Statistical Inference II

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HYPOTHESIS TESTING IS CONCERNED WITH STATISTICAL TESTING OF POSTULATES (USUALLY CONCERNING PARAMETERS) IN AN EMPIRICAL WAY, i.e. FROM THE DATA.

THIS COURSE AIMS TO INTRODUCE MODERN IDEAS IN HYPOTHESIS TESTING.

- HYP. TESTING REVIEW
  - SIGNIFICANCE / HYPOTHESIS TESTS — 1 - 2
  - LINK WITH CONFIDENCE INTERVALS
- REPRODUCIBILITY & REPLICABILITY
  - CRISIS OF MODERN SCIENCE — 2
  - LAW OF SELECTION
- MULTIPLE TESTING
  - GLOBAL TESTS IN HIGH-DIMENSIONS — 3
  - METHODS FOR FWER / FDR CONTROL — 4
- POST SELECTION INFERENCE
  - CLOSED TESTING
  - SIMULTANEOUS CONTROL OF FALSE DISCOVERY PROPORTIONS. — 5

HYPOTHESIS TESTING : A REVIEW

REFERENCES:

- DAVISON (2003) 7.3 p 325 - 352
- COX (2006) 3 p. 30 - 43
  - 6.2.4 p. 103 - 105
- COX & HINKLEY (1976)
- LEHMANN & ROMANO (2006)

} INTEGRATION.

DETERMINISTIC PROOF BY CONTRADICTION

1. ASSUME A PROPOSITION, THE OPPOSITE OF WHAT YOU THINK, i.e. THE OPPOSITE CONCLUSION OF YOUR THEOREM
2. WRITE DOWN A SEQUENCE OF LOGICAL STEPS.
3. DERIVE A CONTRADICTION
4. CONCLUDE THAT THE PROPOSITION IS FALSE (WHICH IMPLIES THAT THE THEOREM IS TRUE)

STOCHASTIC PROOF BY CONTRADICTION

1. SET H_0 (THE PROPOSITION)
2. YOU COLLECT DATA (RANDOM)
3. DERIVE AN APPARENT CONTRADICTION i.e. IF H_0 IS TRUE, THE DATA IS VERY WEIRD
4. REJECT H_0, THIS IS CALLED

4. CONCLUDE THAT THE [illegible]
(WHICH IMPLIES THAT THE THEOREM IS TRUE)

4. REJECT H_0, THIS IS CALLED A "DISCOVERY"

HYPOTHESIS TESTING IS STOCHASTIC BECAUSE WE MIGHT MAKE ERRORS

- TYPE I ERRORS (FALSE DISCOVERIES)
- TYPE II ERRORS (MISSED DISCOVERIES)

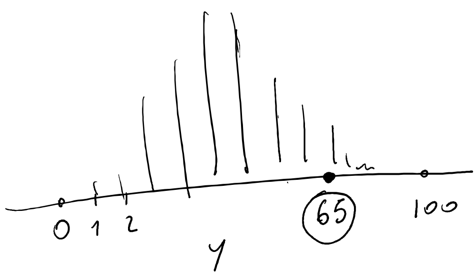
EXAMPLE: SUPPOSE WE HAVE A COIN
WE WANT TO SHOW THAT THE COIN IS BIASED

PROPOSITION: H_0: COIN IS FAIR $(p = 1/2)$

DATA Y = NUMBER OF HEADS IN 100 TRIALS

$$Y \overset{H_0}{\sim} \text{BINOMIAL}(100, 1/2)$$

$$y = 65 \text{ HEADS}$$



P-VALUE = PROBABILITY OF SEEING WHAT YOU SAW - OR SOMETHING MORE EXTREME - GIVEN THAT H_0 IS TRUE

$p_{obs} = 0.0018$ THEN EITHER $\begin{cases} H_0 \text{ IS FALSE} \\ H_0 \text{ IS TRUE BUT WE ARE UNLUCKY AND SAW THIS DATA} \end{cases}$

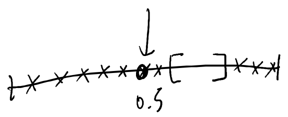
SUPPOSE THAT YOU WANT TO SHOW THAT THE COIN IS FAIR.
HOW TO CHECK THIS BY HYPOTHESIS TESTING?

