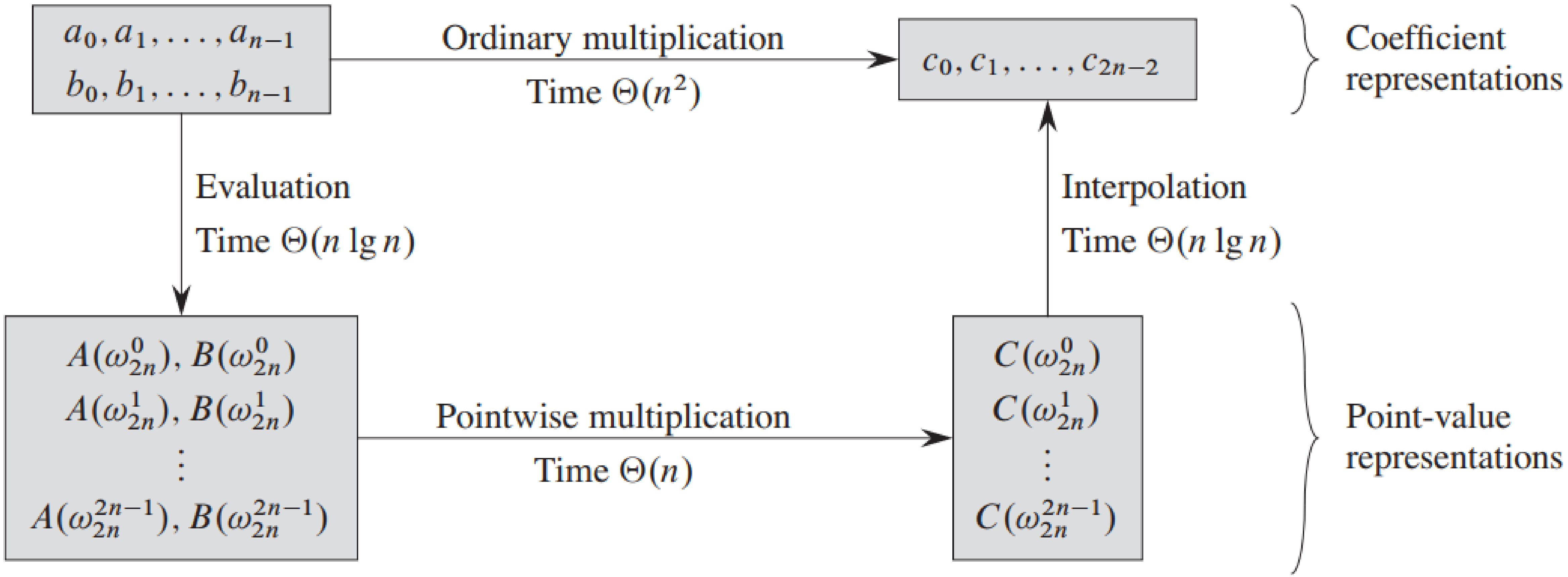
representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

- Can we use the linear time multiplication for polynomials in point-value form to accelerate polynomial multiplication in coefficient form?
  - It depends on whether we can convert a polynomial **quickly** from coefficient form to point-value form (evaluate) and vice versa (interpolate)

Fast Multiplication of Polynomials

- By choosing **complex roots of unity** as the evaluation points, we can produce a point-value representation by taking the **Discrete Fourier Transform (DFT)** of a coefficient vector.
  - Converting between representations takes $\Theta(n \lg n)$ by using DFT (evaluation) and inverse DFT (interpolation).
- The product of two polynomials of degree-bound n is a polynomial of degree-bound $2n$.
  - Before evaluating the input polynomial $A(x)$ and $B(x)$, the degree-bound is doubled to $2n$ by adding n high order coefficient of 0.
  - Thus, we use the **complex 2n$^{\text{th}}$ roots of unity,** denoted by $\boldsymbol{\omega_{2n}}$ terms.

