Multi-Asset Portfolio Stress Testing under Tail Dependence:

A Case Study with Norwegian Allocation

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Executive Summary

This study evaluates the downside risk of a diversified Norwegian portfolio consisting of 60 % equities and 40 % fixed income under multiple market stress scenarios. The analysis integrates Value-at-Risk (VaR) and Expected Shortfall (ES or CVaR) methodologies using both parametric and non-parametric frameworks. Equities are modelled through Historical Simulation, Student-t, and EWMA approaches, while fixed-income exposure is analysed using Principal Component Analysis (PCA) and a Student-t Copula to capture cross-asset dependencies.

The report highlights how tail dependence amplifies losses during market turbulence. The findings demonstrate that:

- For equities, Student-t VaR delivers superior tail calibration.
- For bonds, PCA + Student-t modelling best captures yield-curve shocks.
- The t-Copula framework shows moderate tail dependence ($\nu \approx 17$), producing aggregate portfolio losses of -10.2 % (VaR) and -14.4 % (ES) at the 99.5 % confidence level.

Overall, the study underscores the need for robust tail-risk modelling in portfolio construction and stress testing—critical for institutional investors, asset managers, and private-equity professionals.

Table Contents

1. Portfolio Overview

The base portfolio replicates a Norwegian institutional allocation of 60 % equities (S&P 500 and MSCI EMU) and 40 % sovereign bonds (U.S. and German curves). All FX exposures are hedged.

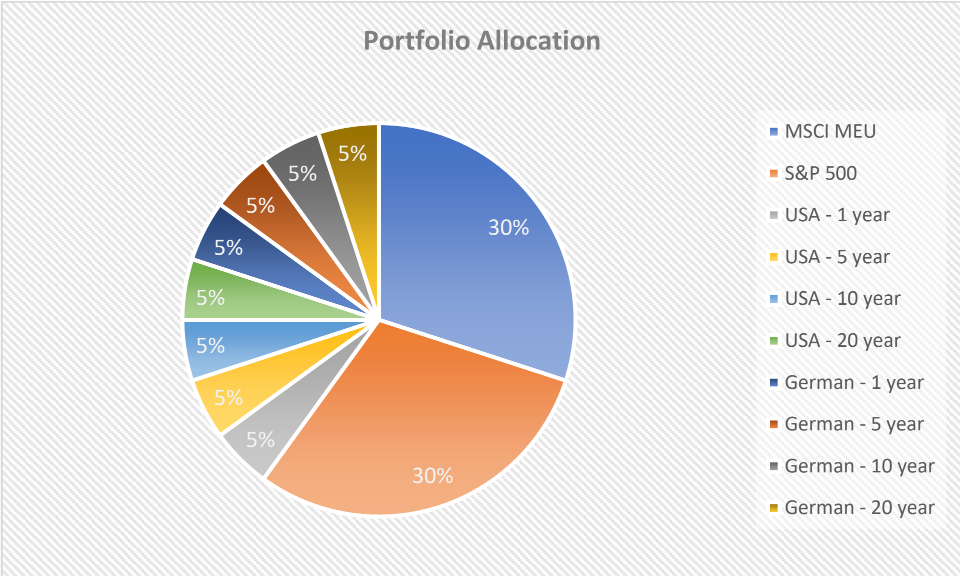

Figure 1. Norwegian Asset Allocation

The assumption of portfolio sensitivity per each additional basis point by maturity and economy is:

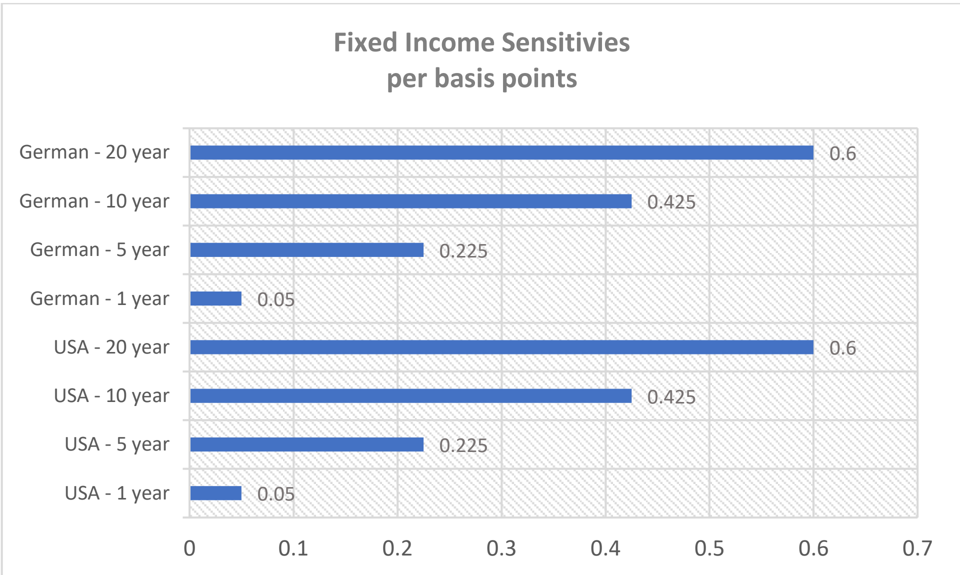

Figure 2. Portfolio Sensitivities to Fixed Income Instruments

The correlation matrix (Figure 3) shows:

- High positive correlation between U.S. and Euro-area equities;
- Low correlation between equities and short-term bonds, reflecting flight-to-quality effects.

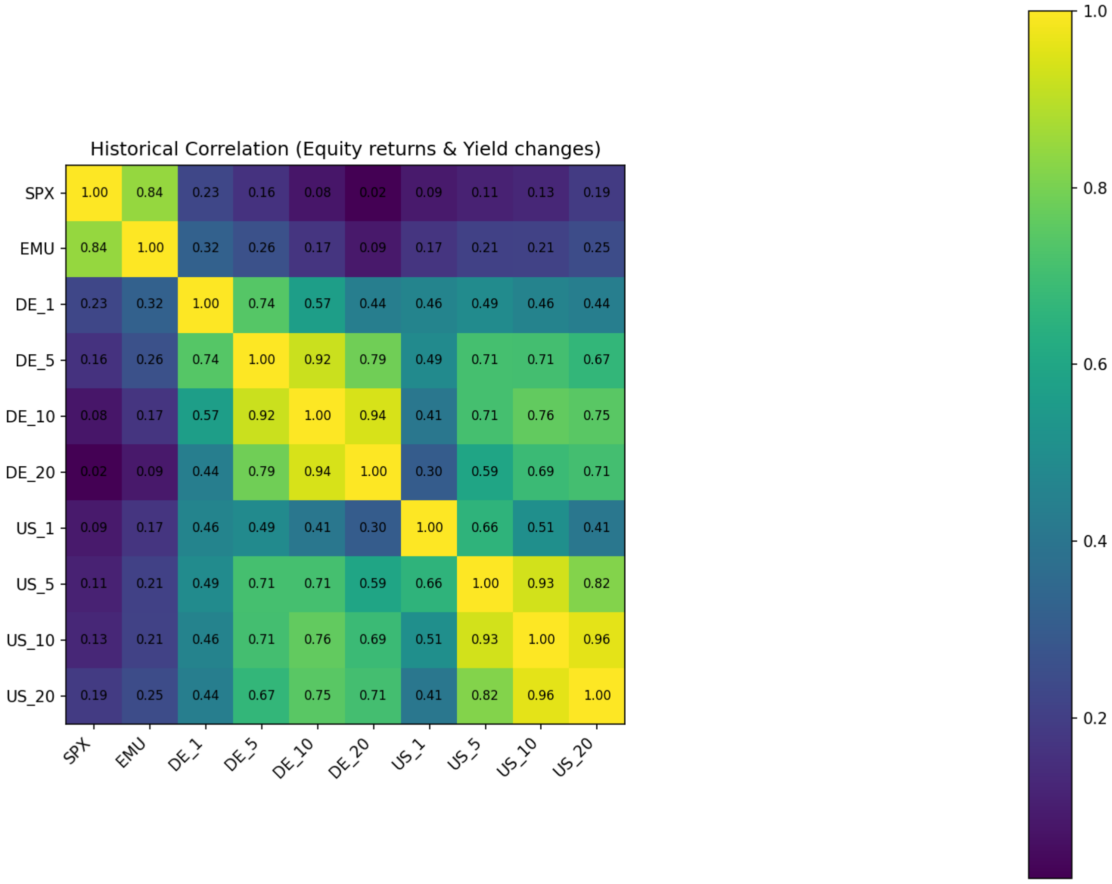

Figure 3. Matrix correlation among equities and fixed income assets

2. Methodology

2.1. Equity Modelling

Historical Simulation (HS)

