

# On the mixed-integer linear-quadratic optimal control with switching cost

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## Abstract

This paper concerns the optimal control of a continuous-time dynamical system via continuous and discrete-valued control variables, where the objective functional also accounts for state-independent switching costs. The class of mixed-integer optimal control problems is interpreted as a bilevel problem, involving both switching times optimization, for a given sequence of modes, and purely continuous optimal control. Additionally, an original non-convex formulation for the switching costs is introduced, in terms of cardinality, inspired by sparse optimization and compressed sensing techniques. We then adopt proximal algorithms to solve the resulting bilevel optimal control problem with composite objective function. An efficient routine for evaluating the proximal operator is developed. Two examples are numerically solved via a proximal gradient method, discussed and compared with the literature. Although this work focuses on switched linear time-varying dynamics and quadratic cost functionals with a specific formulation of the switching costs, the proposed approach may also apply to more general mixed-integer optimal control problems.

**Keywords:** Optimal control, numerical algorithms, hybrid systems.

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# 1 Introduction

Optimal control problems (OCP) involving both continuous and discrete-valued control variables are known as mixed-integer OCPs (MIOCP) and are challenging problems due to their combinatorial nature [22, 16]. We consider MIOCPs constrained by ordinary differential equations and take into account a cost for switching among discrete-valued control inputs in order to avoid chattering [16, 17].

Several approaches exist to deal with such problems. Solving a discretized MIOCP with methods from integer optimization, e.g., branch and bound, suffers the combinatorial complexity [11, 4]. Relaxation of the original MIOCP and subsequent reconstruction of the discrete-valued control variables has been successfully applied in many applications [22, 16, 24]. Moreover, in this framework some constraints can be imposed on the switching structure, but switching costs cannot be easily accounted for [23, 24]. Recently, a similar framework has been proposed in [17] to handle also switching costs and state jumps; it overcomes some limitations but it still requires some sort of penalty term to avoid fractional modes and to correctly identify switches.

On the other hand, different reformulations of the original MIOCP are possible. Based on the idea of a time transformation [9], it is possible to recast a MIOCP into an equivalent OCP without discrete-valued control variables. This approach is referred to as *variable time transformation* [12, 21] and *control parametrization enhancing technique* [18]. Similarly, but in a different spirit, those OCPs with discrete-valued control variables only can be reformulated as switching time optimization problems, whenever a discrete-valued control sequence is given [8, 25]. Analogous ideas have been adopted in a bilevel optimization setting to deal with MIOCPs [30, 14]. These works also consider the mode scheduling problem for generating the optimal discrete-valued control sequence and can account for switching costs via the so called *insertion gradient*, which is based on needle-variations methods [14, 29]. Compared to the approach proposed in this paper, in which a maximal number of switches is fixed, these methods allow to introduce new switches where these likely pay off. On the other hand, this technique requires the evaluation of the insertion gradient on a time grid, which in turn depends on the costate [8]. Also, these methods seem to not perform as fast and as reliably as the methods discussed above [24].

This work aims at introducing a novel approach to deal with such challenging problems. Despite its simplicity, it proves effective. As a proof-of-concept, we focus on linear-quadratic problems, aiming at exploring the proposed approach before further development. In fact, an extension can be readily achieved for switched nonlinear dynamical systems (without continuous-valued control variables) with switching costs, based on [26]. Instead, dealing with nonlinear OCPs with both continuous-valued and discrete-valued control inputs and switching costs requires further work and deeper understanding, especially for what concerns the sensitivity analysis of the lower level problem and possible state-control constraints.

The proposed approach is outlined as follows. By considering a given discrete-valued control sequence, the original MIOCP is transformed into an OCP with the continuous-valued control and the switching times as decision variables. The mode sequence can be inspired by practical intuition about the problem or constructed to fit many combinations, as in [12, 21]. Then, a bilevel optimization problem is formulated, aiming at optimizing the switching times at the upper level and the continuous-valued control function at the lower level. The switching times

are optimized at the upper level, consisting of a nonlinear program with linear constraints. We point out that in general this bilevel problem is not equivalent to the problem it originates from [7]. Switching costs are a sneaky element in the objective function, in that they introduce nonconvexity and discontinuity. Contrary to the framework developed in [17], we propose to deal with switching costs by adopting suitable formulations and by exploiting proximal methods. In doing so, the strenuous part moves to the evaluation of a proximal operator [19]. Switching costs can be expressed via the cardinality function, also known as the  $\ell_0$  norm, abusing of terminology. We highlight that the goodness of this formulation depends on the structure of the discrete-valued control sequence discussed above. For completeness, we mention that cardinality and cardinality-constrained optimization problems have equivalent reformulations as mathematical programs with complementarity constraints [10, 5]. In the linear-quadratic case herein presented, some features are exploited. First of all, the lower level problem turns out to be a time-varying linear-quadratic regulator, which admits an unique global minimum [15]; hence, the bilevel problem is an equivalent reformulation [7]. Secondly, for a given discrete-valued control function, the closed-loop optimally controlled system is a linear time-varying system itself; thus, sensitivity analysis can be performed with a direct derivation.

