Relation: Part 4 - Operations on Relation

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Reference: Rosen, *Discrete Mathematics and Its Applications*, 8ed, 2019, Sec. 9.1

Operations on relation to obtain (another) relation

- All set operations: intersection, union, difference, symmetric difference, complement.
- Inverse
- Composition and powers of relation

Set operations over relation

Since every relation is a set (of tuples), then all set operations are applicable to it.

Definition

Given binary relations $R, R_1, R_2 \subseteq A \times B$:

- **Union.** $R_1 \cup R_2 = \{(a,b) \in A \times B \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\}$
- **Intersection.** $R_1 \cap R_2 = \{(a,b) \in A \times B \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\}$
- **Difference.** $R_1 - R_2 = \{(a,b) \in A \times B \mid (a,b) \in R_1 \text{ but } (a,b) \notin R_2\}$
- **Symmetric difference.** $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$
- **Complement.** $\overline{R} = (A \times B) - R = \{(a,b) \in A \times B \mid (a,b) \notin R\}$

The above definition also applies to n-ary relations if (i) R_1 and R_2 have the same arity; and (ii) the Cartesian product used in $\overline{R}$ is over n sets R is defined.

Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $R_1, R_2 \subseteq A \times B$ where $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$. Compute $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, $R_1 \oplus R_2$, and $\overline{R_2}$

Exercise

Let $R_1 = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant y\}$ and $R_2 = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant y\}$.
Find $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, $R_1 \oplus R_2$, and $\overline{R_1}$.

Inverse of binary relation

Definition

Let $R \subseteq A \times B$ be a binary relation from a set A to a set B.
The **inverse** of R, denoted R^{-1}, is a binary relation from B to A defined as

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

This operation and the symmetric property of relation are similar, but different.

Let $R_1, R_2 \subseteq A \times A$ with $A = \{1, 2, 3, 4\}$, $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$. Find R_1^{-1} and R_2^{-1}.

Exercise

Let $R_1 = \{(a,b) \in \mathbb{R}^2 \mid b > a\}$ and $R_2 = \{(a,b) \in \mathbb{R} \times \mathbb{Z} \mid b = \lfloor a \rfloor\}$. Find R_1^{-1} and R_2^{-1}. Express your answer using an expression with b on the left-hand side.

Composition of binary relations

Definition

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. **Composition** of R and S, denoted $S \circ R$, is

$$S \circ R = \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B\}$$

This operation and the transitive property of relation is similar, but different.

Let $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}, C = \{0, 1, 2\}$. Also, $R \subseteq A \times B$ where $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$, and $S \subseteq B \times C$ where $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$. Find $S \circ R$.

Exercises

- Let $R = \{(a, b) \mid a \text{ is a parent of } b\}$. Then what is $R \circ R$?
- Let $R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}$ and $R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geqslant b\}$. Then what are $R_2 \circ R_1$ and $R_2 \circ R_2$?

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Powers of binary relations

Definition

Let $R \subseteq A \times A$. The binary relations $R^n \subseteq A^2$, $n = 1, 2, 3, \ldots,$ are defined recursively:

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R$$

Let $R \subseteq \{1,2,3,4\}^2$ with $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find R^n, $n = 2,3,4,\ldots$.

Exercise

Let $R = \{(a, b) \in \mathbb{N}^2 \mid 2a < b\}$. Find R^n for $n = 2, 3, 4, \ldots$.

Theorem

Binary relation R over a set A is transitive if and only if $R^n \subseteq R$ for every $n = 1, 2, 3, \ldots$.