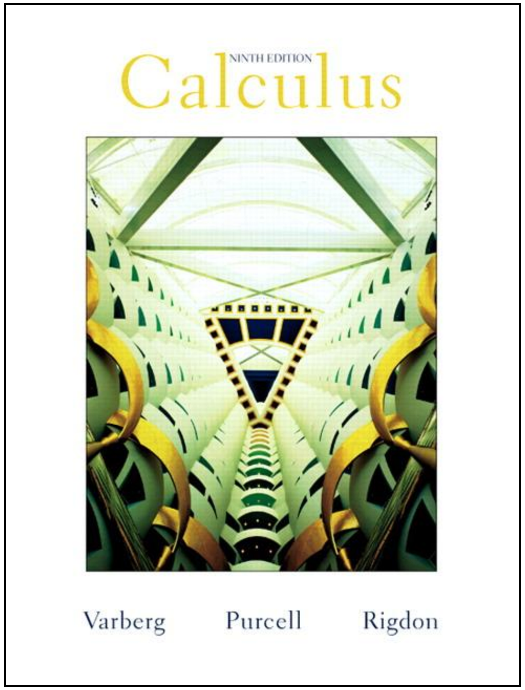
Varberg, Calculus 9e



Chapter 13

Multiple Integrals

Section 13.1

Double Integrals over Rectangles

Double Integral

Definition The Double Integral

Let f be a function of two variables that is defined on a closed rectangle R. If

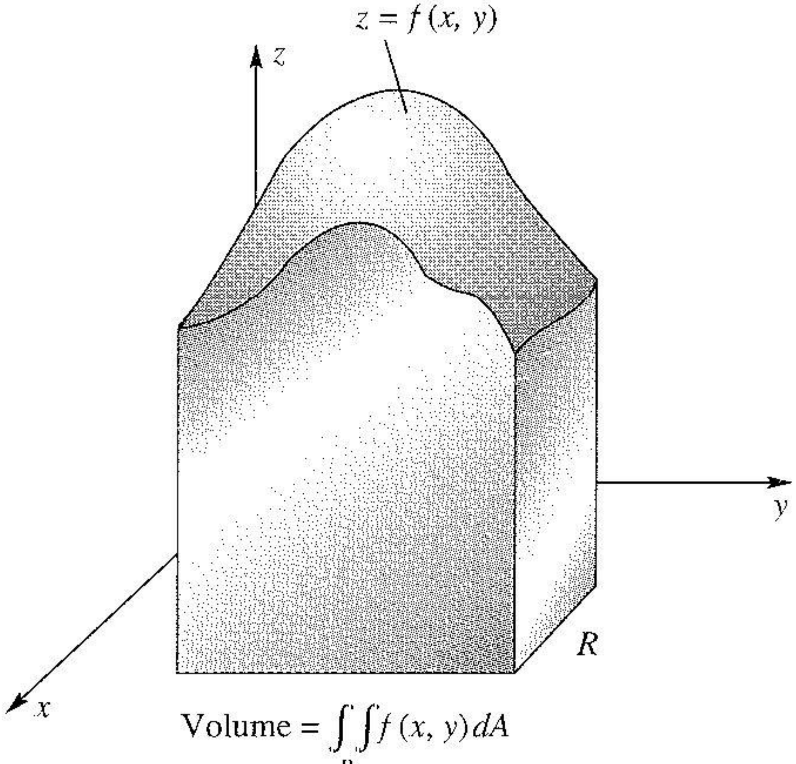
$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k)\, \Delta A_k$$

exists, we say that f is integrable on R. Moreover, $\iint_R f(x, y)\, dA$, called the **double integral** of f over R, is then given by

$$\iint_R f(x, y)\, dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k)\, \Delta A_k$$

Double Integral

- If $f(x, y) \geq 0$, $\iint_R f(x, y)dA$ represents the volume of the solid under $z = f(x,y)$ and above the rectangle R.



The Existence Question

Theorem A **Integrability Theorem**

If f is bounded on the closed rectangle R and if it is continuous there except on a finite number of smooth curves, then f is integrable on R. In particular, if f is continuous on all of R, then f is integrable there.

Properties of Double Integral

1. The double integral is linear; that is,

   a. $$\iint_R kf(x, y)\,dA = k\iint_R f(x, y)\,dA;$$

   b. $$\iint_R [f(x, y) + g(x, y)]\,dA = \iint_R f(x, y)\,dA + \iint_R g(x, y)\,dA.$$

2. The double integral is additive on rectangles (Figure 6) that overlap only on a line segment.

$$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$$

3. The comparison property holds. If $f(x, y) \le g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y)\,dA \le \iint_R g(x, y)\,dA$$

Section 13.2

Iterated Integrals

Iterated Integrals

Now we face in earnest the problem of evaluating $\iint_R f(x, y)\, dA$, where R is the rectangle

$$R = \{(x, y): a \le x \le b, c \le y \le d\}$$

Suppose for the time being that $f(x, y) \ge 0$ on R so that we may interpret the double integral as the volume V of the solid under the surface of Figure 1.

$$V = \iint_R f(x, y)\, dA \tag{1}$$

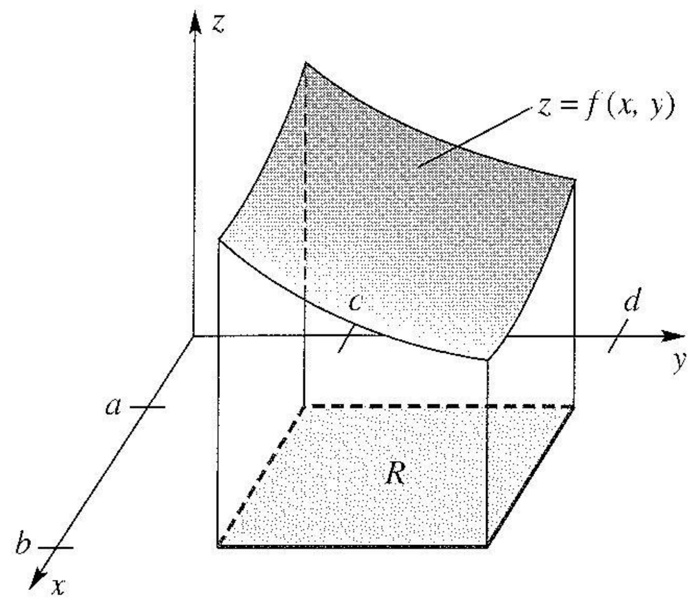
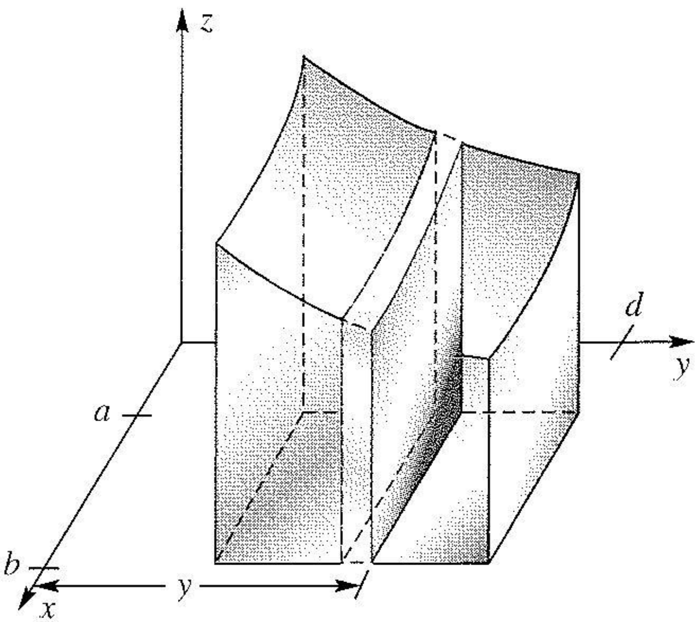


Figure 1

Iterated Integrals

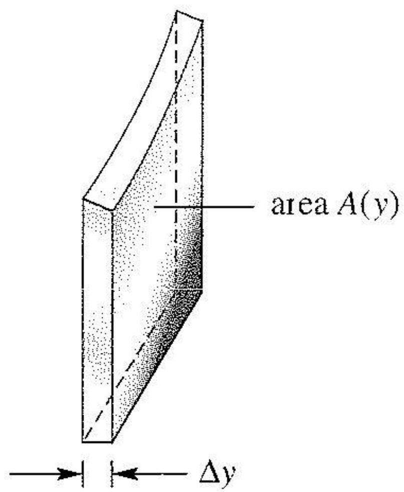
There is another way to calculate the volume of this solid, which at least intuitively seems just as valid. Slice the solid into thin slabs by means of planes parallel to the xz-plane. A typical such slab is shown in Figure 2a. The area of the face of this slab depends on how far it is from the xz-plane; that is, it depends on y. Therefore, we denote this area by $A(y)$ (see Figure 2b).