The problem class of interest is stated in Section 2, and the bilevel formulation is introduced. In Section 3, a method for the lower level OCP solution and sensitivity analysis is detailed. Proximal methods for handling the upper level problem are discussed in Section 4. Numerical experiments in Section 5 demonstrate the soundness of our approach. Finally, Section 6 concludes the paper and suggests directions for future research.

## 2 Problem

Consider a switched linear system with some continuous-valued control variables but with one and only one discrete-valued control variable; this is not a restriction [12, 23]. The former are unconstrained, while the latter takes value from a finite set  $\mathcal{V}$ , i.e.,  $\mathbf{u}(t) \in \mathbb{R}^{n_u}$  and  $v(t) \in \mathcal{V}$  respectively, for  $t \in [0, T]$ , with final time  $T \geq 0$ . Let us assume  $N$  switches happen in the time interval  $[0, T]$ , with  $N$  positive and finite, and that a discrete control sequence  $\{v_i\}_{i=0}^N$  is given. Hence, the discrete control function  $v : [0, T] \rightarrow \mathcal{V}$  can be expressed as  $v(t) = v_i$ , for  $t \in [\tau_i, \tau_{i+1})$ ,  $i = 0, \dots, N$ . Thus, it depends only on the switching times  $\boldsymbol{\tau} := (\tau_0, \dots, \tau_{N+1})^\top$ . As proposed in [26], we define  $\delta_i := \tau_{i+1} - \tau_i$ ,  $i = 0, \dots, N$ , and consider the switching intervals  $\boldsymbol{\delta} := (\delta_0, \dots, \delta_N)^\top$  as decision variables; in fact, for any given discrete control sequence, the vector  $\boldsymbol{\delta}$  uniquely identifies the discrete control function. We set the initial time  $\tau_0 = 0$  and define the final time  $T_\delta := \tau_{N+1}$ ; it holds  $\tau_i = \sum_{j=0}^{i-1} \delta_j$  for any  $i = 1, \dots, N+1$ . The switching intervals  $\boldsymbol{\delta}$  are subject to some constraints, i.e., for fixed  $T$

$$\Delta := \{\boldsymbol{\delta} \in \mathbb{R}^{N+1} \mid \delta_i \geq 0, i = 0, \dots, N \wedge T_\delta = T\}, \quad (1)$$

which requires all switching intervals to be nonnegative and to sum up to the desired final time  $T$ ; notice that the feasible set  $\Delta$  resembles a  $(N+1)$  simplex. Then, the dynamics under consideration can be expressed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_\delta(t)\mathbf{x}(t) + \mathbf{B}_\delta(t)\mathbf{u}(t), \quad t \in [0, T_\delta), \quad (2)$$

with piecewise constant  $\mathbf{A}_\delta(t) = \mathbf{A}(v(t))$ ,  $\mathbf{B}_\delta(t) = \mathbf{B}(v(t))$  for  $t \in [0, T_\delta)$ , being  $\mathbf{A} : \mathcal{V} \rightarrow \mathbb{R}^{n_x \times n_x}$ ,  $\mathbf{B} : \mathcal{V} \rightarrow \mathbb{R}^{n_x \times n_u}$  the mode-dependent matrices describing a switched linear system. The state  $\mathbf{x}(\cdot)$  is subject to coupled linear boundary conditions

$$\mathbf{C}_0 \mathbf{x}(0) + \mathbf{C}_T \mathbf{x}(T) = \mathbf{c} \quad (3)$$

with  $\mathbf{C}_0, \mathbf{C}_T \in \mathbb{R}^{n_c \times n_x}$  and  $\mathbf{c} \in \mathbb{R}^{n_c}$ . Our goal is to find the optimal switching intervals  $\boldsymbol{\delta}^* \in \Delta$  and the optimal continuous-valued control function  $\mathbf{u}^* : [0, T] \rightarrow \mathbb{R}^{n_u}$  minimizing a composite objective function

$$J(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) := L(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) + \gamma S(\boldsymbol{\delta}) \quad (4)$$

where the Lagrange term  $L$  is an integral state-control quadratic penalty,

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) := \frac{1}{2} \int_0^{T_\delta} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}^\top \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix} dt, \quad (5)$$

weighted by a symmetric block-diagonal matrix, with positive semidefinite  $\mathbf{Q} \in \mathbb{R}^{n_x \times n_x}$  and positive definite  $\mathbf{R} \in \mathbb{R}^{n_u \times n_u}$ . The switching cost term  $S$  is defined by

$$S(\boldsymbol{\delta}) := \text{card}(\boldsymbol{\delta}) = |\{i \mid \delta_i \neq 0, i = 0, \dots, N\}|, \quad (6)$$

hence it is nonconvex and penalizes the occurrence of nonempty switching intervals. The non-negative parameter  $\gamma$  rules the relative importance of  $L$  and  $S$ . Some comments are in order.