Historical Simulation is a non-parametric method: it uses actual past returns to infer the potential future distribution of losses — assuming “the past resembles the future”.

If we have a return series r_t(t = 1,…,N), then the empirical distribution of these returns approximates the true unknown distribution of portfolio returns.

Thus, the VaR at confidence level $\alpha$is:

$$\mathrm{VaR}_\alpha = -\mathrm{Quantile}_{1-\alpha}(r_t)$$

and the Expected Shortfall (ES, or CVaR) is:

$$\mathrm{ES}_\alpha = -E[r_t \mid r_t \leq -\mathrm{VaR}_\alpha]$$

Assumptions

- Future losses behave like past ones (stationarity).
- No parametric form; completely data-driven.

Student-t Parametric Model

Parametric VaR assumes returns follow a known probability distribution — but instead of the normal distribution (which underestimates tail risk), we use the Student-t distribution, which allows fat tails.

Empirically, asset returns exhibit:

- Leptokurtosis (fat tails),
- Skewness,
- Volatility clustering (partially handled by other models).

The Student-t distribution with v degrees of freedom has heavier tails than Normal, especially when v is small.

If $r_t \sim t_v(\mu, s)$:

- VaR (α):

$$\mathrm{VaR}_\alpha = -(\mu + s\, t_\nu^{-1}(\alpha))$$

- Expected Shortfall (ES):

$$\mathrm{ES}_\alpha = -\left(\mu + s\,\frac{f_\nu(q_\alpha)}{1-\alpha} \cdot \frac{\nu + q_\alpha^2}{\nu - 1}\right)$$

where $q_\alpha = t_\nu^{-1}(\alpha)$ and f_ν is the t PDF.

Parameter estimation

Use Maximum Likelihood Estimation (MLE) to fit:

$$(\hat{\nu}, \hat{\mu}, \hat{s}) = \arg \max_{\nu,\mu,s} L(r_t \mid \nu, \mu, s)$$

Interpretation

- If $\nu \to \infty$, the Student-t → Normal
- Small ν → fatter tails → higher VaR & ES

EWMA (Exponentially Weighted Moving Average)

EWMA is a conditional volatility model (similar to GARCH(1,1)) assuming that returns are conditionally Normal but volatility evolves over time. Recent shocks matter more than older ones.

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2$$

- λ= smoothing parameter (usually 0.94 for daily, 0.97 for monthly)
- σ_t^2= conditional variance at time t
- r_{t-1}^2= squared return (shock)

Forecasted risk

Assume $r_t \mid \mathcal{F}_{t-1} \sim N(\mu, \sigma_t^2)$.

Then:

$$\text{VaR}_\alpha = -(\mu + \sigma_t z_\alpha)$$

$$\text{ES}_\alpha = -(\mu + \sigma_t \frac{\phi(z_\alpha)}{1-\alpha})$$

where $z_\alpha$and $\phi(z_\alpha)$are the Normal quantile and PDF.

Interpretation

- High volatility → larger σ → larger VaR/ES.
- Adapts dynamically to market stress.

Assumptions

- Conditional normality (but varying σ).
- Returns are serially uncorrelated but conditionally heteroskedastic.

	Historical Simulation	Student t	Exponentially Weighted Moving Average
Strengths	✅ Simple, transparent, intuitive.	✅ Closed-form VaR/ES formulas.	✅ Captures volatility clustering.
	✅ Naturally captures skewness, kurtosis, fat tails, and non-linearity.	✅ Realistically captures heavy-tailed financial returns.	✅ Fast and simple (one parameter λ).
	✅ No need to fit or estimate parameters.	✅ Efficient use of data (parametric inference).	✅ Adaptable — used by RiskMetrics and regulators.
Weaknesses	❌ Assumes history repeats — poor under regime changes or structural breaks.	❌ Still assumes i.i.d. (no dynamic volatility).	❌ Ignores skewness and fat tails (Normal innovations).
	❌ Ignores volatility clustering (conditional heteroskedasticity).	❌ Fitting ν via MLE can be unstable for small samples.	❌ One-sided memory (no long-range persistence modeling).
	❌ Discrete → quantile resolution limited by sample size.	❌ Doesn't adapt to changing volatility unless combined with GARCH/EWMA.	❌ Needs calibration of λ (too low → noisy; too high → sluggish).
	❌ Doesn't easily forecast beyond historical window.		

Table 1. Comparison among three equities models

2.2. Fixed-Income Modelling via PCA

PCA finds an orthogonal set of directions (factors) that maximize variance of your data. Projecting onto the first K directions captures most of the variability with fewer dimensions.

Eigen-decomposition:

$$\Sigma = V\Lambda V^{\top},$$

- $V = [v_1, \dots, v_d]$ has orthonormal eigenvectors ($v_i^{\top} v_j = \delta_{ij}$).
- $\Lambda = \mathrm{diag}(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0)$ are eigenvalues (variances explained by each PC).

Singular Value Decomposition (SVD):

$$X_c = USV^{\top} \text{ with } S = \mathrm{diag}(s_1, \dots, s_r), r = \mathrm{rank}(X_c)$$

Then $\lambda_i = s_i^2/(n-1)$ and the right singular vectors V equal the PCA loadings.

Properties:

- PCs are uncorrelated: $\mathrm{Cov}(Z) = \Lambda$
- Variance explained by PC i is λ_i
- Cumulative variance up to K: $\sum_{i=1}^{K} \lambda_i \,/\, \sum_{i=1}^{d} \lambda_i$

Monthly yield changes across maturities (1–20 years) for U.S. and German sovereign curves are decomposed using PCA:

Principal Component	Variance Explained	Economic Meaning
PC1	78.6 %	Parallel shift
PC2	10.4 %	Slope change
PC3	6.4 %	Curvature

Table 2. PCAs variance explanation of German and American Bonds

These three PCs capture > 95 % of total variance. Fixed-income VaR/ES is then modelled with Normal, Student-t, and EWMA assumptions for yield shocks.

2.3. Portfolio Integration via Student-t Copula

By Abe Sklar's Sklar's theorem, any multivariate joint distribution $H(x_1, \dots, x_d)$ with continuous marginals $F_i$can be written as:

$$H(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$$

where C is a copula (a distribution on $[0,1]^d$) that captures the dependence structure separately from the marginals.

The Student's t-copula introduces a common degrees of freedom parameter ν. When $\nu$is small, the tails of the latent distribution are heavy → implies tail dependence: if one variable is extreme, it increases chance others are extreme.

