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# Sorting in Linear Time

Lower bound for comparison sort, Non comparison sort: Counting Sort, Radix Sort

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# Introduction

- Previously learned sorting algorithms (Insertion sort, Merge sort) sort the elements by comparing them. Such sorting algorithms are called **Comparison-based sorting algorithms.**
  - Include the heapsort, quicksort and other sorting algorithms that use comparison between elements to obtain the sorted version.
- We are going to prove that any comparison-based sorting takes $\Omega(n \lg n)$ comparisons in the worst case to sort n elements.
  - Thus, algorithm that run in $\Theta(n \lg n)$ including merge sort and heap sort are asymptotically optimal
- Some sorting algorithms do not use comparison to determine the sorted order, they run in linear time. For example: Counting sort, radix sort, bucket sort. The $\Omega(n \lg n)$ lower bound does not apply to them.

# Comparison

- From a list of $n$ elements $\langle a_1, a_2, a_3, \dots, a_n \rangle$, given two elements $a_i$ and $a_j$, we test whether they satisfy one of the following conditions to determine their relative order.
  - $a_i < a_j$
  - $a_i \leq a_j$
  - $a_i = a_j$
  - $a_i \geq a_j$
  - $a_i > a_j$

We assume that all comparisons have the form $a_i \leq a_j$

# Decision Tree

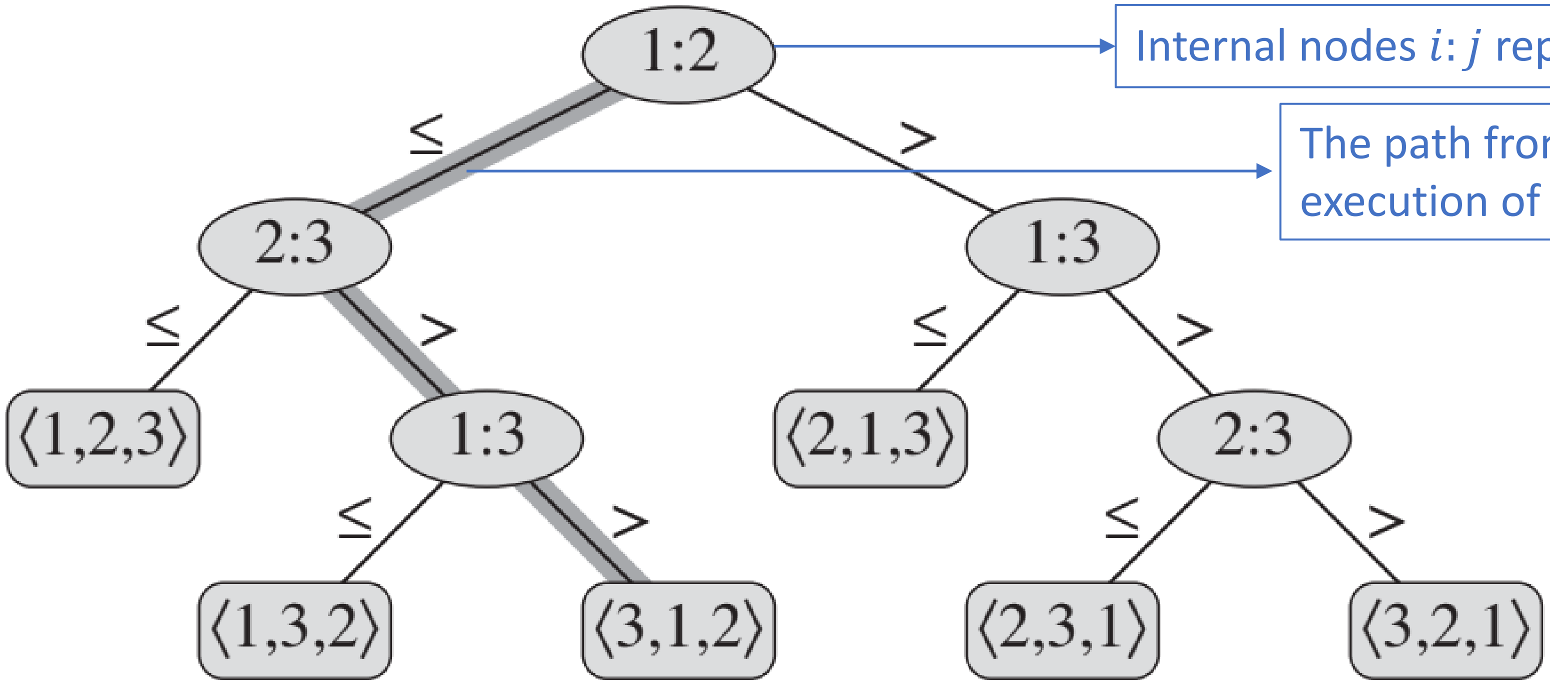


- Comparison sorts can be viewed in terms of **Decision Tree**
  - A full binary tree that represents the comparisons between elements that are performed by a particular sorting algorithm operating on an input of a given size.

