Chapter 9

Infinite Series

Section 9.1

Infinite Sequences

Infinite Sequences

An infinite sequence is a function whose domain is the set of positive integers and whose range is a set of real numbers. We may denote a sequence by a1, a2, a3, …, by $\{a_n\}_{n=1}^{\infty}$, or simply by $\{\boldsymbol{a_n}\}$

explicit formula for the nth term, as in

$$a_n = 3n - 2, \qquad n \geq 1$$

recursion formula

$$a_1 = 1, \qquad a_n = a_{n-1} + 3, \qquad n \geq 2$$

Convergence

Definition

The sequence $\{a_n\}$ is said to **converge** to L, and we write

$$\lim_{n\to\infty} a_n = L$$

if for each positive number ε there is a corresponding positive number N such that

$$n \geq N \Rightarrow |a_n - L| < \varepsilon$$

A sequence that fails to converge to any finite number L is said to **diverge,** or to be divergent.

EXAMPLE 1 Show that if p is a positive integer, then

$$\lim_{n \to \infty} \frac{1}{n^p} = 0$$

SOLUTION This is almost obvious from earlier work, but we can give a formal demonstration. Let an arbitrary $\varepsilon > 0$ be given. Choose N to be any number greater than $\sqrt[p]{1/\varepsilon}$. Then $n \geq N$ implies that

$$|a_n - L| = \left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\left(\sqrt[p]{1/\varepsilon}\right)^p} = \varepsilon \qquad \blacksquare$$

Properties of Limit of Sequences

Theorem A **Properties of Limits of Sequences**

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and k a constant. Then

(i) $\lim_{n\to\infty} k = k$;

(ii) $\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n$;

(iii) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$;

(iv) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$;

(v) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n \neq 0$.

EXAMPLE 2 Find $\lim_{n\to\infty} \dfrac{3n^2}{7n^2 + 1}$.

SOLUTION To decide what is happening to a quotient of two polynomials in n as n gets large, it is wise to divide the numerator and denominator by the largest power of n that occurs in the denominator. This justifies our first step below; the others are justified by appealing to statements from Theorem A as indicated by the circled numbers.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{3n^2}{n^2+1} &= \lim_{n\to\infty} \frac{3}{7 + (1/n^2)} \\
&\overset{5}{=} \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} [7 + (1/n^2)]} \\
&\overset{3}{=} \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} 7 + \lim_{n\to\infty} 1/n^2} \\
&\overset{1}{=} \frac{3}{7 + \lim_{n\to\infty} 1/n^2} = \frac{3}{7+0} = \frac{3}{7}
\end{aligned}
$$

By this time, the limit theorems are so familiar that we will normally jump directly from the first step to the final result. ■

$$\boxed{\text{If } \lim_{x\to\infty} f(x) = L, \text{ then } \lim_{n\to\infty} f(n) = L.}$$

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{1/x}{e^x} = 0$$

Thus,

$$\lim_{n\to\infty} \frac{\ln n}{e^n} = 0$$

That is, $\{(\ln n)/e^n\}$ converges to 0.

Theorem B **Squeeze Theorem**

Suppose that $\{a_n\}$ and $\{c_n\}$ both converge to L and that $a_n \leq b_n \leq c_n$ for $n \geq K$ (K a fixed integer). Then $\{b_n\}$ also converges to L.

EXAMPLE 4 Show that $\displaystyle\lim_{n\to\infty} \frac{\sin^3 n}{n} = 0$.

SOLUTION For $n \geq 1$, $-1/n \leq (\sin^3 n)/n \leq 1/n$. Since $\displaystyle\lim_{n\to\infty}(-1/n) = 0$ and $\displaystyle\lim_{n\to\infty}(1/n) = 0$, the result follows by the Squeeze Theorem. ■

Theorem C

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 5 Show that if $-1 < r < 1$ then $\lim_{n\to\infty} r^n = 0$.

