Math 273: Basic Theory of Ordinary Differential Equations
Computational Problems

Autumn 2024

Problem 1.10: Methods of Integration

Consider methods to integrate numerically the ODE

$$\dot{x} = f(t, x)$$

where $x(t)$ is a scalar-valued function, and we are given $x(0)$. We discretize time with size Δ and examine three methods:

Methods

(a) **Euler's Method:**

$$x(t_{n+1}) = x(t_n) + \Delta \cdot f(t_n, x(t_n))$$

(b) **Midpoint Method:**

$$\begin{aligned} k_{\text{temp}} &= x(t_n) + \frac{\Delta}{2} \cdot f(t_n, x(t_n)) \\ x(t_{n+1}) &= x(t_n) + \Delta \cdot f(t_n + \frac{\Delta}{2}, k_{\text{temp}}) \end{aligned}$$

(c) **Runge-Kutta Method:**

$$\begin{aligned} k_1 &= f(t_n, x(t_n)) \\ k_2 &= f(t_n + \frac{\Delta}{2}, x(t_n) + \frac{\Delta}{2} k_1) \\ k_3 &= f(t_n + \frac{\Delta}{2}, x(t_n) + \frac{\Delta}{2} k_2) \\ k_4 &= f(t_n + \Delta, x(t_n) + \Delta k_3) \\ x(t_{n+1}) &= x(t_n) + \frac{\Delta}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

Task: Compare these methods by solving $\dot{x} = x$ with $x(0) = 1$ (solution: $x(t) = e^t$) using step sizes $\Delta = 1/N$ for $N = 2^k$, $k \in [10, 20]$.

Note: This problem provides excellent insight into numerical stability and convergence. The Runge-Kutta method typically provides the best accuracy, but at the cost of more computational complexity per step.

Problem 2.8: Lotka-Volterra System

Write code to simulate the evolution of the Lotka-Volterra system (predator-prey equations) in 2D phase space:

$$\frac{dx}{dt} = kx - axy$$
$$\frac{dy}{dt} = -ly + bxy$$

Task: Choose values for parameters k, l, a, b and draw several phase curves using both Euler's and Runge-Kutta methods. Start with large step sizes and decrease until phase curves close up.

Note: The Lotka-Volterra system is a classic example where numerical stability is crucial. With large step sizes, the numerical solution may not conserve the system's inherent periodic behavior, leading to spiraling trajectories instead of closed orbits.

Problem 3.9: Van der Pol Oscillator

Consider the differential equation:

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

where μ is a real parameter.

Task: Plot trajectories in the phase plane $(x, \dot{x})$ for various values of μ. Superimpose trajectories with different μ values to observe behavioral changes.

Note: The van der Pol oscillator is a fundamental example of a nonlinear system exhibiting limit cycles. The parameter μ controls the strength of the nonlinearity and damping, leading to rich dynamical behavior.

Problem 7.8: Lorenz System

Consider the system:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Task: Explore the system near the classical parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. Take various initial conditions, including:

- A "cloud" of nearby initial conditions
- Points with large coordinates
- Plot the x-coordinate over time and compare with previous experiments

Note: The Lorenz system is perhaps the most famous example of deterministic chaos in continuous dynamical systems. The sensitive dependence on initial conditions means that even sophisticated numerical methods must be used with care.