**Remark 1.** *Considering a given discrete control sequence is questionable. However, given a finite upper bound for the number of switches, one can build a sequence to capture any solution and to let the solver to search among all the possible switching combinations, discarding some dynamics and adapting the switching sequence without recourse to integer optimization [25]. Other formulations may introduce different feasible sets  $\Delta$ , e.g., considering additional linear equalities due to minor grids [12, 21].*

**Remark 2.** *The final time  $T$  can be considered as fixed to a desired value or as an optimization variable, yielding a free final time MIOCP. In the latter case, the constraint  $T_\delta = T$  in (1) must be excluded from the definition of the feasible set  $\Delta$ .*

**Remark 3.** *The dynamics in (2) represent a linear piecewise time-invariant system. However, matrix-valued  $\mathbf{A}$  and  $\mathbf{B}$  can be considered both time and mode-dependent, with minor changes. In the linear time-varying case, the state transition matrix can be adopted but without exploiting the exponential matrix.*

**Remark 4.** *State and control cost matrices  $\mathbf{Q}$  and  $\mathbf{R}$  in (5) can be considered time and mode-dependent, analogously to  $\mathbf{A}$  and  $\mathbf{B}$ . Furthermore, a mixed state-control cost can be easily introduced. Additionally, a quadratic Mayer term can be introduced in the cost function  $J$  in order to penalize deviations of the initial and final state. These extensions require minor changes and are omitted for brevity, see [6].*

**Remark 5.** *In the spirit of compressed sensing and sparse optimization, the cardinality-cost  $S$  in (6) plays the role of a regularization term in (4). In fact, it cancels out many feasible vectors  $\boldsymbol{\delta}$  with equivalent associated state trajectories.*

Let us consider a single-objective bilevel optimization problem. At the lower level, the switching intervals are fixed and the continuous-valued control is to be optimized; concurrently, the switching times are decision variables at the upper level, which reads

$$\begin{aligned} & \text{Minimize} && J_\delta(\boldsymbol{\delta}) \\ & \text{subject to} && \boldsymbol{\delta} \in \Delta, \end{aligned} \tag{7}$$

with the reduced cost function  $J_\delta(\boldsymbol{\delta}) := J(\mathbf{x}_\delta, \mathbf{u}_\delta, \boldsymbol{\delta})$ , where  $\mathbf{x}_\delta$  and  $\mathbf{u}_\delta$  solve the lower level problem, namely

$$\begin{aligned} & \text{Minimize} && L(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \\ & \text{subject to} && \dot{\mathbf{x}}(t) = \mathbf{A}_\delta(t)\mathbf{x}(t) + \mathbf{B}_\delta(t)\mathbf{u}(t), && t \in [0, T_\delta), \\ & && \mathbf{C}_0\mathbf{x}(0) + \mathbf{C}_T\mathbf{x}(T_\delta) = \mathbf{c}, \end{aligned} \tag{8}$$

for any given  $\boldsymbol{\delta} \in \Delta$ . The term  $\gamma S(\boldsymbol{\delta})$  can be neglected in (8) because it is a constant.

**Remark 6.** As mentioned above, in the linear-quadratic case the lower level problem can be solved to global optimality for any  $\boldsymbol{\delta} \in \Delta$ , thanks to convexity [15, 13]. Nonetheless, even disregarding the switching costs, the upper level problem is nonlinear and nonconvex in general, thus one may obtain locally optimal switching times and continuous-valued control [26, 17]. In fact, most nonlinear optimizers are only able to detect local minima.

**Remark 7.** The ideas just presented show correspondence with those in [6] on the final time optimization in finite horizon LQR. Analogously here, the final time for each operational mode is subject to optimization. Also, these works share the bilevel perspective.