Defining upper tail dependence coefficient:

$$\lambda_U = \lim_{u \to 1^-} \Pr\left(X_1 > F_1^{-1}(u) \mid X_2 > F_2^{-1}(u)\right)$$

A Gaussian copula has $\lambda_U = 0$for continuous marginals (no tail-dependence) while a t-copula has $\lambda_U > 0$ when ν finite.

- That means: under the t-copula, large losses in one factor raise the probability of large losses in others.

Calibration

- Two main steps:
    1. Estimate correlation matrix $R$of latent (Gaussianised) scores (via PIT → Gaussian transform).
    2. Estimate degrees of freedom ν: maximize log-likelihood of multivariate t density given Zsample and R.
- Log-likelihood for multivariate t:

$$\ell(\nu) = \sum_{i=1}^{n} \ln f_{t_d}(z_i; R, \nu)$$

where $z_i$are the latent Gaussianised observations, $d$is dimension, and $f_{t_d}$is the density of the multivariate t distribution with correlation R and dof ν.

A t-Copula combines the marginal distributions of equity returns and bond PC factors, introducing a single degrees-of-freedom parameter (ν) controlling tail dependence:

$$C_\nu(u_1, \dots, u_d) = t_{\nu,\Sigma}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d))$$

Steps:

1. Transform marginals via Probability-Integral Transform (PIT).
2. Estimate correlation matrix Σ from latent z-scores.
3. Calibrate ν by maximizing log-likelihood of multivariate t.

3. Model Results and Backtesting

3.1. S&P 500

Data Summary

- Index: S&P 500
- Historical Period: 31 January 2002 – 31 March 2025
- Frequency: Monthly
- Number of Observations: 279
- Out-of-sample period: 218 months (after a 60-month rolling window)

Backtesting

The backtesting plot shows several VaR breaches clustered around periods of elevated volatility, indicating that static historical quantiles fail to capture shifts in market regimes.

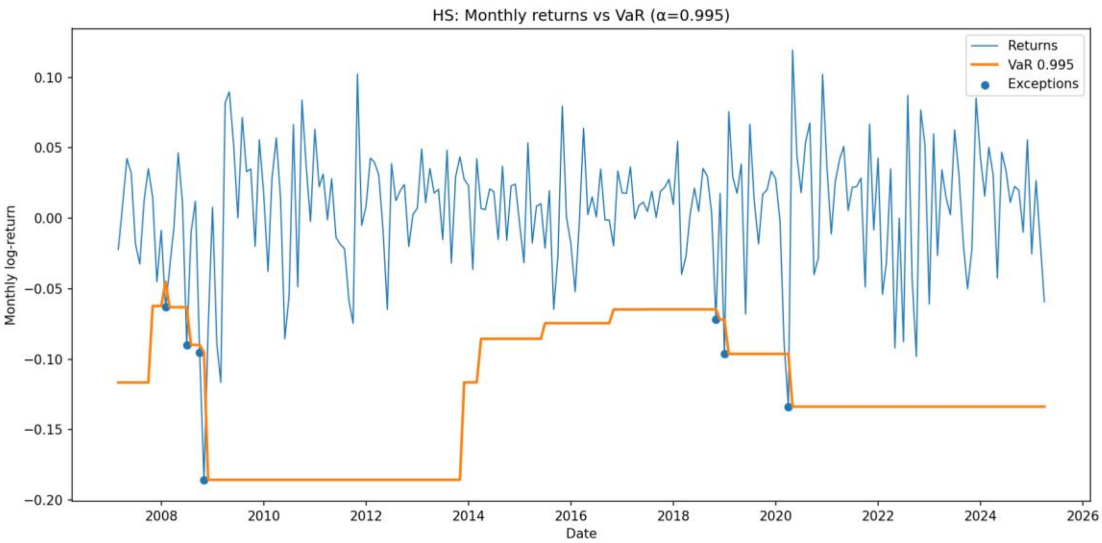

Figure 4. Backtesting results of using Historical Simulation - S&P 500

The backtesting plot shows that exceedances are rare and generally isolated, suggesting a better calibration to the observed tail risk.

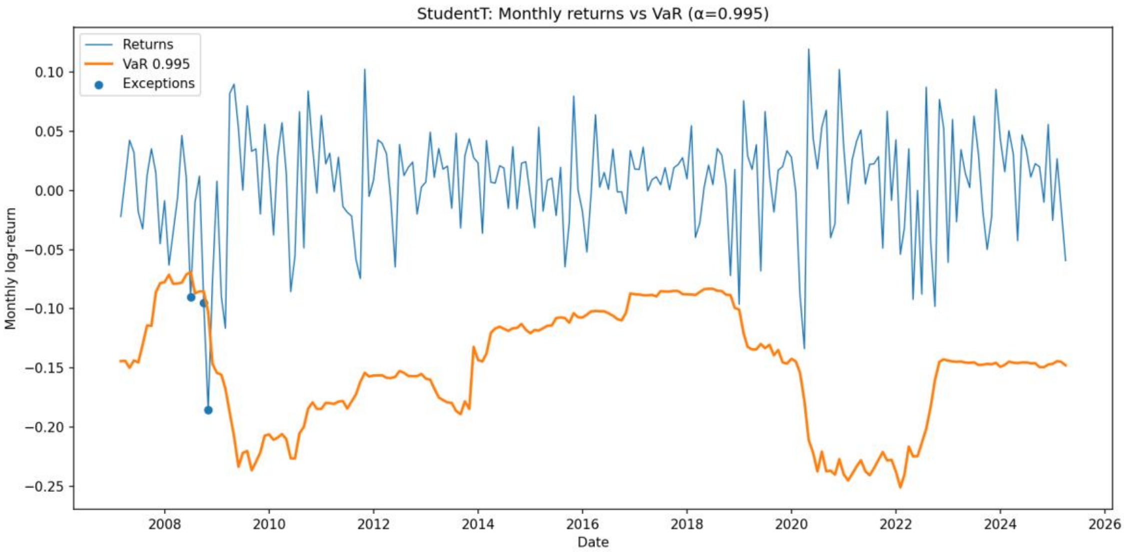

Figure 5. Backtesting results of using Student-t - S&P 500

The backtesting plot shows occasional underestimation of extreme losses during turbulent months.

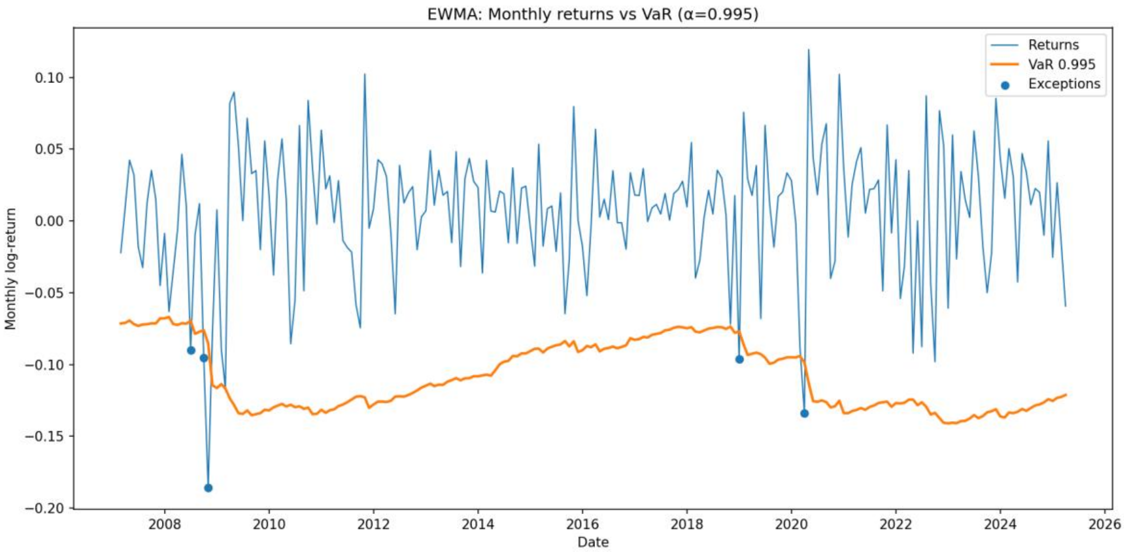

Figure 6. Backtesting results of using EWMA - S&P 500

Goodness-of-fit

The histogram and Q–Q plot confirm that the fitted Student-t distribution adequately captures the heavy tails of monthly returns. Only mild deviations are visible in the extreme quantiles, supporting its superior tail representation relative to the Normal assumption.

