1 Quantum Mechanics Formulae

1.1 Postulates

$$p_{a_i} = p(a_i, |b_j\rangle) = \|P_i\,|b_j\rangle\|^2$$
$$(\Delta A)^2_\psi = \langle A^2\rangle_\psi - \langle A\rangle^2_\psi$$

1.2 Temporal evolution

$$U(t,t_0) = e^{-iH(t-t_0)/\hbar} = \sum_i e^{-iE_i(t-t_0)/\hbar}\,|i\rangle\langle i|$$
$$S: |\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle\,, \quad H: A(t) = U^\dagger(t,t_0) A\, U(t,t_0)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A_H(t)\rangle_\psi = \frac{1}{i\hbar}\langle [A_H, H]\rangle_\psi + \left\langle \frac{\partial A}{\partial t}\right\rangle_\psi$$

1.3 Infinite well

$$[0,L] \Rightarrow \begin{cases} \phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) & x \in [0,L] \\ \phi_n(x) = 0 & x \notin [0,L] \end{cases}$$
$$\left[-\frac{L}{2}, \frac{L}{2}\right] \Rightarrow \begin{cases} \phi_n(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right) & n \text{ odd} \\ \phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) & n \text{ even} \end{cases}$$
$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

1.4 Harmonic oscillator (1D)

$$\text{Gaussian: } \psi(x) = \frac{1}{(2\pi\sigma_x^2)^{1/4}}\exp\left[-\frac{x^2}{4\sigma_x^2}\right]$$
$$\phi_n(x) = C_n H_n(\tilde{x})\exp\left[-\frac{\tilde{x}^2}{2}\right], \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$
$$\tilde{x} = \frac{x}{a_0}, \quad a_0 = \sqrt{\frac{\hbar}{m\omega}}, \quad C_n = \left(\frac{1}{\pi a_0^2}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2}$$
$$H_0(\tilde{x}) = 1 \qquad H_2(\tilde{x}) = 4\tilde{x}^2 - 2$$
$$H_1(\tilde{x}) = 2\tilde{x} \qquad H_3(\tilde{x}) = 8\tilde{x}^3 - 12\tilde{x}$$

1.5 Symmetries

$$D_{\hat{n}}(\phi) = e^{-i\phi\,\hat{n}\cdot\vec{\sigma}/2} = \cos\frac{\phi}{2}\mathbb{1} - i\sin\frac{\phi}{2}\,\hat{n}\cdot\vec{\sigma}$$
$$\hat{n} = \big(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta\big)$$
$$T_a = e^{-ia\hat{k}} = e^{-i\vec{a}\cdot\vec{p}/\hbar} = e^{\alpha(a^\dagger - a)}$$
$$\pi\,|l,m\rangle = (-1)^l\,|l,m\rangle\,, \quad P_a = P_b P_c (-1)^l$$
$$\Theta\,|l,m\rangle = (-1)^m\,|l,-m\rangle$$

$$[T_a, T_{a'}]\,✓, \quad [T_a, \pi]\,✗, \quad \big[D_{\hat{n}}(\phi), D_{\hat{n}'}(\phi)'\big]\,✗, \quad [D_{\hat{n}}(\phi), \pi]\,✓$$

1.6 Angular Momentum

$$J_\pm = J_x \pm J_-, \quad J_x = \frac{1}{2}(J_+ + J_-), \quad J_y = \frac{1}{2i}(J_+ - J_-)$$
$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k, \quad [J_z, J_\pm] = \pm\hbar J_\pm$$
$$[J_+, J_-] = 2\hbar J_z, \quad \big[J^2, J_{z,\pm}\big] = 0$$
$$J_\pm\,|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$
$$J^2\,|j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle$$
$$J_z\,|j,m\rangle = \hbar m\,|j,m\rangle$$

