# Graph: Part 5 - Isomorphism

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# References and acknowledgements

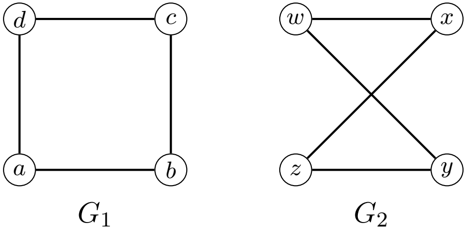
- Materials of these slides are taken from:
  - Kenneth H. Rosen. *Discrete Mathematics and Its Applications*, 8ed. McGraw-Hill, 2019. Section 10.3.
  - Jean Gallier. *Discrete Mathematics Second Edition in Progress*, 2017 [Draft]. Section 4.2, 4.4.
- Figures taken from the above books belong to their respective authors. I do not claim any rights whatsoever.

# Motivation

- Sometimes we want to tell if two given graphs have the same structure (regardless of the identities of their nodes).
- Practical example: two chemical compounds may have the same chemical formula, but differ in structure (hence they are actually different compounds).
- Graph equality is too strong.
    - Graph equality means both graphs have exactly the same set of nodes and edges.

# Example

These two graphs are not equal, but actually have the same structure. (Try redraw $G_2$)



# Mapping between graphs

In the following, a **graph mapping** is a function $f$ that maps a graph $G_1 = (V_1, E_1)$ to a graph $G_2 = (V_2, E_2)$, denoted $f\colon G_1 \to G_2$, where both graphs are directed or both are undirected, and $f = f_v \cup f_e$ such that

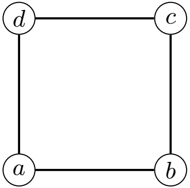
- $f_v$ maps each node $x \in V_1$ to a node $f_v(x) \in V_2$, i.e., if $x \in V_1$, then $f(x) = f_v(x) \in V_2$;
- $f_e$ maps each edge $e \in E_1$ to an edge $f_e(e) \in E_2$, i.e., if $e \in E_1$, then $f(e) = f_e(e) \in E_2$.

Also, we can apply $f_v$ to a set of two nodes as follows: $f_v(\{x, y\}) = \{f_v(x), f_v(y)\}$.

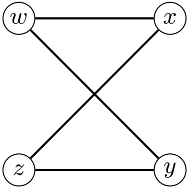
What kind of graph mapping that preserves the structure of $G_1$ in $G_2$?

# Example

The graph mapping $f$ preserve the structure of $G_1$ in $G_2$)




$G_1$



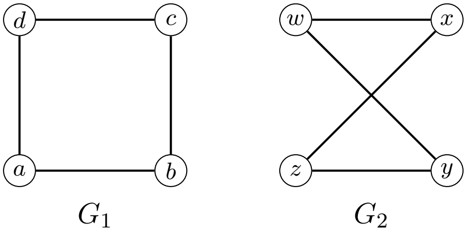

$G_2$

$f = f_v \cup f_e$ where

- $f_v(a) = w$, $f_v(b) = y$, $f_v(d) = x$, $f_v(c) = z$
- $f_e(\{a, b\}) = \{w, y\}$, $f_e(\{a, d\}) = \{w, x\}$, $f_e(\{b, c\}) = \{y, z\}$, $f_e(\{c, d\}) = \{x, z\}$

# Example

The graph mapping $g$ does **not** preserve the structure of $G_1$ in $G_2$)




$g = g_v \cup g_e$ where

- $g_v(a) = z$, $g_v(b) = y$, $g_v(c) = x$, $g_v(d) = w$
- $g_e(\{a, b\}) = \{z, x\}$,
  $g_e(\{a, d\}) = \{z, y\}$,
  $g_e(\{b, c\}) = \{w, x\}$,
  $g_e(\{c, d\}) = \{w, y\}$

# Homomorphism and isomorphism for undirected graphs

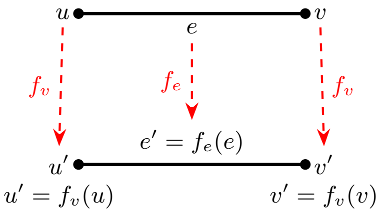


**Definition**

Let $f = f_v \cup f_e$ be a graph mapping from $G_1 = (V_1, E_1, st_1)$ to $G_2 = (V_2, E_2, st_2)$.

$f$ is a **homomorphism** from $G_1$ to $G_2$ iff $st_2(f_e(e)) = f_v(st_1(e))$ for every edge $e \in E_1$

If $f = f_v \cup f_e$ is a homomorphism from $G_1$ to $G_2$ such that both $f_v$ and $f_e$ are bijective, then $f$ is called an **isomorphism** from $G_1$ to $G_2$.




The following must hold for every edge in $E_1$ to have a homomorphism.