The volume ΔV of the slab is given approximately by



Slice by planes y = constant

(a)



The corresponding slab of volume $\approx A(y)\,\Delta y$

(b)

Figure 2

and, recalling our old motto (*slice, approximate, integrate*), we may write

$$V = \int_c^d A(y)\,dy$$

On the other hand, for fixed y we may calculate $A(y)$ by means of an ordinary single integral; in fact,

$$A(y) = \int_a^b f(x, y)\,dx$$

Thus, we have a solid whose cross sectional areas are known to be $A(y)$. The problem of finding the volume of a region whose cross sections are known was treated in Section 5.2. We conclude that

$$V = \int_c^d A(y)\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy \tag{2}$$

The last expression is called an **iterated integral.**

When we equate the expressions for V from (1) and (2), we obtain the result that we want.

$$\boxed{\iint\limits_R f(x, y)\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy}$$

If we had begun the process above by slicing the solid with planes parallel to the yz-plane, we would have obtained another iterated integral, with the integrations occurring in the opposite order.

$$\boxed{\iint\limits_R f(x, y)\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx}$$

EXAMPLE 1 Evaluate $\int_0^3 \left[\int_1^2 (2x + 3y)\, dx \right] dy$.

SOLUTION In the inner integration, y is a constant, so

$$\int_1^2 (2x + 3y)\, dx = \left[x^2 + 3yx \right]_1^2 = 4 + 6y - (1 + 3y) = 3 + 3y$$

Consequently,

$$\int_0^3 \left[\int_1^2 (2x + 3y)\, dx \right] dy = \int_0^3 [3 + 3y]\, dy = \left[3y + \frac{3}{2} y^2 \right]_0^3$$

$$= 9 + \frac{27}{2} = \frac{45}{2}$$

EXAMPLE 2 Evaluate $\int_1^2 \left[\int_0^3 (2x + 3y)\, dy \right] dx$.

SOLUTION Note that we have simply reversed the order of integration from Example 1; we expect the same answer as in that example.

$$\int_0^3 (2x + 3y)\, dy = \left[2xy + \frac{3}{2} y^2 \right]_0^3$$
$$= 6x + \frac{27}{2}$$

Thus,

$$\int_1^2 \left[\int_0^3 (2x + 3y)\, dy \right] dx = \int_1^2 \left[6x + \frac{27}{2} \right] dx = \left[3x^2 + \frac{27}{2} x \right]_1^2$$
$$= 12 + 27 - \left(3 + \frac{27}{2} \right) = \frac{45}{2}$$

From now on, we shall usually omit the brackets in the iterated integral.

Section 13.3

Double Integrals over Nonrectangular Regions

Evaluation of Double Integrals over General Sets

- A set S is ***y*-simple** if there are functions Φ_1 and Φ_2 on $[a,b]$ such that

$$S = \{(x, y): \phi_1(x) \leq y \leq \phi_2(x), a \leq x \leq b\}$$

- A set S is ***x*-simple** if there are functions Ψ_1 (Ψ is the Greek letter psi) and Ψ_2 on $[a,b]$ such that

$$S = \{(x, y): \psi_1(y) \leq x \leq \psi_2(y), c \leq y \leq d\}$$

Evaluation of Double Integrals over General Sets

$$\iint_S f(x, y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx$$

$$\iint_S f(x, y)\, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy$$

EXAMPLE 1 Evaluate the iterated integral

$$\int_3^5 \int_{-x}^{x^2} (4x + 10y)\, dy\, dx$$

SOLUTION We first perform the inner integration with respect to y, temporarily thinking of x as constant (see Figure 9), and obtain

$$\int_3^5 \int_{-x}^{x^2} (4x + 10y)\, dy\, dx = \int_3^5 \left[4xy + 5y^2\right]_{-x}^{x^2} dx$$

$$= \int_3^5 \left[(4x^3 + 5x^4) - (-4x^2 + 5x^2)\right] dx$$

$$= \int_3^5 (5x^4 + 4x^3 - x^2)\, dx = \left[x^5 + x^4 - \frac{x^3}{3}\right]_3^5$$

$$= \frac{10{,}180}{3} = 3393\frac{1}{3}$$

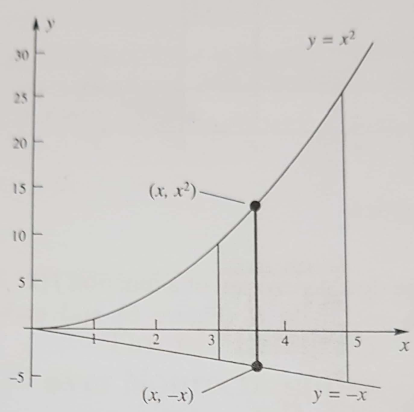


Figure 9

EXAMPLE 2 Evaluate the iterated integral

$$\int_0^1 \int_0^{y^2} 2ye^x \, dx \, dy$$

SOLUTION The region of integration is shown in Figure 10

$$\int_0^1 \int_0^{y^2} 2ye^x \, dx \, dy = \int_0^1 \left[\int_0^{y^2} 2ye^x \, dx \right] dy$$

$$= \int_0^1 \left[2ye^x \right]_0^{y^2} dy = \int_0^1 \left(2ye^{y^2} - 2ye^0 \right) dy$$

$$= \int_0^1 e^{y^2} (2y \, dy) - 2 \int_0^1 y \, dy$$

$$= \left[e^{y^2} \right]_0^1 - 2 \left[\frac{y^2}{2} \right]_0^1 = e - 1 - 2\left(\frac{1}{2} \right) = e - 2$$

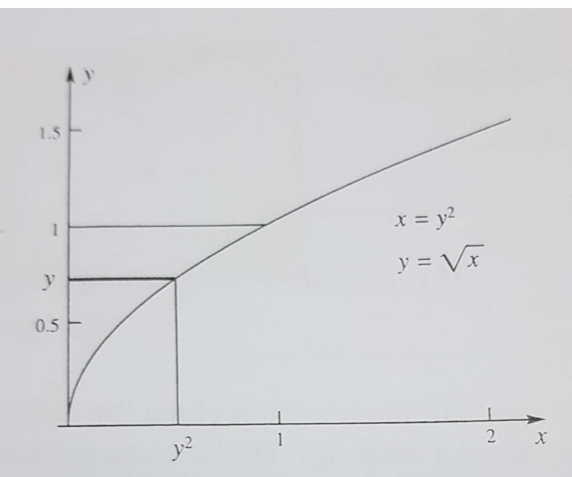


Figure 10

Section 13.4

Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

Let R have the shape shown in Figure 1, which we call a *polar rectangle* and will describe analytically in a moment. Let $z = f(x, y)$ determine a surface over R and suppose that f is continuous and nonnegative. Then the volume V of the solid under this surface and above R (Figure 2) is given by