1.7 Addition of Angular Momentum

$$\mathcal{H}_{J_1}\otimes\mathcal{H}_{J_1} = \mathcal{H}_{J^{(1)}}\oplus\cdots\oplus\mathcal{H}_{J^{(n)}}, \quad \dim(\mathcal{H}_J) = 2J+1$$
$$J = J_1 + J_2, J_1 + J_2 - 1, \ldots, |J_1 - J_2| \geq 0$$

1.8 Pauli matrices (spin-1/2)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\epsilon_{ijk}\sigma_k$$
$$\sigma_{\hat{n}} = \hat{n}\cdot\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$
$$|+\rangle_{\hat{n}} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}, \quad |-\rangle_{\hat{n}} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}$$
$$|+\rangle_x = (1,1)/\sqrt{2}, \quad |-\rangle_x = (1,-1)/\sqrt{2}, \quad \sigma_x\,|\pm\rangle = |\mp\rangle$$
$$|+\rangle_y = (1,i)/\sqrt{2}, \quad |-\rangle_y = (i,1)/\sqrt{2}, \quad \sigma_y\,|\pm\rangle = \pm i\,|\mp\rangle$$
$$|+\rangle_z = (1,0), \quad |-\rangle_z = (0,1), \quad \sigma_z\,|\pm\rangle = \pm\,|\pm\rangle$$

1.9 Spin-1 matrices

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{aligned} |+\hbar\rangle_x &= (1, \sqrt{2}, 1)/2 \\ |-\hbar\rangle_x &= (1, -\sqrt{2}, 1)/2 \\ |0\rangle_x &= (1, 0, -1)/\sqrt{2} \end{aligned}$$
$$J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \begin{aligned} |+\hbar\rangle_y &= (-1, -\sqrt{2}i, 1)/2 \\ |-\hbar\rangle_y &= (-1, \sqrt{2}i, 1)/2 \\ |0\rangle_y &= (1, 0, 1)/\sqrt{2} \end{aligned}$$
$$J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{aligned} |+\hbar\rangle_z &= (1, 0, 0) \\ |-\hbar\rangle_z &= (0, 0, 1) \\ |0\rangle_z &= (0, 1, 0) \end{aligned}$$

1.10 Spin-1 squared matrices

$$J_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \left|\hbar^2\right\rangle &= (1, 0, 1)/\sqrt{2} \\ \left|\hbar^2\right\rangle &= (0, 1, 0) \\ |0\rangle &= (1, 0, -1)/\sqrt{2} \end{aligned}$$
$$J_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \left|\hbar^2\right\rangle &= (1, 0, -1)/\sqrt{2} \\ \left|\hbar^2\right\rangle &= (0, 1, 0) \\ |0\rangle &= (1, 0, 1)/\sqrt{2} \end{aligned}$$
$$J_z^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \left|\hbar^2\right\rangle &= (1, 0, 0) \\ \left|\hbar^2\right\rangle &= (0, 0, 1) \\ |0\rangle &= (0, 1, 0) \end{aligned}$$

1.11 Trigonometry

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \cos\frac{\pi}{3} = \sin\frac{\pi}{6} = \frac{1}{2}$$
$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\sin(\theta\pm\phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi \qquad \cos\frac{\pi}{6} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

1.12 Eigenstuff in $SU(2)$ (RMT)

$$\begin{pmatrix} a & c-id \\ c+id & b \end{pmatrix} = \frac{a+b}{2}\mathbb{1} + \frac{a-b}{2}\sigma_z + c\sigma_x + d\sigma_y$$
$$= \varepsilon\mathbb{1} + \vec{v}\cdot\vec{\sigma}$$
$$\Rightarrow \text{vaps} : \lambda_\pm = \varepsilon \pm \|\vec{v}\|$$

1.13 The good stuff

$$\int_{-\infty}^{\infty} e^{-ax^2} x^2\,\mathrm{d}x = \frac{1}{2a}\sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2} x^4\,\mathrm{d}x = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}}$$
$$\int_0^\infty e^{-ar} r^n\,\mathrm{d}r = \frac{n!}{(a)^{n+1}}, \quad 0 \le n$$

