Formulari de Física Quàntica II

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1 Oscil·lador Harmònic

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \quad \begin{matrix}[\hat{x},\hat{p}] = i\hbar \\ p \leftrightarrow -i\hbar\frac{\partial}{\partial x}\end{matrix}$$

$$a = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}) \quad a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$$

$$[a, a^\dagger] = 1, \hat{N} = a^\dagger a$$

$$a|0\rangle = 0 \quad a|n\rangle = \sqrt{n}|n-1\rangle \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a^n|0\rangle = 0, \forall n \neq 0 \quad {a^\dagger}^n|0\rangle = \sqrt{n!}|n\rangle$$

$$\langle x^2\rangle_n = \frac{\hbar}{2m\omega}(2n+1) \quad \langle x^4\rangle_n = \left(\frac{\hbar}{2m\omega}\right)^2(6n^2+6n+3)$$

$$[a, {a^\dagger}^n] = n{a^\dagger}^{n-1} \quad [a^n, a^\dagger] = na^{n-1}$$

$$[a, f(a^\dagger)] = df(a^\dagger)/da^\dagger \quad [f(a), a^\dagger] = df(a)/da$$

$$\underset{\text{Singlet}}{|\psi\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}} \qquad \underset{\text{Pou}}{E_n = \frac{n^2\hbar^2\pi^2}{8ma^2}}$$

1.1 Estats coherents

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - bx = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{b}{m\omega^2}\right) - \frac{b^2}{2m\omega^2}$$

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$(\langle\alpha|a^\dagger) = \alpha^*\langle\alpha| \quad |\alpha(t)\rangle = e^{-\frac{i\omega t}{2}}|\alpha e^{-i\omega t}\rangle$$

2 Moment Angular

2.1 Total

$$J = L + S$$
$$J^2 = J_x^2 + J_y^2 + J_z^2$$

- **Commutadors**

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \quad [J^2, J_z] = 0$$
$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \quad [L^2, L_z] = 0$$
$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad [S^2, S_z] = 0$$

- ***Ladder***: Anàleg amb S i L

$$J_+ = J_x + iJ_y \qquad J_- = J_x - iJ_y$$
$$J_x = \frac{1}{2}(J_+ + J_-) \qquad J_y = \frac{1}{2i}(J_+ - J_-)$$
$$J_+J_- = J^2 - J_z^2 + \hbar J_z \quad J_-J_+ = J^2 - J_z^2 - \hbar J_z$$

- **Commutadors**

$$[J_+, J_-] = 2\hbar J_z \quad [J_z, J_\pm] = \pm\hbar J_\pm \quad [J^2, J_\pm] = 0$$
$$[J^2, J_z] = 0 \leftrightarrow \text{ Tenen base comuna}$$
$$J_z|j,m\rangle = \hbar m|j,m\rangle$$
$$J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$$
$$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}|j,m\pm1\rangle$$

2.2 Spin

- **Spin** $\frac{1}{2}$: $\{|\uparrow\rangle, |\downarrow\rangle\}$ on $s = \frac{1}{2}, m = \pm\frac{1}{2}$.

$S_i = \frac{\hbar}{2}\sigma_i$ on $i \in \{x, y, z\}$

$$\sigma_x|\pm\rangle = |\mp\rangle \quad \sigma_y|\pm\rangle = \pm i|\mp\rangle \quad \sigma_z|\pm\rangle = \pm|\pm\rangle$$

$$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \quad |+\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \quad |-\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I} \quad \sigma_i^2 = \mathbb{I} \quad \sigma_i\sigma_j = i\sigma_k$$

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\sigma_{\vec{n}} = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$$

$$|+\rangle_{\vec{n}} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\end{pmatrix} \quad |-\rangle_{\vec{n}} = \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\\ \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}}\end{pmatrix}$$

- **Spin** 1: $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ on $s = 1, m = 1, 0, -1$

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \quad J_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\1\\1/\sqrt{2}\end{pmatrix} \quad |0\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \quad |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\-1\\1/\sqrt{2}\end{pmatrix}$$

$$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\i\\-1/\sqrt{2}\end{pmatrix} \quad |0\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \quad |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\-i\\-1/\sqrt{2}\end{pmatrix}$$

$$|+\rangle_z = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |0\rangle_z = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad |-\rangle_z = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