- By assuming that the input elements are distinct, we exclude the comparison "$a_i = a_j$", and the other comparisons are equivalent, so we only consider a comparison in form "$a_i \leq a_j$"

# Decision Tree

- Example:
  - A decision tree for **Insertion Sort** operating on three elements $(a_1, a_2, a_3)$. In each node, $i$: $j$ denotes the comparison between $a_i$ and $a_j$.




Internal nodes $i$: $j$ represents the comparison of $a_i$ and $a_j$

The path from root to the leaf represents the execution of the sorting algorithm

# Decision Tree

- Based on the previous decision tree:
  - Each leaf represents the permutation of $n$ elements.
  - For a comparison sort to be correct, each of the $n!$ permutation on $n$ elements must appear as one of the leaves of the decision tree, and each of these leaf must be **reachable** from the root by a path corresponding the actual execution of the comparison sort.
  - The **worst-case number of comparison** is represented by the height of the decision tree. Why?
  - **A lower bound on the heights of all decision trees** in which each permutation appears as reachable leaf <u>is a lower bound on the running time of any comparison sort algorithm</u>.

# Lower Bound for Comparison Sort

- Theorem:

Any comparison sort algorithm requires $\Omega(n \lg n)$ comparisons **in the worst case**.

- Proof:

Consider a decision tree of height $h$ with $k$ reachable leaves corresponds to a comparison sort of $n$ elements.

Each of $n!$ permutations appear as some leaves, and binary tree of height $h$ has no more than $2^h$ leaves. Thus, we have $n! \leq k \leq 2^h$

$$n! \leq 2^h$$
$$\lg n! \leq h$$
$$h = \Omega(n \lg n)$$

# Sorting in Linear Time

# Counting Sort



- Assumption: each of the input elements is an integer in the range 0 to $k$. When $k = O(n)$, then counting sort runs in $\Theta(n)$ time.

- The basic idea of counting sort:
  - For each input element $x$, determine the number of elements less than $x$.
  - Place the element $x$ directly into its position in the output array.
  - Example: If there are 9 elements less than $x$, then $x$ belongs in output position 9.

# Counting Sort

```
COUNTING-SORT(A, B, k)
    let C[0..k] be a new array
    for i = 0 to k
        C[i] = 0
    for j = 1 to A.length
        C[A[j]] = C[A[j]] + 1
    // C[i] now contains the number of elements equal to i.
    for i = 1 to k
        C[i] = C[i] + C[i - 1]
    // C[i] now contains the number of elements less than or equal to i.
    for j = A.length downto 1
        B[C[A[j]]] = A[j]
        C[A[j]] = C[A[j]] - 1
```

Initialize array C to all zeros

C[i] is incremented if the current value of input array equals to i

Keeping a running sum of the array C

To place each element into its sorted correct position in the input array B

# Counting Sort

- Example: Counting sort on ⟨2,5,3,0,2,3,0,3⟩

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| A | 2 | 5 | 3 | 0 | 2 | 3 | 0 | 3 |

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| C | 2 | 0 | 2 | 3 | 0 | 1 |

(a) After line 5

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| C | 2 | 2 | 4 | 7 | 7 | 8 |

(b) After line 8

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B |   |   |   |   |   |   | 3 |   |

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| C | 2 | 2 | 4 | 6 | 7 | 8 |

(c) Line 10-12

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B |   | 0 |   |   |   |   | 3 |   |

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| C | 1 | 2 | 4 | 6 | 7 | 8 |

(d) Line 10-12

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B |   | 0 |   |   |   | 3 | 3 |   |

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| C | 1 | 2 | 4 | 5 | 7 | 8 |

(e) Line 10-12

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B | 0 | 0 | 2 | 2 | 3 | 3 | 3 | 5 |

(f) Sorted array B

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# Running time of Counting Sort

COUNTING-SORT$(A, B, k)$

```
1   let C[0..k] be a new array
2   for i = 0 to k
3       C[i] = 0
4   for j = 1 to A.length
5       C[A[j]] = C[A[j]] + 1
6   // C[i] now contains the number of elements equal to i.
7   for i = 1 to k
8       C[i] = C[i] + C[i - 1]
9   // C[i] now contains the number of elements less than or equal to i.
10  for j = A.length downto 1
11      B[C[A[j]]] = A[j]
12      C[A[j]] = C[A[j]] - 1
```

