Statistical inference II - lecture 2

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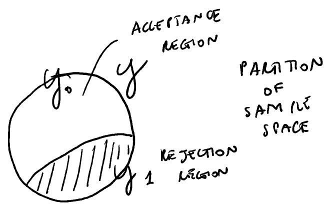
TWO-DECISION PROBLEM

- REJECT / NOT REJECT H_0
- FIX THE PROBABILITY OF REJECTING H_0 WHEN TRUE (PROB. OF TYPE I ERROR) AT **LEVEL** α AND UNDER THIS CONSTRAINT, MAXIMIZE THE POWER (PROB. REJ. H_0 WHEN FALSE)
- REQUIRES AN EXPLICIT FORMULATION OF THE ALTERNATIVE HYPOTHESIS H_1

HYPOTHESIS TESTING

- DECISION PROCEDURE IS CALLED THE **TEST** OF H_0 AGAINST H_1
- SUPPOSE $Y \sim \mathbb{P}_\vartheta$ WITH $\vartheta \in \Theta$
- NULL HYP. $H_0 : \vartheta \in \Theta_0 \subseteq \Theta$

  ALT. " $H_1 : \vartheta \in \Theta_1 \; (= \Theta \setminus \Theta_0)$
- IF AN HYPOTHESIS DETERMINES COMPLETELY THE DISTRIBUTION OF Y IS CALLED **SIMPLE**, OTHERWISE IS **COMPOSITE**



- A TEST $\phi = \phi(Y)$

$$\underset{\text{SAMPLE SPACE}}{\phi : \mathcal{Y}} \longmapsto \{ \underset{\text{NOT REJECT } H_0}{0}, \underset{\text{REJECTION OF } H_0}{1} \}$$

- **CONSTRAINT**:

$$\mathbb{E}_\vartheta(\phi) \le \alpha \quad \forall \vartheta \in \Theta_0$$

$$\sup_{\vartheta \in \Theta_0} \mathbb{E}_\vartheta(\phi) \quad \text{SIZE OF THE TEST}$$

MAXIMIZE POWER $\quad \mathbb{E}_\vartheta(\phi) \quad \vartheta \in \Theta_1$

$$\beta(\vartheta) = \mathbb{E}_\vartheta(\phi) \quad \text{POWER FUNCTION}$$

P-VALUE : SUPPOSE THAT THE REJECTION REGIONS ARE NESTED IN THE SENSE

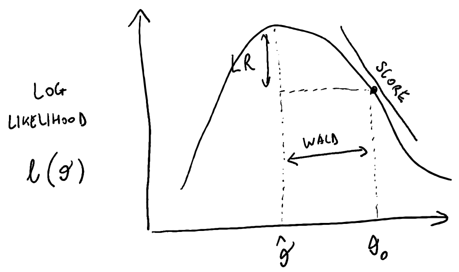
$$\mathcal{Y}_1(\alpha) \subseteq \mathcal{Y}_1(\tilde{\alpha}) \quad \text{IF} \quad \alpha \le \tilde{\alpha}$$

$$p_{obs} = \inf \{ \alpha \in (0,1) : y \in \mathcal{Y}_1(\alpha) \}$$

LIKEHOOD-BASED TESTS

$H_0 : \vartheta = \vartheta_0$

$H_1 : \vartheta \ne \vartheta_0$



$$\left.\begin{array}{lll} \text{WALD} & W_E = & (\hat{\vartheta} - \vartheta_0)^2 \, i(\vartheta_0) \\ \text{LR} & W_L = & 2 \{ \ell(\hat{\vartheta}) - \ell(\vartheta_0) \} \end{array}\right\} \begin{array}{l} \text{ASYMPT. EQUIVALENT} \\ \overset{H_0}{\sim} \chi^2_1 \end{array}$$

LR $\quad W_L = 2\{\ell(\hat{\vartheta}) - \ell(\vartheta_0)\}$

SCORE $\quad W_U = [U(\vartheta_0)]^2 i^{-1}(\vartheta_0)$

$\Bigg\}$ EQUIVALENT $\overset{H_0}{\underset{n\to\infty}{\sim}} \chi^2_1$ + REGULARITY CONDITIONS