SOLUTION If $r = 0$, the result is trivial, so suppose otherwise. Then $1/|r| > 1$, and so $1/|r| = 1 + p$ for some number $p > 0$. By the Binomial Formula,

$$\frac{1}{|r|^n} = (1 + p)^n = 1 + pn + (\text{positive terms}) \geq pn$$

Thus,

$$0 \leq |r|^n \leq \frac{1}{pn}$$

Since $\lim_{n\to\infty}(1/pn) = (1/p)\lim_{n\to\infty}(1/n) = 0$, it follows from the Squeeze Theorem that $\lim_{n\to\infty}|r|^n = 0$ or, equivalently, $\lim_{n\to\infty}|r^n| = 0$. By Theorem C, $\lim_{n\to\infty} r^n = 0$. ▪

Monotonic Sequence Theorem

Theorem D **Monotonic Sequence Theorem**

If U is an upper bound for a nondecreasing sequence $\{a_n\}$, then the sequence converges to a limit A that is less than or equal to U. Similarly, if L is a lower bound for a nonincreasing sequence $\{b_n\}$, then the sequence $\{b_n\}$ converges to a limit B that is greater than or equal to L.

The expression **monotonic sequence** is used to describe either a nondecreasing or nonincreasing sequence

EXAMPLE 6 Show that the sequence $b_n = n^2/2^n$ converges by using Theorem D.

SOLUTION The first few terms of this sequence are

$$\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \frac{9}{16}, \frac{49}{128}, \ldots$$

For $n \geq 3$ the sequence appears to be decreasing ($b_n > b_{n+1}$), a fact that we now establish. Each of the following inequalities is equivalent to the others.

$$\frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}}$$

$$n^2 > \frac{(n+1)^2}{2}$$

$$2n^2 > n^2 + 2n + 1$$

$$n^2 - 2n > 1$$

$$n(n-2) > 1$$

The last inequality is clearly true for $n \geq 3$. Since the sequence is decreasing (a stronger condition than nonincreasing) and is bounded below by zero, the Monotonic Sequence Theorem guarantees that it has a limit.

It would be easy using l'Hôpital's Rule to show that the limit is zero. ■

Section 9.2

Infinite Series

Definition

The infinite series $\sum_{k=1}^{\infty} a_k$ **converges** and has **sum** S if the sequence of partial sums $\{S_n\}$ converges to S. If $\{S_n\}$ diverges, then the series **diverges**. A divergent series has no sum.

Geometric Series A series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

where $a \neq 0$, is called a **geometric series.**

EXAMPLE 1 Show that a geometric series converges, and has sum $S = a/(1-r)$ if $|r| < 1$, but diverges if $|r| \geq 1$.

SOLUTION Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. If $r = 1$, $S_n = na$, which grows without bound, and so $\{S_n\}$ diverges. If $r \neq 1$, we may write

$$S_n - rS_n = (a + ar + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n) = a - ar^n$$

and so

$$\boxed{S_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}r^n}$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ (Section 9.1, Example 5), and thus

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$$

If $|r| > 1$ or $r = -1$, the sequence $\{r^n\}$ diverges, and consequently so does $\{S_n\}$. ■

EXAMPLE 2 Use the result of Example 1 to find the sum of the following two geometric series.

(a) $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$

(b) $0.515151\ldots = \frac{51}{100} + \frac{51}{10{,}000} + \frac{51}{1{,}000{,}000} + \cdots$

SOLUTION

(a) $S = \dfrac{a}{1-r} = \dfrac{\frac{4}{3}}{1-\frac{1}{3}} = \dfrac{\frac{4}{3}}{\frac{2}{3}} = 2$

(b) $S = \dfrac{\frac{51}{100}}{1-\frac{1}{100}} = \dfrac{\frac{51}{100}}{\frac{99}{100}} = \dfrac{51}{99} = \dfrac{17}{33}$

Incidently, the procedure in part (b) suggests how to show that any repeating decimal represents a rational number.

A General Test for Divergence

Theorem A ***n*th-Term Test for Divergence**

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist, then the series diverges.

The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

EXAMPLE 6 Show that the harmonic series diverges.