(1) $$V = \iint_R f(x, y)\, dA$$

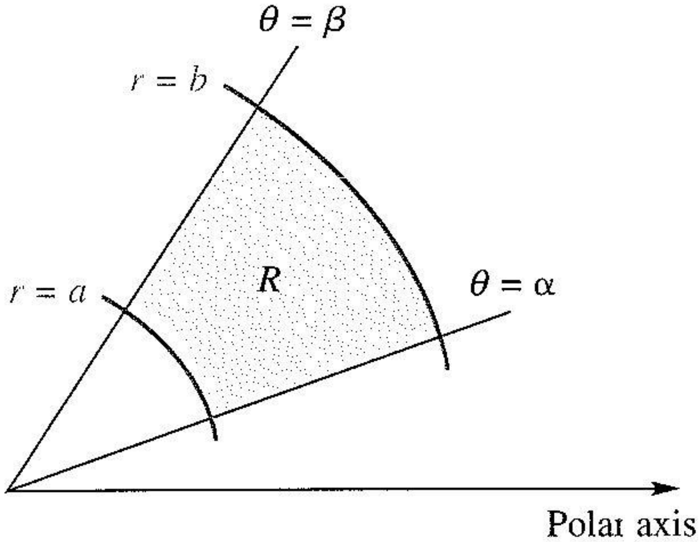


Figure 1

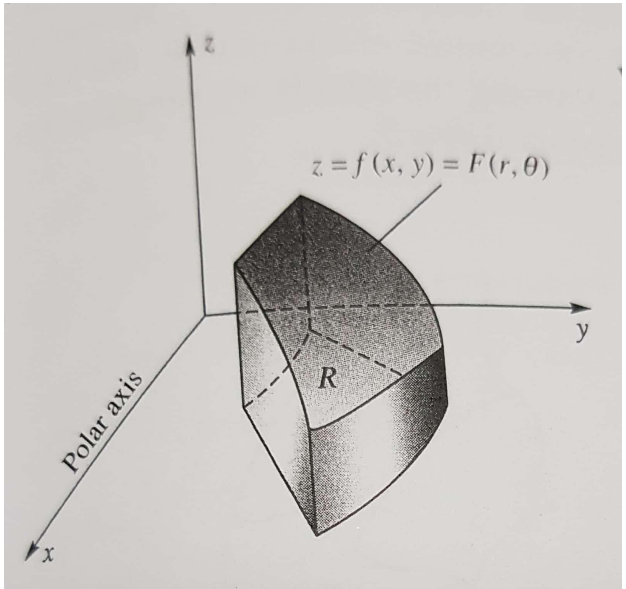


Figure 2

Double Integrals in Polar Coordinates

In polar coordinates, a polar rectangle R has the form

$$R = \{(r, \theta): a \le r \le b, \alpha \le \theta \le \beta\}$$

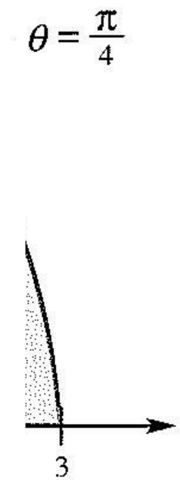
where $a \ge 0$ and $\beta - \alpha \le 2\pi$. Also, the equation of the surface can be written as

$$z = f(x, y) = f(r\cos\theta, r\sin\theta) = F(r, \theta)$$

(2) $$V = \iint_R F(r, \theta)\, r\, dr\, d\theta = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Interated Integrals

EXAMPLE 1 Find the volume V of the solid above the polar rectangle $R = \{(r, \theta): 1 \le r \le 3, 0 \le \theta \le \pi/4\}$ (Figure 4) and under the surface $z = e^{x^2+y^2}$.



SOLUTION Since $x^2 + y^2 = r^2$,

$$V = \iint_R e^{x^2+y^2}\, dA$$

$$= \int_0^{\pi/4} \left[\int_1^3 e^{r^2} r\, dr \right] d\theta$$

$$= \int_0^{\pi/4} \left[\frac{1}{2} e^{r^2} \right]_1^3 d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}(e^9 - e)\, d\theta = \frac{\pi}{8}(e^9 - e) \approx 3181$$

Without the help of polar coordinates, we could not have done this problem. Note how the extra factor of r was just what we needed in order to antidifferentiate e^{r^2}.

General Regions

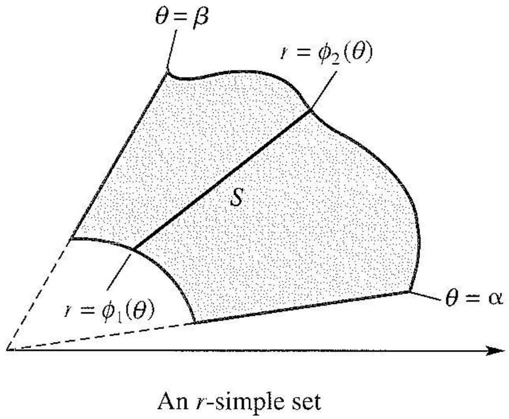


Figure 5

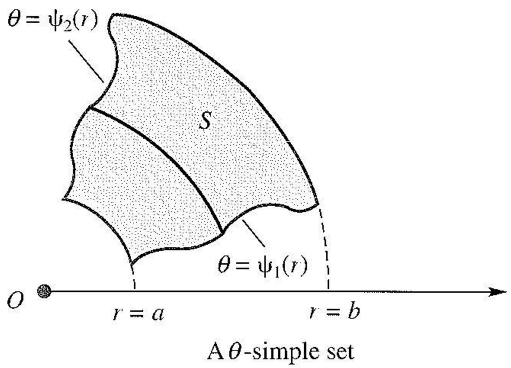


Figure 6

- Set S an ***r-simple*** set if it has the form (Figure 5)

$$S = \{(r, \theta): \phi_1(\theta) \le r \le \phi_2(\theta), \alpha \le \theta \le \beta\}$$

- ***θ-simple*** set if it has the form (Figure 6)

$$S = \{(r, \theta): a \le r \le b, \psi_1(r) \le \theta \le \psi_2(r)\}$$

EXAMPLE 2 Evaluate

$$\iint_S y\, dA$$

where S is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$ (see Figure 7).

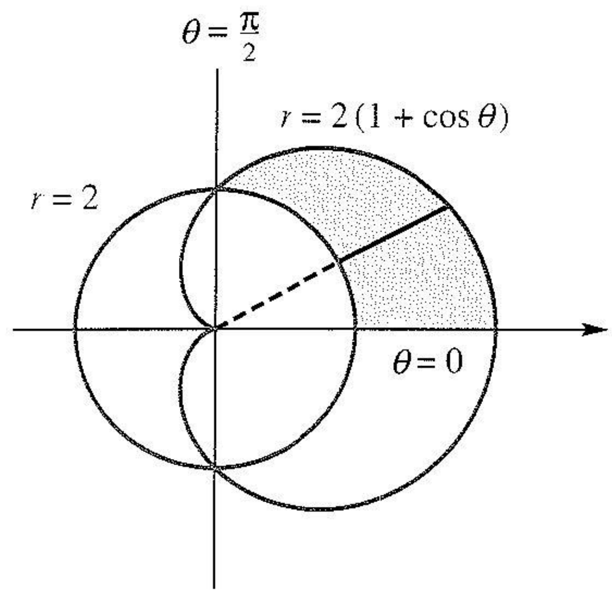


Figure 7

SOLUTION Since S is an r-simple set, we write the given integral as an iterated polar integral, with r as the inner variable of integration. In this inner integration, θ is held fixed; the integration is along the heavy line of Figure 7 from $r = 2$ to $r = 2(1 + \cos\theta)$.

$$\begin{aligned}
\iint_S y\,dA &= \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (r \sin\theta) r\,dr\,d\theta \\
&= \int_0^{\pi/2} \left[\frac{r^3}{3} \sin\theta\right]_2^{2(1+\cos\theta)} d\theta \\
&= \frac{8}{3} \int_0^{\pi/2} \left[(1 + \cos\theta)^3 \sin\theta - \sin\theta\right] d\theta \\
&= \frac{8}{3}\left[-\frac{1}{4}(1 + \cos\theta)^4 + \cos\theta\right]_0^{\pi/2} \\
&= \frac{8}{3}\left[-\frac{1}{4} + 0 - (-4 + 1)\right] = \frac{22}{3}
\end{aligned}$$

EXAMPLE 3 Find the volume of the solid under the surface $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$ (Figure 8).