EXAMPLE

DATA $\quad Y_1, \dots, Y_n$ IID $N(\mu, \sigma^2)$

HYP $\quad H_0 : \mu = \mu_0, (\sigma^2 > 0)$ COMPOSITE NULL HYPOTHESIS

$\quad H_1 : \mu \neq \mu_0 \quad (\sigma^2 > 0)$ COMPOSITE ALTERNATIVE HYPOTHESIS

LOG LIKELIHOOD $\quad \ell(\mu, \sigma^2) = -\frac{1}{2}\left\{ n \log \sigma^2 + \frac{1}{\sigma^2}\sum_{i=1}^n (y_i - \mu)^2 \right\}$

$$W_L = 2\left\{ \max_{\mu, \sigma^2} \ell(\mu, \sigma^2) - \max_{\sigma^2} \ell(\mu_0, \sigma^2) \right\}$$

$$= n \log\left\{ 1 + \frac{T^2}{n-1} \right\} \overset{H_0}{\underset{n\to\infty}{\sim}} \chi^2_1$$

where $T = \frac{\bar{Y} - \mu_0}{\sqrt{S^2/n}} \overset{H_0}{\sim} t_{n-1}$ WITH $S^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2$

NP LEMMA

$H_0 : f = f_0$ SIMPLE

$H_1 : f = f_1$ SIMPLE

THE MOST POWERFUL TEST OF SIZE α HAS CRITICAL REGION

$$\mathcal{Y}_1 = \left\{ y \in \mathcal{Y} : \underbrace{\frac{f_1(y)}{f_0(y)}}_{\text{LIKELIHOOD RATIO}} > t_\alpha \right\}$$

EXAMPLE

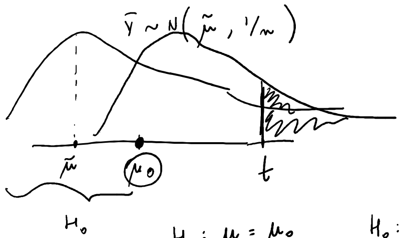
DATA $\quad Y_1, \dots, Y_n$ IID $N(\mu, 1)$

HYP $\quad H_0 : \mu \leq \mu_0$ COMPOSITE

$\quad H_1 : \mu > \mu_0$

REJECT FOR LARGE VALUES

$\bar{Y} > t_\alpha$



$H_0 : \mu = \mu_0 \qquad H_0 : \mu = \mu_0$

$H_1 : \mu > \mu_0 \qquad H_1 : \mu = \mu_1 \quad \mu_1 > \mu_0$

SIZE $\quad \sup_{\mu \leq \mu_0} \mathbb{P}_\mu(\bar{Y} > t_\alpha) = \mathbb{P}_{\mu_0}(\bar{Y} > t_\alpha)$

$$= \mathbb{P}_{\mu_0}\left(\frac{\bar{Y} - \mu_0}{\sqrt{1/n}} > \frac{t_\alpha - \mu_0}{\sqrt{1/n}} \right)$$

$$= \Phi\left(\frac{\mu_0 - t_\alpha}{\sqrt{1/n}} \right) = \alpha$$

so $t_\alpha = \mu_0 + \frac{z_{1-\alpha}}{\sqrt{n}}$

/ $\sqrt{m}$

$$\mathcal{Y}_1 = \left\{ (y_1, \dots, y_m) : \bar{y} > \mu_0 + \frac{z_{1-\alpha}}{\sqrt{m}} \right\}$$

POWER FUNCTION
$$\beta(\mu_1) = \mathbb{P}_{\mu_1}\left(\bar{Y} \geq t_\alpha \right) = \Phi\left(z_{1-\alpha} + \delta \right)$$
$$\delta = \sqrt{m}\,(\mu_1 - \mu_0)$$

NP LEMMA
$$LR = \frac{f_1(Y)}{f_0(Y)} = \exp\left\{ \frac{1}{2} \left(2m\bar{Y}(\mu_1 - \mu_0) - \mu_1^2 + \mu_0^2 \right) \right\}$$

IF $\mu_1 > \mu_0$, LR IS MONOTONE INCREASING IN $\bar{Y}$

THIS TEST IS MP FOR ANY $\mu_1 > \mu_0$ => UNIFORMLY MORE POWERFUL.