SOLUTION We show that S_n grows without bound. Imagine n to be large and write

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \frac{1}{n} \\ &> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n} \end{aligned}$$

It is clear that by taking n sufficiently large we can introduce as many $\frac{1}{2}$'s into the last expression as we wish. Thus, S_n grows without bound, and so $\{S_n\}$ diverges. Hence, the harmonic series diverges. ■

Properties of Convergent Series

Theorem B **Linearity of Convergent Series**

If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, and if c is a constant, then $\sum_{k=1}^{\infty} ca_k$ and $\sum_{k=1}^{\infty}(a_k + b_k)$ also converge, and

(i) $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$;

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

Properties of Convergent Series

Theorem C

If $\sum_{k=1}^{\infty} a_k$ diverges and $c \neq 0$, then $\sum_{k=1}^{\infty} ca_k$ diverges.

Theorem D **Grouping Terms in an Infinite Series**

The terms of a convergent series can be grouped in any way (provided that the order of the terms is maintained), and the new series will converge with the same sum as the original series.

Section 9.3

Positive Series: The Integral Test

Bounded Partial Sums

Theorem A **Bounded Sum Test**

A series Σa_k of nonnegative terms converges if and only if its partial sums are bounded above.

Series and Improper Integrals

Series and Improper Integrals The behavior of $\sum_{k=1}^{\infty} f(k)$ and $\int_1^{\infty} f(x)\,dx$ with respect to convergence is similar and gives a very powerful test.

Theorem B Integral Test

Let f be a continuous, positive, nonincreasing function on the interval $[1, \infty)$ and suppose that $a_k = f(k)$ for all positive integers k. Then the infinite series

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx$$

converges.

EXAMPLE 2 **(p-Series Test)** The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

where p is a constant, is called a ***p*-series.** Show each of the following:

(a) The p-series converges if $p > 1$.

(b) The p-series diverges if $p \leq 1$.

SOLUTION If $p \geq 0$, the function $f(x) = 1/x^p$ is continuous, positive, and nonincreasing on $[1, \infty)$ and $f(k) = 1/k^p$. Thus, by the Integral Test, $\Sigma(1/k^p)$ converges if and only if $\lim_{t\to\infty} \int_1^t x^{-p}\,dx$ exists (as a finite number).

If $p \neq 1$,

$$\int_1^t x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{t^{1-p}-1}{1-p}$$

If $p = 1$,

$$\int_1^t x^{-1}\,dx = [\ln x]_1^t = \ln t$$

Since $\lim_{t\to\infty} t^{1-p} = 0$ if $p > 1$ and $\lim_{t\to\infty} t^{1-p} = \infty$ if $p < 1$, and since $\lim_{t\to\infty} \ln t = \infty$, we conclude that the p-series converges if $p > 1$ and diverges if $0 \leq p \leq 1$.

We still have the case $p < 0$ to consider. In this case, the nth term of $\Sigma(1/k^p)$, that is, $1/n^p$, does not even tend toward 0. Thus, by the nth-Term Test, the series diverges.

Note that the case $p = 1$ gives the harmonic series, which was treated in Section 9.2. Our results here and there are consistent. The harmonic series diverges. ∎

The Tail of a Series

The beginning of a series plays no role in its convergence or divergence. Only the tail is important (the tail really does wag the dog). By the *tail* of a series, we mean

$$a_N + a_{N+1} + a_{N+2} + \cdots$$

where N denotes an arbitrarily large number. Hence, in testing for convergence or divergence of a series, we can ignore the beginning terms or even change them. Clearly, however, the sum of a series does depend on all its terms, including the initial ones.

Section 9.4

Positive Series: Other Tests

Comparing One Series with Another

Theorem A **Ordinary Comparison Test**

Suppose that $0 \leq a_n \leq b_n$ for $n \geq N$.

(i) If Σb_n converges, so does Σa_n.

(ii) If Σa_n diverges, so does Σb_n.

EXAMPLE 1 Does $\sum_{n=1}^{\infty} \frac{n}{5n^2 - 4}$ converge or diverge?

SOLUTION A good guess would be that it diverges, since the nth term behaves like $1/5n$ for large n. In fact,

$$\frac{n}{5n^2 - 4} > \frac{n}{5n^2} = \frac{1}{5} \cdot \frac{1}{n}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{5} \cdot \frac{1}{n}$ diverges since it is one-fifth of the harmonic series (Theorem 9.2C). Thus, by the Ordinary Comparison Test, the given series also diverges.