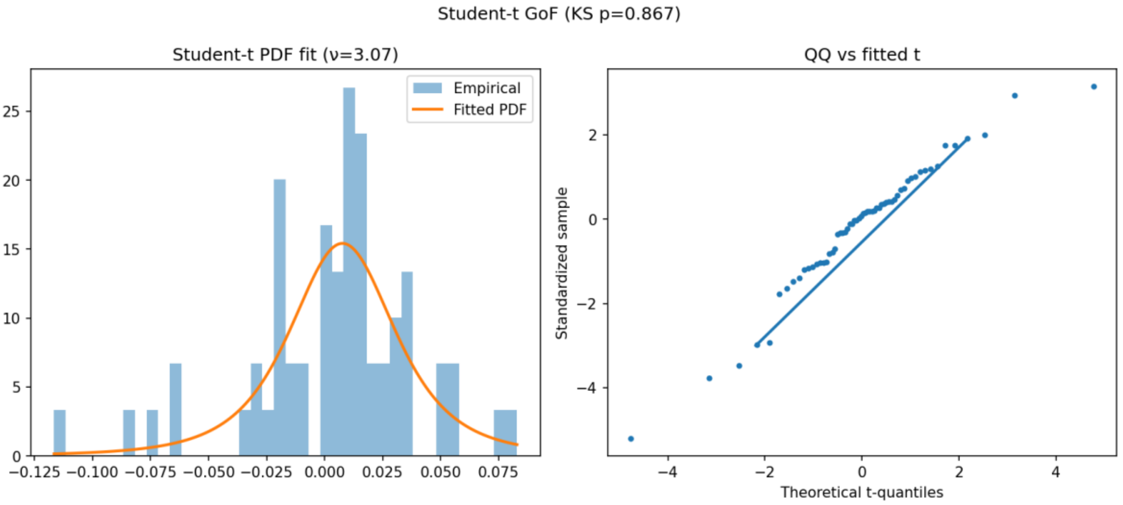

Figure 7. Goodness-of-Fit results of using Student- t - S&P 500

The GoF plots reveal slight excess kurtosis, indicating that the EWMA–Normal assumption still underestimates tail thickness

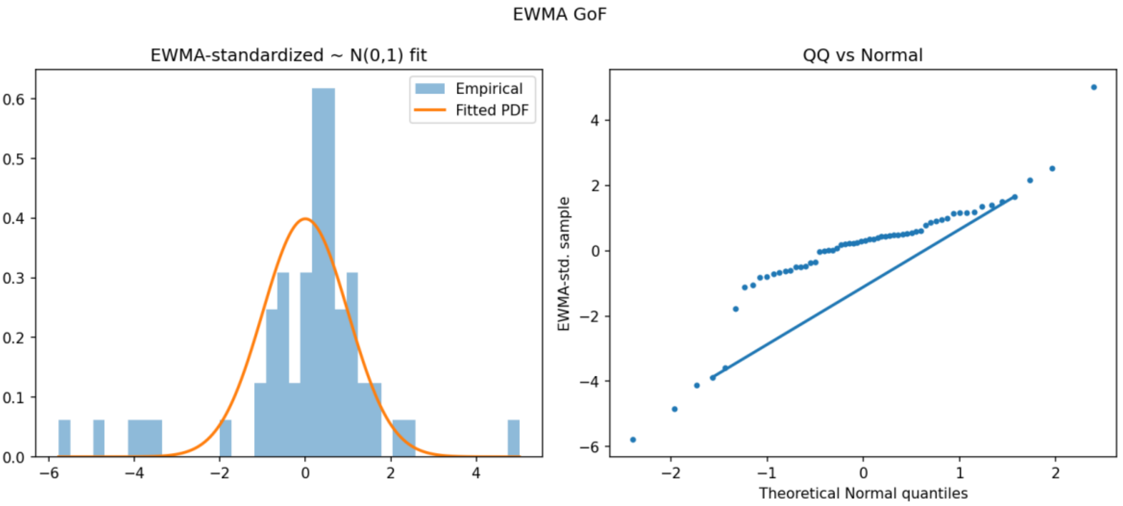

Figure 8. Goodness-of-Fit results of using EWMA - S&P 500

Model	Number of out-of-sample months	Number of breaches	hit_rate	Kupiec_p	Indep_p	CC_p
Historical Simulation	218	7	0.032110092	0.000149479	0.20381471	0.00033654
Student t	218	3	0.013761468	0.13177438	0.023187639	0.024405803
EWMA	218	5	0.02293578	0.006226654	0.088684953	0.005567568

Table 3. Summary table of Backtesting & GoF results – S&P 500

The Kupiec test examines whether the empirical hit rate matches the theoretical 0.5 % expected breach frequency.

- Student-t's p-value (0.132) is the highest, indicating no significant violation of the expected rate.
- HS and EWMA reject the null hypothesis, implying excessive breaches.

The Independence test checks for clustering of exceptions.

- All three models show mild clustering, but Student-t's independence p-value remains within an acceptable range.

The Conditional Coverage test jointly evaluates correct frequency and independence.

- The Student-t model achieves the highest joint p-value (0.0244), outperforming HS (0.0003) and EWMA (0.0056).

Overall, the Student-t VaR provides the best statistical performance among the three models, as it:

- Produces fewer and more evenly distributed VaR breaches.
- Exhibits the highest Kupiec and joint (CC) p-values.
- Accurately reflects heavy-tailed behavior in S&P 500 monthly returns.

3.2. MSCI EMU

Data Summary

- Index: MSCI EMU
- Historical Period: 31 January 2002 – 31 March 2025
- Frequency: Monthly
- Number of Observations: 279
- Out-of-sample period: 218 months (after a 60-month rolling window)

The analysis evaluates the performance of three Value-at-Risk (VaR) methodologies — Historical Simulation (HS), Student-t parametric, and Exponentially Weighted Moving Average (EWMA) — in modeling the downside risk of the MSCI EMU Index.

For each model, both VaR and Expected Shortfall (ES) are computed at the 99.5% confidence level, and performance is assessed through statistical backtesting and goodness-of-fit (GoF) diagnostics.

Backtesting

The backtesting plot shows several VaR breaches clustered around periods of elevated volatility such as during the financial crisis (2008) and Covid (2020), indicating that static historical quantiles fail to capture shifts in market regimes. However, it captures better the downside risk than the other two models.