2.3 Sistema 2 partícules

Amb 2 partícules A i B: $J_\xi = J_\xi^A + J_\xi^B$ on $\xi \in \{x, y, z, \pm\}$

$$\begin{aligned} J^2 = \left(\vec{J}^A + \vec{J}^B\right)^2 &= J_A^2 + J_B^2 + 2J_AJ_B \\ &= J_A^2 + J_B^2 + 2\left(J_x^AJ_x^B + J_y^AJ_y^B + J_z^AJ_z^B\right) \\ &= J_A^2 + J_B^2 + 2J_z^AJ_z^B + J_+^AJ_-^B + J_-^AJ_+^B \end{aligned}$$

3 Mètode Variacional

Donat $H = K + V$ i $\varphi_a(x)$, calculem $\langle H\rangle_{\varphi_a(x)} = E(a)$. Trobem el mínim d'$E$ fent $\frac{dE}{da} = 0$ i el mínim a^* dóna una cota d'energia $E(a^*)$.

3.1 Valors esperats

$$\langle x^2\rangle_{|\psi\rangle} = \int_{-\infty}^{+\infty}\psi^*x^2\psi dx \quad \langle p^2\rangle_{|\psi\rangle} = \int_{-\infty}^{+\infty}\left|-i\hbar\frac{\partial\psi}{\partial x}\right|^2 dx$$

$$\langle V\rangle_{|\psi\rangle} = \int_{-\infty}^{+\infty}\psi^*V(x)\psi dx \quad \langle K\rangle_{|\psi\rangle} = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left|\frac{\partial\psi}{\partial x}\right|^2 dx$$

3.2 Funcions d'ona de prova i K

$$\varphi_\alpha(x) = \sqrt{a}e^{-\alpha|x|} \quad \varphi_\beta(x) = \left(\frac{\beta}{\pi}\right)^{1/4}e^{-1/2\beta x^2}$$
$$\langle K\rangle_\alpha = \frac{\hbar^2\alpha^2}{2m} \qquad \langle K\rangle_\beta = \frac{\hbar^2\beta}{4m}$$
$$\varphi_\gamma(x) = \sqrt{\frac{3}{\gamma^3}}(\gamma - 2|x|) \quad \varphi_\lambda(x) = \left(\frac{4\lambda^3}{\pi}\right)^{1/4}xe^{-1/2\lambda x^2}$$
$$\langle K\rangle_\gamma = \frac{6\hbar^2\alpha^2}{m\gamma^2} \qquad \langle K\rangle_\lambda = \frac{3\hbar^2\lambda}{4m}$$

4 Pertorbacions

Si tenim $H = H_0 + H_1$ on $H_0|\psi_n^{(0)}\rangle = \varepsilon_n^{(0)}|\psi_n^{(0)}\rangle$, aleshores:

$$E_n^{(1)} = \langle\psi_n^{(0)}|H_1|\psi_n^{(0)}\rangle \qquad E_n^{(2)} = \sum_{m\neq n}\frac{|\langle\psi_n^{(0)}|H_1|\psi_m^{(0)}\rangle|^2}{\varepsilon_n^{(0)} - \varepsilon_m^{(0)}}$$

$$E_n^{(r)} = \langle\psi_n^{(0)}|H_1|\psi_n^{(r-1)}\rangle \quad |\psi_n^{(1)}\rangle = \sum_{m\neq n}\frac{\langle\psi_m^{(0)}|H_1|\psi_n^{(0)}\rangle}{\varepsilon_n^{(0)} - \varepsilon_m^{(0)}}|\psi_m^{(0)}\rangle$$

Al cas degenerat, funciona igual, però utilitzem la base $\{|\psi_{n,l}\rangle = |n,\ell\rangle\}$, que és ortogonal $\langle n,\ell'|H_1|n,\ell\rangle = 0$.

5 Integrals i coses

$$\int_0^\infty r^ne^{-\alpha r}dr = \frac{n!}{\alpha^{n+1}} \qquad \int_{-\infty}^{+\infty}e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty}x^me^{-ax^2}dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{a^{\frac{m+1}{2}}} \quad \Gamma\left(m + \frac{1}{2}\right) = \frac{1\cdot3\cdots(2m-1)}{2^m}\sqrt{\pi}$$

$$\text{Virial}: V(x) = ax^n \qquad \begin{matrix} 2\langle T\rangle = n\langle V\rangle \\ 2\langle K\rangle = \left\langle x\frac{dV}{dx}\right\rangle \end{matrix}$$