Comparing One Series with Another

Theorem B **Limit Comparison Test**

Suppose that $a_n \geq 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

If $0 < L < \infty$, then Σa_n and Σb_n converge or diverge together. If $L = 0$ and Σb_n converges, then Σa_n converges.

EXAMPLE 3 Determine the convergence or divergence of each series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{n^3 - 2n^2 + 11}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 19n}}$

SOLUTION We apply the Limit Comparison Test, but we still must decide to what we should compare the nth term. We see what the nth term is like for large n by looking at the largest-degree terms in the numerator and denominator. In the first case, the nth term is like $3/n^2$; in the second, it is like $1/n$.

$$\text{(a)} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n-2)/(n^3 - 2n^2 + 11)}{3/n^2} = \lim_{n\to\infty} \frac{3n^3 - 2n^2}{3n^3 - 6n^2 + 33} = 1$$

$$\text{(b)} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2 + 19n}}{1/n} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2 + 19n}} = 1$$

Since $\Sigma 3/n^2$ converges and $\Sigma 1/n$ diverges, we conclude that the series in (a) converges and the series in (b) diverges.

EXAMPLE 4 Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converge or diverge?

SOLUTION To what shall we compare $(\ln n)/n^2$? If we try $1/n^2$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \div \frac{1}{n^2} = \lim_{n\to\infty} \ln n = \infty$$

The test fails because its conditions are not satisfied. On the other hand, if we use $1/n$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \div \frac{1}{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Again, the test fails. Possibly something between $1/n^2$ and $1/n$ will work, such as $1/n^{3/2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \div \frac{1}{n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$

(The last equality follows from l'Hôpital's Rule.) We conclude from the second part of the Limit Comparison Test that $\Sigma(\ln n)/n^2$ converges (since $\Sigma 1/n^{3/2}$ converges by the p-Series Test). ▪

Comparing a Series with Itself

Theorem C **Ratio Test**

Let Σa_n be a series of positive terms and suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho$$

(i) If $\rho < 1$, the series converges.

(ii) If $\rho > 1$ or if $\lim_{n \to \infty} a_{n+1}/a_n = \infty$, the series diverges.

(iii) If $\rho = 1$, the test is inconclusive.

EXAMPLE 5 Test for convergence or divergence: $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

SOLUTION

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

We conclude by the Ratio Test that the series converges.

EXAMPLE 6 Test for convergence or divergence: $\sum_{n=1}^{\infty} \frac{2^n}{n^{20}}$.

SOLUTION

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^{20}} \frac{n^{20}}{2^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{20} \cdot 2 = 2$$

We conclude that the given series diverges.

Summary To test a series Σa_n of positive terms for convergence or divergence, look carefully at a_n.

1. If $\lim_{n\to\infty} a_n \neq 0$, conclude from the nth-Term Test that the series diverges.
2. If a_n involves $n!$, r^n, or n^n, try the Ratio Test.
3. If a_n involves only constant powers of n, try the Limit Comparison Test. In particular, if a_n is a rational expression in n, use this test with b_n as the quotient of the leading terms from the numerator and denominator.
4. If the tests above do not work, try the Ordinary Comparison Test, the Integral Test, or the Bounded Sum Test.
5. Some series require a clever manipulation or a neat trick to determine convergence or divergence.

Section 9.5

Alternating Series, Absolute Convergence, and Conditional Convergence

A Convergence Test

Theorem A **Alternating Series Test**

Let

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

be an alternating series with $a_n > a_{n+1} > 0$. If $\lim_{n\to\infty} a_n = 0$, then the series converges. Moreover, the error made by using the sum S_n of the first n terms to approximate the sum S of the series is not more than a_{n+1}.

EXAMPLE 2 Show that

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

converges. Calculate S_5 and estimate the error made by using this as a value for the sum of the whole series.