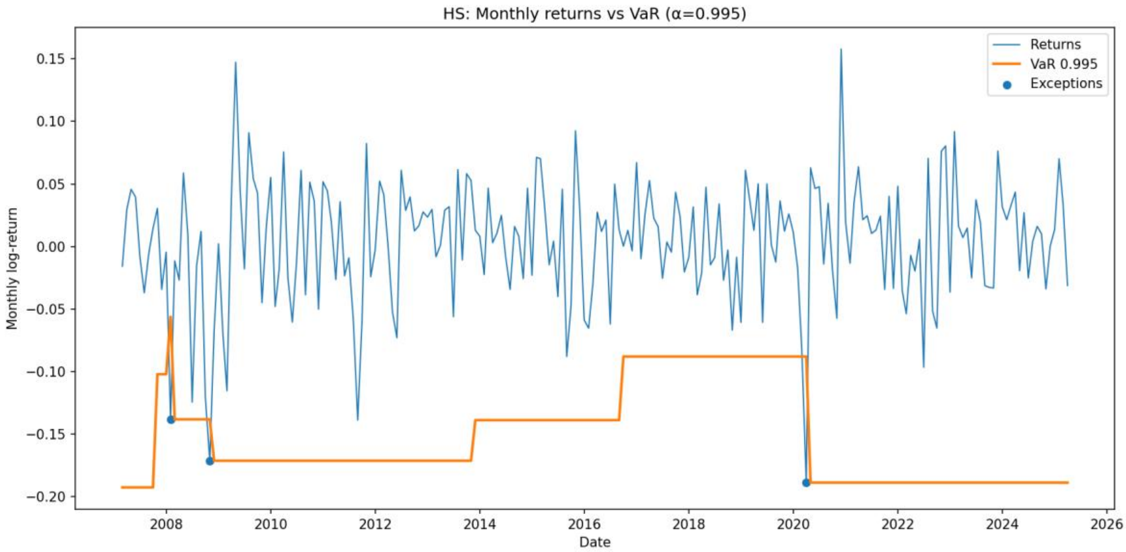

Figure 9. Backtesting results of using Historical Simulation – MSCI EMU

The backtesting plot shows that exceedances during 2008 is higher than the historical simulation, suggesting a lower calibration to the observed tail risk.

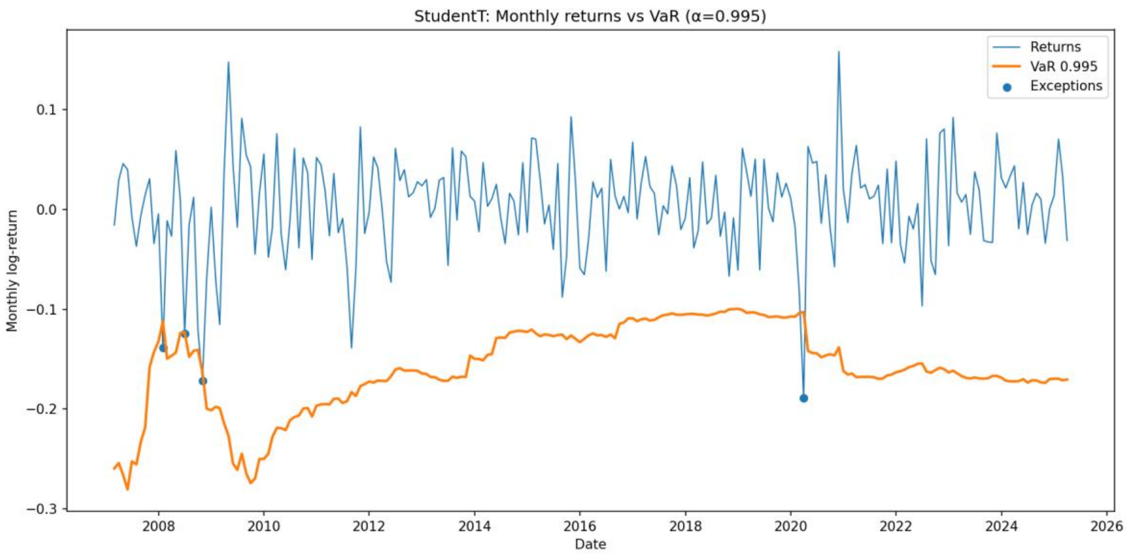

Figure 9. Backtesting results of using Student-t – MSCI EMU

The backtesting plot shows occasional underestimation of extreme losses during turbulent months such during 2008, 2012 and 2020.

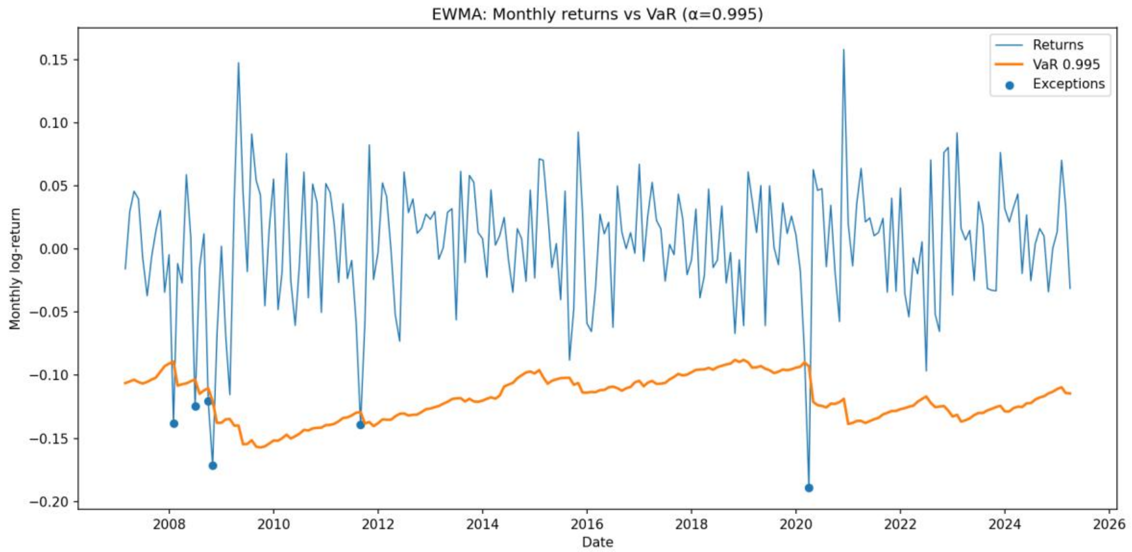

Figure 10. Backtesting results of using EMWA – MSCI EMU

Goodness-of-fit

The histogram and Q–Q plot confirm that the fitted Student-t distribution adequately captures the heavy tails of monthly returns. Only mild deviations are visible in the extreme quantiles, supporting its superior tail representation relative to the Normal assumption.

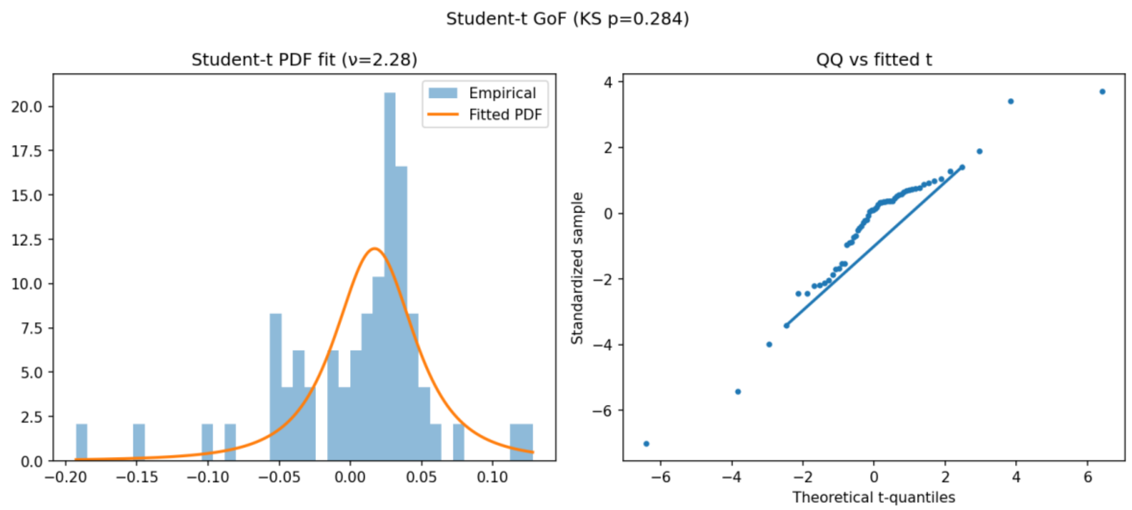

Figure 11. Goodness-of-Fit results of using Student- t – MSCI EMU

The GoF plots reveal slight excess kurtosis, indicating that the EWMA–Normal assumption still underestimates tail thickness

