Relation: Part 7 - Partial Orderings

Adila A. Krisnadhi

Faculty of Computer Science, Universitas Indonesia

Reference: Rosen, *Discrete Mathematics and Its Applications*, 8ed, Sec. 9.6

(Non-strict) partial order and posets

Definition

Let A be a set and $R \subseteq A \times A$ a binary relation on a set A.

- R is a **(non-strict) partial order** iff R is reflexive, antisymmetric, and transitive. Notation $\preccurlyeq$ is often used to denote a partial order.
- If R is a partial order on A, the pair (A, R) or $(A, \preccurlyeq)$ is called a **partially ordered set** or **poset**. Elements of A are also elements of the poset.
- Let (A, R) be a poset and $a, b \in A$.
  - We write $a \preccurlyeq_R b$ whenever $(a, b) \in R$.
  - We write $a \prec_R b$ if $a \preccurlyeq_R b$ and $a \neq b$.
  - Subscript R is omitted if it's clear from the context.
  - We sometimes use $(A, \preccurlyeq)$ to denote a poset with $\preccurlyeq$ as the corresponding partial order.

Example

Is (A, R) a poset where $A = \{1, 2, 3, 4, 5\}$ and
$R = \{(1, 1), (1, 3), (1, 5), (2, 2), (3, 3), (4, 2), (4, 4), (5, 3), (5, 5)\}$?

Example

Is $(\mathbb{Z}, \leqslant)$ a poset?

Example

Is $(\mathbb{N}, <)$ a poset?

Example

Is $(\mathbb{Z}, R)$ a poset where $R = \{(a, b) \in \mathbb{Z}^2 \mid |a| \leqslant |b|\}$?

Exercises

- Is $(\mathbb{R}, \geqslant)$ a poset?
- Let $R = \{(x, y) \in \mathbb{Z}^2 \mid x \leqslant y \text{ and } |x - y| \leqslant 5\}$. Is $(\mathbb{Z}, R)$ a poset?
- R is a binary relation on $\mathbb{N} \times \mathbb{N}$ such that for any two pairs (a, b) and (c, d) of natural numbers, $((a, b), (c, d)) \in R$ iff $a < c$ or $(a = c$ and $b \leqslant d)$. Is $(\mathbb{N} \times \mathbb{N}, R)$ a poset?

Is $(\mathbb{R}, \geqslant)$ a poset?

Is $(\mathbb{Z}, R)$ a poset where $R = \{(x, y) \in \mathbb{Z}^2 \mid x \leqslant y \text{ and } |x - y| \leqslant 5\}$?

R is a binary relation on $\mathbb{N} \times \mathbb{N}$ such that for any two pairs (a, b) and (c, d) of natural numbers, $((a, b), (c, d)) \in R$ iff $a < c$ or ($a = c$ and $b \leqslant d$). Is $(\mathbb{N} \times \mathbb{N}, R)$ a poset?

Strict partial order

Definition

Let A be a set and $R \subseteq A \times A$ a binary relation on a set A.
R is a **strict partial order** iff R is irreflexive, asymmetric, and transitive.

- R is antisymmetric and irreflexive iff R is asymmetric [Show this!].
- So, non-strict and strict partial order only differ in the fact that the former is reflexive, while the latter is irreflexive.
- In a strict partial order $R \subseteq A \times A$, we use the notation $a \prec_R b$ for each $(a, b) \in R$.

Examples

We've seen in previous examples that the following are not posets. Which of those correspond to strict partial order?

1. $(\mathbb{N}, <)$
2. $(\mathbb{Z}, R)$ where $R = \{(a, b) \in \mathbb{Z}^2 \mid |a| \leqslant |b|\}$
3. $(\mathbb{Z}, R)$ where $R = \{(x, y) \in \mathbb{Z}^2 \mid x \leqslant y$ and $|x - y| \leqslant 5\}$

Exercises

Which of the following are strict partial order?

1. $R_1 = \{(x, y) \in S^2 \mid x \text{ is the father of } y\}$ where S is the set of all people.
2. $R_2 = \{(a, b) \in \mathbb{Z}^2 \mid b = a + 2k \text{ for some } k \in \mathbb{Z}^+\}$

Hasse diagram

It is sometimes helpful to visualize a poset using Hasse diagram.