SOLUTION The Alternating Series Test (Theorem A) applies and guarantees convergence.

$$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

$$|S - S_5| \leq a_6 = \frac{1}{6!} \approx 0.0014$$

Absolute Convergence

Theorem B **Absolute Convergence Test**

If $\Sigma |u_n|$ converges, then Σu_n converges.

A series Σu_n is said to **converge absolutely** if $\Sigma |u_n|$ converges.

Theorem C **Absolute Ratio Test**

Let Σu_n be a series of nonzero terms and suppose that

$$\lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \rho$$

(i) If $\rho < 1$, the series converges absolutely (hence converges).

(ii) If $\rho > 1$, the series diverges.

(iii) If $\rho = 1$, the test is inconclusive.

EXAMPLE 4 Show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{n!}$ converges absolutely.

SOLUTION

$$\rho = \lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!} \div \frac{3^n}{n!}$$

$$= \lim_{n\to\infty}\frac{3}{n+1} = 0$$

We conclude from the Absolute Ratio Test that the series converges absolutely (and therefore converges).

Conditional Convergence

A series Σu_n is called conditionally convergent if Σu_n converges but $\Sigma|u_n|$ diverges

EXAMPLE 6 Show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditionally convergent.

SOLUTION $\sum_{n=1}^{\infty}(-1)^{n+1}[1/\sqrt{n}]$ converges by the Alternating Series Test.

However, $\sum_{n=1}^{\infty}1/\sqrt{n}$ diverges, since it is a p-series with $p = \frac{1}{2}$.

Rearrangement Theorem

Theorem D **Rearrangement Theorem**

The terms of an absolutely convergent series can be rearranged without affecting either the convergence or the sum of the series.

Section 9.6

Power Series

Power Series

A **power series in *x*** has the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

The Convergence Set

We call the set on which a power series converges its **convergence set**.

EXAMPLE 2 What is the convergence set for

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n} = 1 + \frac{1}{2}\frac{x}{2} + \frac{1}{3}\frac{x^2}{2^2} + \frac{1}{4}\frac{x^3}{2^3} + \cdots$$

SOLUTION Note that some of the terms may be negative (if x is negative). Let's test for absolute convergence using the Absolute Ratio Test (Theorem 9.5C).

$$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)2^{n+1}} \div \frac{x^n}{(n+1)2^n}\right| = \lim_{n\to\infty}\frac{|x|}{2}\cdot\frac{n+1}{n+2} = \frac{|x|}{2}$$

The series converges absolutely (hence converges) when $\rho = |x|/2 < 1$ and diverges when $|x|/2 > 1$. Consequently, it converges when $|x| < 2$ and diverges when $|x| > 2$.

If $x = 2$ or $x = -2$, the Ratio Test fails. However, when $x = 2$, the series is the harmonic series, which diverges; and when $x = -2$, it is the alternating harmonic series, which converges. We conclude that the convergence set for the given series is the interval $-2 \le x < 2$ (Figure 2). ■

The Convergence Set

Theorem A

The convergence set for a power series $\Sigma a_n x^n$ is always an interval of one of the following three types:

(i) The single point $x = 0$.

(ii) An interval $(-R, R)$, plus possibly one or both end points.

(iii) The whole real line.

In (i), (ii), and (iii), the series is said to have **radius of convergence** 0, R, and ∞, respectively.

Theorem B

A power series $\Sigma a_n x^n$ converges absolutely on the interior of its interval of convergence.

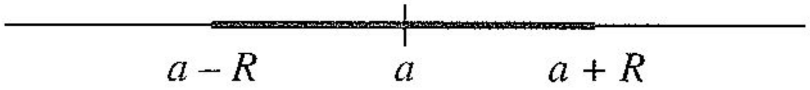
Power Series in *x-a*

Power Series in $x - a$ A series of the form

$$\sum a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

is called a **power series in $x - a$.** All that we have said about power series in x applies equally well for series in $x - a$. In particular, its convergence set is always one of the following kinds of intervals:

1. The single point $x = a$.
2. An interval $(a - R, a + R)$, plus possibly one or both end points (Figure 5).
3. The whole real line.



Convergence set

Figure 5

EXAMPLE 5 Find the convergence set for $\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)^2}$.