Fast Multiplication of Polynomials



Fast Multiplication of Polynomials

- Given the FFT, the following is $\Theta(n \lg n)$ time procedure for multiplying two polynomials $A(x)$ and $B(x)$ of degree-bound n. It is assumed that n is power of 2.
  - **Double degree-bound**: create the coefficient representation of $A(x)$ and $B(x)$ as degree-bound $2n$ polynomials. $(\Theta(n))$
  - **Evaluate**: Compute point-value representations for $A(x)$ and $B(x)$ by applying the FFT of order $2n$ on each polynomial. $(\Theta(n \lg n))$
    - Contain of the value of both polynomials at the $2n^{th}$ roots of unity.
  - **Pointwise multiply**: Compute point-value representation for $C(x) = A(x)B(x)$ by multiplying these value <u>pointwise</u>. $(\Theta(n))$
    - Contain of the value of $C(x)$ at the $2n^{th}$ roots of unity.
  - **Interpolate**: Create the coefficient representation of the polynomial $C(x)$ by applying the FFT on $2n$ point-value pairs to compute the inverse DFT. $(\Theta(n \lg n))$

Fast Multiplication of Polynomials

- Based on the use of FFT, we have the following theorem:

Theorem 30.2
We can multiply two polynomials of degree-bound n in time $\Theta(n \lg n)$, with both the input and output representations in coefficient form. ■

  - Proof: from the steps described in previous slides, the total time needed is $\Theta(n \lg n)$.

Complex Roots of Unity

A ***complex nth root of unity*** is a complex number ω such that

$\omega^n = 1$. (So, it is any complex number, when it multiplied by itself some number of times, yields 1)

There are exactly n complex nth roots of unity: $e^{2\pi ik/n}$ for $k = 0, 1, \ldots, n-1$. To interpret this formula, we use the definition of the exponential of a complex number:

$e^{iu} = \cos(u) + i\sin(u)$. Euler's Formula

e denotes the euler's number and i denotes the imaginary number,

while $u = 0, \frac{1}{n}\tau, \frac{2}{n}\tau, \ldots, \frac{n-1}{n}\tau$ where $\tau = 2\pi$

Complex Roots of Unity

- ω_n is the **principal n^{th} root of unity**, defined as $\omega_n = e^{\frac{2\pi i}{n}}$
- All other complex n^{th} roots of unity are powers of ω_n:

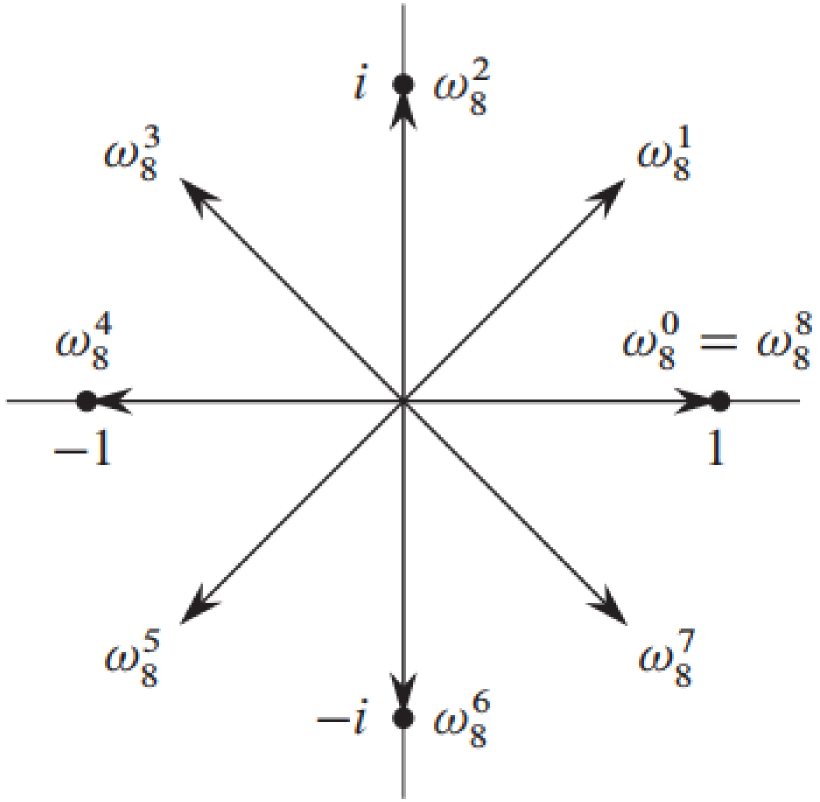
$$\omega_n^0, \omega_n^1, \omega_n^2, \dots, \omega_n^{n-1}$$

