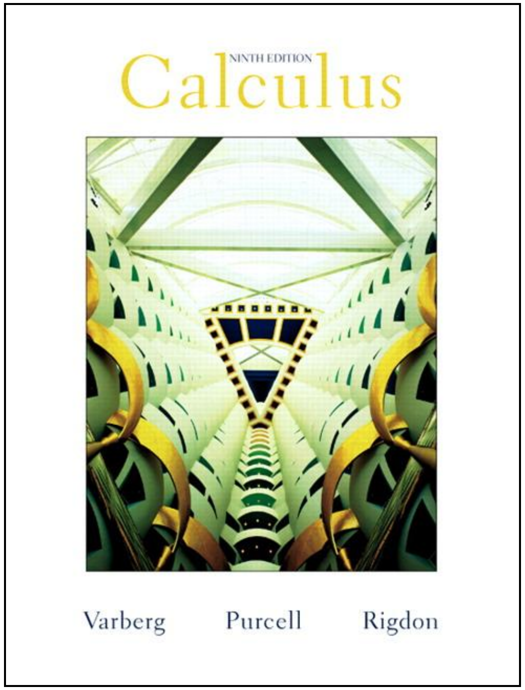
# Varberg, Calculus 9e



# Chapter 12

Derivatives for Functions of Two or More Variables

# Section 12.1

Functions of Two or More Variables

# Function of Two or More Variables

- **Real-valued function of two real variables**, the is a function *f* the assigns to each ordered pair $(x,y)$ in some set *D* of the plane a (unique) real number $f(x,y)$. Examples are

(1) $$f(x, y) = x^2 + 3y^2$$

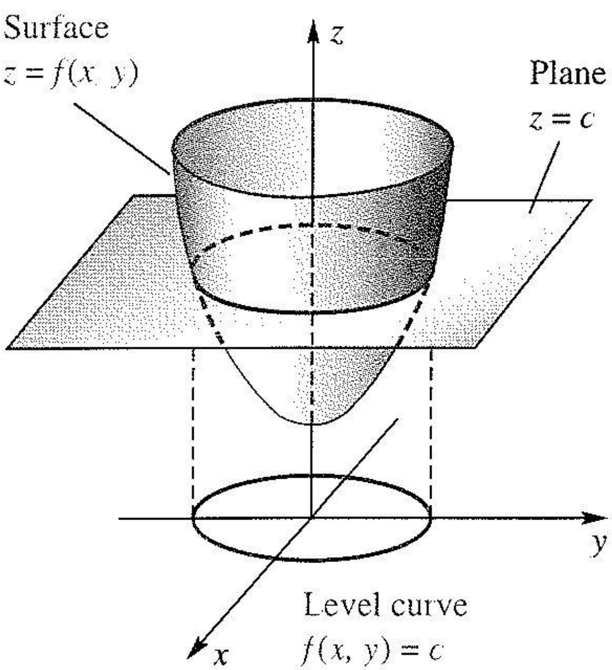
(2) $$g(x, y) = 2x\sqrt{y}$$

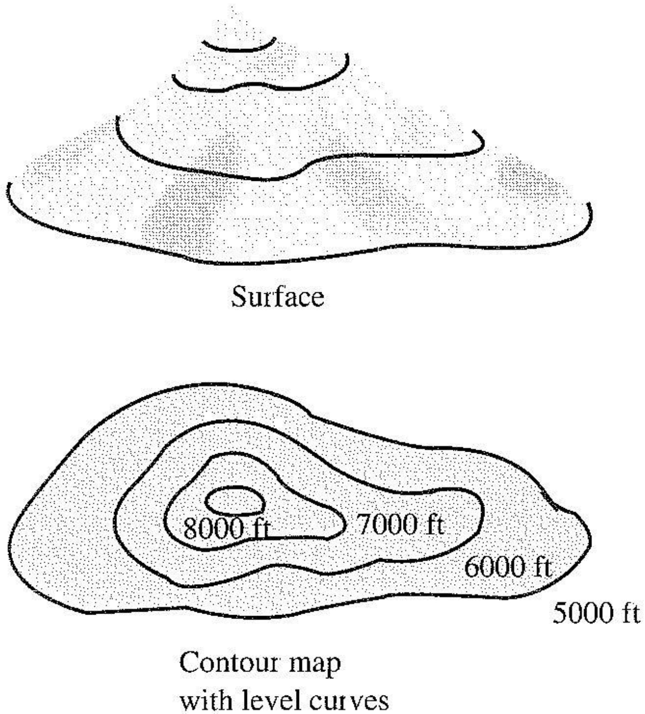
# Function of Two or More Variables

- The set $D$ is called the **domain** , the set of all points $(x,y)$ in the plane for which the function rule makes sense and gives a real number value.
- The **range** of a function is its set of values. If $z = f(x,y)$, we call x an y the **independent variables** and $z$ the **dependent variable**.

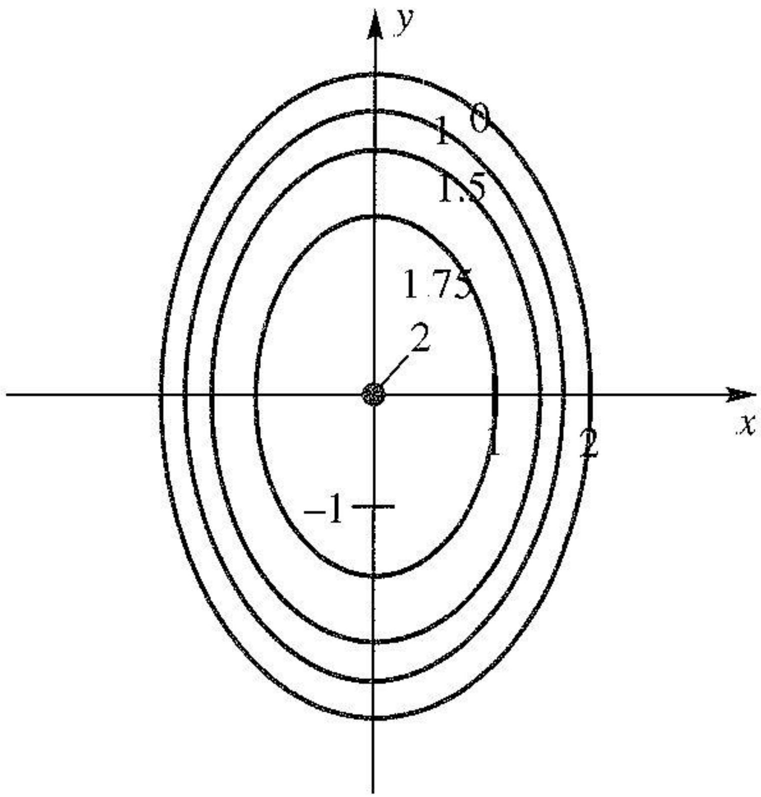
# Level Curves

- The projection of the curve on the *xy*-plane is called a **level curve**, and a collection of such curves is a **contour plot** or a **contour map**.