1.14 Time-dependent perturbation theory

$$H = H_0 + V \quad \Rightarrow |\psi_S(t)\rangle \equiv \sum_n e^{-iE_n t/\hbar} c_n^{(1)}(t)\,|n\rangle$$
$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{n0}t}\,\langle n|V(x,t)|0\rangle\,\mathrm{d}t\,, \quad p_{0\to n} = \left\|c_n^{(1)}\right\|^2$$
$$c_n^{(2)}(t) = -\frac{i}{\hbar}\iint_0^{t,t'} e^{i\omega_{n0}t}\sum_m \langle n|V|m\rangle\,\langle m|V|0\rangle\,\mathrm{d}t\,\mathrm{d}t'$$

1.15 Fock basis

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$$
$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \quad |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}\,|0\rangle$$

Position

$$\begin{aligned}
&x\,|0\rangle \to |1\rangle\,, && x^2\,|0\rangle \to |0\rangle + \sqrt{2}\,|2\rangle \\
&x\,|1\rangle \to |0\rangle + \sqrt{2}\,|3\rangle\,, && x^2\,|1\rangle \to 3\,|1\rangle + \sqrt{6}\,|3\rangle \\
&x\,|2\rangle \to \sqrt{2}\,|1\rangle + \sqrt{3}\,|3\rangle\,, && x^2\,|2\rangle \to \sqrt{2}\,|0\rangle + 5\,|2\rangle + 2\sqrt{3}\,|4\rangle \\
&x\,|3\rangle \to \sqrt{3}\,|2\rangle + \sqrt{4}\,|4\rangle\,, && x^2\,|3\rangle \to \sqrt{6}\,|1\rangle + 7\,|3\rangle + 2\sqrt{5}\,|5\rangle \\
&x\,|4\rangle \to \sqrt{4}\,|3\rangle + \sqrt{5}\,|5\rangle\,, && x^2\,|4\rangle \to 2\sqrt{3}\,|2\rangle + 9\,|4\rangle + \sqrt{30}\,|6\rangle \\
&x\,|5\rangle \to \sqrt{5}\,|4\rangle + \sqrt{6}\,|6\rangle\,, && x^2\,|5\rangle \to 2\sqrt{5}\,|3\rangle + 11\,|5\rangle + \sqrt{42}\,|7\rangle
\end{aligned}$$

Momentum

$$\begin{aligned}
&p\,|0\rangle \to |1\rangle\,, && p^2\,|0\rangle \to -\,|0\rangle + \sqrt{2}\,|2\rangle \\
&p\,|1\rangle \to -\,|0\rangle + \sqrt{2}\,|3\rangle\,, && p^2\,|1\rangle \to -3\,|1\rangle + \sqrt{6}\,|3\rangle \\
&p\,|2\rangle \to -\sqrt{2}\,|1\rangle + \sqrt{3}\,|3\rangle\,, && p^2\,|2\rangle \to \sqrt{2}\,|0\rangle - 5\,|2\rangle + 2\sqrt{3}\,|4\rangle \\
&p\,|3\rangle \to -\sqrt{3}\,|2\rangle + \sqrt{4}\,|4\rangle\,, && p^2\,|3\rangle \to \sqrt{6}\,|1\rangle - 7\,|3\rangle + 2\sqrt{5}\,|5\rangle \\
&p\,|4\rangle \to -\sqrt{4}\,|3\rangle + \sqrt{5}\,|5\rangle\,, && p^2\,|4\rangle \to 2\sqrt{3}\,|2\rangle - 9\,|4\rangle + \sqrt{30}\,|6\rangle \\
&p\,|5\rangle \to -\sqrt{5}\,|4\rangle + \sqrt{6}\,|6\rangle\,, && p^2\,|5\rangle \to 2\sqrt{5}\,|3\rangle - 11\,|5\rangle + \sqrt{42}\,|7\rangle
\end{aligned}$$