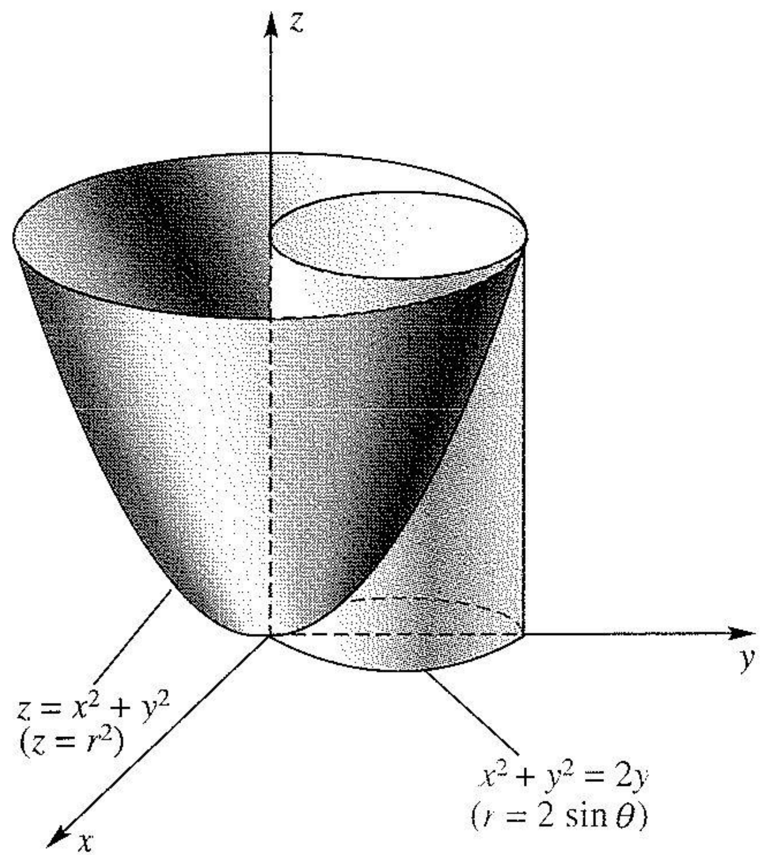


Figure 8

SOLUTION From symmetry, we can double the volume in the first octant. When we use $x = r\cos\theta$ and $y = r\sin\theta$, the equation of the surface becomes $z = r^2$ and that of the cylinder, $r = 2\sin\theta$. Let S denote the region shown in Figure 9. The required volume V is given by

$$V = 2\iint_S (x^2 + y^2)\,dA = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^2 r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\sin\theta} d\theta = 8\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 8\left(\frac{3}{8}\cdot\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

The last integral was evaluated by means of Formula 113 in the table of integrals at the end of the book.

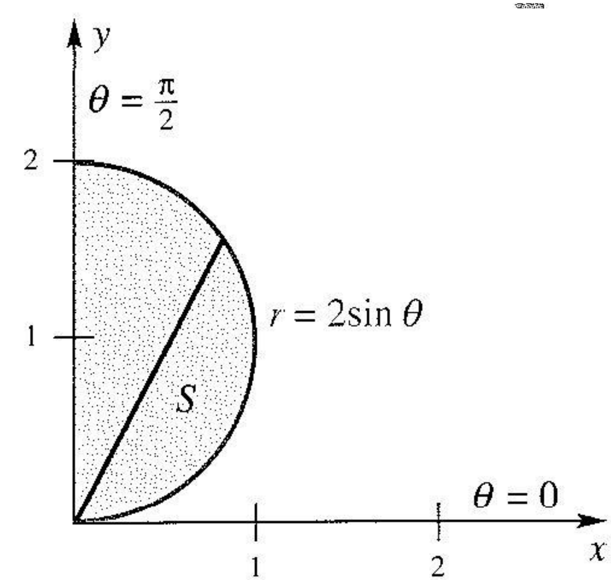


Figure 9

A Probability Integral

EXAMPLE 4 Show that $I = \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

SOLUTION We are going to sneak up on this problem in a roundabout, but decidedly ingenious, way. First recall that

$$I = \int_0^{\infty} e^{-x^2}\,dx = \lim_{b \to \infty} \int_0^{b} e^{-x^2}\,dx$$

Now let V_b be the volume of the solid (Figure 10) that lies under the surface $z = e^{-x^2-y^2}$ and above the square with vertices $(\pm b, \pm b)$. Then

$$\begin{aligned} V_b &= \int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\,dy\,dx = \int_{-b}^{b} e^{-x^2}\left[\int_{-b}^{b} e^{-y^2}\,dy\right]dx \\ &= \int_{-b}^{b} e^{-x^2}\,dx \int_{-b}^{b} e^{-y^2}\,dy = \left[\int_{-b}^{b} e^{-x^2}\,dx\right]^2 = 4\left[\int_0^{b} e^{-x^2}\,dx\right]^2 \end{aligned}$$

It follows that the volume of the region under $z = e^{-x^2-y^2}$ above the whole xy-plane is

$$(1) \qquad V = \lim_{b\to\infty} V_b = \lim_{b\to\infty} 4\left[\int_0^b e^{-x^2}\,dx\right]^2 = 4\left[\int_0^\infty e^{-x^2}\,dx\right]^2 = 4I^2$$

On the other hand, we can also calculate V using polar coordinates. Here V is the limit as $a \to \infty$ of V_a, the volume of the solid under the surface $z = e^{-x^2-y^2} = e^{-r^2}$ above the circular region of radius a centered at the origin (Figure 11).

$$(2) \qquad V = \lim_{a\to\infty} V_a = \lim_{a\to\infty} \int_0^{2\pi} \int_0^a e^{-r^2} r\,dr\,d\theta = \lim_{a\to\infty} \int_0^{2\pi} \left[-\frac{1}{2}e^{-r^2}\right]_0^a d\theta$$

$$= \lim_{a\to\infty} \frac{1}{2} \int_0^{2\pi} \left[1 - e^{-a^2}\right] d\theta = \lim_{a\to\infty} \pi\left[1 - e^{-a^2}\right] = \pi$$

Equating the two values obtained for V in (1) and (2) yields $4I^2 = \pi$, or $I = \frac{1}{2}\sqrt{\pi}$, as desired. ∎

EXAMPLE 5 Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1.$

SOLUTION By symmetry,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Now we make the substitution $u = x/\sqrt{2}$, so $dx = \sqrt{2}\, du$. The limits on the integral remain the same, so we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx &= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2}\sqrt{2}\, du \\
&= \frac{2\sqrt{2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2}\, du \\
&= \frac{2\sqrt{2}}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} = 1
\end{aligned}$$

To get the last line, we used the result of Example 4.

Section 13.5

Applications of Double Integrals

The most obvious application of double integrals is in calculating volumes of solids. This use of double integrals has been amply illustrated, so now we turn to other applications (mass, center of mass, moment of inertia, and radius of gyration).

Consider a flat sheet that is so thin that we may consider it to be two-dimensional. In Section 5.6, we called such a sheet a lamina, but there we considered only laminas of constant density. Here we wish to study laminas of variable density, that is, laminas made of nonhomogeneous material (Figure 1).

Suppose that a lamina covers a region S in the xy-plane, and let the density (mass per unit area) at (x, y) be denoted by $\delta(x, y)$. Partition S into small rectangles $R_1, R_2, \ldots, R_k$, as shown in Figure 2. Pick a point $(\overline{x}_k, \overline{y}_k)$ in R_k. Then the mass of R_k is approximately $\delta(\overline{x}_k, \overline{y}_k)A(R_k)$, and the total mass of the lamina is approximately

$$m \approx \sum_{k=1}^{n} \delta(\overline{x}_k, \overline{y}_k)A(R_k)$$

The actual mass m is obtained by taking the limit of the above expression as the norm of the partition approaches zero, which is, of course, a double integral.

$$\boxed{m = \iint_S \delta(x, y)\, dA}$$

EXAMPLE 1 A lamina with density $\delta(x, y) = xy$ is bounded by the x-axis, the line $x = 8$, and the curve $y = x^{2/3}$ (Figure 3). Find its total mass.