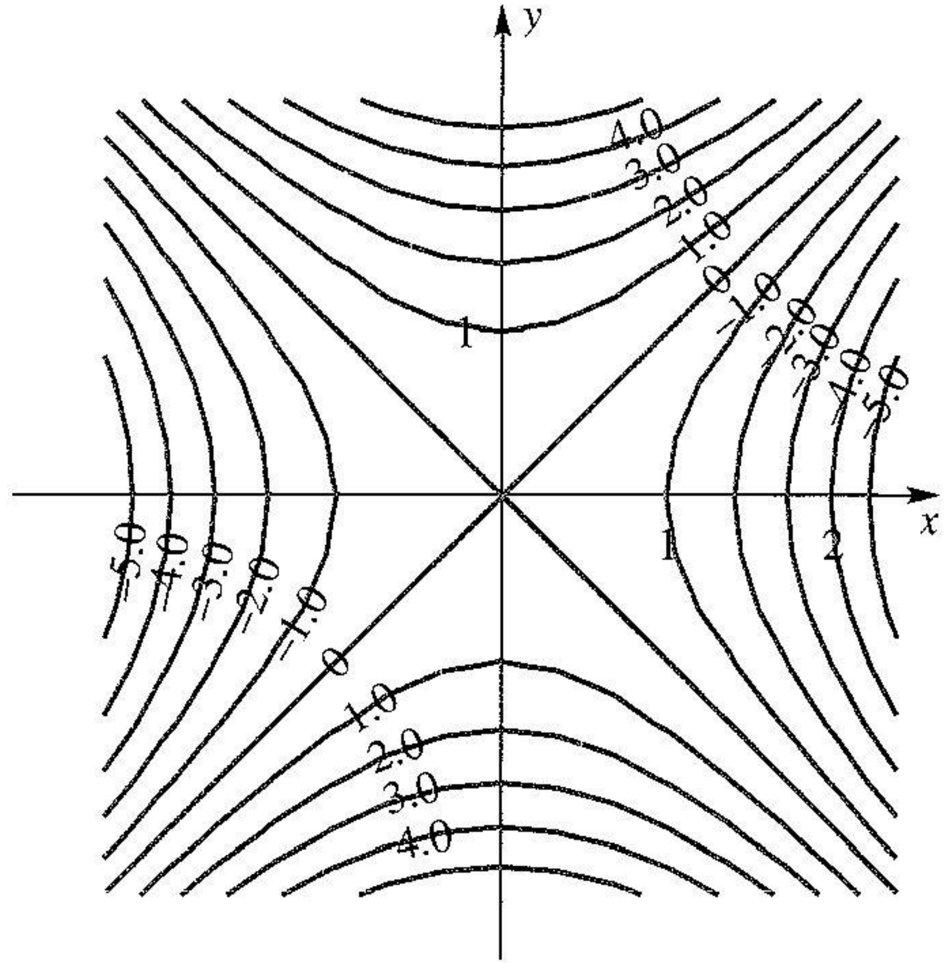


Contour Map $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$




Contour Map $z = y^2 - x^2$

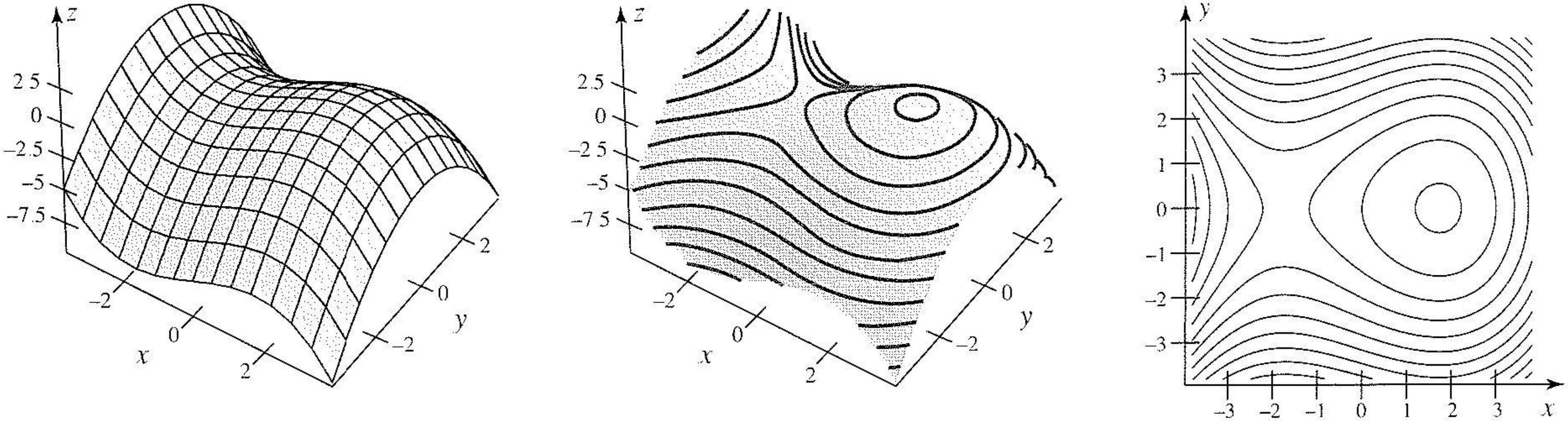


**EXAMPLE 4** Draw contour maps for the surfaces corresponding to $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$ and $z = y^2 - x^2$ (see Examples 2 and 3, and Figures 4 and 5).

**SOLUTION** The level curves of $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$ corresponding to $z = 0, 1, 1.5, 1.75, 2$ are shown in Figure 12. They are ellipses. Similarly, in Figure 13, we show the level curves of $z = y^2 - x^2$ for $z = -5, -4, -3, \ldots, 2, 3, 4$. These curves are hyperbolas unless $z = 0$. The level curve for $z = 0$ is a pair of intersecting lines. ■

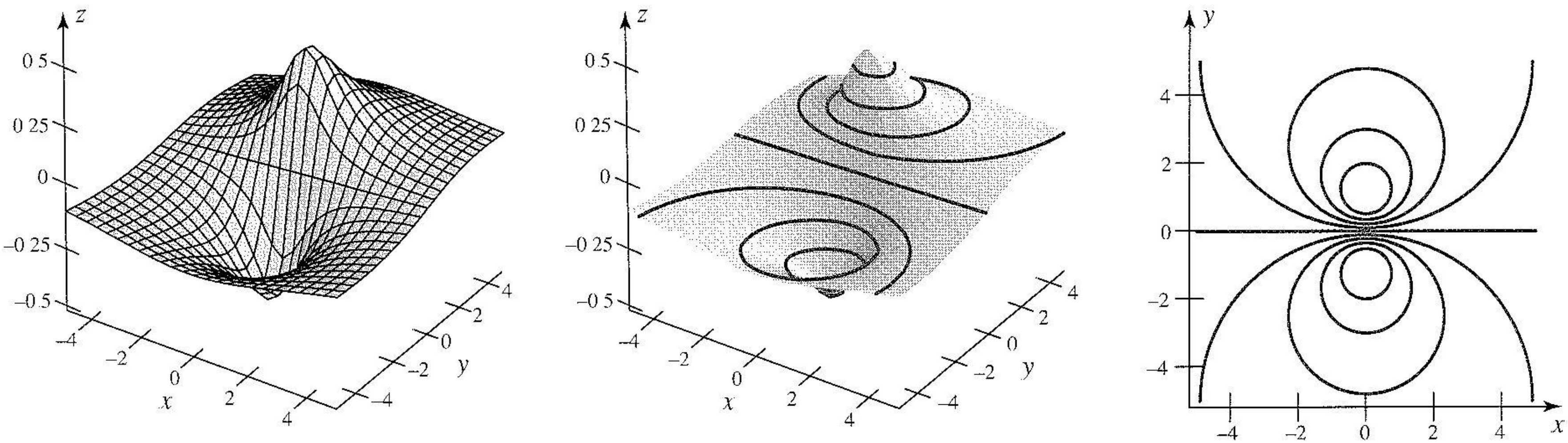
# Computer Graphs and Level Curves

- In Figures 15 through 19, we have drawn five more surfaces



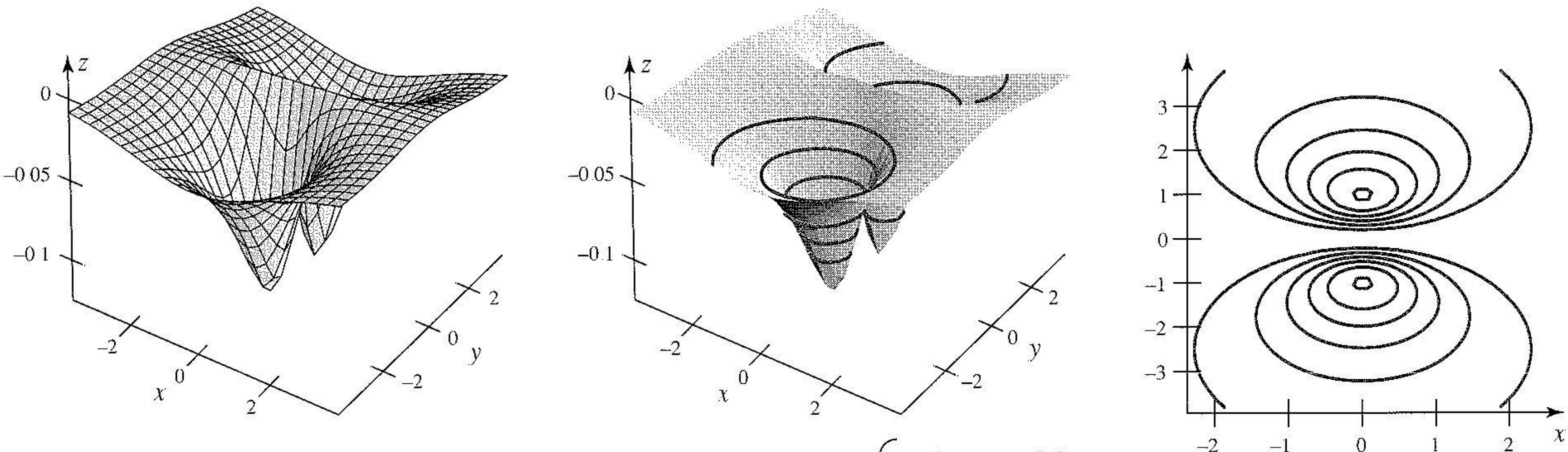

$$z = x - \left(\tfrac{1}{9}\right)x^3 - \left(\tfrac{1}{2}\right)y^2 \quad \begin{cases} -3.8 \le x \le 3.8 \\ -3.8 \le y \le 3.8 \end{cases}$$

Figure 15



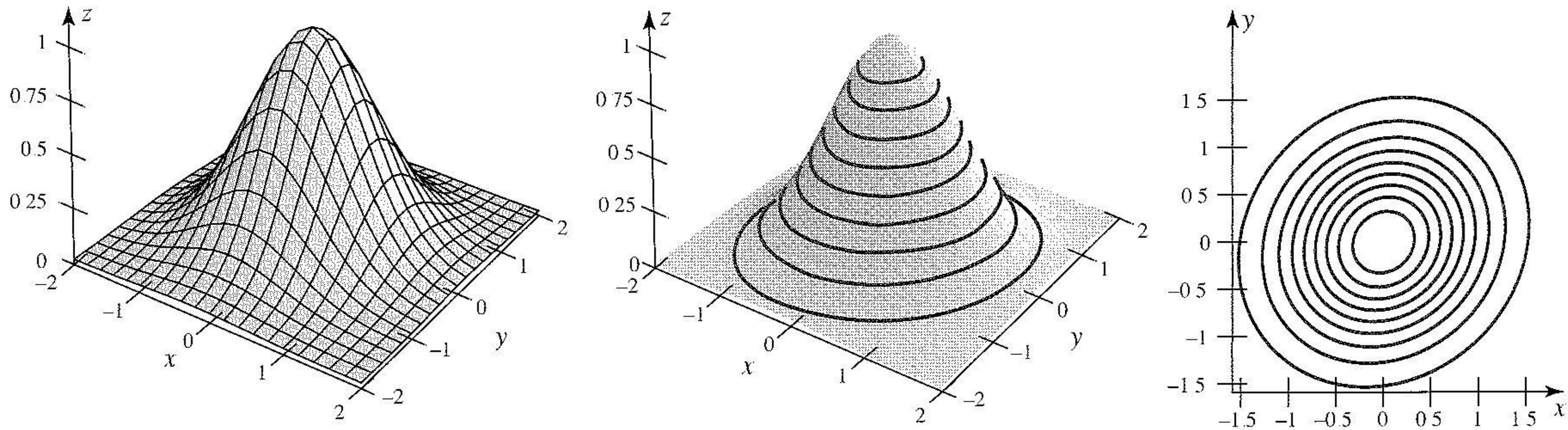

$$z = \frac{y}{(1 + x^2 + y^2)} \qquad \begin{cases} -5 \le x \le 5 \\ -5 \le y \le 5 \end{cases}$$