SOLUTION We apply the Absolute Ratio Test.

$$\begin{aligned}\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+2)^2} \div \frac{(x-1)^n}{(n+1)^2} \right| &= \lim_{n\to\infty} |x-1| \frac{(n+1)^2}{(n+2)^2} \\ &= |x-1|\end{aligned}$$

Thus, the series converges if $|x-1| < 1$, that is, if $0 < x < 2$; it diverges if $|x-1| > 1$. It also converges (even absolutely) at both of the end points 0 and 2, as we see by substitution of these values. The convergence set is the closed interval $[0,2]$ (Figure 6). ■

Section 9.7

Operations on Power Series

Term-by-Term Differentiation and Integration

Theorem A

Suppose that $S(x)$ is the sum of a power series on an interval I; that is,

$$S(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Then, if x is interior to I,

(i) $$S'(x) = \sum_{n=0}^{\infty} D_x(a_n x^n) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

(ii) $$\int_0^x S(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$
$$= a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \frac{1}{4} a_3 x^4 + \cdots$$

EXAMPLE 1 Apply Theorem A to the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad -1 < x < 1$$

to obtain formulas for two new series.

SOLUTION Differentiating term by term yields

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \qquad -1 < x < 1$$

Integrating term by term gives

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x 1\,dt + \int_0^x t\,dt + \int_0^x t^2\,dt + \cdots$$

That is,

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \qquad -1 < x < 1$$

If we replace x by $-x$ in the latter and multiply both sides by -1, we obtain

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1}$$

From Problem 45 of Section 9.5, we learn that this result is valid at the end point $x = 1$ (also see the note in the margin). ■

EXAMPLE 2 Find the power series representation for $\tan^{-1} x$.

SOLUTION Recall that

$$\tan^{-1} x = \int_0^x \frac{1}{1 + t^2}\, dt$$

From the geometric series for $1/(1 - x)$, with x replaced by $-t^2$, we get

$$\frac{1}{1 + t^2} = 1 - t^2 + t^4 - t^6 + \cdots, \qquad -1 < t < 1$$

Thus,

$$\tan^{-1} x = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$

That is,

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1}$$

(By the note in the margin, this also holds at $x = \pm 1$.)

Algebraic Operations

Theorem B

Let $f(x) = \Sigma a_n x^n$ and $g(x) = \Sigma b_n x^n$, with both of these series converging at least for $|x| < r$. If the operations of addition, subtraction, and multiplication are performed on these series as if they were polynomials, the resulting series will converge for $|x| < r$ and represent $f(x) + g(x)$, $f(x) - g(x)$, and $f(x) \cdot g(x)$, respectively. If $b_0 \neq 0$, the corresponding result holds for division, but we can guarantee its validity only for $|x|$ sufficiently small.

Section 9.8

Taylor and Maclaurin Series

Uniqueness Theorem

Theorem A Uniqueness Theorem

Suppose that f satisfies

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots$$

for all x in some interval around a. Then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The power series representation of a function in x - a is called its **Taylor series** after English mathematician Brook Taylor (1685-1731). If a = 0, the corresponding series is called the **Maclaurin series** after the Scottish mathematician Colin Maclaurin (1698-1746)

Convergence of Taylor Series

Theorem B **Taylor's Formula with Remainder**

Let f be a function whose $(n+1)$st derivative $f^{(n+1)}(x)$ exists for each x in an open interval I containing a. Then, for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder (or error) $R_n(x)$ is given by the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

and c is some point between x and a.

Convergence of Taylor Series

Theorem C **Taylor's Theorem**

Let f be a function with derivatives of all orders in some interval $(a - r, a + r)$. The Taylor series

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots$$

represents the function f on the interval $(a - r, a + r)$ if and only if

$$\lim_{n \to \infty} R_n(x) = 0$$

where $R_n(x)$ is the remainder in Taylor's Formula,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

and c is some point in $(a - r, a + r)$.