SOLUTION

$$m = \iint_S xy\,dA = \int_0^8 \int_0^{x^{2/3}} xy\,dy\,dx$$

$$= \int_0^8 \left[\frac{xy^2}{2}\right]_0^{x^{2/3}} dx = \frac{1}{2}\int_0^8 x^{7/3}\,dx$$

$$= \frac{1}{2}\left[\frac{3}{10}x^{10/3}\right]_0^8 = \frac{768}{5} = 153.6$$

Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k} \qquad \bar{y} = \frac{M_x}{m} = \frac{\sum_{k=1}^{n} y_k m_k}{\sum_{k=1}^{n} m_k}$$

$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{\iint_S x\delta(x, y)\, dA}{\iint_S \delta(x, y)\, dA} \qquad \bar{y} = \frac{M_x}{m} = \frac{\iint_S y\delta(x, y)\, dA}{\iint_S \delta(x, y)\, dA}}$$

EXAMPLE 2 Find the center of mass of the lamina of Example 1.

SOLUTION In Example 1, we showed that the mass m of this lamina is $\frac{768}{5}$. The moments M_y and M_x with respect to the y-axis and x-axis are

$$M_y = \iint_S x\delta(x, y)\, dA = \int_0^8 \int_0^{x^{2/3}} x^2 y\, dy\, dx$$

$$= \frac{1}{2}\int_0^8 x^{10/3}\, dx = \frac{12{,}288}{13} \approx 945.23$$

$$M_x = \iint_S y\delta(x, y)\, dA = \int_0^8 \int_0^{x^{2/3}} x y^2\, dy\, dx$$

$$= \frac{1}{3}\int_0^8 x^3\, dx = \frac{1024}{3} \approx 341.33$$

We conclude that

$$\bar{x} = \frac{M_y}{m} = \frac{80}{13} \approx 6.15, \qquad \bar{y} = \frac{M_x}{m} = \frac{20}{9} \approx 2.22$$

$\boxed{\approx}$ Notice in Figure 3 that the center of mass $(\bar{x}, \bar{y})$ is in the upper-right portion of S; but this is to be expected since a lamina with density $\delta(x, y) = xy$ gets heavier as the distance from the x- and y-axes increases.

EXAMPLE 3 Find the center of mass of a lamina in the shape of a quarter-circle of radius a whose density is proportional to the distance from the center of the circle (Figure 4).

SOLUTION By hypothesis, $\delta(x, y) = k\sqrt{x^2 + y^2}$, where k is a constant. The shape of S suggests the use of polar coordinates.

$$m = \iint_S k\sqrt{x^2 + y^2}\, dA = k\int_0^{\pi/2}\int_0^a rr\, dr\, d\theta$$

$$= k\int_0^{\pi/2} \frac{a^3}{3}\, d\theta = \frac{k\pi a^3}{6}$$

Also,

$$M_y = \iint_S xk\sqrt{x^2 + y^2}\, dA = k\int_0^{\pi/2}\int_0^a (r\cos\theta)r^2\, dr\, d\theta$$

$$= k\int_0^{\pi/2} \frac{a^4}{4}\cos\theta\, d\theta = \left[\frac{ka^4}{4}\sin\theta\right]_0^{\pi/2} = \frac{ka^4}{4}$$

We conclude that

$$\bar{x} = \frac{M_y}{m} = \frac{ka^4/4}{k\pi a^3/6} = \frac{3a}{2\pi}$$

Because of the symmetry of the lamina, we recognize that $\bar{y} = \bar{x}$, so no further calculation is needed.

Section 13.6

Surface Area

Surface Area

- Suppose that G is such a surface over the closed and bounded region S in the xy-plane.\
- Surface area of G

$$\boxed{A(G) = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA}$$

EXAMPLE 2 Find the area of the surface $z = x^2 + y^2$ below the plane $z = 9$.

SOLUTION The designated part G of the surface projects onto the circular region S inside the circle $x^2 + y^2 = 9$ (Figure 5). Let $f(x, y) = x^2 + y^2$. Then $f_x = 2x$, $f_y = 2y$, and

$$A(G) = \iint_S \sqrt{4x^2 + 4y^2 + 1}\, dA$$

The shape of S suggests use of polar coordinates.

$$\begin{aligned} A(G) &= \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{8}\left[\frac{2}{3}(4r^2 + 1)^{3/2}\right]_0^3 d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(37^{3/2} - 1)\, d\theta = \frac{\pi}{6}(37^{3/2} - 1) \approx 117.32 \end{aligned}$$

Section 13.7

Triple Integrals in Cartesian Coordinates

Triple Integral

The concept embodied in single and double integrals extends in a natural way to triple and even n-dimensional integrals.

Consider a function f of three variables defined over a box-shaped region B with faces parallel to the coordinate planes. We can no longer graph f (four dimensions would be required), but we can picture B (Figure 1). Form a partition P of B by passing planes through B parallel to the coordinate planes, thus cutting B into small subboxes $B_1, B_2, \ldots, B_n$; a typical subbox, B_k, is shown in Figure 1. On B_k, pick a sample point $(\bar{x}_k, \bar{y}_k, \bar{z}_k)$ and consider the Riemann sum

$$\sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k, \bar{z}_k)\, \Delta V_k$$

$$\sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k, \overline{z}_k)\, \Delta V_k$$

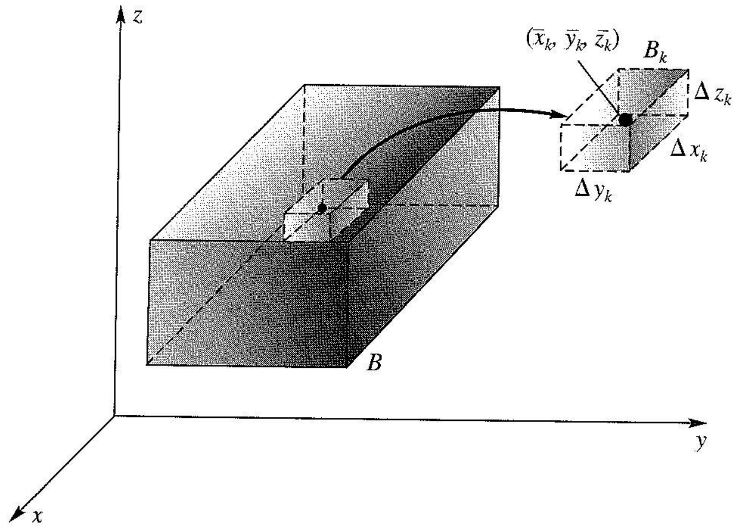


Figure 1

where $\Delta V_k = \Delta x_k \, \Delta y_y \, \Delta z_k$ is the volume of B_k. Let the norm of the partition $\|P\|$ be the length of the longest diagonal of all the subboxes. Then we define the **triple integral** by

$$\boxed{\iiint_B f(x, y, z)\, dV = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k, \overline{z}_k)\, \Delta V_k}$$

provided that this limit exists.

EXAMPLE 1 Evaluate $\iiint\limits_B x^2yz\,dV$, where B is the box

$$B = \{(x, y, z): 1 \le x \le 2, 0 \le y \le 1, 0 \le z \le 2\}$$

SOLUTION

$$\iiint\limits_B x^2yz\,dV = \int_0^2 \int_0^1 \int_1^2 x^2yz\,dx\,dy\,dz$$

$$= \int_0^2 \int_0^1 \left[\frac{1}{3}x^3yz\right]_1^2 dy\,dz = \int_0^2 \int_0^1 \frac{7}{3}yz\,dy\,dz$$

$$= \frac{7}{3}\int_0^2 \left[\frac{1}{2}y^2z\right]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}z\,dz$$

$$= \frac{7}{6}\left[\frac{z^2}{2}\right]_0^2 = \frac{7}{3}$$

There are six possible orders of integration. Every one of them will yield the answer $\frac{7}{3}$.