Definition

For a poset (A, R), a **Hasse diagram** is a (undirected) graph obtained from the graph representation of R as follows:

1. remove all loops;
2. remove all directed edges due to transitivity of R (i.e., remove all 'shortcut' edges)
   - repeatedly remove the edge (a, c) if we already have the edges (a, b) and (b, c)
3. draw the remaining edges "upward" and remove all the arrows (i.e., make the edges undirected).
   - the edge (a, b) goes upward from a to b; so place a below b, then connect a to b with an undirected edge.

Examples

Create a Hasse diagram for (A, R) where

1. $A = \{a, b, c, d, e\}$ and
   $R = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (c,c), (d,d), (e,b), (e,c), (e,d), (e,e)\}$
2. $A = \{a, b, c, d, e\}$ and
   $R = \{(a,a), (a,c), (a,d), (a,e), (b,b), (b,c), (b,d), (b,e), (c,c), (c,d), (c,e), (d,d), (e,e)\}$
3. $A = \{a, b, c, d, e\}$ and $R = \{(a,a), (b,a), (b,b), (c,a), (c,b), (c,c), (d,a), (d,b), (d,d), (e,a), (e,e)\}$
4. $A = \{a, b, c, d, e\}$ and $R = \{(a,a), (a,b), (a,d), (b,b), (b,d), (c,c), (c,d), (d,d), (e,e)\}$

Let (A, R) be a poset with $A = \{a, b, c, d, e\}$ and $R = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (c, c), (d, d), (e, b), (e, c), (e, d), (e, e)\}$. Create its Hasse diagram.

Let (A, R) with $A = \{a, b, c, d, e\}$ and
$R = \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}$.
Create its Hasse diagram.

Let (A, R) with $A = \{a, b, c, d, e\}$ and
$R = \{(a, a), (b, a), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, d), (e, a), (e, e)\}$. Create its Hasse diagram.

Let (A, R) with $A = \{a, b, c, d, e\}$ and
$R = \{(a, a), (a, b), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d), (e, e)\}$. Create its Hasse diagram.

Exercises

Create a Hasse diagram for the poset $\{S, R\}$ if

- $S = \{1, 2, 3, 4, 6, 8, 12\}$ and $R = \{(a, b) \mid a \text{ divides } b\}$
- $S = 2^D$, the power set of $D = \{1, 2, 3\}$, and $R = \{(A, B) \mid A \subseteq B\}$

Comparability

Definition

Let $(S, \preccurlyeq)$ be poset.

- Any two elements $a, b \in S$ are called **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
- Otherwise, we call a and b **incomparable**.

In a poset $(\mathbb{Z}^+, |)$, are 3 and 9 comparable? How about 5 dan 7?

Total order

Definition

Let $(S, \preccurlyeq)$ be a poset. The relation $\preccurlyeq$ is called a **total order** or **linear order** iff every two elements of S are comparable.
If $\preccurlyeq$ is a total order, S is called a **totally ordered set** or **linearly ordered set** or **chain**.

Examples

For each of the following pair of set and relation, decide if the set is totally ordered by the relation: (i) $(\mathbb{Z}, \leqslant)$; (ii) $(\mathbb{Z}, >)$; (iii) $(\mathbb{Z}^+, |)$

Exercises

1. If 2^S is the power set of a set S, when is 2^S totally ordered by $\subseteq$ and when is it not?
2. Let $R \subseteq (\mathbb{N} \times \mathbb{N})^2$ be a binary relation such that for any two pairs (a, b) and (c, d) in $\mathbb{N}^2$, $((a, b), (c, d)) \in R$ iff either $a < c$; or $a = c$ and $b \leqslant d$. Is $\mathbb{N}^2$ totally ordered by R?
3. Give a total order that allows us to list all words in English dictionary in the usual alphabetical order.

If 2^S is the power set of a set S, when is 2^S totally ordered by $\subseteq$ and when is it not?