Figure 16



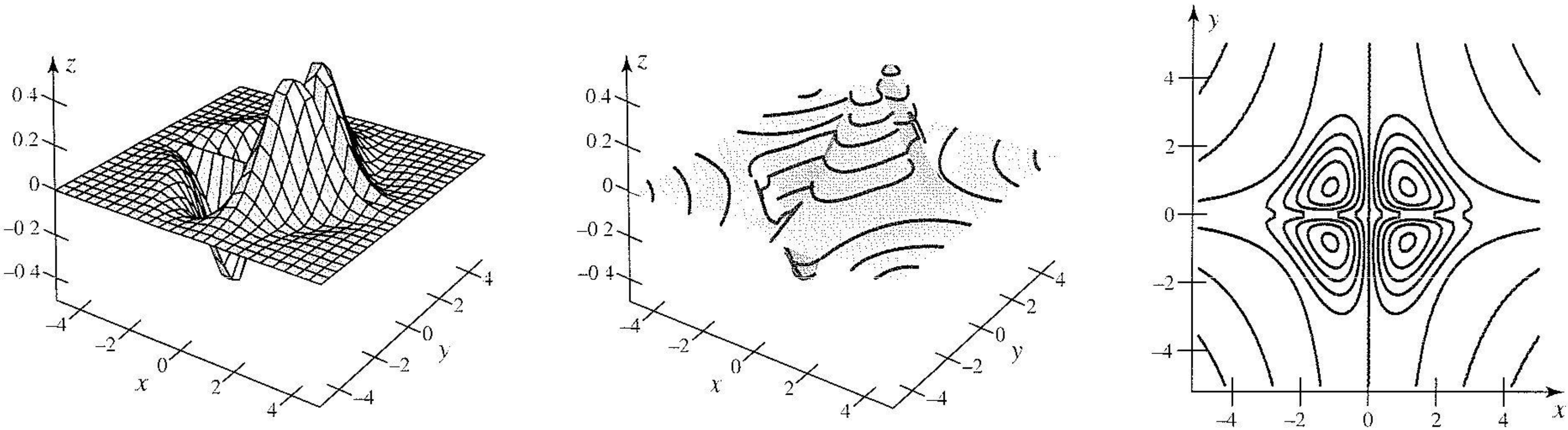

$$z = -1 + \cos\left(\frac{y}{1 + x^2 + y^2}\right) \quad \begin{cases} -3.8 \le x \le 3.8 \\ -3.8 \le y \le 3.8 \end{cases}$$

Figure 17




$$z = e^{-x^2 - y^2 + xy/4} \qquad \begin{cases} -2 \leq x \leq 2 \\ -2 \leq y \leq 2 \end{cases}$$

Figure 18




$$z = e^{-(x^2 + y^2)/4}\sin(x\sqrt{|y|}) \qquad \begin{cases} -5 \le x \le 5 \\ -5 \le y \le 5 \end{cases}$$

Figure 19

# Section 12.2

Partial Derivatives

# Partial Derivatives

- Suppose that $f$ is a function of two variables $x$ and $y$. If y is held constant, say $y = y_0$, then f($x$,$y_0$) is a function of the single variable $x$. Its derivative at $x = x_0$ is called the **partial derivative of *f* with respect to *x*** at ($x_0$, $y_0$)

$$\boxed{f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}}$$

Similarly, the partial derivative of $f$ with respect to $y$ at $(x_0, y_0)$ is denoted by $f_y(x_0, y_0)$ and is given by

$$\boxed{f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}}$$

**EXAMPLE 1** Find $f_x(1, 2)$ and $f_y(1, 2)$ if $f(x, y) = x^2y + 3y^3$.

**SOLUTION** To find $f_x(x, y)$, we treat $y$ as a constant and differentiate with respect to $x$, obtaining

$$f_x(x, y) = 2xy + 0$$

Thus,

$$f_x(1, 2) = 2 \cdot 1 \cdot 2 = 4$$

Similarly, we treat $x$ as a constant and differentiate with respect to $y$, obtaining

$$f_y(x, y) = x^2 + 9y^2$$

and so

$$f_y(1, 2) = 1^2 + 9 \cdot 2^2 = 37$$

# Partial Derivatives

The symbol $\partial$ is special to mathematics and is called the partial derivative sign. The symbols $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ represent linear operators, much like the linear operators $D_x$ and $\dfrac{d}{dx}$ that we encountered in Chapter 2.

**EXAMPLE 2** If $z = x^2 \sin(xy^2)$, find $\partial z/\partial x$ and $\partial z/\partial y$.

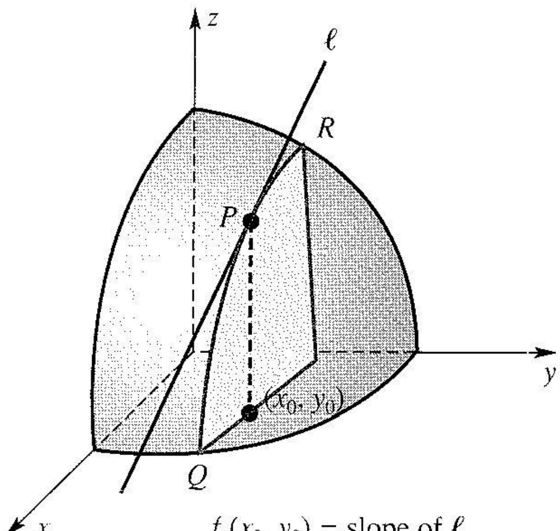
**SOLUTION**

$$\begin{aligned}
\frac{\partial z}{\partial x} &= x^2 \frac{\partial}{\partial x}[\sin(xy^2)] + \sin(xy^2)\frac{\partial}{\partial x}(x^2) \\
&= x^2 \cos(xy^2)\frac{\partial}{\partial x}(xy^2) + \sin(xy^2)\cdot 2x \\
&= x^2 \cos(xy^2)\cdot y^2 + 2x\sin(xy^2) \\
&= x^2y^2 \cos(xy^2) + 2x\sin(xy^2)
\end{aligned}$$

$$\frac{\partial z}{\partial y} = x^2\cos(xy^2)\cdot 2xy = 2x^3y\cos(xy^2)$$

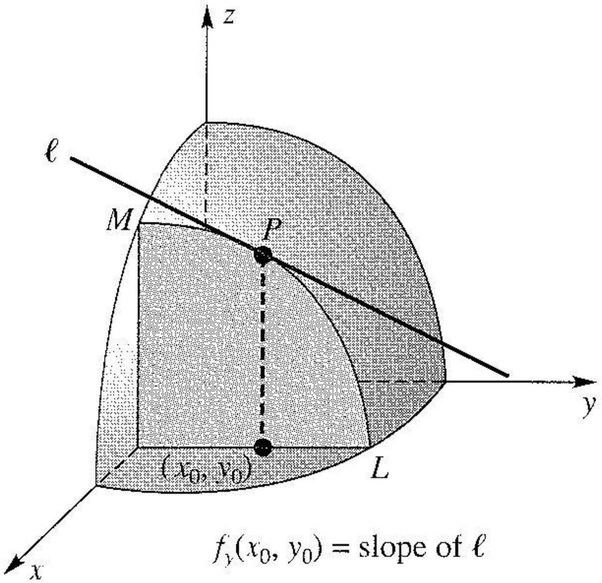
# Geometric an Physical Interpretations

**Geometric and Physical Interpretations** Consider the surface whose equation is $z = f(x, y)$. The plane $y = y_0$ intersects this surface in the plane curve $QPR$ (Figure 1), and the value of $f_x(x_0, y_0)$ is the slope of the tangent line to this curve at $P(x_0, y_0, f(x_0, y_0))$. Similarly, the plane $x = x_0$ intersects the surface in the plane curve $LPM$ (Figure 2), and $f_y(x_0, y_0)$ is the slope of the tangent line to this curve at $P$.




$f_x(x_0, y_0) =$ slope of $\ell$

Figure 1




$f_y(x_0, y_0) =$ slope of $\ell$

Figure 2

**EXAMPLE 3** The surface $z = f(x, y) = \sqrt{9 - 2x^2 - y^2}$ and the plane $y = 1$ intersect in a curve as in Figure 1. Find parametric equations for the tangent line at $(\sqrt{2}, 1, 2)$.

**SOLUTION**

$$f_x(x, y) = \tfrac{1}{2}(9 - 2x^2 - y^2)^{-1/2}(-4x)$$

and so $f_x(\sqrt{2}, 1) = -\sqrt{2}$. This number is the slope of the tangent line to the curve at $(\sqrt{2}, 1, 2)$; that is, $-\sqrt{2}/1$ is the ratio of rise to run along the tangent line. It follows that this line has direction vector $\langle 1, 0, -\sqrt{2} \rangle$ and, since it goes through $(\sqrt{2}, 1, 2)$,

$$x = \sqrt{2} + t, \qquad y = 1, \qquad z = 2 - \sqrt{2}t$$

provide the required parametric equations.

# Higher Partial Derivatives

**Higher Partial Derivatives** Since a partial derivative of a function of $x$ and $y$ is, in general, another function of these same two variables, it may be differentiated partially with respect to either $x$ or $y$, resulting in four **second partial derivatives** of $f$.

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

**EXAMPLE 5** Find the four second partial derivatives of

$$f(x, y) = xe^y - \sin(x/y) + x^3y^2$$

**SOLUTION**

$$f_x(x, y) = e^y - \frac{1}{y}\cos\left(\frac{x}{y}\right) + 3x^2y^2$$

$$f_y(x, y) = xe^y + \frac{x}{y^2}\cos\left(\frac{x}{y}\right) + 2x^3y$$

$$f_{xx}(x, y) = \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + 6xy^2$$

$$f_{yy}(x, y) = xe^y + \frac{x^2}{y^4}\sin\left(\frac{x}{y}\right) - \frac{2x}{y^3}\cos\left(\frac{x}{y}\right) + 2x^3$$

$$f_{xy}(x, y) = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\left(\frac{x}{y}\right) + 6x^2y$$

$$f_{yx}(x, y) = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\left(\frac{x}{y}\right) + 6x^2y$$

# Higher Partial Derivatives

- Partial derivatives of the third and higher orders are defined analogously, and the notation for them is similar.