General Regions

Let S be a z-simple set (vertical lines intersect S in a single line segment), and let S_{xy} be its projection in the xy-plane (Figure 3). Then

$$\iiint_S f(x, y, z)\, dV = \iint_{S_{xy}} \left[\int_{\psi_1(x, y)}^{\psi_2(x, y)} f(x, y, z)\, dz \right] dA$$

If, in addition, S_{xy} is a y-simple set (as shown in Figure 3), we can rewrite the outer double integral as an iterated integral.

$$\boxed{\iiint_S f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x, y)}^{\psi_2(x, y)} f(x, y, z)\, dz\, dy\, dx}$$

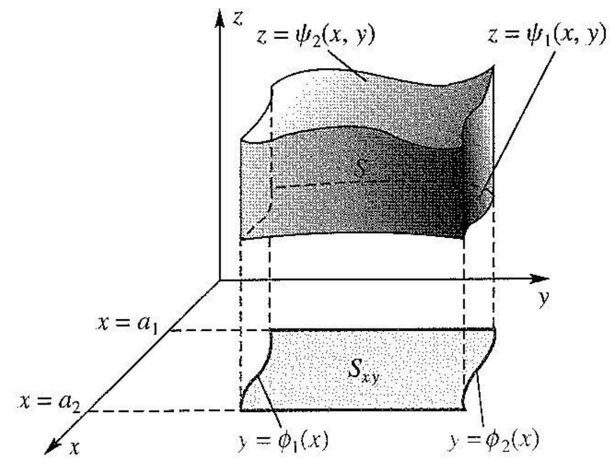


Figure 3

EXAMPLE 2 Evaluate the iterated integral

$$\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\,dz\,dy\,dx$$

SOLUTION

$$\begin{aligned}\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\,dz\,dy\,dx &= \int_{-2}^{5}\int_{0}^{3x}\left(\int_{y}^{x+2} 4\,dz\right)dy\,dx \\ &= \int_{-2}^{5}\int_{0}^{3x}[4z]_{y}^{x+2}\,dy\,dx \\ &= \int_{-2}^{5}\int_{0}^{3x}(4x - 4y + 8)\,dy\,dx \\ &= \int_{-2}^{5}[4xy - 2y^2 + 8y]_{0}^{3x}\,dx \\ &= \int_{-2}^{5}(-6x^2 + 24x)\,dx = -14\end{aligned}$$

EXAMPLE 3 Evaluate the triple integral of $f(x, y, z) = 2xyz$ over the solid region S in the first octant that is bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $z = 0$, $y = x$, and $y = 0$.

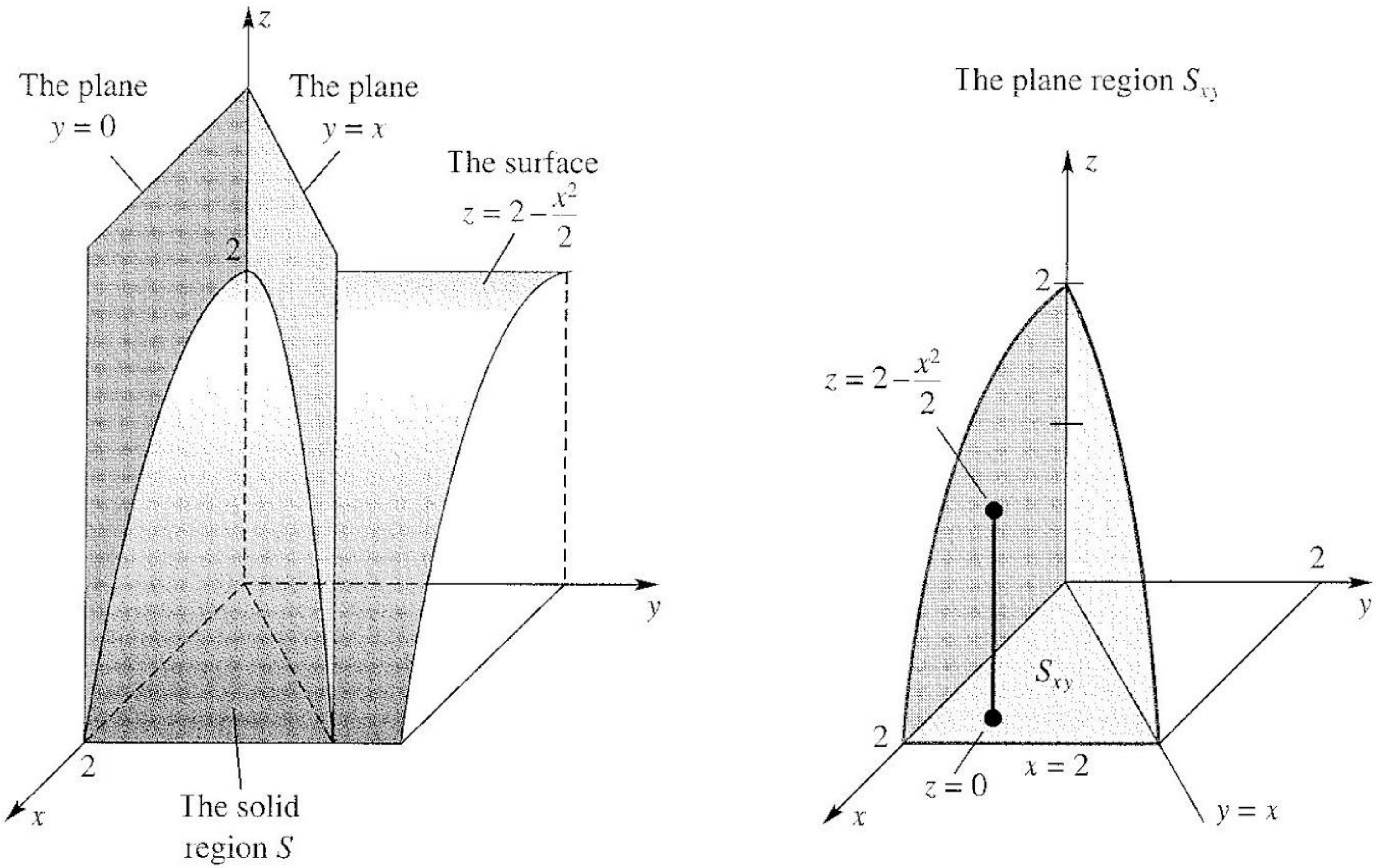


Figure 4

SOLUTION The solid region S is shown in Figure 4. The triple integral

$$\iiint_S 2xyz\, dV$$

can be evaluated by an iterated integral.

Note first that S is a z-simple set and that its projection S_{xy} in the xy-plane is y-simple (also x-simple). In the first integration, x and y are fixed; we integrate along a vertical line from $z = 0$ to $z = 2 - x^2/2$. The result is then integrated over the set S_{xy}.

$$\begin{aligned}
\iiint_S 2xyz\, dV &= \int_0^2 \int_0^x \int_0^{2-x^2/2} 2xyz\, dz\, dy\, dx \\
&= \int_0^2 \int_0^x [xyz^2]_0^{2-x^2/2}\, dy\, dx \\
&= \int_0^2 \int_0^x \left(4xy - 2x^3y + \frac{1}{4}x^5y\right) dy\, dx \\
&= \int_0^2 \left(2x^3 - x^5 + \frac{1}{8}x^7\right) dx = \frac{4}{3}
\end{aligned}$$

Section 13.8

Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

Cylindrical Coordinates Figure 1 serves to remind us of the meaning of cylindrical coordinates and displays the symbols that we use. Cylindrical and Cartesian (rectangular) coordinates are related by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$

As a result, the function $f(x, y, z)$ transforms to

$$f(x, y, z) = f(r\cos\theta, r\sin\theta, z) = F(r, \theta, z)$$

when written in cylindrical coordinates.