Let $R \subseteq (\mathbb{N} \times \mathbb{N})^2$ be a binary relation such that for any two pairs (a, b) and (c, d) in $\mathbb{N}^2$, $((a, b), (c, d)) \in R$ iff either $a < c$; or $a = c$ and $b \leqslant d$. Is $\mathbb{N}^2$ totally ordered by R?

Give a total order that allows us to list all words in English dictionary in the usual alphabetical order.

Least, greatest, minimal, maximal element(s) of a poset

Definition

Let $(S, \preccurlyeq)$ be a poset and $c \in S$ an element of the poset.

- If $c \preccurlyeq d$ for every $d \in S$, then c is the **least/smallest element** of $(S, \preccurlyeq)$.
- If $d \preccurlyeq c$ for every $d \in S$, then c is the **greatest/largest element** of $(S, \preccurlyeq)$.
- If there is no $d \in S$ with $d \prec c$, then c is a **minimal element** of $(S, \preccurlyeq)$
- If there is no $d \in S$ with $c \prec d$, then c is a **maximal element** of $(S, \preccurlyeq)$

Example

Let (A, R) be a poset with $A = \{a, b, c, d, e\}$ and
$R = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (c, c), (d, d), (e, b), (e, c), (e, d), (e, e)\}$. Give, if any, its least, greatest, minimal, and maximal elements.

Example

Let (A, R) with $A = \{a, b, c, d, e\}$ and $R = \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,d),(c,e),(d,d),(e,e)\}$. Give, if any, its least, greatest, minimal, and maximal elements.

Example

Let (A, R) with $A = \{a, b, c, d, e\}$ and
$R = \{(a, a), (b, a), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, d), (e, a), (e, e)\}$. Give, if any, its least, greatest, minimal, and maximal elements.

Example

Let (A, R) with $A = \{a, b, c, d, e\}$ and
$R = \{(a, a), (a, b), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d), (e, e)\}$. Give, if any, its least, greatest, minimal, and maximal elements.

Exercises

1. Determine, if they exist, the least, greatest, minimal, and maximal elements of the following posets.
   a. $(2^S, \subseteq)$ for any set S.
   b. $(\{2, 4, 5, 10, 12, 20, 25\}, |)$
   c. $(\mathbb{Z}^+, |)$
2. Give an example of an infinite poset that has a greatest and least element.
3. Show that a greatest element of a poset is unique if it exists.

Give, if any, the least, greatest, minimal, and maximal elements of $(2^S, \subseteq)$ for any set S.

Give, if any, the least, greatest, minimal, and maximal elements of $(\{2, 4, 5, 10, 12, 20, 25\}, |)$

Give, if any, the least, greatest, minimal, and maximal elements of $(\mathbb{Z}^+, |)$.

Give an example of an infinite poset that has a greatest and least element.

Show that a greatest element of a poset is unique if it exists.

Well-ordered set

Definition

A poset $(S, \preccurlyeq)$ is called a **well-ordered set** iff $\preccurlyeq$ is a total order and every nonempty subset of S has a least element according to the ordering given by $\preccurlyeq$.

Example

Is $(\mathbb{Z}^+, \leqslant)$ a well-ordered set?

Example

Is $(\mathbb{Z}^+ \times \mathbb{Z}^+, \preccurlyeq)$ a well-ordered set where $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $b_1 \leqslant b_2$?

Example

Is $(\mathbb{Z}, \leqslant)$ a well-ordered set?

Exercise

Define a relation R on $\mathbb{Z}$ such that $\mathbb{Z}$ becomes well-ordered.

Generalized induction (well-ordered induction)

Principle of induction can be generalized to show that a statement holds for every element of a well-ordered set (not just for elements of $\mathbb{N}$).

- Since $\mathbb{N}$ is a well-ordered set (via the ordering by $\leqslant$), what you learned in Discrete Mathematics I course is just a special case of this generalization.

Theorem (Principle of well-ordered induction)

Let S be a well-ordered set (with $\preccurlyeq$ as the corresponding total order). Then the statement $P(x)$ is true for all $x \in S$ if the following statement holds.