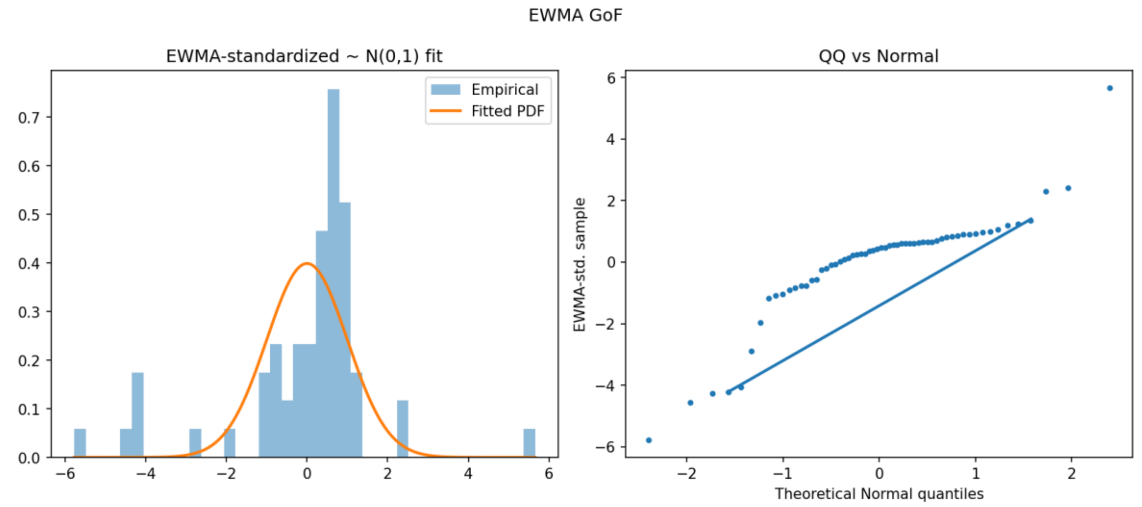

Figure 12. Goodness-of-Fit results of using EWMA – MSCI EMU

Model	Out of sample	Breaches	hit_rate	Kupiec_p	Indep_p	CC_p
HS	218	3	0.013761468	0.13177438	0.771796579	0.307956953
Student t	218	4	0.018348624	0.031598497	0.698302699	0.092070916
EWMA	218	6	0.027522936	0.00103775	0.139932679	0.00155103

Table 4. Summary table of Backtesting & GoF results – MSCI EMU

The Kupiec test examines whether the empirical hit rate matches the theoretical 0.5 % expected breach frequency.

- Historical simulation's p-value (0.132) is the highest, indicating no significant violation of the expected rate.
- Student-t and EWMA reject the null hypothesis, implying excessive breaches.

The Independence test checks for clustering of exceptions.

- All three models show mild clustering, but Historical simulation's independence p-value remains within an acceptable range.

The Conditional Coverage test jointly evaluates correct frequency and independence.

- The Historical simulation model achieves the highest joint p-value (0.3079), outperforming Student-t (0.09) and EWMA (0.001).

Overall, the Historical simulation VaR provides the best statistical performance among the three models, as it:

- Produces fewer and more evenly distributed VaR breaches.
- Exhibits the highest Kupiec and joint (CC) p-values.
- Accurately reflects heavy-tailed behavior in MSCI EMU monthly returns.

3.3. US and German Bonds

Backtesting

The backtesting plot shows several VaR breaches clustered around periods of interest rates jumps like during Covid (2022), indicating that static historical quantiles fail to capture shifts in market regimes.

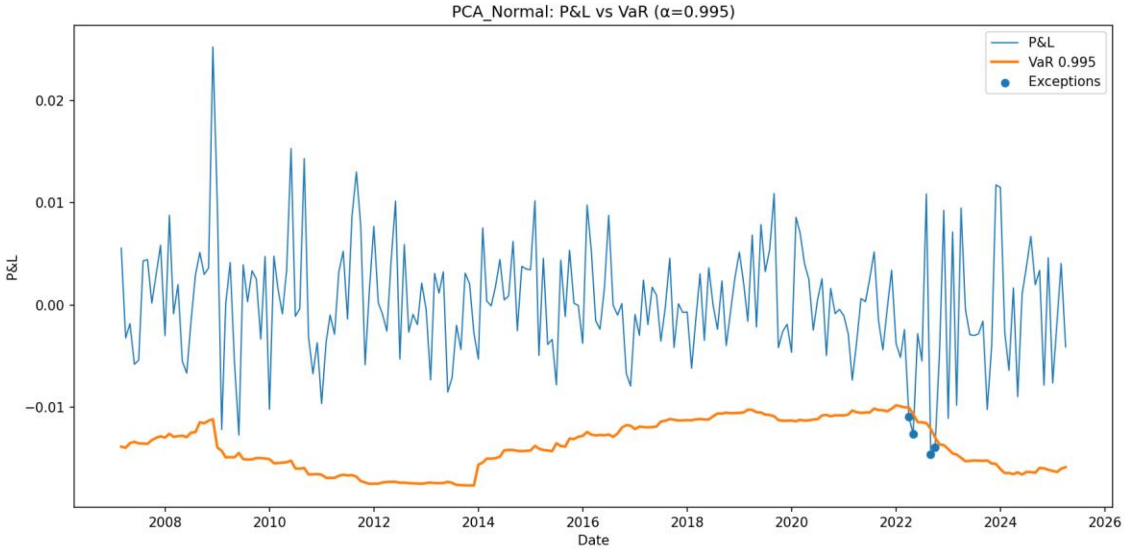

Figure 13. Backtesting results of using Normal Distribution – US & German bonds

The backtesting plot shows this distribution captures better the downside risk of the portfolio, just breaches twice during 2022.

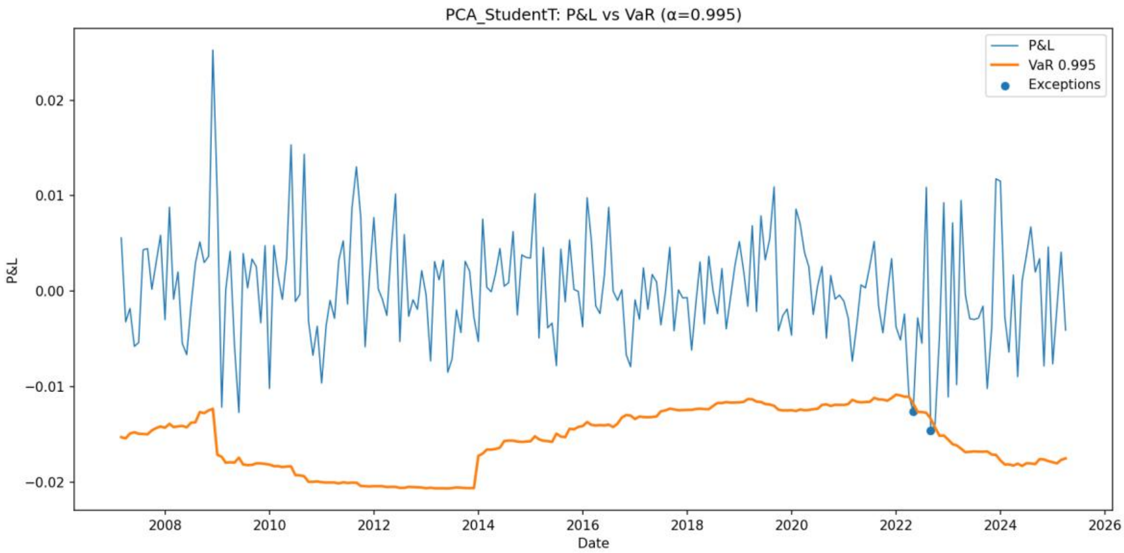

Figure 14. Backtesting results of using Student-t – US & German bonds

The backtesting plot shows occasional underestimation of extreme losses during Covid period, however, it slightly underperforms in comparison to Student-t.