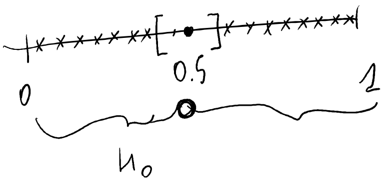
DATA $y = 5001$ HEADS IN $n = 10000$ TRIALS

$\hat{p} = 0.5001$ $\quad CI = [0.49 ; 0.509]$

$\boxed{H_0: p \neq 0.5 \qquad H_1: p = 0.5}$ ← NO POWER

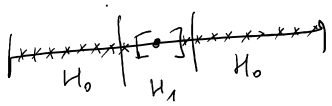
H_0





$$\boxed{H_0: \quad p \in [0, 0{,}49) \cup (0{,}51, 1] \quad \text{vs} \quad \underline{H_1 \quad p \in [0{,}49, 0{,}51]}}$$

"EQUIVALENCE TESTING"



<u>ARGUMENT FROM IGNORANCE</u>: (LOGICAL FALLACY) ASSERTS THAT A PROPOSITION IS TRUE BECAUSE IT HAS NOT ~~Y~~YET BEEN PROVED FALSE

<u>LACK OF EVIDENCE TO REJECT H_0 DOESN'T IMPLY THAT H_0 IS TRUE</u>.

<u>SIMPLE</u> SIGNIFICANCE TESTS

DATA: y REALIZATION OF Y

NULL HYPOTHESIS H_0: <u>FULLY</u> SPECIFIES THE DISTRIBUTION OF Y

TEST STATISTIC $T = t(Y)$ LARGE VALUES OF T REPRESENT A DEPARTURE FROM H_0

(OBSERV) P-VALUE $p_{obs} = \mathbb{P}_0 (T \geq t_{obs})$

UNDER H_0 — $t(y)$ OBSERVED TEST STATISTIC

P-VALUE NULL DISTRIBUTION

$p_{obs} = 1 - F_0(t_{obs})$ WHERE $F_0(t) = \mathbb{P}_0(T \leq t)$

NULL CDF OF T

CONTINUOUS & INVERTIBLE

RANDOM VARIABLE

RANDOM VARIABLE $P = 1 - F_0(T)$

NULL DISTRIBUTION OF P IS UNIFORM$(0,1)$

<u>Proof</u>

$\forall u \in (0,1)$

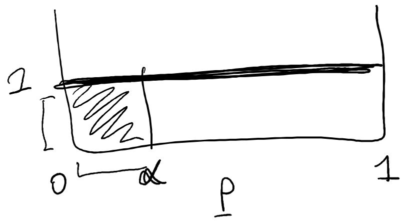
$$\underbrace{\mathbb{P}_0(P \leq u)}_{\text{(NULL) CDF OF } P} = \mathbb{P}_0\left(1 - F_0(T) \leq u\right)$$

$$= \mathbb{P}_0\left(1 - u \leq F_0(T)\right)$$

$$= \mathbb{P}_0\left(F_0^{-1}(1-u) \leq T\right)$$

$$= 1 - F_0\left(F_0^{-1}(1-u)\right) = u \quad \square$$

UNDER H_0



$$\mathbb{P}_0(P \leq \alpha) = \int_0^{\alpha} U(0,1) = \alpha$$

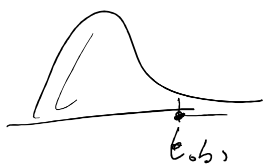
ONE AND TWO-SIDED TESTS

TEST STATISTIC T WITH CONTINUOUS DISTRIBUTION

EXTREME (SMALL OR LARGE) VALUES INDICATE DEPARTURE FROM H_0

$$p^-_{obs} = \mathbb{P}_0(T \leq t_{obs}) \qquad p^+_{obs} = \mathbb{P}_0(T \geq t_{obs})$$

$$p_{obs} = 2 \cdot \underbrace{\min\left(p^-_{obs}, p^+_{obs}\right)}_{q_{obs}}$$



<u>Proof</u>

$P^- = 1 - P^+$

$P^+ \overset{H_0}{\sim} U(0,1)$

$Q = \min(1 - P^+, P^+) \overset{H_0}{\sim} U(0, 1/2)$

$P = 2Q \overset{H_0}{\sim} U(0,1) \quad \square$

α ...

$$p = 2Q \overset{H_0}{\sim} U(0,1) \quad \square$$

DISCRETE NULL DISTRIBUTION

$p_{obs} = q_{obs} = \min(p^-_{obs}, p^+_{obs})$ PLUS THE ACHIEVABLE P-VALUE FROM THE OTHER TAIL NEAREST BUT NOT EXCEEDING q_{obs}