"(Inductive step) For every $y \in S$, if $P(x)$ is true for all $x \prec y$, then $P(y)$ is true."

Note: we don't need to establish the base case of induction because when the inductive step is proved and x_0 is the least element of a well-ordered set, then there is no x in the set for which $x \prec x_0$. Hence, the premise of the induction step vacuously true, and consequently $P(x_0)$ must be true. vacuously true

Prove that every integer $n \geqslant 2$ can be written as a product of one or more primes using well-ordered induction.

Lower bounds and upper bounds

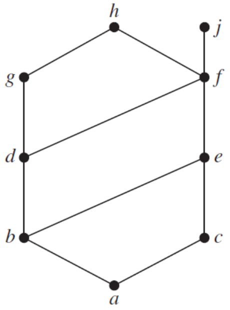
Definition

Let $(S, \preccurlyeq)$ be a poset and $A \subseteq S$ a subset of S.

- A **lower bound**) of A is any element $\ell \in S$ such that $\ell \preccurlyeq a$ for all $a \in A$
- An **upper bound**) of A is any element $u \in S$ such that $a \preccurlyeq u$ for all $a \in A$
- An element $\ell \in S$ is called the **greatest lower bound** (**glb**) of A iff ℓ is a lower bound of A and for every lower bound z of A, $z \preccurlyeq \ell$.
- An element $u \in S$ is called the **least upper bound** (**lub**) of A iff ℓ is an upper bound of A and for every upper bound z of A, $u \preccurlyeq z$.

- glb dan lub of A are unique, if they exist. [Show this for exercise.]
- A set A can have lower/upper bounds without having a lub/glb.

Example



Rosen, Fig. 7, p.657

Find the lower and upper bounds of $\{a, b, c\}$, $\{j, h\}$, $\{a, c, d, f\}$, and $\{b, d, g\}$ in the poset with this Hasse diagram. Give its lub and glb if they exist.

Example

In the poset $(\mathbb{Z}^+, |)$, find the glb and lub of $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist.

Exercise

Let S be an arbitrary set, 2^S its the power set, and $A, B, C \subseteq S$ arbitrary subsets of S. Find the glb and lub of $\{A, B, C\}$ if they exist in the poset $(2^S, \subseteq)$.

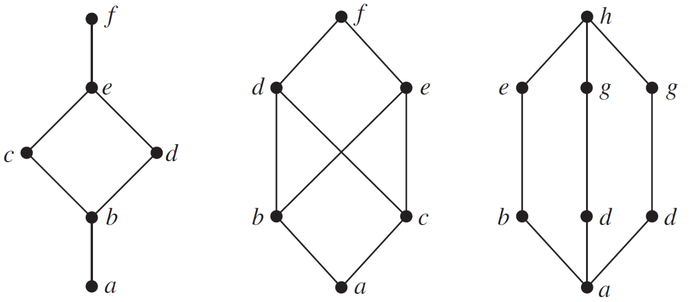
Lattice

Definition

A poset $(S, \preccurlyeq)$ is called a **lattice** iff for every pair of elements $a, b \in S$, the set $\{a, b\}$ has both a lub and a glb

Example

Which of the posets with the following Hasse diagrams are lattices?



Rosen, Fig. 8, p.658

Example

Is $(\{1, 2, 3, 4, 5\}, |)$ a lattice?

Example

Is $(\{1, 2, 4, 6, 8\}, |)$ a lattice?

Exercises

1. Is $(\mathbb{Z}^+, |)$ a lattice?
2. Is $(2^S, \subseteq)$ a lattice for every set S?
3. Which of these two are lattices: $(\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}, \geqslant)$ and $(\{x \in \mathbb{R} \mid 0 < x < 1\}, \geqslant)$? What are their least and greatest elements, if any?
4. Is every totally ordered set a lattice?

Is $(\mathbb{Z}^+, |)$ a lattice?

Is $(2^S, \subseteq)$ for every set S a lattice?

Which of these two are lattices: $(\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}, \geqslant)$ and $(\{x \in \mathbb{R} \mid 0 < x < 1\}, \geqslant)$?
What are their least and greatest elements, if any?

Is every totally ordered set a lattice?