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right] = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x} = f_{xyy}$$

# More Than Two Variables

**More Than Two Variables** Let $f$ be a function of three variables, $x$, $y$, and $z$. The **partial derivative of $f$ with respect to $x$** at $(x, y, z)$ is denoted by $f_x(x, y, z)$ or $\partial f(x, y, z)/\partial x$ and is defined by

$$f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

**EXAMPLE 6** If $f(x, y, z) = xy + 2yz + 3zx$, find $f_x$, $f_y$, and $f_z$.

**SOLUTION** To get $f_x$, we think of $y$ and $z$ as constants and differentiate with respect to the variable $x$. Thus,

$$f_x(x, y, z) = y + 3z$$

To find $f_y$, we treat $x$ and $z$ as constants and differentiate with respect to $y$:

$$f_y(x, y, z) = x + 2z$$

Similarly,

$$f_z(x, y, z) = 2y + 3x$$

# Section 12.3

Limits and Continuity

# Limits

**Definition** Limit of a Function of Two Variables

To say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that for each $\varepsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$, provided that $0 < \|(x, y) - (a, b)\| < \delta$.

Note several aspects of this definition.

1. The path of approach to $(a, b)$ is irrelevant. This means that if different paths of approach lead to different $L$-values then the limit does not exist.
2. The behavior of $f(x, y)$ at $(a, b)$ is irrelevant; the function does not even have to be defined at $(a, b)$. This follows from the restriction $0 < \|(x, y) - (a, b)\|$.

# Limits

**Theorem A**

If $f(x, y)$ is a polynomial, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

and if $f(x, y) = p(x, y)/q(x, y)$, where $p$ and $q$ are polynomials, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = \frac{p(a, b)}{q(a, b)}$$

provided $q(a, b) \neq 0$. Furthermore, if

$$\lim_{(x,y)\to(a,b)} p(x, y) = L \neq 0 \quad \text{and} \quad \lim_{(x,y)\to(a,b)} q(x, y) = 0$$

then

$$\lim_{(x,y)\to(a,b)} \frac{p(x, y)}{q(x, y)}$$

does not exist.

**EXAMPLE 1** Evaluate the following limits if they exist:

(a) $\lim_{(x,y)\to(1,2)} (x^2 y + 3y)$ and (b) $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + 1}{x^2 - y^2}$

**SOLUTION**

(a) The function whose limit we seek is a polynomial, so by Theorem A

$$\lim_{(x,y)\to(1,2)} (x^2 y + 3y) = 1^2 \cdot 2 + 3 \cdot 2 = 8$$

(b) The second function is a rational function, but the limit of the denominator is equal to 0, while the limit of the numerator is 1. Thus, by Theorem A, this limit does not exist.

**EXAMPLE 2** Show that the function $f$ defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

has no limit at the origin (Figure 3).

has no limit at the origin (Figure 3).

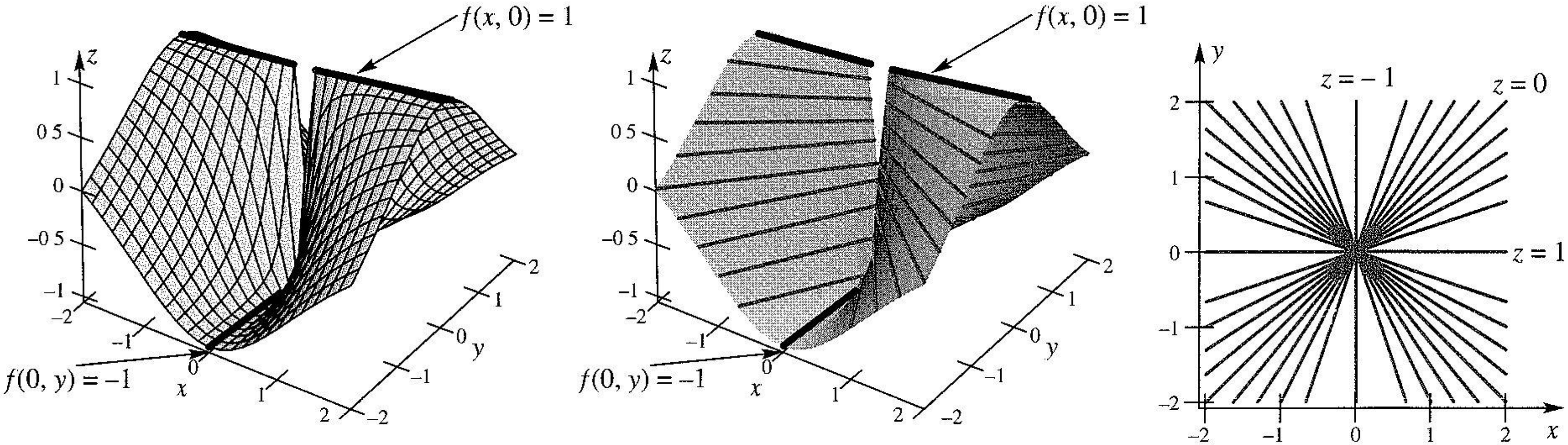



Figure 3

**SOLUTION** The function $f$ is defined everywhere in the $xy$-plane except at the origin. At all points on the $x$-axis different from the origin, the value of $f$ is

$$f(x, 0) = \frac{x^2 - 0}{x^2 + 0} = 1$$

Thus, the limit of $f(x, y)$ as $(x, y)$ approaches $(0, 0)$ along the $x$-axis is

$$\lim_{(x,0)\to(0,0)} f(x, 0) = \lim_{(x,0)\to(0,0)} \frac{x^2 - 0}{x^2 + 0} = +1$$

Similarly, the limit of $f(x, y)$ as $(x, y)$ approaches $(0, 0)$ along the $y$-axis is

$$\lim_{(0,y)\to(0,0)} f(0, y) = \lim_{(0,y)\to(0,0)} \frac{0 - y^2}{0 + y^2} = -1$$

Thus, we get different values depending on how $(x, y) \to (0, 0)$. In fact, there are points arbitrarily close to $(0, 0)$ at which the value of $f$ is 1 and other points equally close at which the value of $f$ is $-1$. Therefore, the limit cannot exist at $(0, 0)$. ▩

# Continuity at a Point

**Continuity at a Point** To say that $f(x, y)$ is **continuous** at the point $(a, b)$, we require the following: (1) $f$ has a value at $(a, b)$, (2) $f$ has a limit at $(a, b)$, and (3) the value of $f$ at $(a, b)$ is equal to the limit there. In summary, we require that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

**Theorem B** **Composition of Functions**

If a function $g$ of two variables is continuous at $(a, b)$ and a function $f$ of one variable is continuous at $g(a, b)$, then the composite function $f \circ g$, defined by $(f \circ g)(x, y) = f(g(x, y))$, is continuous at $(a, b)$.

**Theorem C** **Equality of Mixed Partials**

If $f_{xy}$ and $f_{yx}$ are continuous on an open set $S$, then $f_{xy} = f_{yx}$ at each point of $S$.

# Section 12.4

Differentiability

# Differentiability

- A function $f$ is **locally linear at** a if there is a constant $m$ such that

$$f(a+h) = f(a) + hm + h\varepsilon(h)$$

where $\varepsilon(h)$ is a function satisfying $\lim_{h \to 0} \varepsilon(h) = 0$

**Definition** Local Linearity for a Function of Two Variables

We say that $f$ is **locally linear** at $(a, b)$ if

$$f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) + h_1\varepsilon_1(h_1,h_2) + h_2\varepsilon_2(h_1,h_2)$$

where $\varepsilon_1(h_1, h_2) \to 0$ as $(h_1, h_2) \to 0$ and $\varepsilon_2(h_1, h_2) \to 0$ as $(h_1, h_2) \to 0$.

# Differentiability

**Definition** Differentiability for a Function of Two or More Variables

The function $f$ is **differentiable** at $\mathbf{p}$ if it is locally linear at $\mathbf{p}$. The function $f$ is differentiable on an open set $R$ if it is differentiable at every point in $R$.

The vector $(f_x(\mathbf{p}), f_y(\mathbf{p})) = f_x(\mathbf{p})\mathbf{i} + f_y(\mathbf{p})\mathbf{j}$ is denoted $\nabla f(\mathbf{p})$ and is called the **gradient** of $f$. Thus, $f$ is differentiable at $\mathbf{p}$ if and only if

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \nabla f(\mathbf{p}) \cdot \mathbf{h} + \varepsilon(\mathbf{h}) \cdot \mathbf{h}$$

where $\varepsilon(\mathbf{h}) \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$. The operator $\nabla$ is read "del" and is often called the **del operator.**

In the sense described above, *the gradient becomes the analog of the derivative*. We point out several aspects of our definitions.

1. The derivative $f'(x)$ is a number, whereas the gradient $\nabla f(\mathbf{p})$ is a vector.
2. The products $\nabla f(\mathbf{p}) \cdot \mathbf{h}$ and $\varepsilon(\mathbf{h}) \cdot \mathbf{h}$ are dot products.
3. The definitions of differentiability and gradient are easily extended to any number of dimensions.

# Differentiability

**Theorem A**

If $f(x, y)$ has continuous partial derivatives $f_x(x, y)$ and $f_y(x, y)$ on a disk $D$ whose interior contains $(a, b)$, then $f(x, y)$ is differentiable at $(a, b)$.

# Differentiability

If the function $f$ is differentiable at $\mathbf{p}_0$, then, when $\mathbf{h}$ has small magnitude

$$f(\mathbf{p}_0 + \mathbf{h}) \approx f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot \mathbf{h}$$

Letting $\mathbf{p} = \mathbf{p}_0 + \mathbf{h}$, we find that the function $T$ defined by

$$T(\mathbf{p}) = f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)$$

should be a good approximation to $f(\mathbf{p})$ if $\mathbf{p}$ is close to $\mathbf{p}_0$. The equation $z = T(\mathbf{p})$ defines a plane that approximates $f$ near $\mathbf{p}_0$. Naturally, this plane is called the **tangent plane.** See Figure 4.