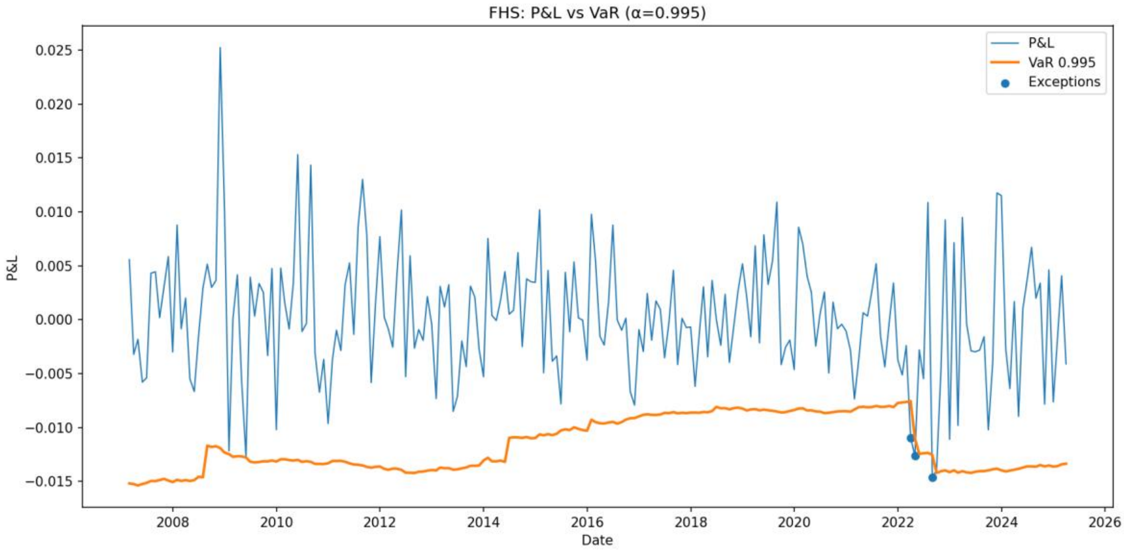

Figure 15. Backtesting results of using EMWA– US & German bonds

Goodness-of-fit

It has a bumpy distribution which means it fails to capture change regimes.

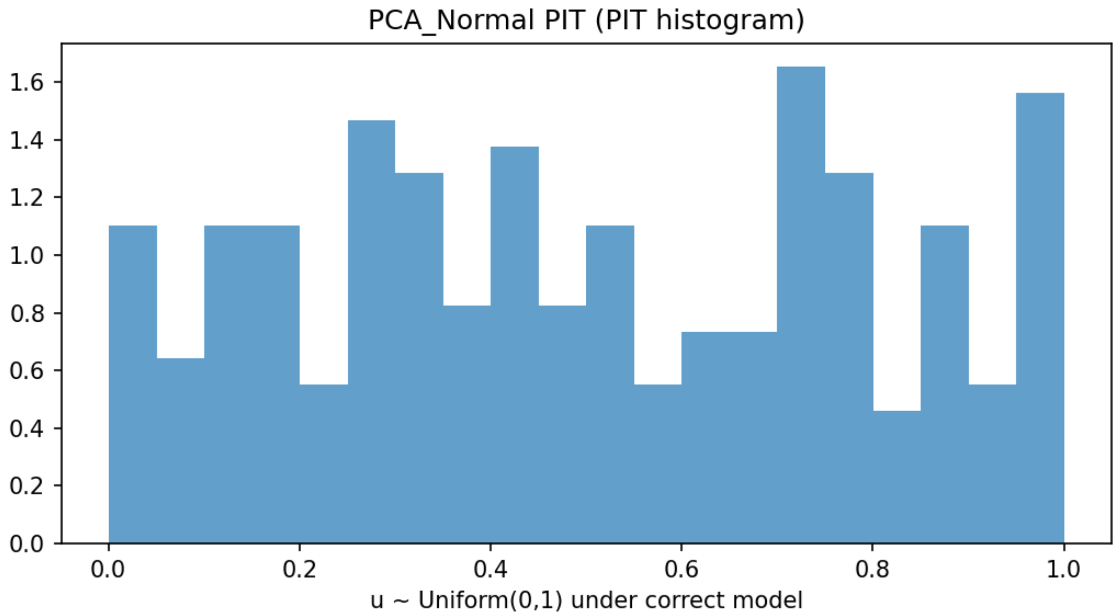

Figure 16. Goodness-of-Fit results of using Normal – US & German bonds

It has a bumpy distribution which means it fails to capture change regimes.

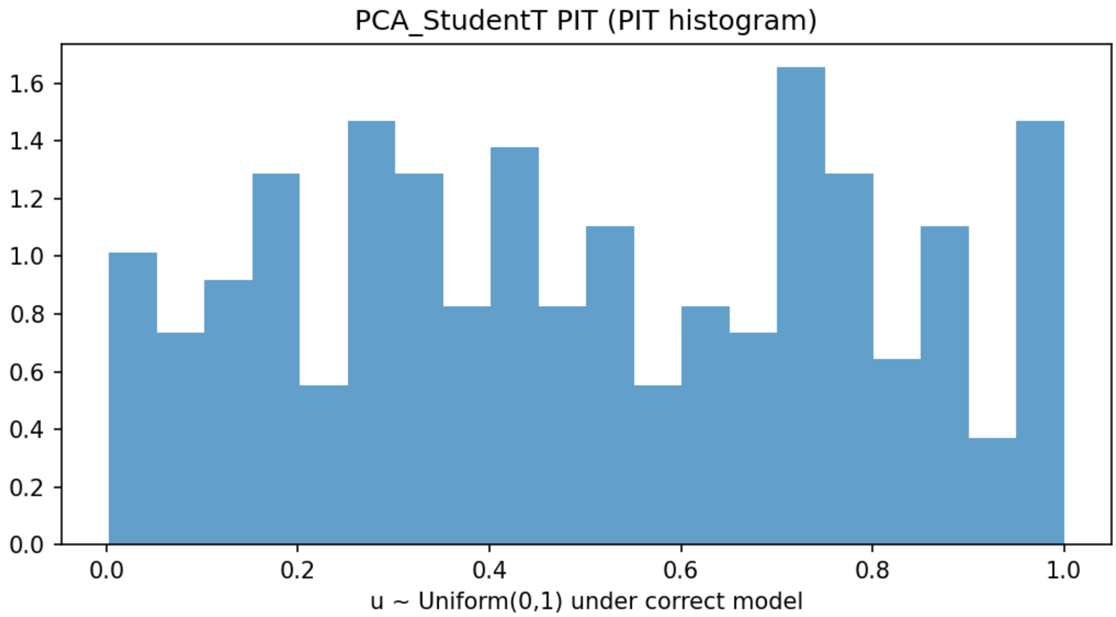

Figure 17. Goodness-of-Fit results of using Student t – US & German bonds

Model	Out-of-sample months	Breaches	hit_rate	Kupiec_p	Indep_p	CC_p	PIT_KS_p
EWMA	218	3	0.0138	0.1318	0.0232	0.0244	
PCA_Normal	218	4	0.0183	0.0316	0.0006	0.0003	0.9078
PCA_Student-t	218	2	0.0092	0.4341	0.8470	0.7229	0.9727

Table 5. Summary table of Backtesting & GoF results – US & German bonds

The Kupiec test examines whether the empirical hit rate matches the theoretical 0.5 % expected breach frequency.