- Proof for n complex n^{th} roots of unity:

$$\left(\omega_n^k\right)^n = \left(e^{\frac{2\pi ik}{n}}\right)^n = \left(e^{\pi i}\right)^{2k} = (-1)^{2k} = 1$$

By Euler's Formula, $e^{\pi i} = \cos\pi + i\sin\pi = -1$

Complex Roots of Unity



In this example, the values of 8 complex 8^{th} roots of unity are:

$$\omega_8^0 = e^{\frac{2\pi i 0}{8}} = \mathbf{1},$$
$$\omega_8^1 = e^{\frac{2\pi i}{8}} = e^{\frac{\pi}{4}i} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} + \left(\frac{1}{2}\sqrt{2}\right)\boldsymbol{i},$$
$$\omega_8^2 = e^{\frac{4\pi i}{8}} = e^{\frac{\pi}{2}i} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = \boldsymbol{i},$$
$$\omega_8^3 = e^{\frac{6\pi i}{8}} = e^{\frac{3\pi}{4}i} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{2}\sqrt{2} + \left(\frac{1}{2}\sqrt{2}\right)\boldsymbol{i},$$
$$\omega_8^4 = -\mathbf{1},$$
$$\omega_8^5 = -\frac{1}{2}\sqrt{2} - \left(\frac{1}{2}\sqrt{2}\right)\boldsymbol{i},$$
$$\omega_8^6 = \cdots, \text{ and}$$
$$\omega_8^7 = \cdots.$$

Other references may mention $\omega_8^1, \ldots, \omega_8^8$ as the 8 complex 8^{th} roots of unity. It is the same since $\omega_8^0 = \omega_8^8$

Figure 30.2 The values of $\omega_8^0, \omega_8^1, \ldots, \omega_8^7$ in the complex plane, where $\omega_8 = e^{2\pi i/8}$ is the principal 8th root of unity.

Complex Roots of Unity

- Cancellation Lemma

  For any integers $n \geq 0, k \geq 0, d > 0$, $\omega_{dn}^{dk} = \omega_n^k$

  Proof: $\omega_{dn}^{dk} = \left(e^{\frac{2\pi i d k}{dn}}\right) = \left(e^{2\pi \frac{i}{n}}\right)^k$

  Example: $\omega_8^2 = \omega_4^1$

- A Corollary

  For any even integer $n > 0$, $\omega_n^{\frac{n}{2}} = \omega_2 = -1$

  Proof: $\omega_n^{\frac{n}{2}} = \left(e^{\frac{2\pi i}{n}}\right)^{\frac{n}{2}} = \left(e^{2\pi i/2}\right) = \omega_2 = -1$

Complex Roots of Unity

- Halving Lemma

If $n > 0$ is even, the squares of the n complex n^{th} roots of unity are the $\frac{n}{2}$ complex $\left(\frac{n}{2}\right)^{th}$ roots of unity.