UMP

$Y_1, \dots, Y_m$ IID $N(\mu, \sigma^2)$

$H_0: \quad \mu \in (-\infty, -\Delta] \cup [\Delta, +\infty)$

$H_1: \quad \mu \in (-\Delta, \Delta)$

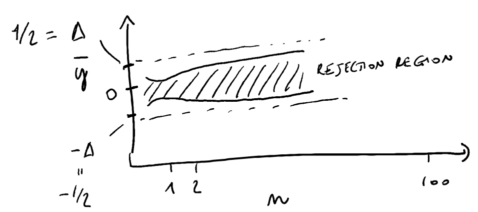
FOR SOME $\Delta > 0$.

$$T = m\bar{Y}^2 \sim \chi_1^2(m\mu^2) \qquad \text{NON-CENTRAL CHI-SQUARE}$$

SIZE
$$\sup_{\mu \in (-\infty, -\Delta] \cup [\Delta, +\infty)} \mathbb{P}_\mu(T \leq t_\alpha) = \mathbb{P}\left(\chi_1^2(m\Delta^2) \leq t_\alpha \right)$$

REJ. REGION
$$\mathcal{Y}_1 = \left\{ (y_1, \dots, y_m) : -\sqrt{t_\alpha / m} \leq \bar{y} \leq +\sqrt{t_\alpha / m} \right\}$$

WHERE t_α IS THE α-QUANTILE OF $\chi_1(m\Delta^2)$



WELLEK (2010)

UMP(UNBIASED)

$Y_1, \dots, Y_m$ IID $N(\mu, 1)$

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

$$\mathcal{Y}_1^- \cup \mathcal{Y}_1^+$$

$$\mathcal{Y}_1^- = \left\{ (y_1, \dots, y_m) : \bar{y} \leq \mu_0 + \frac{z_\alpha}{\sqrt{m}} \right\} \qquad \mathcal{Y}_1^+ = \left\{ (y_1, \dots, y_m) : \bar{y} > \mu_0 + \frac{z_{1-\alpha}}{\sqrt{m}} \right\}$$

HAS SIZE 2α

NO UMP TEST EXISTS. BUT THIS IS THE UMP TEST IN THE CLASS OF UNBIASED TEST.

UNBIASED TEST : A TEST ϕ IS UNBIASED FOR $H_0 : \vartheta \in \Theta_0$ AGAINST $H_1 : \vartheta \in \Theta_1$ OF SIZE α IF

$\sup_{\vartheta \in \Theta_0} E_\vartheta(\phi) = \alpha$ AND $\mathbb{E}_\vartheta(\phi) \geq \alpha \quad \forall \vartheta \in \Theta_1$.

LOCALLY MOST POWERFUL TEST

$$f_0(y) = f_0(y; \vartheta_0)$$

$$f_1(y) = f_0(y; \vartheta_0 + \varepsilon) \quad \text{FOR SMALL } \varepsilon$$

$$\log \frac{f_1(Y)}{f_0(Y)} = \log \frac{f(Y; \vartheta_0 + \varepsilon)}{f(Y; \vartheta_0)} \approx \varepsilon \underbrace{\frac{\partial \log f(Y; \vartheta_0)}{\partial \vartheta_0}}_{U(\vartheta_0)} + \dots$$

THIS IMPLIES THAT THE LOCALLY MOST POWERFUL TEST HAS CRITICAL REGION

$$\mathcal{Y}_1 = \left\{ (y_1, \dots, y_n) : u(\vartheta_0) \geq i(\vartheta_0)^{1/2} z_{1-\alpha} \right\}$$

EXAMPLE

DATA $\quad Y_1, \dots, Y_n$ IID CAUCHY $\dfrac{1}{\pi\left(1 + (y - \vartheta)^2\right)}$

$$H_0 : \vartheta = \vartheta_0$$

SCORE FOR Y_1 $\quad U_1(\vartheta_0) = -\dfrac{\partial}{\partial \vartheta_0} f(Y_1; \vartheta_0) = \dfrac{2(Y_1 - \vartheta_0)}{1 + (Y_1 - \vartheta_0)^2}$

INFORMATION FOR Y_1 $\quad i_1(\vartheta_0) = \mathbb{E}_{\vartheta_0}\left[\left\{\dfrac{\partial \log f(Y_1; \vartheta_0)}{\partial \vartheta_0}\right\}^2\right] = \dfrac{1}{2}$

