Modern Inference (Statistical Learning) - University of Milano-Bicocca

Homework 3

To submit via e-mail by 10/04/2020 h 14:00.
You will have to submit your solution by the deadline. Feel free to choose the format for your solution (.txt, .tex, .pdf etc.), the nicer the better, and the number of files in attachment (one or more, but not too many). Answer with clarity and precision. All R code must be reproducible. For theoretical questions, try to provide a well-reasoned mathematical argument. Simulations can help form your intuition; but a purely empirical answer will only receive partial points. It is encouraged to discuss the problem sets with others, but every group needs to turn in a unique write-up. Use of sources (people, books, internet and so on) without citing them in homework sets results in failing grade.

1 Higher Criticism

We would like to understand the finite sample properties of the higher criticism (HC). Throughout this problem, we will work with $m = 10^6$ independent observations $X_i \sim N(\mu_i, 1)$ (or p-values $p_i = 1 - \Phi(X_i)$). Under the global null H_0, all the μ_is are equal to zero while under H_1, $n^{1-\beta}$ of the μ_is are equal to $\sqrt{2r(\beta)\log n}$ where

$$r(\beta) = 1.2\rho^*(\beta) + 0.1$$

and

$$\rho^*(\beta) = \begin{cases} \beta - 1/2 & 1/2 \le \beta \le 3/4 \\ (1-\sqrt{1-\beta})^2 & 3/4 \le \beta \le 1 \end{cases}$$

is the critical threshold above which the HC test should theoretically have full asymptotic power. Throughout, we work with values of $\beta \in \{0.5, 0.55, 0.6, \ldots, 1\}$, and a significance level set to $\alpha = 0.05$.

1. Use Use Monte-Carlo simulations to compute an approximate threshold for the higher criticism statistic

$$T_{\text{HC}} = \max_{0 \le i \le m/2} \sqrt{m} \frac{(i/m) - p_{(i)}}{\sqrt{p_{(i)}(1-p_{(i)})}}$$

   where $p_{(1)} \le \ldots \le p_{(n)}$ are the sorted p-values. For what values of the index i is this maximum typically achieved?

2. In the setup described above, compute the power of the HC statistic through simulations. Is the power close to one? Has the asymptotics kicked in?

2 Holm

Prove that Holm's method controls the FWER at level α.

3 Weighted Bonferroni

Assume that you assign each of m hypotheses a data independent weight $w_i > 0$ such that $\sum_{i=1}^m w_i = m$. Prove that the following procedure controls the FWER at level α: reject H_i if $p_i/w_i \le \alpha/m$.

4 Online Bonferroni

Online multiple testing refers to the setting in which a potentially infinite stream of hypotheses $H_1, H_2, \ldots$, (respectively $p_1, p_2, \ldots$) is tested one by one over time.

At each step $t \in \mathbb{N}$, one must decide whether to reject the current null hypothesis H_t or not, without knowing the outcomes of all the future tests. Typically, we reject the null hypothesis H_t when p_t is smaller than some threshold α_t.

Specifically, for FWER level α, given an infinite nonnegative sequence $\{\gamma_t\}_{t=1}^{\infty}$ that sums to one, the online Bonferroni method tests individual hypothesis H_t at level $\alpha_t = \alpha\gamma_t$. Prove that for $\gamma_t = \frac{6}{(\pi t)^2}$ the online Bonferroni method controls the PFER at level α. Controlling the PFER implies controlling the FWER. Could you suggest an online procedure that uniformly improves Bonferroni and control the FWER at α under the assumption of independence for the p-values $p_1, p_2, \ldots$?

5 Two-step Bonferroni

Suppose we wish to test $m > 1$ hypotheses $H_1, \ldots, H_m$. In this problem, we are interested in a procedure which operates in two-step:

Step 1: Select a set $\mathcal{S} \subseteq \mathcal{H} = \{H_1, \ldots, H_m\}$ of "interesting" hypotheses; consider the selection rule:

$$\mathcal{S} = \{H_i \in \mathcal{H} : p_i \leq \alpha\}$$

for some pre-specified $\alpha \in (0, 1)$

Step 2: Apply a multiple testing method to test only the selected hypotheses in S; consider the Bonferroni method at level α

$$\mathcal{R} = \{H_i \in \mathcal{S} : p_i \leq \frac{\alpha}{|\mathcal{S}|}\}$$

where $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$.

Calculate the FWER of this Bonferroni two-step procedure by assuming that all m hypotheses are true and $p_1, \ldots, p_m$ are i.i.d. Uniform(0,1). Would you expect FWER control at level α? Explain why and why not.