Proof By the cancellation lemma, we have $(\omega_n^k)^2 = \omega_{n/2}^k$, for any nonnegative integer k. Note that if we square all of the complex nth roots of unity, then we obtain each $(n/2)$th root of unity exactly twice, since

$$\begin{aligned}
(\omega_n^{k+n/2})^2 &= \omega_n^{2k+n} \\
&= \omega_n^{2k}\omega_n^n \\
&= \omega_n^{2k} \\
&= (\omega_n^k)^2 .
\end{aligned}$$

Example: $(\omega_8^2)^2 = \omega_4^2 = e^{\pi i} = -1$

Complex Roots of Unity

- Summation Lemma

For any even integer $n \geq 1$ and nonzero integer k not divisible by n,

$$\Sigma_{i=0}^{n-1}\left(\omega_n^k\right)^i = 0$$

Proof:

$$\begin{aligned}
\sum_{j=0}^{n-1}\left(\omega_n^k\right)^j &= \frac{(\omega_n^k)^n - 1}{\omega_n^k - 1} \\
&= \frac{(\omega_n^n)^k - 1}{\omega_n^k - 1} \\
&= \frac{(1)^k - 1}{\omega_n^k - 1} \\
&= 0 .
\end{aligned}$$

Because we require that k is not divisible by n, and because $\omega_n^k = 1$ only when k is divisible by n, we ensure that the denominator is not 0. ■

The DFT

- Back to the problem of evaluating polynomial $A(x) = \Sigma_{i=0}^{n-1} a_i x^i$ of degree-bound n at $\omega_n^0, \omega_n^1, \omega_n^2, \dots, \omega_n^{n-1}$.
  - The coefficient representation for A is $(a_0, a_1, \dots, a_{n-1})$
  - The degree-bound n refers to $2n$ since we perform double degree-bound

- Let $y_k = A(x_k) = A\left(\omega_n^k\right)$ for $k = 0,1,2, \dots, n-1$, then we have

$$A(x_k) = \Sigma_{i=0}^{n-1} a_i \left(\omega_n^k\right)^i$$

- The vector $y = (y_0, y_1, \dots, y_{n-1})$ is the **Discrete Fourier Transform (DFT)** of the coefficient vector $(a_0, a_1, \dots, a_{n-1})$.
  - It is also written as $y = DFT_n(a)$.

The DFT

- Example: Compute DFT for $p = (0,1,2,3)$.

The FFT

- By using a method known as the **Fast Fourier Transform (FFT)**, the computation of $y = DFT_n(a)$ takes $\Theta(n \lg n)$.
  - FFT takes the advantages of <u>the special properties of complex roots of unity</u>
  - We assume that n is an exact power of 2

- The divide and conquer strategy for FFT:
  - Two new polynomials degree-bound $\frac{n}{2}$ are generated: the even-indexed coefficient ($A^{[0]}(x)$) and odd-indexed coefficient ($A^{[1]}(x)$).
    - $A^{[0]}(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{\frac{n}{2}-1}$
    - $A^{[1]}(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{\frac{n}{2}-1}$
  - Then it follows that $A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$

The FFT

- By using FFT, the problem of evaluating A at $\omega_n^0, \omega_n^1, \omega_n^2, \dots, \omega_n^{n-1}$ reduces to:
  - Evaluating the degree-bound $\frac{n}{2}$ polynomials $A^{[0]}(x)$ and $A^{[1]}(x)$ at the points $(\omega_n^0)^2, (\omega_n^1)^2, (\omega_n^2)^2, \dots, (\omega_n^{n-1})^2$.
    - It recursively evaluate the polynomials $A^{[0]}$ and $A^{[1]}$ of degree-bound $\frac{n}{2}$ at the $\frac{n}{2}$ complex $\left(\frac{n}{2}\right)^{th}$ roots of unity
    - Following the halving lemma, the n element DFT_n computation is divided into two $\frac{n}{2}$ element $DFT_{n/2}$ computation.
  - Combining the result into $A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$

The FFT

RECURSIVE-FFT(a)

1 $n = a.length$ // n is a power of 2
2 **if** $n == 1$
3 **return** a

Represents the basis of the recursion

4 $\omega_n = e^{2\pi i/n}$
5 $\omega = 1$

Make sure that ω is updated properly, together with line 13.

