# Graph: Part 4 - Representation

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# References and acknowledgements

- Materials of these slides are taken from:
  - Kenneth H. Rosen. *Discrete Mathematics and Its Applications*, 8ed. McGraw-Hill, 2019. Section 10.3.
  - Jean Gallier. *Discrete Mathematics Second Edition in Progress*, 2017 [Draft]. Section 8.2.
  - Robin J. Wilson. *Introductio to Graph Theory*, 4ed, 1996. Chapter 1.
- Figures taken from the above books belong to their respective authors. I do not claim any rights whatsoever.

# Representing graph

Besides the usual visualization, it is often useful to represent graphs using other types of structures. Here, we shall discuss

- adjacency list;
- adjacency matrix;
- incidence matrix.

# Adjacency list

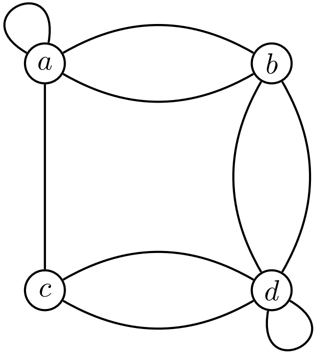
**Definition**

Let $G = (V, E)$ be a graph. The **adjacency list** of $G$ is a list $L$ indexed using nodes in $V$ and given a node $x$, $L(x)$ contains the list of all neighbors of $x$ (where duplicates are allowed).

- Adjacency list can be used for both directed and undirected graphs
- If $G$ is undirected, then $L(x)$ contains $y$ iff $L(y)$ contains $x$.
- If $G$ has $n$ loops at node $x$, then $L(x)$ contains $n$ copies of $x$.
- The more edges a graph can have, the longer each neighbor list tends to be.

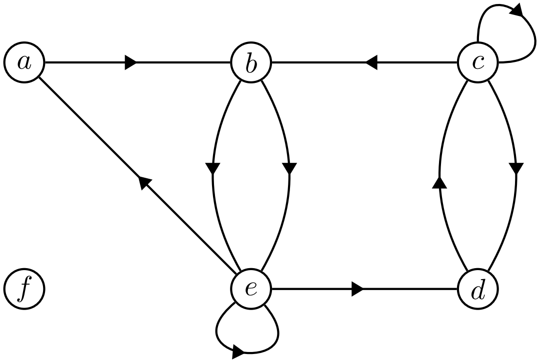
# Example

Give the adjacency list of the following graph.



# Example

Give the adjacency list of the following graph.



# Adjacency matrix of undirected graphs



**Definition**

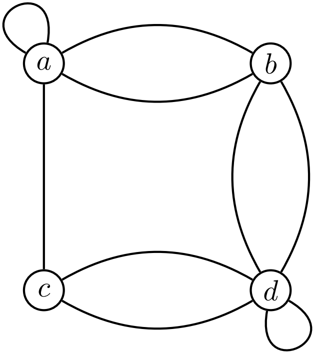
Let $G = (V, E, st)$ be an undirected graph with $n$ nodes $v_1, \ldots, v_n$. An **adjacency matrix** $\mathbf{A}$ (or $\mathbf{A}_G$) of $G$ is the $n \times n$ matrix $[a_{ij}]$ whose $(i, j)$th entry is

$$a_{ij} = \begin{cases} 2k & \text{if } i = j \text{ and } k = |\{e \in E \mid st(e) = \{v_i\}\}| \\ k & \text{if } i \neq j \text{ and } k = |\{e \in E \mid st(e) = \{v_i, v_j\}\}| \end{cases}$$

- Each loop at node $v_i$ adds 2 to the entry $a_{ii}$.
- Each edge between $v_i$ and $v_j$ adds 1 to each of the entry $a_{ij}$ and $a_{ji}$.
- In particular, for a simple graph without loops and parallel edges, the adjacency matrix contains only zero or one and its diagonal entries are all zero.
- Adjacency matrix of undirected graphs are symmetric on the main diagonal.
- Sum over the $i$th row = sum over the $i$th column = $\deg(v_i)$

# Example

Give the adjacency matrix of the following graph.