### 3 Lower Level Problem

The time-varying linear-quadratic problem (8) is well known in literature; here it is briefly discussed to introduce notation and highlight crucial features. From Pontryagin's minimum principle [20], there exists an adjoint function  $\boldsymbol{\lambda} : [0, T_\delta] \rightarrow \mathbb{R}^{n_x}$  and a multiplier  $\boldsymbol{\eta} \in \mathbb{R}^{n_c}$ , such that a solution to (8) satisfies the first-order necessary optimality conditions [13]:

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}\mathbf{x}(t) - \mathbf{A}_\delta(t)^\top \boldsymbol{\lambda}(t), \quad t \in [0, T_\delta), \tag{9a}$$

$$\mathbf{0} = \mathbf{R}\mathbf{u}(t) + \mathbf{B}_\delta(t)^\top \boldsymbol{\lambda}(t), \quad t \in [0, T_\delta), \tag{9b}$$

$$\boldsymbol{\lambda}(0) = \mathbf{C}_0^\top \boldsymbol{\eta}, \tag{9c}$$

$$\boldsymbol{\lambda}(T_\delta) = \mathbf{C}_T^\top \boldsymbol{\eta}, \tag{9d}$$

along with dynamics (2) and boundary conditions (3). One can algebraically solve for the continuous-valued control  $\mathbf{u}$  in (9b) obtaining a linear two-point boundary value problem. The controlled state  $\mathbf{z} := (\mathbf{x}, \boldsymbol{\lambda})$  has linear homogeneous dynamics, namely  $\dot{\mathbf{z}}(t) = \mathbf{H}_\delta(t)\mathbf{z}(t)$ , with the piecewise constant Hamilton matrix  $\mathbf{H}_\delta$  defined for any  $t \in [0, T_\delta]$  by

$$\mathbf{H}_\delta(t) := \begin{bmatrix} \mathbf{A}_\delta(t) & -\mathbf{B}_\delta(t)\mathbf{R}^{-1}\mathbf{B}_\delta(t)^\top \\ -\mathbf{Q} & -\mathbf{A}_\delta(t)^\top \end{bmatrix}. \tag{10}$$

By means of the state transition matrix  $\Phi_\delta(t, t_0)$ ,  $t \geq t_0$ , based on  $\mathbf{z}(t_0)$ , one can write  $\mathbf{z}(t) = \Phi_\delta(t, t_0)\mathbf{z}(t_0)$ , see [15, 6]. Then, considering the final controlled state given as  $\mathbf{z}(T_\delta)$ , boundary conditions (3) and transversality conditions (9c)–(9d) form a linear system, namely  $\mathbf{A}_s(\delta)\mathbf{s}(\delta) = \mathbf{b}_s$ , with  $\mathbf{A}_s(\delta)$  given by

$$\mathbf{A}_s(\delta) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & -\mathbf{C}_0^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_T^\top \\ \mathbf{C}_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \\ \mathbf{C}_T & \mathbf{0} \end{bmatrix} [\Phi_\delta(T_\delta, 0) \quad \mathbf{0}]$$

and  $\mathbf{b}_s = (\mathbf{0}, \mathbf{0}, \mathbf{c})$ . For any given  $\delta$ , the unique solution vector  $\mathbf{s}(\delta)$  collects the (globally optimal) initial controlled state  $\mathbf{z}(0)$  and the associated multiplier  $\boldsymbol{\eta}$ , encapsulating the whole evolution of the closed-loop controlled system.

In order to efficiently solve the upper level problem, some information about the lower level solution sensitivity with respect to the upper level decision variables is needed. In [25, 26] the cost function, gradient and Hessian are efficiently computed for switched autonomous linear and nonlinear systems. In order to take advantage of that work, we make the key observation that, from the upper level viewpoint, the lower level controlled system is autonomous; furthermore, it is even linear and homogeneous. However, there is the need to slightly generalize the problem and extend the results of [26]. In fact, in our mixed-integer case, the initial controlled state  $\mathbf{z}(0)$  is not fixed but depends on the switching times  $\delta$  via the aforementioned linear system.

### 3.1 State Evolution and Derivatives

For any given  $\delta \in \Delta$ , the controlled state  $\mathbf{z}$  at any time  $t \in [\tau_\ell, \tau_{\ell+1}]$ ,  $\ell = 0, \dots, N$ , can be expressed as  $\mathbf{z}(t) = \Phi_\delta(t, \tau_i)\mathbf{z}(\tau_i)$ , with  $0 \leq i \leq \ell$ . Notice that  $\mathbf{H}_\delta$  in (10) is piecewise constant as are  $\mathbf{A}_\delta$  and  $\mathbf{B}_\delta$ ; then, let us denote  $\mathbf{H}_i$  the constant Hamilton matrix in the time interval  $[\tau_i, \tau_{i+1})$ , for  $i = 0, \dots, N$ . Hence, the state transition matrix reads

$$\Phi_\delta(t, \tau_i) = e^{\mathbf{H}_\ell(t-\tau_\ell)} e^{\mathbf{H}_{\ell-1}\delta_{\ell-1}} \dots e^{\mathbf{H}_i\delta_i}. \quad (11)$$