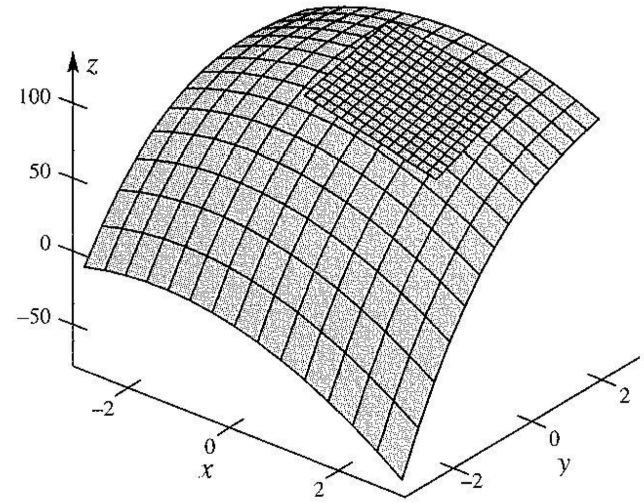



Figure 4

**EXAMPLE 1** Show that $f(x, y) = xe^y + x^2y$ is differentiable everywhere and calculate its gradient. Then find the equation of the tangent plane at $(2, 0)$.

**SOLUTION** We note first that

$$\frac{\partial f}{\partial x} = e^y + 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = xe^y + x^2$$

Both of these functions are continuous everywhere and so, by Theorem A, $f$ is differentiable everywhere. The gradient is

$$\nabla f(x, y) = (e^y + 2xy)\mathbf{i} + (xe^y + x^2)\mathbf{j} = \langle e^y + 2xy, xe^y + x^2 \rangle$$

Thus,

$$\nabla f(2, 0) = \mathbf{i} + 6\mathbf{j} = \langle 1, 6 \rangle$$

and the equation of the tangent plane is

$$\begin{aligned} z &= f(2, 0) + \nabla f(2, 0) \cdot \langle x - 2, y \rangle \\ &= 2 + \langle 1, 6 \rangle \cdot \langle x - 2, y \rangle \\ &= 2 + x - 2 + 6y = x + 6y \end{aligned}$$

# Rules for Gradients

**Theorem B** **Properties of $\nabla$**

The gradient operator $\nabla$ satisfies

1. $\nabla[f(\mathbf{p}) + g(\mathbf{p})] = \nabla f(\mathbf{p}) + \nabla g(\mathbf{p})$
2. $\nabla[\alpha f(\mathbf{p})] = \alpha \nabla f(\mathbf{p})$
3. $\nabla[f(\mathbf{p})g(\mathbf{p})] = f(\mathbf{p})\,\nabla g(\mathbf{p}) + g(\mathbf{p})\nabla f(\mathbf{p})$

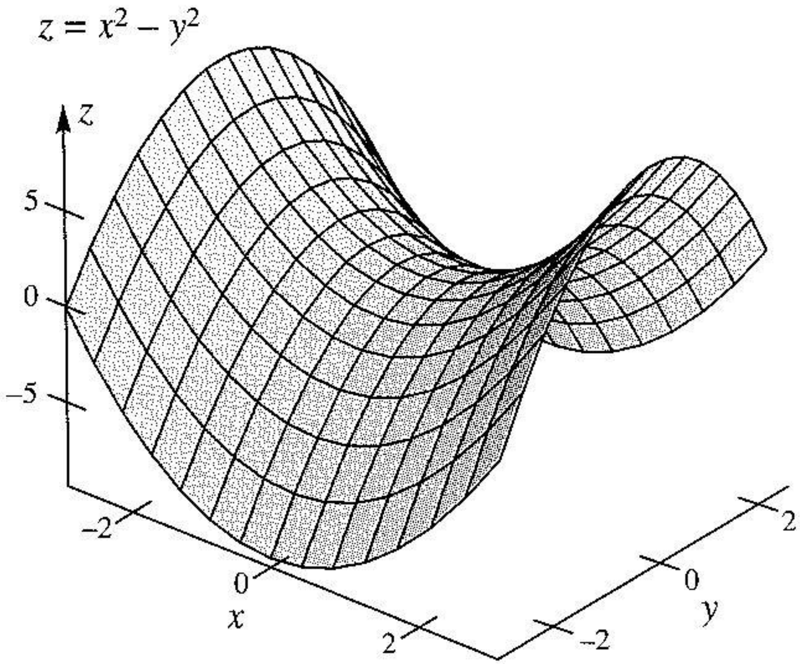
# Continuity versus Differentiability

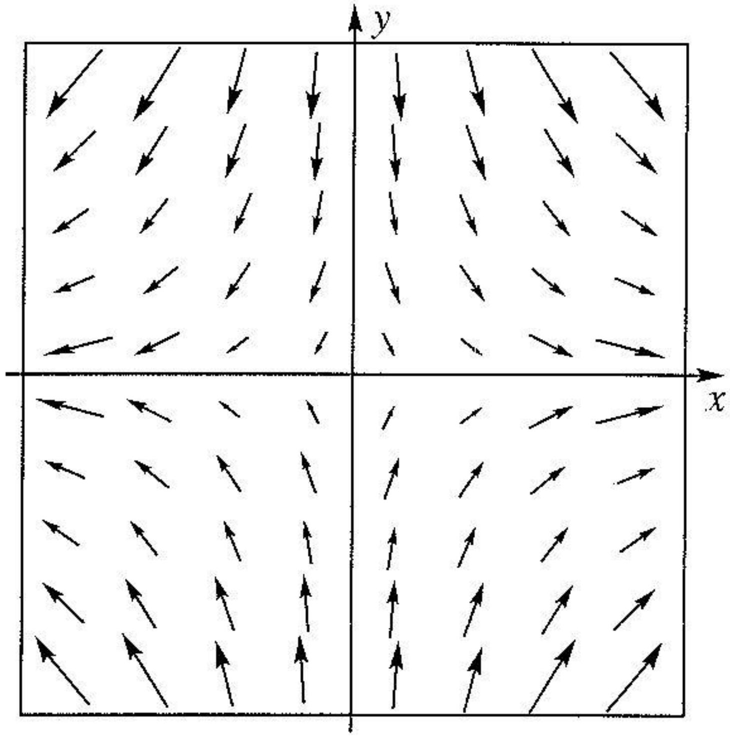
**Theorem C**

If $f$ is differentiable at $\mathbf{p}$, then $f$ is continuous at $\mathbf{p}$.

# The Gradient Field

- The gradient $\nabla f$ associates with each point **p** in the domain of $f$ a vector $\nabla f(p)$. The set of all these vectors is called the **gradient field** for $f$.



# Section 12.5

Directional Derivatives and Gradients

# Directional Derivative

**Definition**

For any unit vector $\mathbf{u}$, let

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})}{h}$$

This limit, if it exists, is called the **directional derivative** of $f$ at $\mathbf{p}$ in the direction $\mathbf{u}$.

Thus, $D_{\mathbf{i}} f(\mathbf{p}) = f_x(\mathbf{p})$ and $D_{\mathbf{j}} f(\mathbf{p}) = f_y(\mathbf{p})$. Since $\mathbf{p} = (x, y)$, we also use the notation $D_{\mathbf{u}} f(x, y)$. Figure 1 gives the geometric interpretation of $D_{\mathbf{u}} f(x_0, y_0)$. The vector $\mathbf{u}$ determines a line $L$ in the $xy$-plane through $(x_0, y_0)$. The plane through $L$ perpendicular to the $xy$-plane intersects the surface $z = f(x, y)$ in a curve $C$. Its tangent at the point $(x_0, y_0, f(x_0, y_0))$ has slope $D_{\mathbf{u}} f(x_0, y_0)$. Another useful interpretation is that $D_{\mathbf{u}} f(x_0, y_0)$ measures the rate of change of $f$ with respect to distance in the direction

$$f_y(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{j}) - f(\mathbf{p})}{h}$$

# Connection with the Gradient

**Theorem A**

Let $f$ be differentiable at $\mathbf{p}$. Then $f$ has a directional derivative at $\mathbf{p}$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p})$$

That is,

$$D_{\mathbf{u}}f(x, y) = u_1 f_x(x, y) + u_2 f_y(x, y)$$

**EXAMPLE 1** If $f(x, y) = 4x^2 - xy + 3y^2$, find the directional derivative of $f$ at $(2, -1)$ in the direction of the vector $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$.

**SOLUTION** The unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ is $\left(\frac{4}{5}\right)\mathbf{i} + \left(\frac{3}{5}\right)\mathbf{j}$. Also, $f_x(x, y) = 8x - y$ and $f_y(x, y) = -x + 6y$; thus, $f_x(2, -1) = 17$ and $f_y(2, -1) = -8$. Consequently, by Theorem A,

$$D_{\mathbf{u}}f(2, -1) = \left\langle \tfrac{4}{5}, \tfrac{3}{5} \right\rangle \cdot \left\langle 17, -8 \right\rangle = \tfrac{4}{5}(17) + \tfrac{3}{5}(-8) = \tfrac{44}{5}$$

**EXAMPLE 2** Find the directional derivative of the function $f(x, y, z) = xy \sin z$ at the point $(1, 2, \pi/2)$ in the direction of the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

**SOLUTION** The unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ is $\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Also, $f_x(x, y, z) = y \sin z$, $f_y(x, y, z) = x \sin z$, and $f_z(x, y, z) = xy \cos z$, and so $f_x(1, 2, \pi/2) = 2$, $f_y(1, 2, \pi/2) = 1$, and $f_z(1, 2, \pi/2) = 0$. We conclude that

$$D_{\mathbf{u}}f\left(1, 2, \frac{\pi}{2}\right) = \frac{1}{3}(2) + \frac{2}{3}(1) + \frac{2}{3}(0) = \frac{4}{3}$$

# Maximum Rate of Change

**Maximum Rate of Change** For a given function $f$ at a given point $\mathbf{p}$, it is natural to ask in what direction the function is changing most rapidly, that is, in what direction is $D_{\mathbf{u}}f(\mathbf{p})$ the largest? From the geometric formula for the dot product (Section 11.3), we may write

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p}) = \|\mathbf{u}\|\|\nabla f(\mathbf{p})\| \cos\theta = \|\nabla f(\mathbf{p})\| \cos\theta$$

where $\theta$ is the angle between $\mathbf{u}$ and $\nabla f(\mathbf{p})$. Thus, $D_{\mathbf{u}}f(\mathbf{p})$ is maximized when $\theta = 0$ and minimized when $\theta = \pi$. We summarize as follows.