- Student-t p-value (0.4341) is the highest, indicating no significant violation of the expected rate.
- Normal distribution reject the null hypothesis, implying excessive breaches.

The Independence test checks for clustering of exceptions.

- Student-t is the only that doesn’t show cluster breaches.

The Conditional Coverage test jointly evaluates correct frequency and independence.

- The Student-t model achieves the highest joint p-value (0.7229), outperforming Normal (0.0003) and EWMA (0.0244).

The Kolmogorov–Smirnov test shows the residual almost fit perfectly for Student-t.

In conclusion, the Student-t VaR provides the best statistical performance among the three models, as it:

- Produces fewer and more evenly distributed VaR breaches.
- Exhibits the highest Kupiec, Cluster and joint (CC) p-values.

4. Portfolio Stress Testing and Monte Carlo Simulation

Combining the marginal models within the calibrated t-Copula (ν = 17.47) and simulating 100 000 scenarios yields:

- VaR (99.5 %) = -10.22 %
- Expected Shortfall = -14.38 %

Negative correlation between equities and PC1 confirms flight-to-quality dynamics during stress episodes.

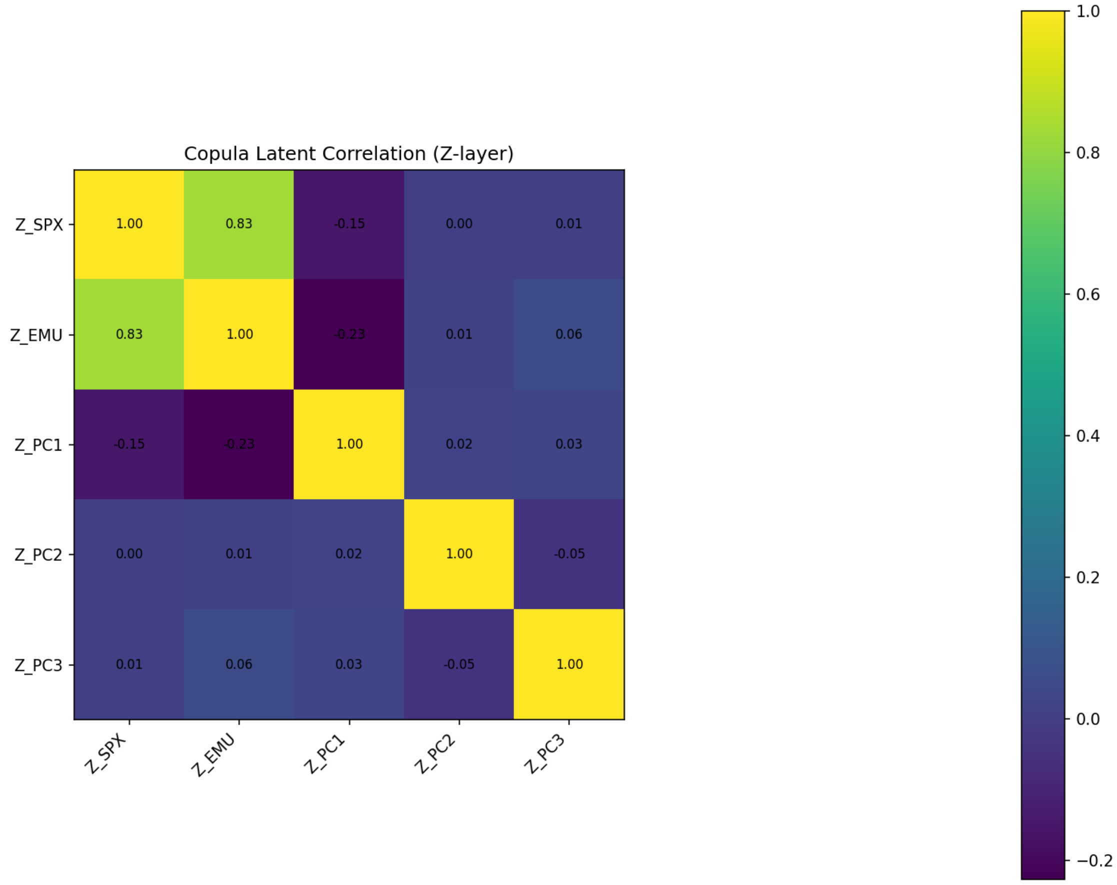

Figure 18. Correlation Matrix of Latent z-Scores

After making 100,000 simulations, the VaR and Expected Shortfall (ES) at 99.5% are monthly losses of -10.22% and -14.375% respectively.

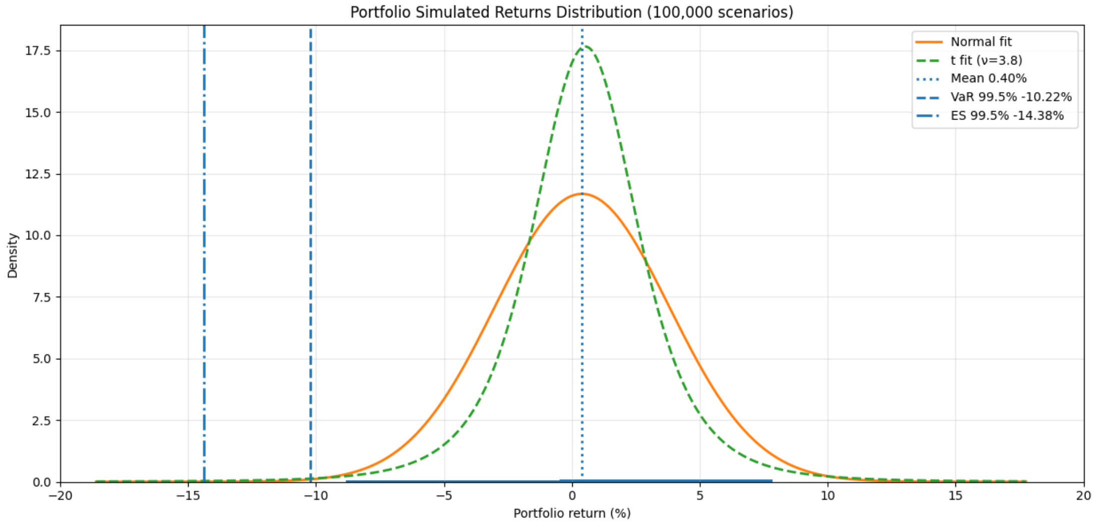

Figure 19. Portfolio Returns Distribution in 100K simulations

5. Conclusion

The integration of student-t and historical simulation marginals and a t-Copula dependence structure offers a realistic view of joint tail events in multi-asset portfolios. For investment managers and private equity professionals, this approach enhances risk-adjusted decision-making by quantifying the magnitude and probability of extreme losses across asset classes.

Future extensions may include dynamic copula models or macro-factor stress linkages to further bridge quantitative risk modelling and strategic asset allocation.