Defining  $\mathcal{E}_i := e^{\mathbf{H}_i\delta_i}$  and denoting  $\mathbf{z}_i = \mathbf{z}(\tau_i)$ , it holds  $\mathbf{z}_{i+1} = \mathcal{E}_i\mathbf{z}_i$  for  $i = 0, \dots, N$ . With direct derivation, the controlled state sensitivity to  $\delta$  can be expressed recursively, through the chain rule, as

$$\frac{\partial \mathbf{z}_{i+1}}{\partial \delta_j} = \mathcal{E}_i \frac{\partial \mathbf{z}_i}{\partial \delta_j} + \frac{d\mathcal{E}_i}{d\delta_j} \mathbf{z}_i \quad (12)$$

for  $j = 0, \dots, N$ , where  $d\mathcal{E}_i/d\delta_i = \mathbf{H}_i e^{\mathbf{H}_i\delta_i} = \mathbf{H}_i \mathcal{E}_i$  from the definition, otherwise  $d\mathcal{E}_i/d\delta_j = \mathbf{0}$  for  $i \neq j$ . Sensitivity of initial state  $\mathbf{z}_0$  with respect to  $\delta$  can be computed by solving an additional linear system [6], namely

$$\mathbf{A}_s(\delta) \frac{\partial \mathbf{s}}{\partial \delta_i}(\delta) = -\frac{\partial \mathbf{A}_s}{\partial \delta_i}(\delta) \mathbf{s}(\delta), \quad (13)$$

for which one could store a factorization of  $\mathbf{A}_s(\delta)$ , e.g., the LU decomposition. In turn, constructing the right hand side vector requires the sensitivity of the state transition matrix  $\Phi_\delta(T_\delta, 0)$ .

Because of (11), matrix  $\Phi(\tau_b, \tau_a)$  does not depend on  $\delta_j$ ,  $j = 0, \dots, N$ , if either  $j < a$  or  $j \geq b$ ; instead, if  $a \leq j < b$ , expansion of  $\Phi(\tau_b, \tau_a)$  based on (11) and direct differentiation yield:

$$\begin{aligned} \frac{\partial \Phi_\delta}{\partial \delta_j}(\tau_b, \tau_a) &= \frac{\partial}{\partial \delta_j} [\Phi(\tau_b, \tau_{j+1}) \Phi(\tau_{j+1}, \tau_j) \Phi(\tau_j, \tau_a)] \\ &= \Phi(\tau_b, \tau_{j+1}) \mathbf{H}_j \Phi(\tau_{j+1}, \tau_a) . \end{aligned} \quad (14)$$

Once the state sensitivity is available, the cost function can be easily analyzed.

### 3.2 Cost Function and Gradient

Let us define the piecewise constant, symmetric controlled state-cost matrix

$$\mathbf{\Pi}_\delta(t) := \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_\delta(t)^\top \mathbf{R}^{-1} \mathbf{B}_\delta(t) \end{bmatrix} \quad (15)$$

for  $t \in [0, T_\delta]$ , based on (9b); denote  $\mathbf{\Pi}_i = \mathbf{\Pi}_\delta(t)$  for  $t \in [\tau_i, \tau_{i+1})$ ,  $i = 0, \dots, N$ . Then, for any given  $\delta$ , the reduced Lagrange cost function  $L_\delta(\delta) := L(\mathbf{x}_\delta, \mathbf{u}_\delta, \delta)$  can be expressed from (5) and (15) as

$$L_\delta(\delta) = \frac{1}{2} \int_0^{T_\delta} \mathbf{z}(t)^\top \mathbf{\Pi}_\delta(t) \mathbf{z}(t) dt = \sum_{i=0}^N \frac{1}{2} \mathbf{z}_i^\top \Upsilon_i \mathbf{z}_i \quad (16)$$

where  $\Upsilon_i := \int_0^{\delta_i} e^{\mathbf{H}_i^\top t} \mathbf{\Pi}_i e^{\mathbf{H}_i t} dt$  for  $i = 0, \dots, N$ . Notice that each matrix  $\Upsilon_i$  is symmetric and depends on  $\delta_i$  only. From the fundamental theorem of calculus, it follows that  $d\Upsilon_i/d\delta_i = e^{\mathbf{H}_i^\top \delta_i} \mathbf{\Pi}_i e^{\mathbf{H}_i \delta_i} = \mathcal{E}_i^\top \mathbf{\Pi}_i \mathcal{E}_i$ . Furthermore, we point out that matrices  $\mathcal{E}_i$  and  $\Upsilon_i$ ,  $i = 0, \dots, N$ , can be computed pairwise by means of a single exponential matrix evaluation [28, 26]. For  $i = 0, 1, \dots, N$ , from (16) through the chain rule, direct differentiation yields

$$\frac{\partial L_\delta}{\partial \delta_i}(\delta) = \frac{1}{2} \mathbf{z}_{i+1}^\top \mathbf{\Pi}_i \mathbf{z}_{i+1} + \sum_{j=0}^N \mathbf{z}_j^\top \Upsilon_j \frac{\partial \mathbf{z}_j}{\partial \delta_i}, \quad (17)$$

where the identity  $\mathbf{z}_{i+1} = \mathcal{E}_i \mathbf{z}_i$  is used.