6 $a^{[0]} = (a_0, a_2, \ldots, a_{n-2})$
7 $a^{[1]} = (a_1, a_3, \ldots, a_{n-1})$

The coefficient vectors for polynomials $A^{[0]}$ and $A^{[1]}$

8 $y^{[0]} =$ RECURSIVE-FFT$(a^{[0]})$
9 $y^{[1]} =$ RECURSIVE-FFT$(a^{[1]})$

Recursive $DFT_{\frac{n}{2}}$ computations

10 **for** $k = 0$ **to** $n/2 - 1$
11 $y_k = y_k^{[0]} + \omega\, y_k^{[1]}$
12 $y_{k+(n/2)} = y_k^{[0]} - \omega\, y_k^{[1]}$

Combine the result of the recursive $DFT_{n/2}$ calculations

13 $\omega = \omega\, \omega_n$
14 **return** y // y is assumed to be a column vector

The FFT

Lines 11–12 combine the results of the recursive $\mathrm{DFT}_{n/2}$ calculations. For $y_0, y_1, \ldots, y_{n/2-1}$, line 11 yields

$$\begin{aligned} y_k &= y_k^{[0]} + \omega_n^k y_k^{[1]} \\ &= A^{[0]}(\omega_n^{2k}) + \omega_n^k A^{[1]}(\omega_n^{2k}) \\ &= A(\omega_n^k) && \text{(by equation (30.9))} . \end{aligned}$$

$$\boxed{A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)}$$

For $y_{n/2}, y_{n/2+1}, \ldots, y_{n-1}$, letting $k = 0, 1, \ldots, n/2 - 1$, line 12 yields

$$\begin{aligned} y_{k+(n/2)} &= y_k^{[0]} - \omega_n^k y_k^{[1]} \\ &= y_k^{[0]} + \omega_n^{k+(n/2)} y_k^{[1]} && \text{(since } \omega_n^{k+(n/2)} = -\omega_n^k\text{)} \\ &= A^{[0]}(\omega_n^{2k}) + \omega_n^{k+(n/2)} A^{[1]}(\omega_n^{2k}) \\ &= A^{[0]}(\omega_n^{2k+n}) + \omega_n^{k+(n/2)} A^{[1]}(\omega_n^{2k+n}) && \text{(since } \omega_n^{2k+n} = \omega_n^{2k}\text{)} \\ &= A(\omega_n^{k+(n/2)}) && \text{(by equation (30.9))} . \end{aligned}$$

The FFT

To determine the running time of procedure RECURSIVE-FFT, we note that exclusive of the recursive calls, each invocation takes time $\Theta(n)$, where n is the length of the input vector. The recurrence for the running time is therefore

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) . \end{aligned}$$

Thus, we can evaluate a polynomial of degree-bound n at the complex nth roots of unity in time $\Theta(n \lg n)$ using the fast Fourier transform.