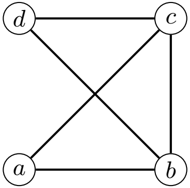
$$
\begin{aligned}
st_2(f_e(e)) &= st_2(e') = \{u', v'\} \\
&= \{f_v(u), f_v(v)\} = f_v(\{u, v\}) = f_v(st_1(e))
\end{aligned}
$$

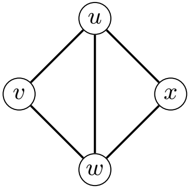
# Remarks

- **Homomorphism condition**: for every edge $e \in E_1$, the endpoints of the edge $f_e(e) \in E_2$ are exactly the nodes in $V_2$ obtained from the mapping of the endpoints of $e$.
- For simple graphs, homomorphism condition from $G_1$ to $G_2$ can be stated as follows: for every two nodes $x, y$ in $G_1$, $x$ and $y$ are connected by an edge $e$ in $G_1$ iff $f_v(x)$ and $f_v(y)$ are connected by the edge $f_e(e)$ in $G_2$.
- For a homomorphism $f = f_v \cup f_e$, $f_v$ and $f_e$ may not necessarily be injective or surjective.
- **Isomorphism condition** $=$ homomorphism condition $+$ $f_v$ and $f_e$ are bijections.
- If $f = f_v \cup f_e$ is an isomorphism, then $f^{-1} = f_v^{-1} \cup f_e^{-1}$ is also an isomorphism.
- Two graphs $G_1$ and $G_2$ are **isomorphic** iff there exists an isomorphism from $G_1$ to $G_2$.

# Example

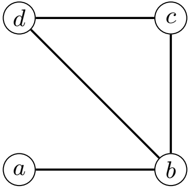
Is there an isomorphism from $G_1$ (left) to $G_2$ (right)? If not, does a homomorphism between them exist?

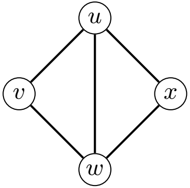


# Example

Is there an isomorphism from $G_1$ (left) to $G_2$ (right)? If not, does a homomorphism between them exist?



# Graph invariant

- A **graph invariant** is a property of graphs that **necessarily** holds in all isomorphic graphs.
- "Necessary" means that if $P$ is a graph invariant and $G_1$ and $G_2$ are isomorphic graphs, then $G_1$ and $G_2$ must both have the same $P$.
  - If a graph invariant does **not** hold on both graphs, then these graphs are not isomorphic.
- Be careful: having a graph invariant on two graphs does **not** imply that the two graphs are isomorphic.

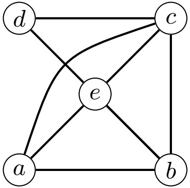
# Examples of graph invariant

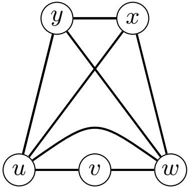
- Number of nodes.
- Number of edges.
- Number of nodes of degree $k$ for all $k$.
- Number of paths of length $k$ for all $k$.
- Bipartiteness.
- Connectedness.
- Number of connected components.
- Vertex connectivity.
- Edge connectivity.
- Planarity.
- $k$-colorability for all $k$
- Eulerian-ness.
- Hamiltonian-ness.
- and many others ...

Example of use: Two isomorphic graphs must have the same number of nodes. But, if two graphs have the same number of nodes, they are not necessarily isomorphic.

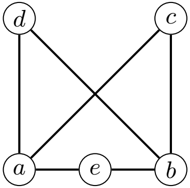
# Exercise

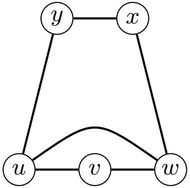
Are $G_1$ (left) and $G_2$ (right) isomorphic? If not, is there a homomorphism between them?



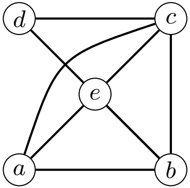



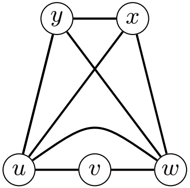

Are $G_3$ (left) and $G_4$ (right) isomorphic? If not, is there a homomorphism between them?



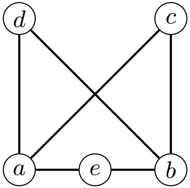



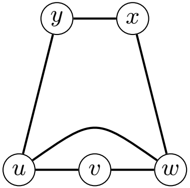
# Exercise



# Exercise



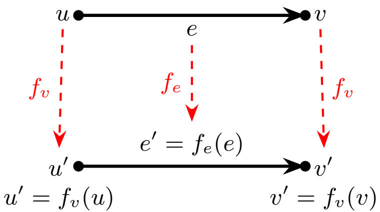



# Homomorphism and isomorphism for digraphs

**Definition**

Let $f = f_v \cup f_e$ be a graph mapping from a digraph $G_1 = (V_1, E_1, s_1, t_1)$ to a digraph $G_2 = (V_2, E_2, s_2, t_2)$.

- $f$ is a **homomorphism** from $G_1$ to $G_2$ iff for every edge $e \in E_1$, $s_2(f_e(e)) = f_v(s_1(e))$ and $t_2(f_e(e)) = f_v(t_1(e))$
- if $f = f_v \cup f_e$ is a homomorphism from $G_1$ to $G_2$ such that both $f_v$ and $f_e$ are bijective, then $f$ is called an **isomorphism** from $G_1$ to $G_2$.




The following must hold for every edge in $E_1$ to have a homomorphism.

$$s_2(f_e(e)) = s_2(e') = u' = f_v(u) = f_v(s_1(e))$$
$$t_2(f_e(e)) = t_2(e') = v' = f_v(v) = f_v(t_1(e))$$

# Remarks

- Homomorphism and isomorphism for digraphs are analogous to that of undirected graphs, except that edge direction needs to be taken into account.
- Most graph invariants for undirected graphs can be adapted to digraphs by considering their directed version.

# Example

Are these graphs isomorphic? If not, is there a homomorphism between them?

$$
\begin{tikzcd}
d \arrow[r] \arrow[d] & c \arrow[d] \\
a \arrow[r] & b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r] \arrow[d] & w \arrow[d] \\
u & v \arrow[l]
\end{tikzcd}
$$

# Example

Are these graphs isomorphic? If not, is there a homomorphism between them?