Finally, notice that second derivatives can also be obtained, as in [25, 26]; however, they are not reported here for brevity, nor exploited in the numerical results.

## 4 Upper Level Problem

The upper level problem (7) has a nonlinear separable objective function (4), consisting of the smooth convex term  $L$  and the nonconvex term  $S$ , and it is subject to a simplex constraint (1). If switching costs are included, i.e.,  $\gamma > 0$ , we propose to solve (7) via a proximal gradient method [3, 19], which is based on a recursive update:

$$\delta^{k+1} \in \text{prox}_\alpha^{S, \Delta} \left( \delta^k - \alpha \nabla L(\delta^k) \right) . \quad (18)$$

This requires that the gradient of  $L$  with respect to  $\delta$  and the (possibly constrained, set-valued) proximal operator of  $S$  are provided as oracles. An analytic expression of the former has been derived in Section 3, while the latter is defined as

$$\text{prox}_\alpha^{S,\Delta}(\delta) := \arg \min_{\mathbf{p} \in \Delta} \left( \alpha \text{card}(\mathbf{p}) + \frac{1}{2} \|\mathbf{p} - \delta\|^2 \right) \quad (19)$$

for any  $\delta \in \mathbb{R}^{N+1}$ , any feasible set  $\Delta$  and any positive scalar stepsize  $\alpha$  [19]. Hence, the proximal operator is an optimization problem itself and this needs to be solved at least once per each iteration of the adopted first-order method. Thus, the whole proposed approach, as well as proximal algorithms in general, benefit from the availability of efficient routines for solving the proximal problem. To our knowledge, the cardinality function has attracted little research effort compared to other sparsity-inducing penalties, especially in the sparse optimization and image processing community [1]. Furthermore, the presence of a simplex-constraint in the proximal operator seems unusual. For free time MIOCPs, only nonnegativity is required in (1) and the proximal point  $\pi$  of  $\delta$  has a closed-form expression, namely  $\pi_i \ni 0$  if  $\delta_i \leq \sqrt{2\alpha}$  and  $\pi_i \ni \delta_i$  if  $\delta_i \geq \sqrt{2\alpha}$ ,  $i = 0, \dots, N$ . Instead, in the case of fixed time MIOCPs, this is not possible, mainly because of the additional equality in the simplex constraint in (1). However, an efficient procedure can be devised by exploiting the problem structure. In fact, the combinatorial nature of the proximal problem can be overcome by noticing that, for any fixed number of zero elements, it turns into a simplex-constrained least-squares problem; the zero elements correspond to the lowest entries of  $\delta$ . Then, through the Lagrange function of the constrained continuous problem, it is possible to express proximal point  $\pi^{[m]}$  and multiplier  $\lambda^{[m]}$  as a function of the number of zeros  $m \in \{0, \dots, N\}$  (if  $T > 0$ , otherwise  $m = N + 1$ ). Assuming, without loss of generality, vector  $\delta$  sorted in ascending order, the cost  $c^{[m]}$  associated with each value of  $m$  can be evaluated:

$$\pi_i^{[m]} = \begin{cases} 0 & \text{if } i < m, \\ \delta_i + \lambda^{[m]} & \text{if } i \geq m \end{cases}, \quad i = 0, \dots, N, \quad (20)$$

$$\lambda^{[m]} = \frac{1}{N + 1 - m} \left( T - \sum_{i=m}^N \delta_i \right), \quad (21)$$

$$c^{[m]} = \alpha(N + 1 - m) + \frac{1}{2} \left\| \pi^{[m]} - \delta \right\|^2. \quad (22)$$

A feasible, optimal value  $m^*$  satisfies  $\delta_{m^*} + \lambda^{[m^*]} > 0$  and  $m^* = \arg \min_m c^{[m]}$ , and the resulting proximal point is  $\pi = \pi^{[m^*]}$ . The implemented algorithm<sup>1</sup> exhibits quasilinear time complexity and requires approximately 70  $\mu\text{s}$  for an instance with 100 entries. Some comments are in order.

**Remark 8.** *At some iterations of the accelerated proximal gradient method [3], infeasible switching intervals may be passed to the lower level problem. In fact, even for a convex feasible set  $\Delta$ , the recursion can generate a point not lying in  $\Delta$  [3, 19]. If needed, an additional call to the proximal operator or a fallback to the proximal gradient method may overcome this issue, possibly degrading the convergence properties of the accelerated method.*

<sup>1</sup>The source code is deposited on Zenodo at DOI: [10.5281/zenodo.2567457](https://doi.org/10.5281/zenodo.2567457).



