3 Formulari física quàntica II

3.1 Estats de Fock

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$$

$$N = a^\dagger a, \quad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(N + \frac{1}{2}\right)$$

$$\begin{aligned} a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle, \quad & a^\dagger\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle \\ N\left|n\right\rangle &= n\left|n\right\rangle, & H\left|n\right\rangle &= \hbar\omega\left(n+\frac{1}{2}\right)\left|n\right\rangle \end{aligned}$$

$$(\Delta A)^2_\psi = \left\langle A^2\right\rangle_\psi - \left\langle A\right\rangle^2_\psi$$

$$\left|\psi(t)\right\rangle = e^{-iHt/\hbar}\left|\psi(0)\right\rangle \qquad \text{(Evolució temporal)}$$

3.2 Commutadors

$$\left[a, a^\dagger\right] \equiv 1 \Leftrightarrow [x,p] = i\hbar$$

$$\left[f(a), a^\dagger\right] = \frac{\mathrm{d}f(a)}{\mathrm{d}a}, \quad \left[a, f(a^\dagger)\right] = \frac{\mathrm{d}f(a^\dagger)}{\mathrm{d}a^\dagger}$$

$$[A+B, C] = [A,C] + [B,C], \quad [A, BC] = [A,B]C + B[A,C]$$

3.3 Estats coherents

$$\begin{aligned} a\left|\alpha\right\rangle &= \alpha\left|\alpha\right\rangle, \quad f(a)\left|\alpha\right\rangle = f(\alpha)\left|\alpha\right\rangle \\ \left|\alpha\right\rangle &= D(\alpha)\left|0\right\rangle = e^{a^\dagger\alpha - a\alpha^\star}\left|0\right\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \end{aligned}$$

$$e^{A+B} = e^{-[A,B]/2}e^A e^B \qquad \text{(Baker–Campbell–Hausdorff)}$$

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha, \quad D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^\star$$

$$\left|\alpha(t)\right\rangle = e^{-i\omega t/2}\left|e^{-i\omega t}\alpha\right\rangle \qquad \text{(Evolució temporal)}$$

$$\begin{aligned} \left\langle x\right\rangle_t &= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\{\alpha(t)\} = x_0\cos(\omega t) - \frac{p_0}{m\omega}\sin(\omega t) \\ \left\langle p\right\rangle_t &= \sqrt{2\hbar m\omega}\,\mathrm{Im}\{\alpha(t)\} = p_0\cos(\omega t) - m\omega x_0\sin(\omega t) \end{aligned}$$

$$(\Delta x)_t = (\Delta x)_0 = \sqrt{\frac{\hbar}{2m\omega}}, \quad (\Delta p)_t = (\Delta p)_0 = \sqrt{\frac{\hbar m\omega}{2}}$$

3.4 Translacions i momentum kicks

$$\begin{aligned} \alpha \in \mathbb{C}\backslash\mathbb{R} &\Rightarrow D(\alpha) = e^{ip_0\hat{x}/\hbar} \qquad &\text{(Mom. kick de } p_0\text{)} \\ \alpha \in \mathbb{R} &\Rightarrow D(\alpha) = e^{-ix_0\hat{p}/\hbar} &\text{(Translació de } x_0\text{)} \end{aligned}$$

3.5 Moment angular

$$J_\pm = J_x \pm J_-, \quad J_x = \frac{1}{2}(J_+ + J_-), \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$\begin{aligned} [J_i, J_j] &= i\hbar\epsilon_{ijk}J_k, \quad & [J_z, J_\pm] &= \pm\hbar J_\pm \\ [J_+, J_-] &= 2\hbar J_z, & \left[J^2, J_{z,\pm}\right] &= 0 \end{aligned}$$

$$\begin{aligned} J_\pm\left|j,m\right\rangle &= \hbar\sqrt{j(j+1) - m(m\pm 1)}\left|j, m\pm 1\right\rangle \\ J^2\left|j,m\right\rangle &= \hbar^2 j(j+1)\left|j,m\right\rangle \\ J_z\left|j,m\right\rangle &= \hbar m\left|j,m\right\rangle \end{aligned}$$

