Advanced Counting: Generating Functions

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Reference: Rosen, Ed.8, Ch.8

Generating functions

Definition

An **(ordinary) generating function for the sequence** of (possibly infinitely many) real numbers $a_0, a_1, \ldots, a_k, \ldots$ is an infinite series of the form

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

- In the context of counting, generating functions are also called **formal power series** (**FPS**)
- Value of x is usually ignored, but some operations on generating functions can only be well-defined if the series converges. So, if needed, x is assumed to be close to 0.

Examples

Write the generating functions of:

- $3, 3, 3, \ldots$
- $1, 2, 3, 4, \ldots$
- $1, 2, 4, 8, \ldots$
- the finite sequence $1, 1, 1, 1, 1, 1$

Binomial is a polynomial containing two terms, e.g., $x + y$, $1 + 2x$, $2x + 3yz$

Theorem

Binomial theorem For every binomial $x + y$ and $n \in \mathbb{N}$, it holds that

$$
\begin{aligned}
(x+y)^n &= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \ldots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k}
\end{aligned}
$$

The notation $\binom{n}{k}$ denotes the **combination k of n without replacement** and is also called a **binomial coefficient**.

Example

$$(x+y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$$

Given $m \in \mathbb{Z}^+$, what is the generating function of the sequence $\{a_k\}$ where $a_k = \binom{m}{k}$ and $k = 0, 1, \ldots, m$?

If $|x| < 1$, is $f(x) = 1/(1-x)$ the generating function of the infinite sequence $1, 1, 1, 1, \ldots$?

Addition and multiplication of generating functions

Theorem

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k x^k$. *Then,*

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \qquad \text{and} \qquad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Suppose $f(x) = 1/(1-x)^2$. Compute $a_0, a_1, a_2, \ldots$ in the expansion of $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Extended binomial coefficient

Generating functions sometimes need to use binomial theorem where the exponents are not positive integers.

Definition

Let $u \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, the **extended binomial coefficient** $\binom{u}{k}$ is

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)(u-2)\cdots(u-k+1)}{k!} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

What's the difference with the standard binomial coefficient?

Compute $\binom{-2}{3}$ and $\binom{1/2}{3}$

Show that for $n > 0$, $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$

Theorem (Extended binomial theorem)

Suppose $u, x \in \mathbb{R}$ *and* $|x| < 1$. *Then,*

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Find the generating function for $(1+x)^{-n}$ and $(1-x)^{-n}$ untuk $n \in \mathbb{Z}^+$.

Assume: $n \in \mathbb{Z}^+$, $k = 0, 1, 2, \ldots$

$$a_k = \binom{n}{k} \quad \rightsquigarrow G(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + x^n$$

$$a_k = \binom{n}{k}c^k \quad \rightsquigarrow G(x) = (1+cx)^n = \sum_{k=0}^{n} \binom{n}{k} c^k x^k = 1 + \binom{n}{1}cx + \binom{n}{2}c^2x^2 + \ldots + c^n x^n$$

$$a_k = \begin{cases} \binom{n}{k/r} & \text{if } r \mid k \\ 0 & \text{if } r \not\mid k \end{cases} \quad \rightsquigarrow G(x) = (1+x^r)^n = \sum_{k=0}^{n} \binom{n}{k} x^{rk} = 1 + \binom{n}{1}x^r + \binom{n}{2}x^{2r} + \ldots + x^{rn}$$

Examples of generating function $G(x)$ for the recurrence relation $\{a_k\}$ (2)

$$a_k = \begin{cases} 1 & \text{if } k \leqslant n \\ 0 & \text{if } k > n \end{cases} \quad \rightsquigarrow G(x) = \frac{1 - x^{n+1}}{1 - x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n$$

$$a_k = 1 \quad \rightsquigarrow G(x) = \frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots$$

$$a_k = c^k \quad \rightsquigarrow G(x) = \frac{1}{1 - cx} = \sum_{k=0}^{\infty} c^k x^k = 1 + cx + c^2 x^2 + \ldots$$

$$a_k = \begin{cases} 1 & \text{if } r \mid k \\ 0 & \text{if } r \nmid k \end{cases} \quad \rightsquigarrow G(x) = \frac{1}{1 - x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \ldots$$

Examples of generating function $G(x)$ for the recurrence relation $\{a_k\}$ (3)

$$a_k = k + 1 \quad \rightsquigarrow G(x) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$a_k = \binom{n+k-1}{k} = \binom{n+k-1}{n-1} \quad \rightsquigarrow G(x) = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \dots$$

$$a_k = (-1)^k\binom{n+k-1}{k} = (-1)^k\binom{n+k-1}{n-1} \quad \rightsquigarrow G(x) = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}(-1)^k x^k = 1 - \binom{n}{1}x + \binom{n+1}{2}x^2 - \binom{n+2}{3}x^3 + \dots$$

Examples of generating function $G(x)$ for the recurrence relation $\{a_k\}$ (4)

$$a_k = \binom{n+k-1}{k} a^k \quad \rightsquigarrow G(x) = \frac{1}{(1+ax)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k x^k$$

$$= \binom{n+k-1}{n-1} a^k \qquad = 1 + \binom{n}{1} ax + \binom{n+1}{2} a^2 x^2 + \binom{n+2}{3} a^3 x^3 + \dots$$

$$a_k = \frac{1}{k!} \quad \rightsquigarrow G(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$a_k = \frac{(-1)^{k+1}}{k} \quad \rightsquigarrow G(x) = \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$