Rectifiability of metric spaces via arbitrarily small perturbations

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Rectifiable metric spaces

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- S is purely n-unrectifiable if every n-rectifiable subset of S has $\mathcal{H}^n$ measure zero. If $\mathcal{H}^n(X) < \infty$ then $X = U \cup R$, U purely n-unrectifiable and R n-rectifiable.

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- Classically (when $X = \mathbb{R}^m$), a fundamental description of rectifiable sets is given by the Besicovitch-Federer projection theorem: $\mathcal{H}^n(S) < \infty$, S purely n-unrectifiable $\Rightarrow$ almost every n-dimensional orthogonal projection of S has Lebesgue measure zero.

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- In (infinite dimensional) Banach spaces: Projection $=$ continuous linear $T\colon B \to \mathbb{R}^n$ (of full rank).
- "Almost every" projection? Prescribe a collection of null sets. Standard examples exist in the theory of GMT in Banach spaces.

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for a non "Aronszajn" null set of projections.

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- BF is completely false in infinite dimensional spaces.

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- This is a complete metric space and so we can consider a typical 1-Lipschitz function (i.e. residual/comeagre in the sense of Baire category: a set that contains a countable intersection of open dense sets).
- "A typical 1-Lipschitz function" is a suitable candidate to replace "almost every projection".

A new characterisation

Theorem (B)

Let $S \subset X$ be purely n-unrectifiable with $\mathcal{H}^n(S) < \infty$ and

$$\liminf_{r \to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \qquad (*)$$

for $\mathcal{H}^n$-a.e. $x \in S$.

For any $m \in \mathbb{N}$, a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies

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- If $S \subset \mathbb{R}^{m'}$, $(*)$ is not necessary.
- Using deep results of the structure of Lebesgue null sets announced by Csörnyei-Jones, $(*)$ is never necessary.
- If $\mathcal{H}^s(S) < \infty$ with $s \notin \mathbb{N}$, then a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies $\mathcal{H}^s(f(S)) = 0$.

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- This direction is simpler: uses Kirchheim's description of rectifiable metric spaces.

Idea of the proof of the main direction

Given $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$, we must make arbitrarily small modifications to obtain a $\tilde{f}$ such that $\mathcal{H}^n(\tilde{f}(S))$ is arbitrarily small. These modifications **cannot** increase the Lipschitz constant.

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$$\mathrm{Tan}_{f(x)} f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$
- If $S \subset \mathbb{R}^{m'}$, or using the announcement of Csörnyei-Jones, the theorem can be proved without assuming $(*)$. Similarly, the consequence is true for the case $s \notin \mathbb{N}$.

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- Have a weak tangent field: $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

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  - $\dim V_x = n-1 \Rightarrow$ can reduce $\mathcal{H}^n(f(S))$ to an arbitrarily small value.

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- Applying the Euclidean squeezing argument to f gives a $\tilde{f}$ with a huge Lipschitz constant (because of the ralationship between $\|.\|_2$ and $\|.\|_\infty$ in $\mathbb{R}^m$).
- If we are more careful we can obtain something more useful.

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- $(*)$ is not necessary under the same conditions as before, and have the corresponding statement for $\mathcal{H}^s(S)$, $s \notin \mathbb{N}$.

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- If S is a subset of a Banach space B with an *unconditional basis* (ℓ_1, $L^p(\mu)$ $1 < p < \infty$, c_0,...) then σ can be chosen to be a genuine perturbation.

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- Generalises a result of H. Pugh who proved this for Ahlfors regular subsets of Euclidean space. The construction relies on BF.