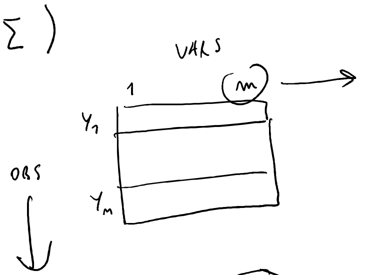
TEST STATISTIC $\quad U(\vartheta_0) = 2 \displaystyle\sum_{i=1}^{n} \frac{(Y_i - \vartheta_0)}{1 + (Y_i - \vartheta_0)^2}$ HAS MEAN 0 AND VARIANCE $n/2$

UMP I(NVARIANT)

$$Y_1, \dots, Y_n \text{ IID } N_m(\mu, \Sigma)$$

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$



Σ UNKNOW, $\boxed{n > m}$

HOTELLING'S $T^2 = n(\bar{Y} - \mu_0)^T S^{-1} (\bar{Y} - \mu_0)$

HIGH DIMENSIONAL DATA

$m >> n$

$\frac{1}{\ } \sum^n (Y_i - \bar{Y})(Y_i - \bar{Y})^T$

HOTELLINGS $T^2 = n(\bar{y} - \mu_0)^T S^{-1} (\bar{y} - \mu_0)$

DATA

$n >> m$

$$S = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})(y_i - \bar{y})^T$$

$$T^2 \overset{H_0}{\sim} \frac{m(n-1)}{n-m} F_{m, n-m}$$

SNEDECOR OF FISHER DISTRIBUTION

NO UMP EXISTS. BUT T^2 IS MOST POWERFUL IN THE CLASS OF TESTS THAT ARE INVARIANT TO LINEAR TRANSFORMATIONS

$$X = AY + b \qquad \underset{m \times m}{A} \text{ MATRIX NON-SINGULAR}$$

$m = 1 \qquad T^2 = (t)^2$

RELATION OF HYPOTHESIS TESTING WITH INTERVAL ESTIMATION

CONFIDENCE INTERVALS, OR MORE GENERALLY CONFIDENCE SETS CAN BE PRODUCED BY TESTING EVERY POSSIBLE VALUE $\vartheta \in \Theta$ AND TAKING THOSE VALUES NOT REJECTED AT LEVEL α.

DUALITY

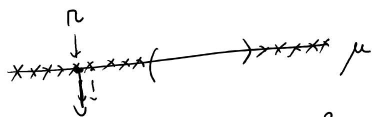
FOR EACH $\vartheta_0 \in \Theta$, WE HAVE $\mathcal{Y}_0(\vartheta_0)$ ACCEPTANCE REGION OF A TEST OF SIZE α FOR $H_0 : \vartheta = \vartheta_0$

THE SET OF VALUES

$$S(y) = \{ \vartheta \in \Theta : Y \in \mathcal{Y}_0(\vartheta) \}$$

CONTAINS THE TRUE PARAMETER WITH PROBABILITY $\geq 1-\alpha$

Proof.
$$\mathbb{P}_\vartheta(\vartheta \in S(Y)) = \mathbb{P}_\vartheta(Y \in \mathcal{Y}_0(\vartheta)) \geq 1-\alpha \quad \forall \vartheta$$



$H_r : \mu = r \quad , \quad r \in \mathbb{R}$

EXAMPLE

n_1 OBS FROM $N(\mu_1, 1) \longrightarrow \bar{y}_1$ BY SUFFICIENCY

n_2 OBS FROM $N(\mu_2, 1) \longrightarrow \bar{y}_2$

PARAMETER OF INTEREST $\quad \vartheta = \mu_2 / \mu_1$

$H_0 : \vartheta = \vartheta_0$

$$\frac{\bar{Y}_2 - \vartheta_0 \bar{Y}_1}{\sqrt{1/m_2 + \vartheta_0/m_1}} \overset{H_0}{\sim} N(0,1)$$

$$CI = \left\{ \vartheta \in \mathbb{R} : \frac{(\bar{Y}_2 - \vartheta \bar{Y}_1)^2}{1/m_2 + \vartheta/m_1} \leq \underbrace{C_{1-\alpha}}_{\substack{1-\alpha \\ \text{quantile} \\ \text{of } \chi^2_1}} \right\}$$

CONFIDENCE LIMITS ARE ROOTS OF THIS QUADRATIC EQUATION.