Graph: Part 7 - Euler and Hamilton Paths

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References and acknowledgements

- Materials of these slides are taken from:
  - Kenneth H. Rosen. *Discrete Mathematics and Its Applications*, 8ed. McGraw-Hill, 2019. Section 10.5
  - Jean Gallier. *Discrete Mathematics Second Edition in Progress*, 2017 [Draft]. Section 8.3.
- Figures taken from the above books belong to their respective authors. I do not claim any rights whatsoever.

Motivation

Picture of Euler and Konigsberg

Euler circuit and path

Definition

Let G be a (directed/undirected) graph. A path/circuit in G is an **Euler** path iff it passes through every edge in G exactly once.

- G may have loops or parallel edges.
- An Euler path/circuit may pass through a vertex more than once, i.e., it is simple, but not necessarily node-simple.

Example

Necessary and sufficient condition for existence of Euler circuits/paths

Theorem

- *A graph has an Euler circuit if and only if all of its vertices have an even degree.*
- *A graph has an Euler path, but not Euler circuit, if and only if it has exactly two vertices with an odd degree.*

Constructing Euler circuit

Suppose G is a connected graph with all vertices of even degree. The following steps give us an Euler circuit in G.

1. Start with $\pi :=$ any simple circuit in G
2. Set $H :=$ the graph obtained from G by removing the edges from π.
3. While H still has edges, do the following:
   a. Set $\pi' :=$ be a simple circuit in H whose initial vertex, say u, was also passed through by π.
   b. Set $H :=$ the graph obtained from H by removing edges in π' as well as all isolated vertices.
   c. Set π a circuit obtained by inserting π' at the u's position.
4. Return π (as an Euler circuit).

The above algorithm can also be used to find an Euler path (if an Euler circuit does not exist) by initializing π in step 1 to a simple path between the only two vertices with an odd degree.

Motivation

Picture of William Rowan Hamilton and original Hamilton puzzle.

Hamilton circuit and path

Definition

Let G be a (directed/undirected) graph. A path/circuit in G is a **Hamilton** path/circuit iff it passes through every vertex in G exactly once, except possibly the initial vertex if it is a circuit.

- G may have loops or parallel edges.
- A Hamilton path/circuit is obviously node-simple, hence also simple (never passes an edge more than once).
- But a Hamilton path/circuit may not necessarily pass all edges in G.

Example

Sufficient conditions of Hamilton circuits and paths

Theorem (Dirac's)

If G is a simple undirected graph with $n \geqslant 3$ vertices such that the degree of every vertex is at least $n/2$, then G has a Hamilton circuit.

Theorem (Ore's)

If G is a simple undirected graph with $n \geqslant 3$ vertices such that $\deg(u) + \deg(v) \geqslant n$ for every pair of vertices u and v in G that are not adjacent, then G has a Hamilton circuit.

- The above thereoms are sufficient conditions for a connected simple graph to have a Hamilton circuit.
    - If G satisfies the premise of the theorems, then G has a Hamilton circuit
    - But, not every graph that has a Hamilton circuit satisfies the premise of the above theorems, e.g.,C_5.
- Necessary condition for the existence of a Hamilton circuit in a graph is still unknown.