3.6 Spin–1/2

$$S_i = \frac{\hbar}{2}\sigma_i, \quad \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k, \quad \begin{aligned} \sigma_x\left|\pm\right\rangle &= \left|\mp\right\rangle \\ \sigma_y\left|\pm\right\rangle &= \pm i\left|\mp\right\rangle \\ \sigma_z\left|\pm\right\rangle &= \pm\left|\pm\right\rangle \end{aligned}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_{\hat{n}} = \hat{n}\cdot\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

3.7 Spin–1

$$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_\pm = \sqrt{2}\hbar\begin{pmatrix} 0 & 1_+ & 0 \\ 1_- & 0 & 1_+ \\ 0 & 1_- & 0 \end{pmatrix}$$

$$S_+S_- = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_-S_+ = \hbar^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow S^2 = S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+) = 2\hbar^2\mathbb{1}$$

3.8 Sistemes de dues partícules

$$\begin{aligned} J_z &= S_z^A + S_z^B \\ J^2 &= \left(\vec{S}^A + \vec{S}^B\right)^2 = S_A^2 + S_B^2 + 2S_z^A S_z^B + S_+^A S_-^B + S_-^A S_+^B \end{aligned}$$

$$\text{spin} = \frac{p}{2}, \quad \dim = p + 1 \qquad \text{(Young Tableaux, } SU(2)\text{)}$$

4 Formulari física quàntica II

4.1 Estats de Fock

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right), \quad N = a^\dagger a$$
$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \quad a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle, \quad N\left|n\right\rangle = n\left|n\right\rangle$$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega\left(N + \frac{1}{2}\right)$$

$$2\left\langle K\right\rangle = \left\langle x\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle \qquad \text{(Teroema del virial)}$$

4.2 Mètode variacional

$$E_1 \leq \min\left\{\frac{\left\langle\psi_\alpha|H|\psi_\alpha\right\rangle}{\|\psi_\alpha\|}\right\}$$

$$\left\langle K\right\rangle = \frac{\hbar^2}{2m}\int\left|\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|^2\mathrm{d}x = -\frac{\hbar^2}{2m}\int\psi^\star(x)\,\psi''(x)\,\mathrm{d}x$$
$$\left\langle V\right\rangle = \int\psi^\star(x)\,V(x)\,\psi(x)\,\mathrm{d}x$$

4.3 Pou infinit

$$\left[-\frac{L}{2},\frac{L}{2}\right] \Rightarrow \begin{cases}\phi_n(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right), & n \text{ senar}\\ \phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), & n \text{ parell}\end{cases}, \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

4.4 Pertorbacions

$$H = H_0 + H_1$$
$$E_n^{(0)} = \left\langle n|H_0|n\right\rangle, \quad E_n^{(1)} = \left\langle n|H_1|n\right\rangle, \quad E_n^{(2)} = \sum_{m\neq n}\frac{\left|\left\langle n|H_1|m\right\rangle\right|^2}{E_n^{(0)} - E_m^{(0)}}$$

4.5 Posició (Fock)

$$\begin{aligned}
x^2\left|0\right\rangle &= \left|0\right\rangle + \sqrt{2}\left|2\right\rangle\\
x^2\left|1\right\rangle &= 3\left|1\right\rangle + \sqrt{6}\left|3\right\rangle\\
x^2\left|2\right\rangle &= \sqrt{2}\left|0\right\rangle + 5\left|2\right\rangle + 2\sqrt{3}\left|4\right\rangle\\
x^3\left|0\right\rangle &= 3\left|1\right\rangle + \sqrt{6}\left|3\right\rangle\\
x^3\left|1\right\rangle &= 3\left|0\right\rangle + 6\sqrt{2}\left|2\right\rangle + 2\sqrt{6}\left|4\right\rangle\\
x^3\left|2\right\rangle &= 6\sqrt{2}\left|1\right\rangle + 9\sqrt{3}\left|3\right\rangle + 2\sqrt{15}\left|5\right\rangle\\
x^4\left|0\right\rangle &= 3\left|0\right\rangle + 6\sqrt{2}\left|2\right\rangle + 2\sqrt{6}\left|4\right\rangle\\
x^4\left|1\right\rangle &= 15\left|1\right\rangle + 10\sqrt{6}\left|3\right\rangle + 2\sqrt{30}\left|5\right\rangle\\
x^4\left|2\right\rangle &= 6\sqrt{2}\left|0\right\rangle + 39\left|2\right\rangle + 28\sqrt{3}\left|4\right\rangle + 6\sqrt{10}\left|6\right\rangle
\end{aligned}$$