**Theorem B**

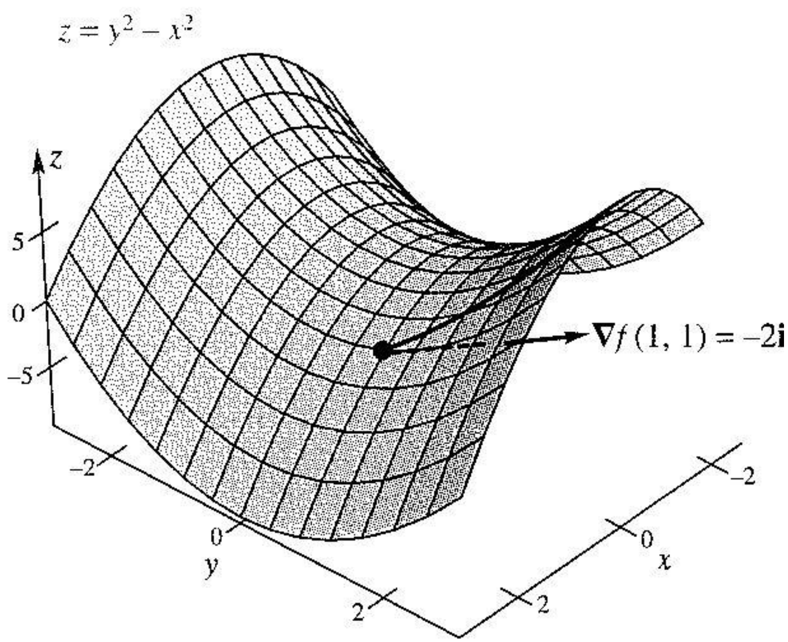
A function increases most rapidly at $\mathbf{p}$ in the direction of the gradient (with rate $\|\nabla f(\mathbf{p})\|$) and decreases most rapidly in the opposite direction (with rate $-\|\nabla f(\mathbf{p})\|$).

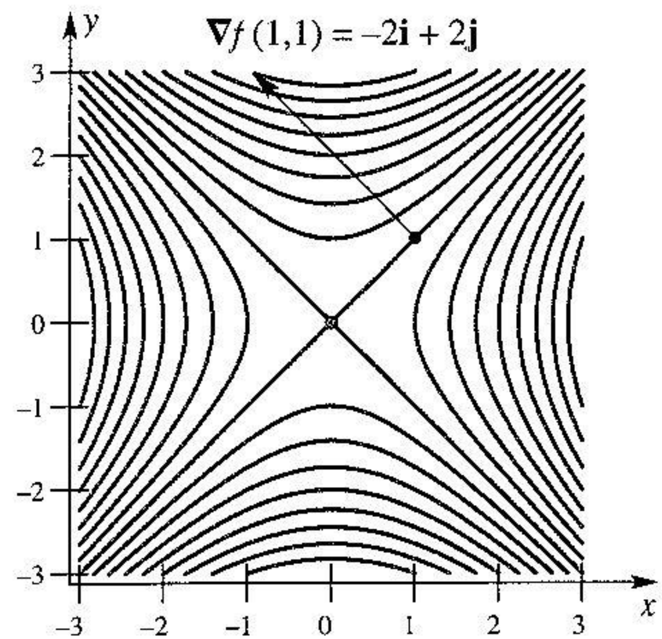
**EXAMPLE 3** Suppose that a bug is located on the hyperbolic paraboloid $z = y^2 - x^2$ at the point $(1, 1, 0)$, as in Figure 2. In what direction should it move for the steepest climb and what is the slope as it starts out?

**SOLUTION** Let $f(x, y) = y^2 - x^2$. Since $f_x(x, y) = -2x$ and $f_y(x, y) = 2y$,

$$\nabla f(1, 1) = f_x(1, 1)\mathbf{i} + f_y(1, 1)\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$$

Thus, the bug should move from $(1, 1, 0)$ in the direction $-2\mathbf{i} + 2\mathbf{j}$, where the slope will be $\|-2\mathbf{i} + 2\mathbf{j}\| = \sqrt{8} = 2\sqrt{2}$. ■



# Level Curves and Gradients

**Theorem C**

The gradient of $f$ at a point $P$ is perpendicular to the level curve of $f$ that goes through $P$.

**EXAMPLE 4** For the paraboloid $z = x^2/4 + y^2$, find the equation of its level curve that passes through the point $P(2, 1)$ and sketch it. Find the gradient vector of the paraboloid at $P$, and draw the gradient with its initial point at $P$.

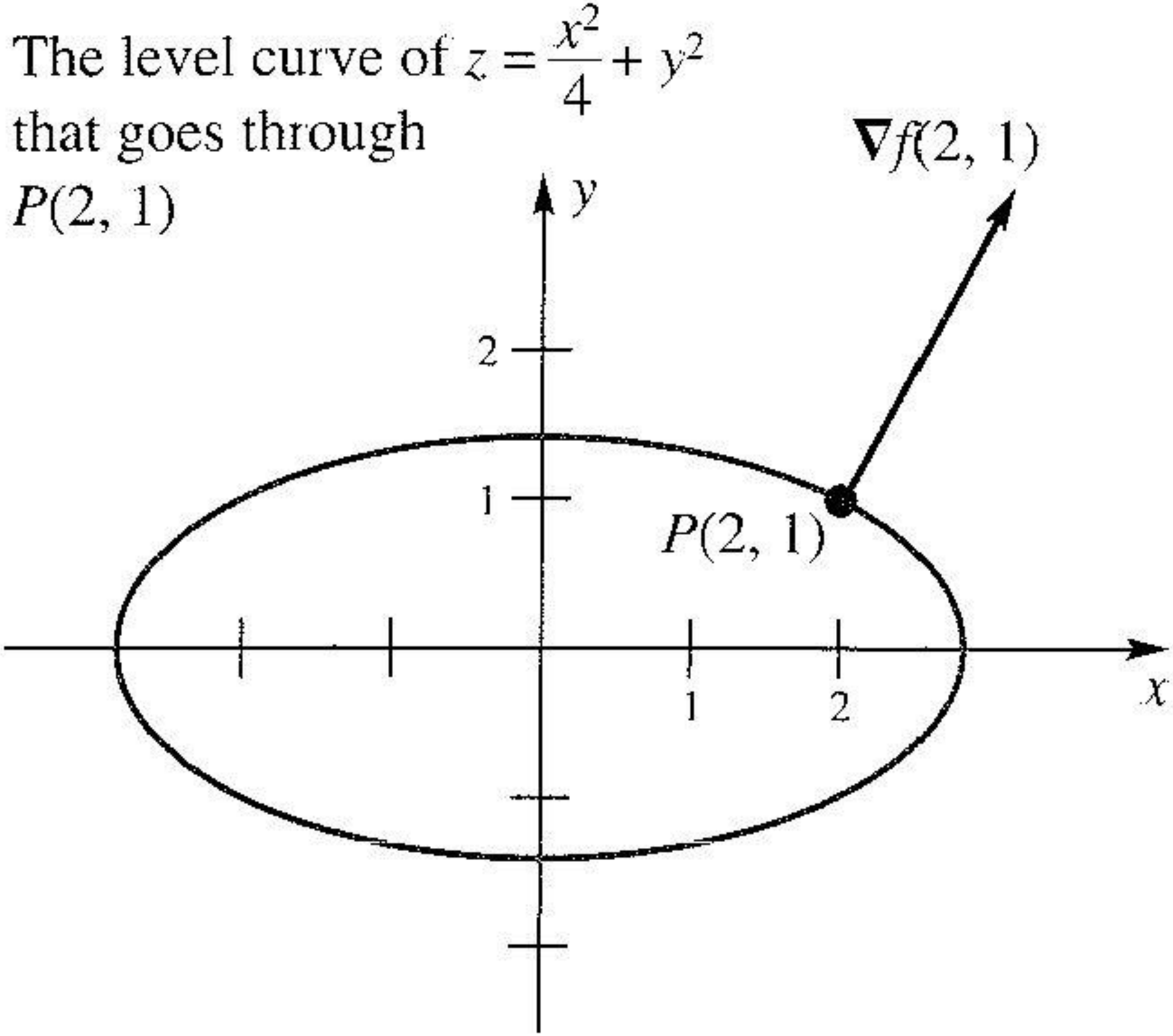
**SOLUTION** The level curve of the paraboloid that corresponds to the plane $z = k$ has the equation $x^2/4 + y^2 = k$. To find the value of $k$ belonging to the level curve through $P$, we substitute $(2, 1)$ for $(x, y)$ and obtain $k = 2$. Thus, the equation of the level curve that goes through $P$ is that of the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

Next let $f(x, y) = x^2/4 + y^2$. Since $f_x(x, y) = x/2$ and $f_y(x, y) = 2y$, the gradient of the paraboloid at $P(2, 1)$ is

$$\nabla f(2, 1) = f_x(2, 1)\mathbf{i} + f_y(2, 1)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

The level curve and the gradient at $P$ are shown in Figure 4.



The level curve of $z = \frac{x^2}{4} + y^2$
that goes through
$P(2, 1)$
$\nabla f(2, 1)$
$y$
2
1
$P(2, 1)$
1
2
$x$

# Section 12.6

The Chain Rule

# First Version

**First Version** If $z = f(x, y)$, where $x$ and $y$ are functions of $t$, then it makes sense to ask for $dz/dt$, and there ought to be a formula for it.

**Theorem A** **Chain Rule**

Let $x = x(t)$ and $y = y(t)$ be differentiable at $t$, and let $z = f(x, y)$ be differentiable at $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at $t$ and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

**EXAMPLE 1** Suppose that $z = x^3y$, where $x = 2t$ and $y = t^2$. Find $dz/dt$.

**SOLUTION**

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (3x^2y)(2) + (x^3)(2t) \\ &= 6(2t)^2(t^2) + 2(2t)^3(t) \\ &= 40t^4\end{aligned}$$

# Second Version

**Second Version** Suppose next that $z = f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$. Then it makes sense to ask for $\partial z/\partial s$ and $\partial z/\partial t$.

