Rectifiability of metric spaces via arbitrarily small perturbations

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Rectifiable metric spaces

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- S is purely n-unrectifiable if every n-rectifiable subset of S has $\mathcal{H}^n$ measure zero.
- Classically (when $X = \mathbb{R}^m$), a fundamental description of rectifiable sets is given by the Besicovitch-Federer projection theorem: $\mathcal{H}^n(S) < \infty$, S purely n-unrectifiable $\Rightarrow$ almost every n-dimensional orthogonal projection of S has Lebesgue measure zero.

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- B, Csörnyei, Wilson: In any separable infinite dimensional Banach space, there exists a purely 1-unrectifiable S with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for **every** projection.

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- B, Csörnyei, Wilson: In any separable infinite dimensional Banach space, there exists a purely 1-unrectifiable S with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for **every** projection.
  - BF is completely false in infinite dimensional spaces.

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- "A typical L-Lipschitz function" is a suitable candidate to replace "almost every projection".

A new characterisation of rectifiable metric spaces I

Theorem (B)

Let $S \subset X$ be purely n-unrectifiable with $\mathcal{H}^n(S) < \infty$ and

$$\liminf_{r \to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \qquad (*)$$

for $\mathcal{H}^n$-a.e. $x \in S$.

For any $L \geq 0$ and $m \in \mathbb{N}$, a typical $f \in \mathrm{Lip}(X, \mathbb{R}^m, L)$ satisfies

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- If $S \subset \mathbb{R}^{m'}$, $(*)$ is not necessary.
- Using deep results of the structure of Lebesgue null sets announced by Csörnyei-Jones, $(*)$ is never necessary.

A new characterisation of rectifiable metric spaces II

As with BF, we get a strong converse.

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- This direction is simpler. Uses Kirchheim's description of rectifiable metric spaces.

Idea of the proof of the main direction

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- Locally, can squeeze f in all directions orthogonal to V_x.
    - Since there are no curves in these directions, this can be done without perturbing f very much.
    - This can reduce $\mathcal{H}^n(f(S))$ to an arbitrarily small value.
- Care must be taken to not increase the Lipschitz constant of f.

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The technique is very general and also obtains the following results:

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$$|d(x,y) - \|\sigma(x) - \sigma(y)\|| < \epsilon \quad \forall x, y \in S$$

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- Have suitable converse.
- $(*)$ is not necessary under the same conditions as before.

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- Generalises a result of Pugh who proved this for Ahlfors regular subsets of Euclidean space.