# Adjacency matrix of directed graphs



**Definition**

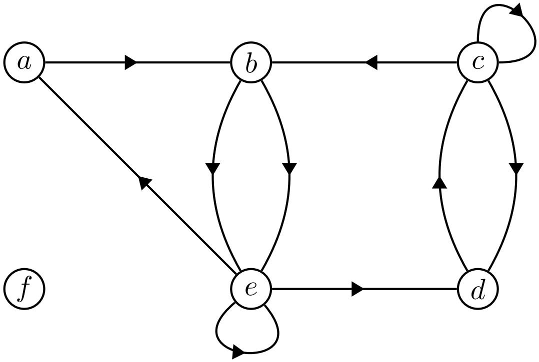
Let $G = (V, E, s, t)$ be a digraph with $n$ nodes $v_1, \ldots, v_n$. An **adjacency matrix** $\mathbf{A}$ (or $\mathbf{A}_G$) of $G$ is the $n \times n$ matrix $[a_{ij}]$ whose $(i, j)$th entry is

$$a_{ij} = |\{e \in E \mid s(e) = v_i, t(e) = v_j\}|$$

- $(i, j)$th entry is the number of edges from $v_i$ to $v_j$
- Sum over the $i$th row $= \deg^+(v_i)$, i.e., outdegree of $v_i$.
- Sum over the $i$th column $= \deg^-(v_i)$, i.e., indegree of $v_i$.
- Diagonal entry $a_{ii}$ is the number of loops at $v_i$.
- Adjacency matrix of directed graphs may not be symmetric.

# Example

Give the adjacency matrix of the following graph.



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**Definition**

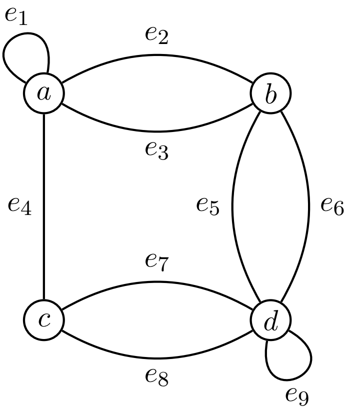
Let $G = (V, E, st)$ be an undirected graph with $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, n\}$. The **incidence matrix** of $\mathbf{D}$ (or $\mathbf{D}_G$) of $G$ is the $m \times n$ matrix $[d_{ij}]$ whose $(i, j)$th entry is:

$$d_{ij} = \begin{cases} 2 & \text{if } \{v_i\} = st(e_j) \\ 1 & \text{if } \{v_i\} \subset st(e_j) \\ 0 & \text{otherwise} \end{cases}$$

- $d_{ij}$ is 2 if $e_j$ is a loop at $v_i$ and 1 if $e_j$ is not loop, but incident at $v_i$
- Sum over the $i$th row $= \deg(v_i)$
- Sum over the $j$th column $= 2$.

# Example

Give the incidence matrix of the following graph



# Incidence matrix of directed graphs

## Definition

Let $G = (V, E, s, t)$ be a digraph with $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, n\}$. The **incidence matrix** of $\mathbf{D}$ (or $\mathbf{D}_G$) of $G$ is the $m \times n$ matrix $[d_{ij}]$ whose $(i, j)$th entry is:

$$d_{ij} = \begin{cases} 2 & \text{if } s(e_j) = t(e_j) = v_i \\ 1 & \text{if } s(e_j) = v_i \neq t(e_j) \\ -1 & \text{if } t(e_j) = v_i \neq s(e_j) \\ 0 & \text{otherwise} \end{cases}$$

- If $e_j$ is a loop, $d_{ij}$ is 2. Otherwise, $d_{ij}$ is 1 if $v_i$ is the source of $e_j$, and -1 if $v_i$ is the target of $e_j$
- Outdegree $\deg^+(v_i) = \sum_{d_{ij}=1} d_{ij} + \frac{1}{2} \sum_{dij=2} d_{ij}$. Indegree $\deg^-(v_i) = \sum_{dij=-1} |d_{ij}| + \frac{1}{2} \sum_{dij=2} d_{ij}$
- Sum over the $j$th column is 2 if $e_j$ is a loop and 0 otherwise.

# Example

Give the incidence matrix of the following graph