**Remark 9.** *If switching costs are omitted, as in [25, 26], one can resort to standard nonlinear optimization methods such as, e.g., interior point and sequential quadratic programming.*

## 5 Numerical Results

For the numerical investigations we adopt the `fista` routine as from the publicly available FOM package [2], implementing an accelerated proximal gradient method [3], with default optional parameters (nonmonotone algorithm with backtracking, `maxiter` = 1000, `eps` = 1e-5). As initial guess, we consider the solution to the problem with  $\gamma = 0$ , obtained with the `fmincon` routine in MATLAB [27], which starts from a feasible initial guess with equal entries, with value  $T/(N + 1)$ , and whose optional parameters are set as follows: interior point method with BFGS Hessian approximation; specified cost function gradient; optimality, constraint violation and step tolerance set to  $10^{-10}$ . All examples are implemented in MATLAB 2018b [27] and run on Ubuntu 16.04, with Intel Core i7-8700 3.2 GHz and 16 GB of RAM.

Table 1 summarizes the optimization process, performed by either `fmincon` or `fista`, in terms of number of iterations, computation time and time spent (in percentage) for evaluating function  $L$  and its gradient or function  $S$  and its proximal operator. Also, it matches switching cost and cardinality<sup>2</sup> of the optimal vector of switching intervals.

### 5.1 An Academic Example Involving Switching Costs

Consider the switched system (without continuous-valued control) from [17, Example 1] described by

$$\dot{x}(t) = \begin{cases} +1 & \text{if } v(t) = 1 \\ -1 & \text{if } v(t) = 2 \end{cases}, \quad t \in [0, T],$$

with fixed final time  $T = 5$ , initial state  $x(0) = 0$  and state cost  $Q = 1$ . Also, consider the fixed discrete-valued control sequence comprising  $N = 24$  switches and starting as  $\{1, 2, 1, 2, \dots\}$ , and a switching cost  $\gamma \in \{0, 0.1, 0.5, 1\}$ . The system state and dynamics are augmented to fit (2), as in [25]. The optimal state trajectories are reported in Fig. 1. The tracking error grows with the switching cost  $\gamma$ , while the number of switches decreases. Although these results are similar to those obtained in [17], no timing is reported in that work. As one may expect from first-order methods, the `fista` routine requires many iterations before convergence. The optimization process seems to converge faster for higher values of  $\gamma$ , see Tab. 1. We argue that, when the switching cost prevails, the proximal operator of the cardinality-based regularization term removes many degrees of freedom by setting many decision variables to zero.

### 5.2 Switched Viscous-Elastic System

Consider a one-dimensional, horizontal system composed by  $n = 10$  point-masses, connected in series by elements consisting of an elastic spring and a viscous damper in parallel. Position

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<sup>2</sup>When  $\gamma = 0$ , we set  $\text{card}(\delta) := |\{\delta_i | \delta_i > 10^{-4}, i = 0, \dots, N\}|$ .

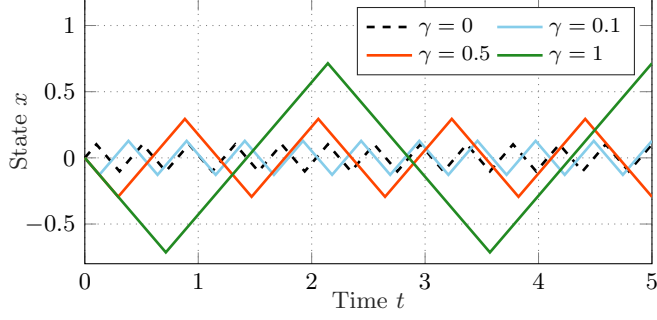


Figure 1: State  $x$  versus time  $t$ , for  $\gamma \in \{0, 0.1, 0.5, 1\}$ .

$x_i$  and speed  $s_i = \dot{x}_i$  describe the state of the  $i$ -th point-mass, for  $i = 1, \dots, n$ . The first mass is connected to  $x_0 = 0$  through a spring-damper element. A horizontal, continuous-valued force  $u$  acts on the  $n$ -th point-mass. Each point-mass has mass  $m_i = 1$ ; each spring has elastic constant  $k_i = 1$  and null equilibrium length; each damper has mode-dependent viscous coefficient  $b_i(v) = 1$ , if  $v = i$ , otherwise  $b_i(v) = 0.1$ , for  $i = 1, \dots, n$  and  $v = 1, \dots, n$ . Consider the final time  $T = 20$  and the sequence  $\{1, \dots, n, 1, \dots\}$ , with  $N = 3n - 1 = 29$  switches. Dynamics are expressed as in (2); for the  $i$ -th point-mass,  $1 < i < n$ , it holds (omitting time  $t$ )

$$m_i \dot{s}_i = -k_i(x_i - x_{i-1}) - b_i(v)(s_i - s_{i-1}) - k_{i+1}(x_i - x_{i+1}) - b_{i+1}(v)(s_i - s_{i+1}). \quad (23)$$