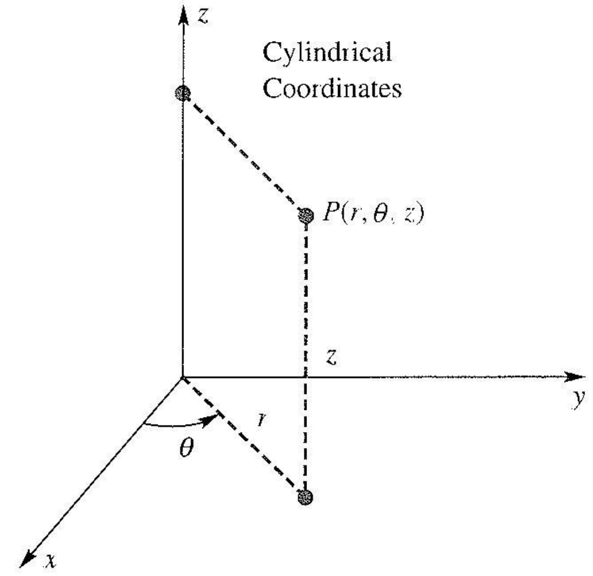


Figure 1

Cylindrical Coordinates

Suppose now that we wish to evaluate $\iiint\limits_S f(x, y, z)\, dV$, where S is a solid region. Consider partitioning S by means of a cylindrical grid, where the typical volume element has the shape shown in Figure 2. Since this piece (called a *cylindrical wedge*) has volume $\Delta V_k = \bar{r}_k \,\Delta r_k \,\Delta\theta_k \,\Delta z_k$, the sum that approximates the integral has the form

$$\sum_{k=1}^{n} F(\bar{r}_k, \bar{\theta}_k, \bar{z}_k)\bar{r}_k \,\Delta z_k \,\Delta r_k \,\Delta\theta_k$$

Taking the limit as the norm of the partition tends to zero leads to a new integral and suggests an important formula for changing from Cartesian to cylindrical coordinates in a triple integral.

Cylindrical Coordinates

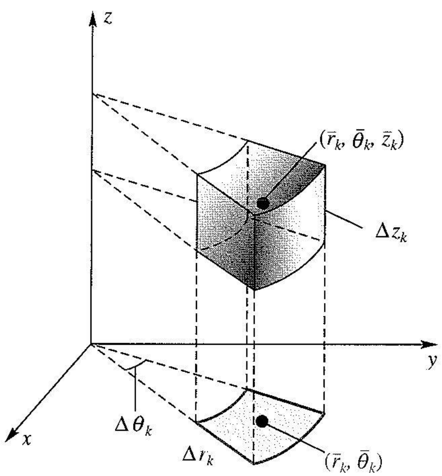


Figure 2

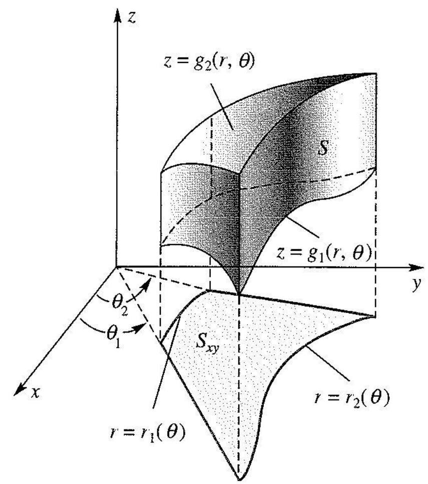


Figure 3

Let S be a z-simple solid and suppose that its projection S_{xy} in the xy-plane is r-simple, as shown in Figure 3. If f is continuous on S, then

$$\boxed{\iiint_S f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

EXAMPLE 2 Find the volume of the solid region S in the first octant bounded above by the paraboloid $z = 4 - x^2 - y^2$, and laterally by the cylinder $x^2 + y^2 = 2x$, as shown in Figure 5.

SOLUTION In cylindrical coordinates, the paraboloid is $z = 4 - r^2$ and the cylinder is $r = 2\cos\theta$. The z-variable runs from the xy-plane up to the paraboloid, that is, from 0 to $4 - r^2$. Figure 6 shows the "footprint" of the solid in the xy-plane; this figure suggests that for a fixed θ, r goes from 0 to $2\cos\theta$. Finally, θ goes from 0 to $\pi/2$. Thus,

$$V = \iiint_S 1\,dV = \int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r(4 - r^2)\,dr\,d\theta = \int_0^{\pi/2}\left[2r^2 - \frac{1}{4}r^4\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2}(8\cos^2\theta - 4\cos^4\theta)\,d\theta$$

$$= 8\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 4\cdot\frac{3}{8}\cdot\frac{\pi}{2} = \frac{5\pi}{4}$$

We used Formula 113 from the table of integrals at the end of the book to make the last calculation. ■

Spherical Coordinates

Spherical Coordinates Figure 7 serves to remind us of the meaning of spherical coordinates, which were introduced in Section 11.9. There we learned that the equations

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi$$

relate spherical coordinates and Cartesian coordinates. Figure 8 exhibits the volume element in spherical coordinates (called a *spherical wedge*). Though we omit the details, it can be shown that the volume of the indicated spherical wedge is

$$\Delta V = \bar{\rho}^2 \sin \bar{\phi} \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

where $(\bar{\rho}, \bar{\theta}, \bar{\phi})$ is an appropriately chosen point in the wedge.

Partitioning a solid S by means of a spherical grid, forming the appropriate sum, and taking the limit leads to an iterated integral in which $dz\, dy\, dx$ is replaced by $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$.

Spherical Coordinates

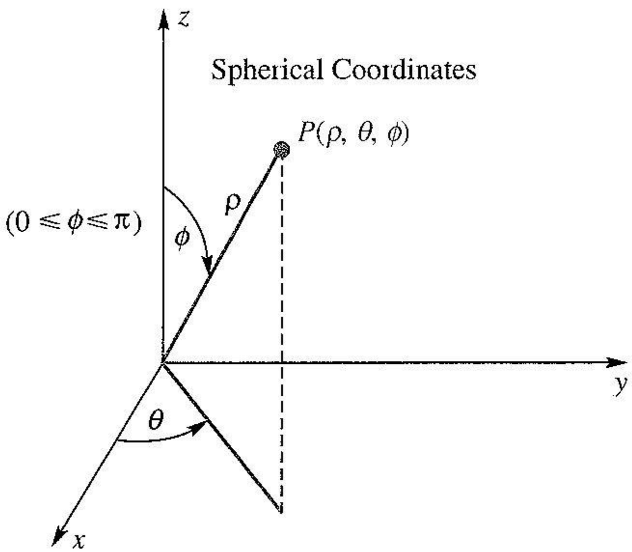


Figure 7

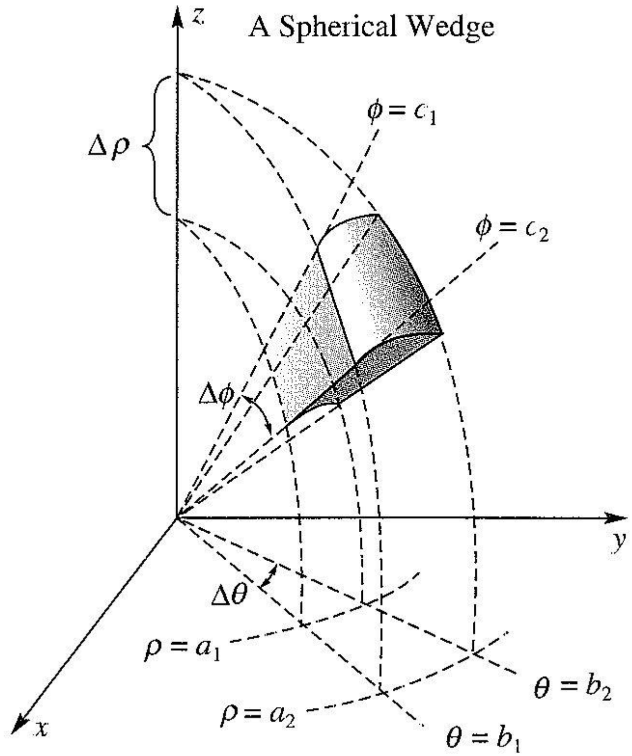


Figure 8

$$\iiint_S f(x, y, z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

EXAMPLE 4 Find the volume and center of mass of the homogeneous solid S that is bounded above by the sphere $\rho = a$ and below by the cone $\phi = \alpha$, where a and α are constants (Figure 9).