The FFT

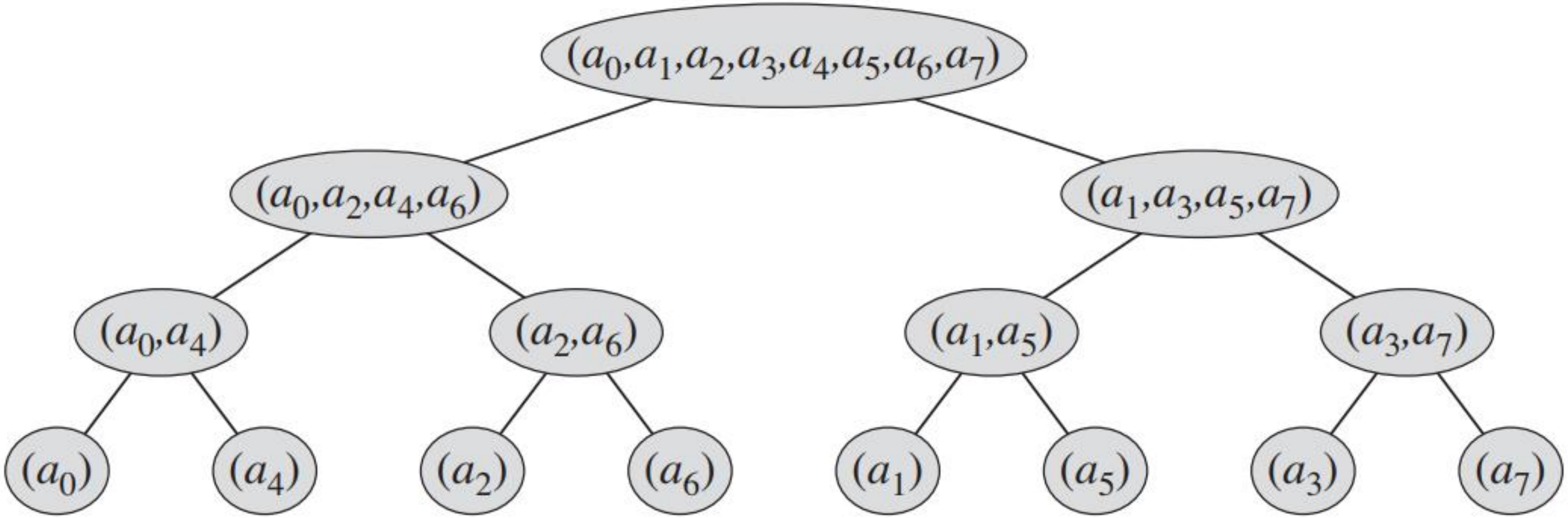


Figure 30.4 The tree of input vectors to the recursive calls of the RECURSIVE-FFT procedure. The initial invocation is for $n = 8$.

Interpolation at The Complex Roots of Unity

- Interpolate by writing DFT as a matrix equation $y = V_n a$ as follows and then looking at the form of the **matrix inverse**.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix} .$$

The (k, j) entry of V_n is ω_n^{kj}, for $j, k = 0, 1, \ldots, n-1$. The exponents of the entries of V_n form a multiplication table.

For the inverse operation, which we write as $a = \mathrm{DFT}_n^{-1}(y)$, we proceed by multiplying y by the matrix V_n^{-1}, the inverse of V_n.

Interpolation at The Complex Roots of Unity

Given the inverse matrix V_n^{-1}, we have that $\mathrm{DFT}_n^{-1}(y)$ is given by

$$a_j = \frac{1}{n} \sum_{k=0}^{n-1} y_k \omega_n^{-kj}$$

- Recall how to evaluate a polynomial: $y_k = A(x_k) = \Sigma_{i=0}^{n-1} a_i \left(\omega_n^k\right)^i$. Both equations have similar form:
  - The role of a and y are switched
  - ω_n is replaced with ω_n^{-1}
  - Each element of the result is divided by n
- Thus, we can compute DFT_n^{-1} in $\Theta(n \lg n)$ as well.

The Convolution Theorem

- By using FFT and inverse FFT, transformation from coefficient representation to point-value representation (and the other direction) take $\Theta(n \lg n)$. In the context of polynomial multiplication, we have the following **Convolution Theorem**.

For any two vectors a and b of length n, where n is a power of 2,

$$a \otimes b = \mathrm{DFT}_{2n}^{-1}(\mathrm{DFT}_{2n}(a) \cdot \mathrm{DFT}_{2n}(b)) ,$$