**Theorem B Chain Rule**

Let $x = x(s, t)$ and $y = y(s, t)$ have first partial derivatives at $(s, t)$, and let $z = f(x, y)$ be differentiable at $(x(s, t), y(s, t))$. Then $z = f(x(s, t), y(s, t))$ has first partial derivatives given by

1. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$;

2. $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$.

**EXAMPLE 4** If $z = 3x^2 - y^2$, where $x = 2s + 7t$ and $y = 5st$, find $\partial z/\partial t$ and express it in terms of $s$ and $t$.

**SOLUTION**

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (6x)(7) + (-2y)(5s) \\
&= 42(2s + 7t) - 10st(5s) \\
&= 84s + 294t - 50s^2t
\end{aligned}$$

Of course, if we substitute the expressions for $x$ and $y$ into the formula for $z$ and then take the partial derivative with respect to $t$, we get the same answer:

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial}{\partial t}[3(2s + 7t)^2 - (5st)^2] \\
&= \frac{\partial}{\partial t}[12s^2 + 84st + 147t^2 - 25s^2t^2] \\
&= 84s + 294t - 50s^2t
\end{aligned}$$

# Implicit Function

**Implicit Functions** Suppose that $F(x, y) = 0$ defines $y$ implicitly as a function of $x$, for example, $y = g(x)$, but that the function $g$ is difficult or impossible to determine. We can still find $dy/dx$. One method for doing this, implicit differentiation, was discussed in Section 2.7. Here is another method.

Let's differentiate both sides of $F(x, y) = 0$ with respect to $x$ using the Chain Rule. We obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for $dy/dx$ yields the formula

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$

**EXAMPLE 6** Find $dy/dx$ if $x^3 + x^2y - 10y^4 = 0$ using

(a) the Chain Rule, and (b) implicit differentiation.

**SOLUTION**

(a) Let $F(x, y) = x^3 + x^2y - 10y^4$. Then

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{3x^2 + 2xy}{x^2 - 40y^3}$$

(b) Differentiate both sides with respect to $x$ to obtain

$$3x^2 + x^2\frac{dy}{dx} + 2xy - 40y^3\frac{dy}{dx} = 0$$

Solving for $dy/dx$ gives the same result as we obtained with the Chain Rule.

If $z$ is an implicit function of $x$ and $y$ defined by the equation $F(x, y, z) = 0$, then differentiation of both sides with respect to $x$, holding $y$ fixed, yields

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If we solve for $\partial z/\partial x$ and note that $\partial y/\partial x = 0$, we get the first of the formulas below. A similar calculation holding $x$ fixed and differentiating with respect to $y$ produces the second formula.

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \qquad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}}$$

**EXAMPLE 7** If $F(x, y, z) = x^3 e^{y+z} - y\sin(x - z) = 0$ defines $z$ implicitly as a function of $x$ and $y$, find $\partial z/\partial x$.

**SOLUTION**

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{3x^2 e^{y+z} - y\cos(x - z)}{x^3 e^{y+z} + y\cos(x - z)}$$

# Section 12.7

Tangent Planes and Approximations

# Tangent Planes

**Definition**

Let $F(x, y, z) = k$ determine a surface, and suppose that $F$ is differentiable at a point $P(x_0, y_0, z_0)$ of this surface, with $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$. Then the plane through $P$ perpendicular to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane** to the surface at $P$.

**Theorem A** **Tangent Planes**

For the surface $F(x, y, z) = k$, the equation of the tangent plane at $(x_0, y_0, z_0)$ is $\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$; that is,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

In particular, for the surface $z = f(x, y)$, the equation of the tangent plane at $(x_0, y_0, f(x_0, y_0))$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**EXAMPLE 1** Find the equation of the tangent plane (Figure 3) to $z = x^2 + y^2$ at the point $(1, 1, 2)$.

**SOLUTION** Let $f(x, y) = x^2 + y^2$, and note that $\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$. Thus, $\nabla f(1, 1) = 2\mathbf{i} + 2\mathbf{j}$, and from Theorem A, the required equation is

$$z - 2 = 2(x - 1) + 2(y - 1)$$

or

$$2x + 2y - z = 2$$

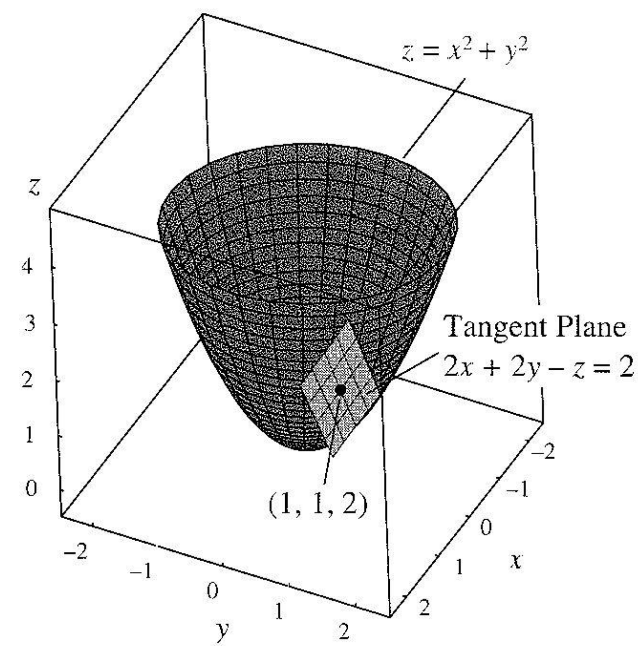


Figure 3

# Differentials and Approximations

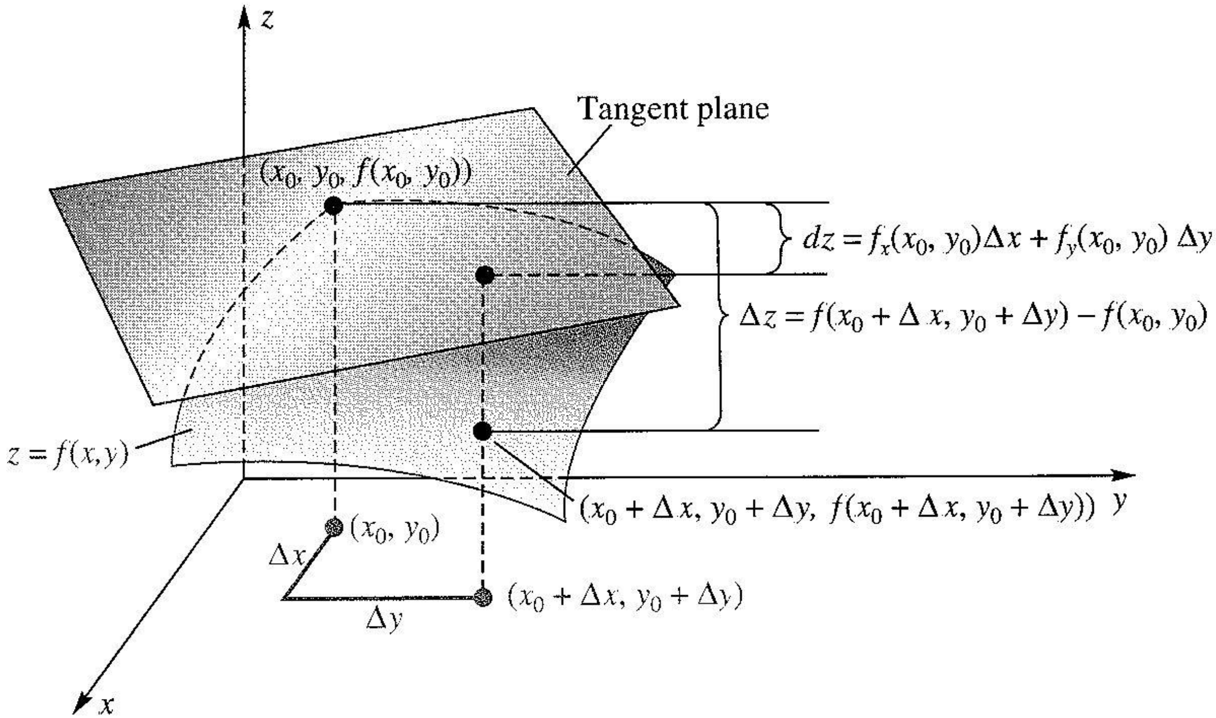
**Definition**

Let $z = f(x, y)$, where $f$ is a differentiable function, and let $dx$ and $dy$ (called the differentials of $x$ and $y$) be variables. The **differential of the dependent variable,** $dz$, also called the total **differential of** $f$ and written $df(x, y)$, is defined by

$$dz = df(x, y) = f_x(x, y)\,dx + f_y(x, y)\,dy = \nabla f \cdot \langle dx, dy \rangle$$

# Differentials and Approximations

The significance of $dz$ arises from the fact that if $dx = \Delta x$ and $dy = \Delta y$ represent small changes in $x$ and $y$, respectively, then $dz$ will be a good approximation to $\Delta z$, the corresponding change in $z$. This is illustrated in Figure 5 and, while $dz$ does not appear to be a very good approximation to $\Delta z$, you can see that it will get better and better as $\Delta x$ and $\Delta y$ get smaller.



**EXAMPLE 3** Let $z = f(x, y) = 2x^3 + xy - y^3$. Compute $\Delta z$ and $dz$ as $(x, y)$ changes from $(2, 1)$ to $(2.03, 0.98)$.

**SOLUTION**

$$\begin{aligned}
\Delta z &= f(2.03, 0.98) - f(2, 1) \\
&= 2(2.03)^3 + (2.03)(0.98) - (0.98)^3 - [2(2)^3 + 2(1) - 1^3] \\
&= 0.779062 \\
dz &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y \\
&= (6x^2 + y)\,\Delta x + (x - 3y^2)\,\Delta y
\end{aligned}$$

At $(2, 1)$ with $\Delta x = 0.03$ and $\Delta y = -0.02$,

$$dz = (25)(0.03) + (-1)(-0.02) = 0.77$$

# Section 12.8

Maxima and Minima

# Maxima and Minima

**Definition**

Let $f$ be a function with domain $S$, and let $\mathbf{p}_0$ be a point in $S$.