SOLUTION The volume V is given by

$$\begin{aligned} V &= \int_0^{\alpha} \int_0^{2\pi} \int_0^{a} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\alpha} \int_0^{2\pi} \left(\frac{a^3}{3}\right) \sin \phi \, d\theta \, d\phi \\ &= \frac{2\pi a^3}{3} \int_0^{\alpha} \sin \phi \, d\phi = \frac{2\pi a^3}{3}(1 - \cos \alpha) \end{aligned}$$

It follows that the mass m of the solid is

$$m = kV = \frac{2\pi a^3 k}{3}(1 - \cos \alpha)$$

where k is the constant density.

From symmetry, the center of mass is on the z-axis; that is, $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate M_{xy}.

$$\begin{aligned}
M_{xy} = \iiint_S kz\,dV &= \int_0^{\alpha}\int_0^{2\pi}\int_0^{a} k(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\alpha}\int_0^{2\pi}\int_0^{a} k\rho^3\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\alpha}\int_0^{2\pi} \frac{1}{4}ka^4\sin\phi\cos\phi\,d\theta\,d\phi \\
&= \int_0^{\alpha} \frac{1}{2}\pi ka^4\sin\phi\cos\phi\,d\phi = \frac{1}{4}\pi a^4 k\sin^2\alpha
\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{z} &= \frac{\frac{1}{4}\pi a^4 k\sin^2\alpha}{\frac{2}{3}\pi a^3 k(1-\cos\alpha)} = \frac{3a\sin^2\alpha}{8(1-\cos\alpha)} \\
&= \frac{3}{8}a(1+\cos\alpha)
\end{aligned}$$

Section 13.9

Change of Variables in Multiple Integrals

The Change of Variable Formula for Double Integrals

For double integrals, such as $\iint_R f(x, y)\,dx\,dy$, the procedure is similar: We must take into account

1. the integrand $f(x, y)$,
2. the differential $dx\,dy$, and
3. the region of integration.

The main result is given in the next theorem.

The Change of Variable Formula for Double Integrals

Theorem A **Change of Variables for Double Integrals**

Suppose G is a one-to-one transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps the bounded region S in the uv-plane onto the bounded region R in the xy-plane. If G is of the form $G(u, v) = (x(u, v), y(u, v))$, then

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(x(u, v), y(u, v))\,|J(u, v)|\, du\, dv$$

where $J(u, v)$, called the **Jacobian,** is equal to the determinant

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

EXAMPLE 3 Evaluate $\iint_R \cos(x - y)\sin(x + y)\,dA$, where R is the triangle with vertices $(0,0)$, $(\pi, -\pi)$, and (π, π).

SOLUTION Let $u = x - y$ and $v = x + y$. Solving for x and y gives $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(v - u)$. The region R can be specified as

$$-x \le y \le x$$

$$0 \le x \le \pi$$

Substituting u and v gives

$$-\frac{1}{2}(u + v) \le \frac{1}{2}(v - u) \le \frac{1}{2}(u + v)$$

$$0 \le \frac{1}{2}(u + v) \le \pi$$

which reduces to

$$u \ge 0, \quad v \ge 0$$

$$0 \le u + v \le 2\pi$$

This is the region S in the uv-plane (see Figure 6). The Jacobian for this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

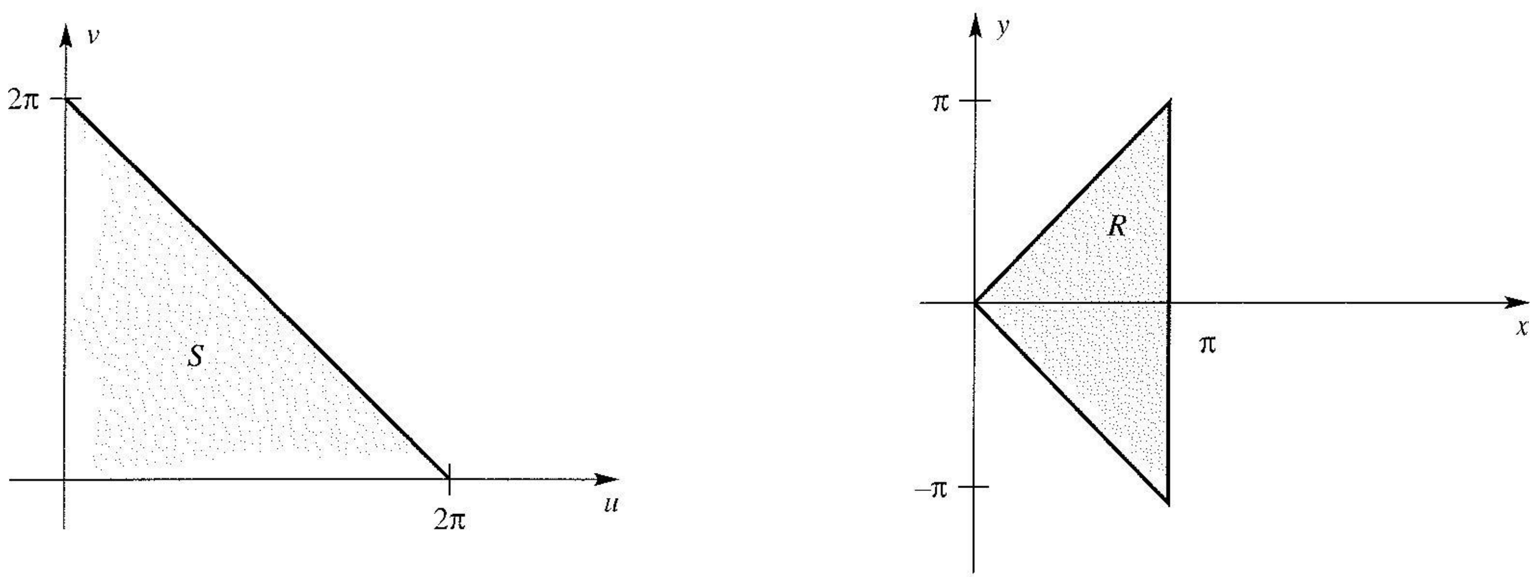


Figure 6

Thus

$$
\begin{aligned}
\iint_R \cos(x - y)\sin(x + y)\, dA &= \iint_S \cos u \sin v \left|\frac{1}{2}\right| dv\, du \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi - u} \cos u \sin v\, dv\, du \\
&= \frac{1}{2}\int_0^{2\pi} \cos u\,(1 - \cos(2\pi - u))\, du \\
&= \frac{1}{2}\int_0^{2\pi} \cos u\,(1 - \cos u)\, du \\
&= \frac{1}{2}\int_0^{2\pi} (\cos u - \cos^2 u)\, du \\
&= \frac{1}{2}\int_0^{2\pi} \left(\cos u - \frac{1 + \cos 2u}{2}\right) du \\
&= \frac{1}{2}\int_0^{2\pi} \left(\cos u - \frac{1}{2} - \frac{1}{2}\cos 2u\right) du \\
&= \frac{1}{2}\left[\sin u - \frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{2\pi} = -\frac{1}{2}\pi \quad \blacksquare
\end{aligned}
$$

The Change of Variable Formula for Triple Integrals

The Change of Variable Formula for Triple Integrals Theorem A generalizes to triple (and even higher-dimensional) integrals. If G is a one-to-one transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which maps the bounded region S in uvw-space onto the bounded region R in the xyz-plane, and if G is of the form $G(u, v) = (x(u, v, w), y(u, v, w), z(u, v, w))$, then

$$\iiint_R f(x, y, z)\,dx\,dy\,dz = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \times |J(u, v, w)|\,du\,dv\,dw$$

where $J(u, v, w)$ is the determinant

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

End of Chapter 13