EXAMPLE

$$H_0 : T \sim \text{Poisson}(2)$$
$$t_{obs} = 3$$

$$p^+_{obs} = \mathbb{P}_0(T \geq t_{obs}) = \sum_{t=t_{obs}}^{\infty} \frac{\mu^t e^{-\mu}}{t!}$$

$$p^-_{obs} = \mathbb{P}_0(T \leq t_{obs}) = \dots \sum_{t=0}^{t_{obs}} \quad //$$

t	0	1	2	3	4	5
$\mathbb{P}_0(T \geq t)$	1	0,865	0,594	0,323	0,143	...
$\mathbb{P}_0(T \leq t)$	$\boxed{0,135}$	0,406	0,677	0,857	0,947	..

$$0,323 = q_{obs}$$

$$p_{obs} = 0,323 + 0,135 = 0,458$$

SIGN TEST (NONPARAMETRIC DISTRIBUTION-FREE TEST)

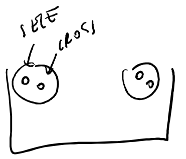
DATA : $Y_1, \dots, Y_n$ I.I.D. F (CDF) CONTINUOUS BUT UNKNOWN

NULL HYP H_0 : F IS SYMMETRIC AROUND 0

$$: F(-y) + F(y) = 1$$

TEST STAT $\quad T = \sum_{i=1}^{m} \mathbb{1}\{Y_i > 0\} \overset{H_0}{\sim} \text{Binomial}(m, 1/2)$

EXAMPLE



	CROSS	SELF	D
1	0	0	
2			
15			

$$D_i = \underbrace{6{,}125 \quad -8{,}375 \quad 1 \quad 2, \ldots\ldots -6}_{13 \text{ POSITIVE} \quad 2 \text{ NEGATIVE}}$$

P-VALUE = 0,007385

$$\hat{\mathbb{P}}(\text{DIFF. IS POSITIVE}) = \frac{13}{15}$$

0.86 $[0{,}59; 0{,}98]$

ADEQUACY OF POISSON MODEL $\quad$ GOODNESS OF FIT TEST.
LACK OF FIT TEST

DATA $\quad Y_1, \ldots, Y_m$ IID

NULL HYP. $\quad H_0: Y_1, \ldots, Y_m$ IID POISSON(μ) $\quad \mu$ UNKNOWN

TEST STATISTIC $\quad$ DISPERSION INDEX $\quad T = \dfrac{\sum_{i=1}^{m} (Y_i - \bar{Y})^2}{\bar{Y}} \overset{H_0}{\approx} \chi^2_{m-1}$

$S = \sum_{i=1}^{m} Y_i$ $\quad$ SUFFICIENT STATISTIC (UNDER H_0)

CONDITIONAL DISTRIBUTION OF DATA GIVEN $S = n$

$$\frac{n!}{\prod_{i=1}^{m} y_i!} \frac{1}{m^n}$$

MULTINOMIAL WITH n TRIALS AND PROB. $(1/m, 1/m, \ldots, 1/m)$

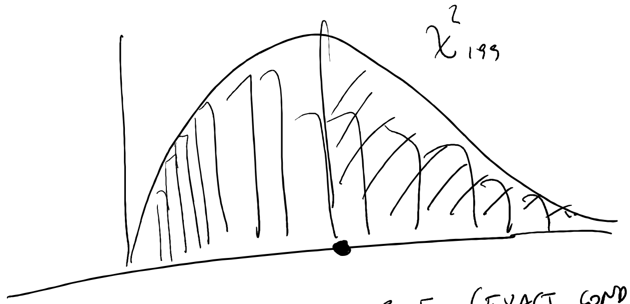
EXAMPLE: VON BORTKIEWICZ'S HORSE KICK DATA

DEATHS	0	1	2	3	4
FREQ	109	65	22	3	1

$m = 200$

$n = 122$

$t_{obs} = 199,3$



$p_{obs} = 0,505$ (EXACT COND)

$= 0.48 \; (\chi^2_{199})$

$\boxed{\text{KOLGOMOROV - SMIRNOV TEST}}$

H_0: $\underline{Y_1, \dots, Y_m}$ IID F_0 KNOWN & CONTINUOS

$$\hat{F}(y) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{Y_i \leq y\}$$ EMPIRICAL CDF