EXAMPLE 1 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

SOLUTION

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

Thus,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

and this is valid for all x, provided we can show that

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = 0$$

Now, $|f^{(n+1)}(x)| = |\cos x|$ or $|f^{(n+1)}(x)| = |\sin x|$, and so

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

But $\lim\limits_{n\to\infty} x^n/n! = 0$ for all x, since $x^n/n!$ is the nth term of a convergent series (see Example 3 and Problem 29 of Section 9.6). As a consequence, we see that $\lim\limits_{n\to\infty} R_n(x) = 0$. ■

Convergence of Taylor Series

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

EXAMPLE 3 Find the Maclaurin series for $f(x) = \cosh x$ in two different ways, and show that it represents $\cosh x$ for all x.

SOLUTION

Method 1. This is the direct method.

$$\begin{aligned} f(x) &= \cosh x & f(0) &= 1 \\ f'(x) &= \sinh x & f'(0) &= 0 \\ f''(x) &= \cosh x & f''(0) &= 1 \\ f'''(x) &= \sinh x & f'''(0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

Thus,

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}$$

provided we can show that $\lim_{n \to \infty} R_n(x) = 0$ for all x.

Now let B be an arbitrary number and suppose that $|x| \le B$. Then

$$|\cosh x| = \left|\frac{e^x + e^{-x}}{2}\right| \le \frac{e^x}{2} + \frac{e^{-x}}{2} \le \frac{e^B}{2} + \frac{e^B}{2} = e^B$$

By similar reasoning, $|\sinh x| \leq e^B$. Since $f^{(n+1)}(x)$ is either $\cosh x$ or $\sinh x$, we conclude that

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}\right| \leq \frac{e^B|x|^{n+1}}{(n+1)!}$$

The latter expression tends to zero as $n \to \infty$, just as in Example 1.

Method 2. We use the fact that $\cosh x = (e^x + e^{-x})/2$. From Example 3 of Section 9.7,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

The previously obtained result follows by adding these two series and dividing by 2. ∎

EXAMPLE 4 Find the Maclaurin series for $\sinh x$ and show that it represents $\sinh x$ for all x.

SOLUTION We do both jobs at once when we differentiate the series for $\cosh x$ (Example 3) term by term and use Theorem 9.7A.

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots}$$

The Binomial Series

Theorem D **Binomial Series**

For any real number p and for $|x| < 1$,

$$(1 + x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \cdots$$

If p is a positive integer, $\binom{p}{k} = 0$ for $k > p$, and so the Binomial Series collapses to a series with finitely many terms, the usual Binomial Formula.

EXAMPLE 5 Represent $(1 - x)^{-2}$ in a Maclaurin series for $-1 < x < 1$.

SOLUTION By Theorem D,

$$(1 + x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Thus,

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Naturally, this agrees with a result we obtained by a different method in Example 1 of Section 9.7.

EXAMPLE 7 Compute $\int_0^{0.4} \sqrt{1 + x^4}\, dx$ to five decimal places.

SOLUTION From Example 6,

$$\sqrt{1 + x^4} = 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{5}{128}x^{16} + \cdots$$

Thus,

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx = \left[x + \frac{x^5}{10} - \frac{x^9}{72} + \frac{x^{13}}{208} + \cdots \right]_0^{0.4} \approx 0.40102$$

Summary We conclude our discussion of series with a list of the important Maclaurin series we have found. These series will be useful in doing the problem set, but, what is more significant, they find application throughout mathematics and science.

Important Maclaurin Series

1. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad -1 < x < 1$

2. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \cdots \qquad -1 < x \le 1$

3. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} + \cdots \qquad -1 \le x \le 1$

4. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$

5. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots$

6. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots$

7. $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \dfrac{x^9}{9!} + \cdots$

8. $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dfrac{x^8}{8!} + \cdots$

9. $(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \binom{p}{4}x^4 + \cdots \qquad -1 < x < 1$

Section 9.9

The Taylor Approximation to a Function

The Taylor Polynomial of Order 1

We called such a line the linear approximation to f near a and we found it to be

$$\boxed{P_1(x) = f(a) + f'(a)(x - a)}$$

The Taylor Polynomial of Order n

$$\boxed{P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Maclaurin Polynomials

$$\boxed{f(x) \approx P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n}$$

End of Chapter 9