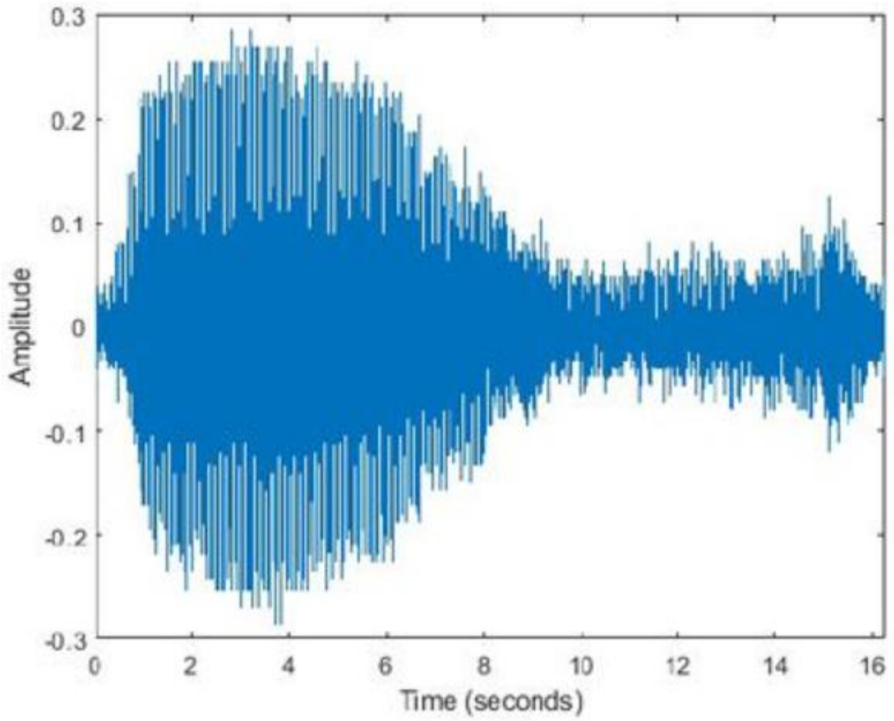
where the vectors a and b are padded with 0s to length $2n$ and $\cdot$ denotes the componentwise product of two $2n$-element vectors. ■

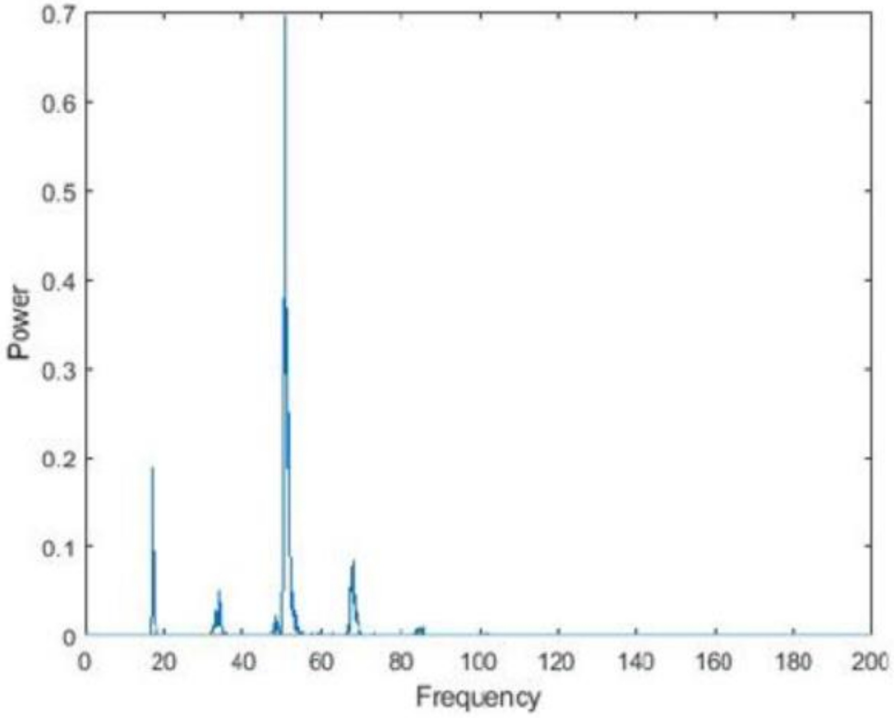
The Convolution Theorem

- Example: Find $(1+x)(1+x+x^2)$ using DFT

Applications

- In signal processing: to convert from time domain to frequency domain.

