Homework 2 - Part A

① Suppose that one letter is to be selected at random from the 42 letter in the sentence:
"The shortest distance between 2 points is a taxi."
If Y denotes the number of letters in the word in which the selected letter apears, what is the value of E(Y)?

Probability of selecting a letter from a word in the sentence $\doteq \dfrac{\text{\# of letters in the word}}{\text{total \# of letters in sentence}}$

$$E(Y) = \left[\frac{3}{42}\times 3 + \frac{8}{42}\times 8 + \frac{8}{42}\times 8 + \frac{7}{42}\times 7 + \frac{3}{42}\times 3 + \frac{6}{42}\times 6 + \frac{2}{42}\times 2 + \frac{1}{42}\times 1 + \frac{4}{42}\times 4 \right]$$

$$= 1/42 \times [3^2 + 8^2 + 8^2 + 7^2 + 3^2 + 6^2 + 2^2 + 1^2 + 4^2]$$

$$= 252/46$$

$$= 6$$

② Suppose that X and Y have a continuous joint distribution for which the joint pdf is:
$f(x,y) = 12y^2$ for $0 \leq y \leq x \leq 1$

Checking for PDF $f(x,y)$:

$$F(x,y) = \int_0^1 \int_0^x 12y^2 \, dy \, dx = \int_0^1 \left[\frac{12y^3}{3}\right]_0^x dx$$

$$= \int_0^1 4x^3 \, dx$$

$$= \left[\frac{4}{4}x^4\right]_0^1$$

$$= 1$$

~~$E(xy) = \int_0^1 \int_0^x xy f(x) f(y) \, dx$~~

$$E(xy) = \int_0^1 \int_0^x xy \, f(x) f(y) \, dy \, dx$$

$$= \int_0^1 \int_0^x xy \, 12y^2 \, dy \, dx$$

~~$= \int_0^1 x \left[\frac{12}{4}y^4\right]_0^x dx$~~

$$= 1/2$$

③ Suppose that 3 RV X_1, X_2, X_3 form a random sample from the uniform Distribution on the interval $[0,1]$.
Find $E[(X_1 - 2X_2 + X_3)^2]$

Since X_1, X_2 and X_3 are RV from a random sample from $[0,1]$, their expected values are:

$$E(X_1) = E(X_2) = E(X_3) = 0.5$$

$$E[(X_1 - 2X_2 + X_3)^2] = [E(X_1) - 2E(X_2) + E(X_3)]^2$$

$$= [0.5 - 2(0.5) + 0.5]^2$$

$$= 0$$

④ X has PDF

$f(x) = e^{-x}, \ x > 0$

$Y = e^{3x/4}$

Find $E(Y)$

$E(Y) = \int_0^{\infty} e^{3x/4} \cdot e^{-x}\, dx$ ⇒ WOLFRAM ALPHA: integrate e^(3x/4) * e^(-x) from x=0 to infinity

$= 4$

⑤ X is the outcome of rolling a fair die.

$Y = g(X) = 2X^2 + 1$

Find $E(Y)$

Probability of the outcome of rolling a fair die:

$P(X) = 1/n$ for an n-sided die

$E(Y) = \sum_{x=1}^{n} (2x^2 + 1) 1/n$

Assuming the die has 6 sides, $n = 6$

$E(Y) = \sum_{x=1}^{6} (2x^2 + 1) \frac{1}{6}$ ~~1+2+3+4+5+6~~

~~$x^2 + (x^2/3 + 1/6)$~~

$= \left(\sum_{x=1}^{6} 2x^2\right) + 1 = \frac{2}{6}\left(\sum_{x=1}^{6} x^2\right) + 1$

1 4 9 16 25 36

$= \frac{2}{6}\left((1^2) + (2^2) + 3^2 + 4^2 + 5^2 + 6^2)\right) + 1$

$=$

$= 2/_4 \cdot 91 + 1$

6) X has PDF

$f(x) = 2(1-x), \quad 0 < x < 1$

$Y = (2X+1)$

Find $E(Y^2)$

Checking for PDF: $F(x) = \int_0^1 2(1-x)\,dx = 1$

$E(Y^2) = \int_0^1 2(2x+1)^2(1-x)\,dx$

$= 2\int_0^1 (x^2+4x+1)(1-x)\,dx$

$= 2\int_0^1 (4x^2+4x+1-4x^3-4x^2-x)\,dx$

$= 2\int_0^1 (-4x^3+3x+1)\,dx$

$= 2\left[-x^4 + \frac{3}{2}x^2 + x\right]_0^1$

$= 2\left[-1 + \frac{3}{2} + 1\right]$

$= 3$

7) Remember the binomial theorem:

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for $\in \mathbb{Z}^+$

Show that $E((ax+b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(x^{n-i})$

$\cancel{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}}$

$E[(ax+b)^n] = E\left[\sum_{k=0}^{n} \binom{n}{k} (ax)^{n-k} b^k\right]$

$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k E(x^{n-k})$

Since a and b are constant

(8) proportion of defective parts in large shipment is p.
A random sample of n parts is seleted from the shipment
Let X denote the number of defective parts in the sample.
Find $E(X-Y)$.
If the sample size is 20 and p is 5%, what is $E(X-Y)$?
Proportion of defective parts $= p$

$E(X) = np$

$E(X-Y) = E[X-(n-X)]$

$= E[2X-n)$

$n = 20, p = 0.05$

$E(X-Y) = E(2 \times 20 \times 0.05 - 20)$

$= 2 - 20$

$= -18$

# of defectivar parts in the sample $= 20 \times 0.05$ ...

~~To summarize it is~~

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