Advanced Counting: Counting and Solving Recurrence Relations with Generating Functions

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Reference: Rosen, Ed.8, Ch.8

Advanced counting using generating functions

- Generating functions are frequently used to count the number of combinations of objects.
- E.g.: Counting the number of r-combination from a set of n elements where repetition/replacement is allowed and other conditions imposed.
  - This problem is equivalent to counting the number of solutions to the equation:

    $$e_1 + e_2 + \ldots + e_n = C$$

    where C is a constant and every $e_i \in \mathbb{N}$ as well as other conditions.

Count the number of possible solutions of $e_1 + e_2 + e_3 = 17$ with $e_1, e_2, e_3 \in \mathbb{N}$ and $2 \leqslant e_1 \leqslant 5$, $3 \leqslant e_2 \leqslant 6$, and $4 \leqslant e_3 \leqslant 7$.

How many ways are there to distribute 8 identical cookies to three children if every child receives at least 2 and at most 4 cookies?

How many ways to insert 1 dollar tokens, 2 dollar tokens and 5 dollar tokens to a vending machine to pay stuff priced at r dollars if the order of insertion is (a) not important, and (b) important.

Solve the recurrence relation $a_k = 3a_{k-1}$ with $a_0 = 2$ using generating functions.

Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ with $a_1 = 9$ using generating functions.

Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.