Starting from a far-from-equilibrium configuration ( $x_i = i$ ,  $s_i = 0$ , for  $i = 1, \dots, n$ ), the optimal control problem consists in reaching  $s_5(T) = s_{10}(T) = 0$  while minimizing a quadratic cost on both positions  $x_i$ ,  $i = 1, \dots, n$ , and continuous-valued control  $u$  (both with unitary cost weight). Solutions for  $\gamma \in \{0, 20, 50, 75, 100\}$  are depicted in Fig. 2. Notice that, even with continuous state and adjoint, the continuous-valued control would jump if the control matrix  $\mathbf{B}_\delta$  changed at the switching times, due to (9b).

Evaluating the smooth cost function  $L$  and its gradient takes most of the computation time, see Tab. 1. In fact, this corresponds to solving an instance of the lower level problem (8). Some enhancements may be possible, by exploiting the linear structure of the problem and pre-computing some matrix operations [26]. Lastly, by comparing the CPU times for  $\gamma = 0$  and  $\gamma > 0$  in Tab. 1, we stress that, due to the combinatorial nature, approaches based on extensive search do not seem competitive, as shown in [11].

## 6 Conclusions

We presented an alternative, novel approach for dealing with mixed-integer optimal control problems constrained by ordinary differential equations and accounting for switching costs. Our original approach mingles ideas from bilevel programming, optimal control and sparse optimization, allowing to tackle the challenges offered by switching costs in MIOCPs. Numerical investigations on linear-quadratic problems have demonstrated the viability of the approach.

Future research needs to extend the present work to a more general class of problems and to address some questions: can sensitivity analysis be exploited to incorporate nonlinear dynam-

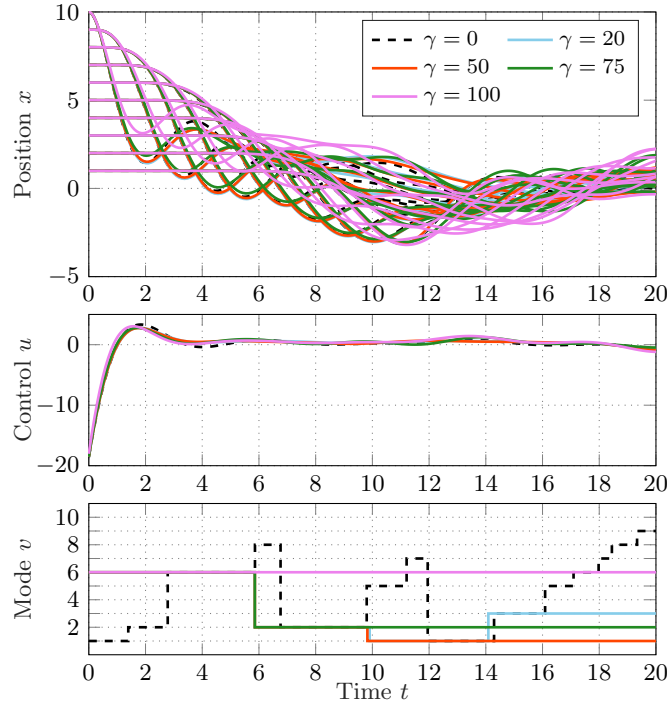


Figure 2: Positions  $\mathbf{x}$ , continuous-valued  $u$  and discrete-valued  $v$  optimal controls versus time  $t$ , for  $\gamma \in \{0, 20, 50, 75, 100\}$ .

Table 1: Performance Profile for the Example Problems

Example	Sw. cost $\gamma$ [-]	card $\delta$ [-]	Iter. [-]	CPU time [s]	CPU time $L / S$ [%]
Academic [17]	0	25	72	0.43	75.6 / -
	0.1	22	876	5.98	98.3 / 0.7
	0.5	9	369	1.80	97.9 / 1.0
	1	4	70	0.29	93.5 / 2.5
Switched viscous- elastic system	0	14	62	1.93	90.8 / -
	20	4	73	1.28	97.9 / 0.8
	50	3	131	2.01	98.5 / 0.6
	75	2	26	0.39	95.2 / 1.7
	100	1	2	0.04	76.4 / 6.3

ics? How can state-dependent and sequence-dependent switching costs be handled? Also, it is appealing to adopt second-order methods, e.g., proximal Newton-type methods.

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