Line 2-3: $\Theta(k)$

Line 4-5: $\Theta(n)$

Line 7-8: $\Theta(k)$

Line 10-12: $\Theta(n)$

Total: $\boldsymbol{\Theta(n + k)}$

Counting sort is usually used when $k = O(n)$

Counting sort beats the lower bound for comparison sort $\Omega(n \lg n)$

# Counting Sort is Stable



- Another important point of Counting sort is that it is **stable –** numbers with the same value appear in the output array in the same order as they do in the input array.
  - It break ties between two elements by the rule that whichever element appears first in the input array, appears first in the output array.
- The property of stability is important only when satellite data are carried around with the element being sorted.
- Counting sort is often used as subroutine in radix sort.

# Exercise

- Use the procedure of Counting-Sort to illustrate the sorting process on the array $A = \langle 6,0,2,0,1,3,4,6,1,3,2 \rangle$.

# Radix Sort

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- For a set of $d$ digit numbers, this algorithm sort on the <u>least significant digit</u> first. For each digit, it use stable sorting algorithm.

| | | | |
|---|---|---|---|
| 329 | 720 | 720 | 329 |
| 457 | 355 | 329 | 355 |
| 657 | 436 | 436 | 436 |
| 839 | 457 | 839 | 457 |
| 436 | 657 | 355 | 657 |
| 720 | 329 | 457 | 720 |
| 355 | 839 | 657 | 839 |

# Radix Sort

- The following is the procedure of Radix Sort. It assumes that each element in the $n$-elements array $A$ has $d$ digits. The sorting process starts from the lowest-order digit (1) to the highest-order digit ($d$).

```
RADIX-SORT(A, d)
for i = 1 to d
    use a stable sort to sort array A on digit i
```

- Lemma

Given $n$ $d$-digit numbers in which each digit can take on up to $k$ possible values, RADIX-SORT correctly sorts these numbers in $\Theta(d(n+k))$ time if the stable sort it uses takes $\Theta(n+k)$ time.

# Radix Sort



When $d$ is constant and $k = O(n)$, we can make radix sort run in linear time. More generally, we have some flexibility in how to break each key into digits.

- Lemma

Given $n$ $b$-bit numbers and any positive integer $r \leq b$, RADIX-SORT correctly sorts these numbers in $\Theta((b/r)(n + 2^r))$ time if the stable sort it uses takes $\Theta(n + k)$ time for inputs in the range 0 to $k$.

# Radix Sort

***Proof*** For a value $r \leq b$, we view each key as having $d = \lceil b/r \rceil$ digits of $r$ bits each. Each digit is an integer in the range 0 to $2^r - 1$, so that we can use counting sort with $k = 2^r - 1$. (For example, we can view a 32-bit word as having four 8-bit digits, so that $b = 32$, $r = 8$, $k = 2^r - 1 = 255$, and $d = b/r = 4$.) Each pass of counting sort takes time $\Theta(n + k) = \Theta(n + 2^r)$ and there are $d$ passes, for a total running time of $\Theta(d(n + 2^r)) = \Theta((b/r)(n + 2^r))$. ■

- Given the values of $n$ and $b$, we wish to choose the value of $r$, with $r \leq b$, that minimize the expression $\left(\frac{b}{r}\right)(n + 2^r)$.
  - If $b < \lfloor \lg n \rfloor$, then choose any value of $r \leq b$ will result in $\Theta(n + 2^r)$.
  - If $b \geq \lfloor \lg n \rfloor$, then choose $r = \lfloor \lg n \rfloor$ gives the best time to within a constant factor

# Radix Sort



- Radix Sort can be used to sort records of information that are keyed by multiple fields.
  - For example, to sort the date (year, month, day), ID (with specific representation of each digit).

- Note that radix sort which uses counting sort as the intermediate stable sorting algorithm does not sort in place. Thus, when primary memory is at a premium, an in-place sorting algorithm (such as quick sort) may be preferable.

# Summary

- Comparison sorts are sorting algorithms that use the comparison of the input elements to obtain the sorted version.
- Any comparison-sorts take $\Omega(n \lg n)$ in the worst case.

- Non-comparison sorts do not compare the element, for example, counting sort get the correct position of each element by counting how many elements are less than or equal to it.
  - Counting sort is stable sorting algorithm, and it does not sort in place.

- Radix Sort use a stable sorting algorithm to sort its elements from the least significant digit to most significant one. Counting sort or quick sort can be the options.

# References


- Lecturer Slides by Bapak L. Yohanes Stefanus
- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.


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