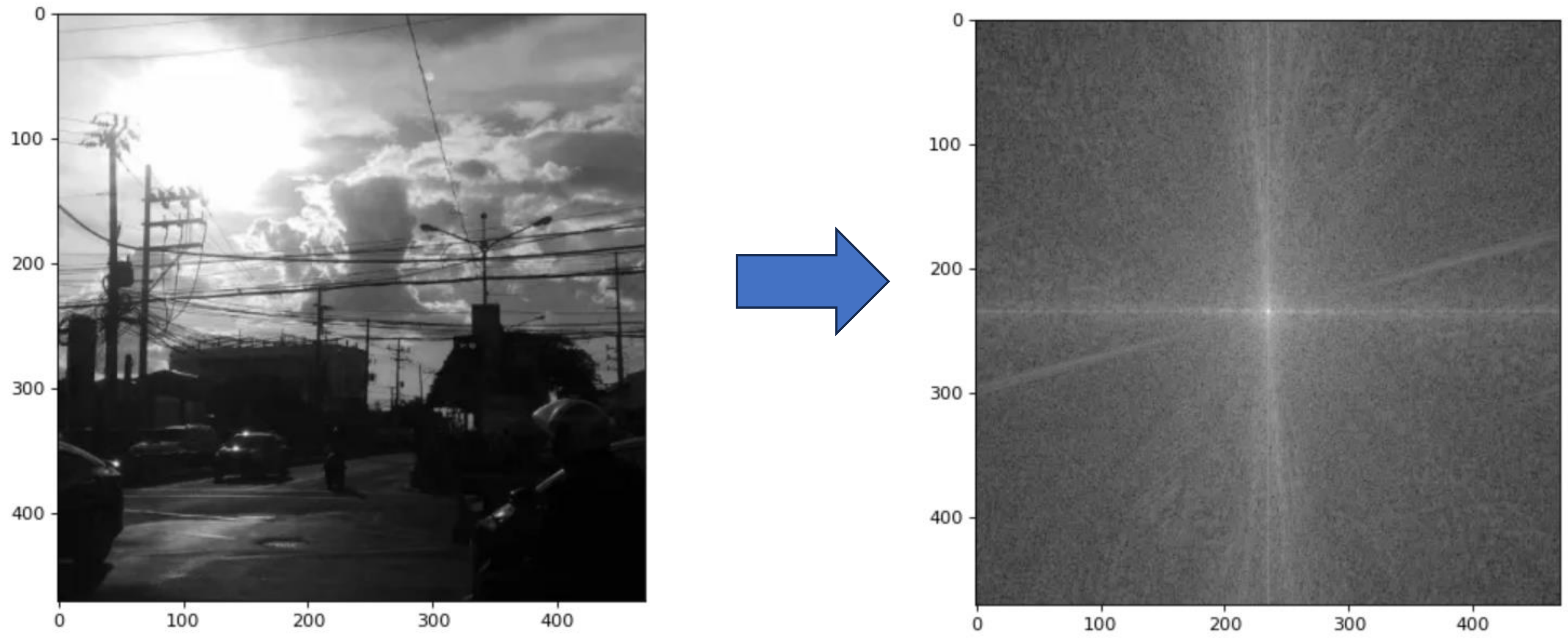




- More on [Fast Fourier Transform (FFT) - MATLAB & Simulink (mathworks.com)]

Applications

- In image processing: to convert from an image to frequency distribution map.



- More on [Image Processing with Python — Application of Fourier Transformation | by Tonichi Edeza | Towards Data Science]

Summary

- FFT speed up the standard polynomial multiplication from $\Theta(n^2)$ to $\Theta(n \lg n)$.
  - The point-wise multiplication contribute the most to linear time multiplication, but converting from point-wise to coefficient representation is not as simple as the other way around (Lagrange formula needs $\Theta(n^2)$)
  - By using DFT, the transformation from coefficient to point-wise representation (and vice versa) takes only $\Theta(n \lg n)$
- The convolution for polynomial multiplication also can be modified such that it takes $\Theta(n \lg n)$.
- FFT is useful for many applications, especially for ..

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