(i) $f(\mathbf{p}_0)$ is a **global maximum value** of $f$ on $S$ if $f(\mathbf{p}_0) \geq f(\mathbf{p})$ for all $\mathbf{p}$ in $S$.

(ii) $f(\mathbf{p}_0)$ is a **global minimum value** of $f$ on $S$ if $f(\mathbf{p}_0) \leq f(\mathbf{p})$ for all $\mathbf{p}$ in $S$.

(iii) $f(\mathbf{p}_0)$ is a **global extreme value** of $f$ on $S$ if $f(\mathbf{p}_0)$ is either a global maximum value or a global minimum value.

We obtain definitions for **local maximum value** and **local minimum value** if in (i) and (ii) we require only that the inequalities hold on $N \cap S$, where $N$ is some neighborhood of $\mathbf{p}_0$. $f(\mathbf{p}_0)$ is a **local extreme value** of $f$ on $S$ if $f(\mathbf{p}_0)$ is either a local maximum value or a local minimum value.

**Theorem A** **Max–Min Existence Theorem**

If $f$ is continuous on a closed bounded set $S$, then $f$ attains both a (global) maximum value and a (global) minimum value there.

# Where Do Extreme Values Occurs?

**Theorem B** **Critical Point Theorem**

Let $f$ be defined on a set $S$ containing $\mathbf{p}_0$. If $f(\mathbf{p}_0)$ is an extreme value, then $\mathbf{p}_0$ must be a critical point; that is, either $\mathbf{p}_0$ is

1. a boundary point of $S$; or
2. a stationary point of $f$; or
3. a singular point of $f$.

**EXAMPLE 1** Find the local maximum or minimum values of $f(x, y) = x^2 - 2x + y^2/4$.

**SOLUTION** The given function is differentiable throughout its domain, the $xy$-plane. Thus, the only possible critical points are the stationary points obtained by setting $f_x(x, y)$ and $f_y(x, y)$ equal to zero. But $f_x(x, y) = 2x - 2$ and $f_y(x, y) = y/2$ are zero only when $x = 1$ and $y = 0$. It remains to decide whether $(1, 0)$ gives a maximum or a minimum or neither. We will develop a simple tool for this soon, but for now we must use a little ingenuity. Note that $f(1, 0) = -1$ and

$$\begin{aligned} f(x, y) &= x^2 - 2x + \frac{y^2}{4} = x^2 - 2x + 1 + \frac{y^2}{4} - 1 \\ &= (x - 1)^2 + \frac{y^2}{4} - 1 \geq -1 \end{aligned}$$

Thus, $f(1, 0)$ is actually a global minimum for $f$. There are no local maximum values. ■

# Sufficient Conditions of Extrema

**Theorem C** **Second Partials Test**

Suppose that $f(x, y)$ has continuous second partial derivatives in a neighborhood of $(x_0, y_0)$ and that $\nabla f(x_0, y_0) = \mathbf{0}$. Let

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

Then

1. if $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum value;
2. if $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum value;
3. if $D < 0$, then $f(x_0, y_0)$ is not an extreme value ($(x_0, y_0)$ is a saddle point);
4. if $D = 0$, then the test is inconclusive.

**EXAMPLE 3** Find the extrema, if any, of the function $F$ defined by $F(x, y) = 3x^3 + y^2 - 9x + 4y$.

**SOLUTION** Since $F_x(x, y) = 9x^2 - 9$ and $F_y(x, y) = 2y + 4$, the critical points, obtained by solving the simultaneous equations $F_x(x, y) = F_y(x, y) = 0$, are $(1, -2)$ and $(-1, -2)$.

Now $F_{xx}(x, y) = 18x$, $F_{yy}(x, y) = 2$, and $F_{xy} = 0$. Thus, at the critical point $(1, -2)$,

$$D = F_{xx}(1, -2) \cdot F_{yy}(1, -2) - F_{xy}^2(1, -2) = 18(2) - 0 = 36 > 0$$

Furthermore, $F_{xx}(1, -2) = 18 > 0$ and so, by Theorem C(2), $F(1, -2) = -10$ is a local minimum value of $F$.

In testing the given function at the other critical point, $(-1, -2)$, we find that $F_{xx}(-1, -2) = -18$, $F_{yy}(-1, -2) = 2$, and $F_{xy}(-1, -2) = 0$, which makes $D = -36 < 0$. Thus, by Theorem C(3), $(-1, -2)$ is a saddle point and $F(-1, -2)$ is not an extremum.

**EXAMPLE 4** Find the minimum distance between the origin and the surface $z^2 = x^2y + 4$.

**SOLUTION** Let $P(x, y, z)$ be any point on the surface. The square of the distance between the origin and $P$ is $d^2 = x^2 + y^2 + z^2$. We seek the coordinates of $P$ that make $d^2$ (and hence $d$) a minimum.

Since $P$ is on the surface, its coordinates satisfy the equation of the surface. Substituting $z^2 = x^2y + 4$ in $d^2 = x^2 + y^2 + z^2$, we obtain $d^2$ as a function of two variables $x$ and $y$.

$$d^2 = f(x, y) = x^2 + y^2 + x^2y + 4$$

To find the critical points, we set $f_x(x, y) = 0$ and $f_y(x, y) = 0$, obtaining

$$2x + 2xy = 0 \quad \text{and} \quad 2y + x^2 = 0$$

By eliminating $y$ between these equations, we get

$$2x - x^3 = 0$$

Thus, $x = 0$ or $x = \pm\sqrt{2}$. Substituting these values in the second of the equations, we obtain $y = 0$ and $y = -1$. Therefore, the critical points are $(0, 0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$. (There are no boundary points.)

To test each of these, we need $f_{xx}(x, y) = 2 + 2y$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = 2x$, and

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4 + 4y - 4x^2$$

Since $D(\pm\sqrt{2}, -1) = -8 < 0$, neither $(\sqrt{2}, -1)$ nor $(-\sqrt{2}, -1)$ yields an extremum. However, $D(0, 0) = 4 > 0$ and $f_{xx}(0, 0) = 2 > 0$; so $(0, 0)$ yields the minimum distance. Substituting $x = 0$ and $y = 0$ in the expression for $d^2$, we find $d^2 = 4$.

The minimum distance between the origin and the given surface is 2. ▒

# Section 12.9

The Method of Lagrange Multipliers

# Geometric Interpretation of the Method

**Theorem A** **Lagrange's Method**

To maximize or minimize $f(\mathbf{p})$ subject to the constraint $g(\mathbf{p}) = 0$, solve the system of equations

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}) \quad \text{and} \quad g(\mathbf{p}) = 0$$

for $\mathbf{p}$ and $\lambda$. Each such point $\mathbf{p}$ is a critical point for the constrained extremum problem, and the corresponding $\lambda$ is called a Lagrange multiplier.

**EXAMPLE 1** What is the greatest area that a rectangle can have if the length of its diagonal is 2?

**SOLUTION** Place the rectangle in the first quadrant with two of its sides along the coordinate axes; then the vertex opposite the origin has coordinates $(x, y)$, with $x$ and $y$ positive (Figure 3). The length of its diagonal is $\sqrt{x^2 + y^2} = 2$, and its area is $xy$.

Thus, we may formulate the problem to be that of maximizing $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 4 = 0$. The corresponding gradients are

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = y\,\mathbf{i} + x\,\mathbf{j}$$

$$\nabla g(x, y) = g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

$$\nabla g(x, y) = g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

Lagrange's equations thus become

(1) $$y = \lambda(2x)$$

(2) $$x = \lambda(2y)$$

(3) $$x^2 + y^2 = 4$$

which we must solve simultaneously. If we multiply the first equation by $y$ and the second by $x$, we get $y^2 = 2\lambda xy$ and $x^2 = 2\lambda xy$, from which

(4) $$y^2 = x^2$$

From (3) and (4), we find that $x = \sqrt{2}$ and $y = \sqrt{2}$; and by substituting these values in (1), we obtain $\lambda = \frac{1}{2}$. Thus, the solution to equations (1) through (3), keeping $x$ and $y$ positive, is $x = \sqrt{2}$, $y = \sqrt{2}$, and $\lambda = \frac{1}{2}$.

We conclude that the rectangle of greatest area with diagonal 2 is the square having sides of length $\sqrt{2}$. Its area is 2. A geometric interpretation of this problem is shown in Figure 4.

**EXAMPLE 3** Find the minimum of $f(x, y, z) = 3x + 2y + z + 5$ subject to the constraint $g(x, y, z) = 9x^2 + 4y^2 - z = 0$.

**SOLUTION** The gradients of $f$ and $g$ are $\nabla f(x, y, z) = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g(x, y, z) = 18x\mathbf{i} + 8y\mathbf{j} - \mathbf{k}$. To find the critical points, we solve the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0$$

for $(x, y, z, \lambda)$, in which $\lambda$ is a Lagrange multiplier. This is equivalent, in the present problem, to solving the following system of four simultaneous equations in the four variables $x$, $y$, $z$, and $\lambda$.

(1) $$3 = 18x\lambda$$

(2) $$2 = 8y\lambda$$

(3) $$1 = -\lambda$$

(4) $$9x^2 + 4y^2 - z = 0$$

From (3), $\lambda = -1$. Substituting this result in equations (1) and (2), we get $x = -\frac{1}{6}$ and $y = -\frac{1}{4}$. By putting these values for $x$ and $y$ in equation (4), we obtain $z = \frac{1}{2}$. Thus, the solution of the foregoing system of four simultaneous equations is $\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}, -1\right)$, and the only critical point is $\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right)$. Therefore, the minimum of $f(x, y, z)$, subject to the constraint $g(x, y, z) = 0$, is $f\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right) = \frac{9}{2}$. (How do we know that this value is a minimum rather than a maximum?) ■

# End of Chapter 12