TEST STAT $$T = \| \hat{F} - F_0 \|_\infty = \sup_y \left| \hat{F}(y) - F_0(y) \right|$$

KOLMOGOROV 1933 GIORNALE DELL'ISTITUTO ITALIANO DEGLI ATTUARI

$$\underline{\lim_{m \to \infty}} \; \mathbb{P}_0 \left(T > \frac{c}{\sqrt{m}} \right) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp\left(-2k^2 c^2\right)$$

MONTE CARLO APPROACH.

$$\begin{array}{cccccc} Y & & Y^1 & Y^2 & \dots & Y^B \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ T_{obs} & & T^1 & T^2 & & T^B \end{array}$$

$$p_{obs} = \frac{1 + \sum_{b=1}^{B} \{ T^b \geq t_{obs} \}}{1 + B}$$

EXAMPLES.

HEIGHT DIFFERENCES ARE $N\left(\hat{\mu} = 2.6,\ \hat{\sigma}^2 = 4.7^2\right)$

$p_{obs} = \underline{0,447}$

DATA (15 obs) → TEST IF DATA ARE FROM N → NOT REJECT → t TEST

→ REJECT → NON PARAM. TEST

PERMUTATION TWO-SAMPLE TEST

$Y_1, \dots, Y_k$ IID F

$Y_{k+1}, \dots, Y_m$ IID G

$H_0: F = G$

UNDER H_0, THE ORDERED STATISTICS $Y_{(1)} \le \dots \le Y_{(m)}$ IS THE SUFFICIENT STATISTIC.

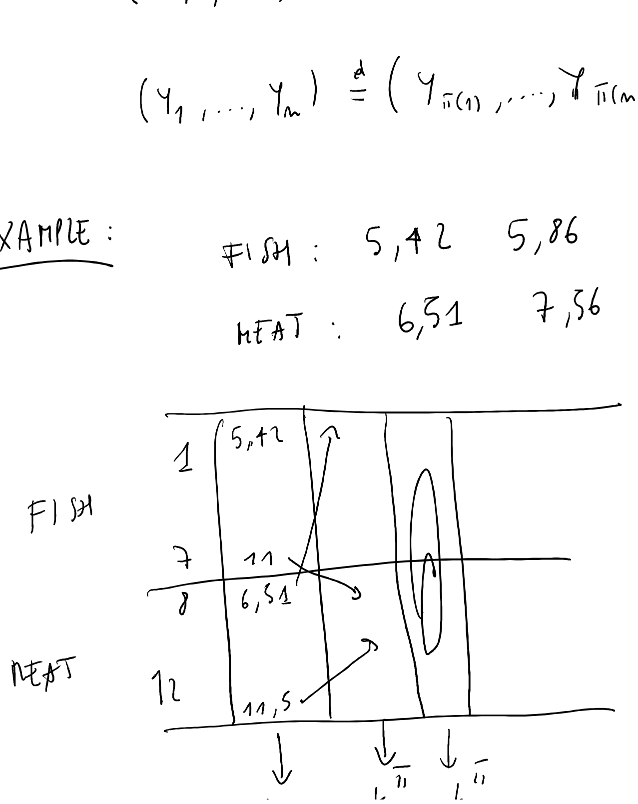
$(Y_1, \dots, Y_m)$ IS EXCHANGEABLE UNDER H_0

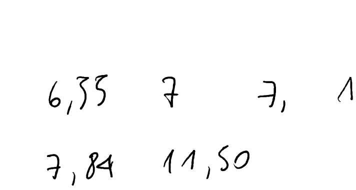
$$(Y_1, \dots, Y_m) \overset{d}{=} (Y_{\pi(1)}, \dots, Y_{\pi(m)}) \quad \forall \pi$$

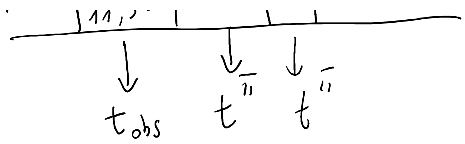
EXAMPLE:

FISH: 5,42 5,86 6,16 6,35 7 7,11

MEAT: 6,51 7,56 7,61 7,84 11,50







$$\mathbb{P}_0\left(T \geq t_{obs} \mid Y_{(1)}, \dots, Y_{(n)}\right) = \frac{1}{n!} \sum_{\pi} \mathbb{1}\left\{T^{\pi} \geq t_{obs}\right\}$$