4.6 Integrals

$$\frac{\mathrm{d}|x|}{\mathrm{d}x} = 2\vartheta(x) - 1, \quad \frac{\mathrm{d}\vartheta(x)}{\mathrm{d}x} = \delta(x), \quad \int\delta(x-x_0)f(x)\,\mathrm{d}x = f(x_0)$$

$$\int_0^\infty e^{-ax^2}x^m\,\mathrm{d}x = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)}{a^{(m+1)/2}}, \quad \Gamma(n+1) = n\Gamma(n), \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$\int_{-\infty}^\infty e^{-ax^2}x^2\,\mathrm{d}x = \frac{1}{2a}\sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^\infty e^{-ax^2}x^4\,\mathrm{d}x = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}}$$
$$\int_0^\infty e^{-ax}x^n\,\mathrm{d}x = \frac{n!}{a^{n+1}}, \quad 0 \leq n$$

4.7 Moment angular

$$J_\pm = J_x \pm J_-, \quad J_x = \frac{1}{2}(J_+ + J_-), \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$\begin{aligned}
[J_i, J_j] &= i\hbar\epsilon_{ijk}J_k, & [J_z, J_\pm] &= \pm\hbar J_\pm\\
[J_+, J_-] &= 2\hbar J_z, & \left[J^2, J_{z,\pm}\right] &= 0
\end{aligned}$$

$$\begin{aligned}
J_\pm\left|j,m\right\rangle &= \hbar\sqrt{j(j+1) - m(m\pm1)}\left|j,m\pm1\right\rangle\\
J^2\left|j,m\right\rangle &= \hbar^2 j(j+1)\left|j,m\right\rangle\\
J_z\left|j,m\right\rangle &= \hbar m\left|j,m\right\rangle
\end{aligned}$$

4.8 Spin–1/2

$$S_i = \frac{\hbar}{2}\sigma_i, \quad \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k, \quad \begin{aligned}\sigma_x\left|\pm\right\rangle &= \left|\mp\right\rangle\\ \sigma_y\left|\pm\right\rangle &= \pm i\left|\mp\right\rangle\\ \sigma_z\left|\pm\right\rangle &= \pm\left|\pm\right\rangle\end{aligned}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad S_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad S_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$\left|+\right\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \left|+\right\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad \left|+\right\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$$
$$\left|-\right\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \quad \left|-\right\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \quad \left|-\right\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$$

4.9 Spin–1

$$S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad S_\pm = \sqrt{2}\hbar\begin{pmatrix}0 & 1_+ & 0\\ 1_- & 0 & 1_+\\ 0 & 1_- & 0\end{pmatrix}$$

$$S_+S_- = \hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad S_-S_+ = \hbar^2\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$\Rightarrow S^2 = S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+) = 2\hbar^2\mathbb{1}$$

$$\left|+\right\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\ 1\\ 1\sqrt{2}\end{pmatrix}, \quad \left|0\right\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}, \quad \left|-\right\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\ -1\\ 1\sqrt{2}\end{pmatrix}$$
$$\left|+\right\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\ i\\ -1\sqrt{2}\end{pmatrix}, \quad \left|0\right\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}, \quad \left|-\right\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\ -i\\ -1\sqrt{2}\end{pmatrix}$$
$$\left|+\right\rangle_z = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \quad \left|0\right\rangle_z = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \quad \left|-\right\rangle_z = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

4.10 Eigenstuff a 2 × 2

$$\begin{pmatrix}a & c+id\\ c-id & b\end{pmatrix} = \frac{a+b}{2}\mathbb{1} + \frac{a-b}{2}\sigma_z + c\sigma_x + d\sigma_y = \varepsilon\mathbb{1} + \|\vec{v}\|\frac{\vec{v}}{\|\vec{\sigma}\|}\vec{\sigma}$$
$$vaps: \lambda_\pm = \varepsilon \pm \|\vec{v}\|$$
$$veps: \left|\pm\right\rangle = \begin{pmatrix}-\frac{-a+b\pm\sqrt{a^2-2ba+b^2+4c^2+4d^2}}{2(c-id)} & 1\end{pmatrix}$$