Black Holes

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Throughout this work, the geometrised unit system, $G = c = 1$, has been used.

1 Exercise 1

Einstein's equations are given by:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1}$$

To check that, in the vacuum ($T_{\mu\nu} = 0$), this set of equations reduces to $R_{\mu\nu} = 0$, let us contract the two indices, bearing in mind that we are working with the usual 4D space-time ($g^{\mu\nu}g_{\mu\nu} = 4$):

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R + \Lambda g^{\mu\nu}g_{\mu\nu} = R - 2R + 4\Lambda = 8\pi g^{\mu\nu}T_{\mu\nu} = 8\pi T \Longrightarrow R = -8\pi T + 4\Lambda. \tag{2}$$

Substituting the last expression into (1), we obtain an analogous and famous form of the Einstein equations:

$$R_{\mu\nu} + 4\pi g_{\mu\nu}T - 2\Lambda g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \Longrightarrow R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}. \tag{3}$$

Then, if $T_{\mu\nu} = 0$, then also $T = g^{\mu\nu}T_{\mu\nu} = 0$, so the above equation is just $R_{\mu\nu} = \Lambda g_{\mu\nu}$. This result makes sense: in the absence of energy-matter in the universe, the Ricci tensor is only associated with the expansion of the universe.

Finally, if we have a non-expanding vacuum universe (which is thought to be related to dark energy, which is not included in $T_{\mu\nu}$), it is verified that $R_{\mu\nu} = 0$.

2 Exercise 2

Limiting ourselves to Einstein system, Birkhoff 's theorem says: *Any spherically symmetric vacuum solution is static, which implies that it must be Schwarzschild.*

To demonstrate this, we need to obtain the Ricci tensor, that is $R_{\mu\nu} = 0$ in the vacuum:

$$R_{\beta\nu} = R^{\mu}{}_{\beta\mu\nu} = \Gamma^{\mu}{}_{\beta\nu,\mu} - \Gamma^{\mu}{}_{\beta\mu,\nu} + \Gamma^{\mu}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\mu}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\beta\mu}. \tag{4}$$

As we can see here, all we need are the Christoffel symbols:

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}), \tag{5}$$

which only depends on the components of the metric. So, with $g_{\mu\nu}$, we can obtain $R_{\mu\nu}$.

Let us take a general metric with spherical symmetry in the vacuum ($R_{\mu\nu} = 0$) which is given by the following expression:

$$ds^2 = -e^{2A(t,r)}dt^2 + e^{2B(t,r)}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2, \tag{6}$$

so the non-zero components of the metric tensor are $g_{tt}(t,r) = -e^{2A(t,r)}$, $g_{rr}(t,r) = e^{2B(t,r)}$, $g_{\theta\theta}(r) = r^2$, and $g_{\phi\phi}(r,\theta) = r^2sin^2\theta$. With this, we can obtain Christoffel's symbols, defining $\partial_r A = A', \partial_r B = B', \partial_t A = \dot{A}$, and $\partial_t B = \dot{B}$ (the generalised expressions that we will obtain are only valid for this particular case):

1. For $\mu = t$ at (5):

   - $\boxed{\Gamma^t{}_{tt}} = \frac{1}{2}g^{tt}g_{tt,t} = \frac{1}{2}(-e^{-2A})(-2\dot{A}e^{2A}) = \dot{A}$.
   - $\boxed{\Gamma^t{}_{tr}} = \frac{1}{2}g^{tt}(g_{tt,r} + g_{tr,t} - g_{tr,t}) = \frac{1}{2}(-e^{-2A})(-2A'e^{2A}) = A'$.
     It can be generalised to $\Gamma^\alpha{}_{\alpha\gamma} = \frac{1}{2}g^{\alpha\alpha}g_{\alpha\alpha,\gamma}$.
   - $\boxed{\Gamma^t{}_{rr}} = \frac{1}{2}g^{tt}(g_{rt,r} + g_{tr,r} - g_{rr,t}) = -\frac{1}{2}g^{tt}g_{rr,t} = \frac{1}{2}(-e^{-2A})(-2\dot{B}e^{2B}) = \dot{B}e^{2(B-A)}$.
     It can be generalised to $\Gamma^\alpha{}_{\beta\beta} = -\frac{1}{2}g^{\alpha\alpha}g_{\beta\beta,\alpha}$ if $\alpha \neq \beta$.
   - $\Gamma^t{}_{t\theta} = \frac{1}{2}g^{tt}g_{tt,\theta} = 0$; $\Gamma^t{}_{t\phi} = \frac{1}{2}g^{tt}g_{tt,\phi} = 0$.
   - $\Gamma^t{}_{r\theta} = \frac{1}{2}g^{tt}(g_{tr,\theta} + g_{t\theta,r} - g_{r\theta,t}) = 0$.
     It can be generalised to $\Gamma^\alpha{}_{\beta\gamma} = 0$ if $(\alpha \neq \beta) \neq \gamma$.
   - $\Gamma^t{}_{\theta\theta} = -\frac{1}{2}g^{tt}g_{\theta\theta,t} = 0$; $\Gamma^t{}_{\phi\phi} = 0$.
   - $\Gamma^t{}_{\theta\phi} = 0$; $\Gamma^t{}_{r\phi} = 0$.

2. For $\mu = r$ at (5):

   - $\boxed{\Gamma^r{}_{rr}} = \frac{1}{2}g^{rr}g_{rr,r} = \frac{1}{2}e^{-2B}(2B'e^{2B}) = B'$.
   - $\boxed{\Gamma^r{}_{rt}} = \frac{1}{2}g^{rr}g_{rr,t} = \dot{B}$.
   - $\Gamma^r{}_{r\theta} = 0$; $\Gamma^r{}_{r\phi} = 0$.
   - $\boxed{\Gamma^r{}_{tt}} = -\frac{1}{2}g^{rr}g_{tt,r} = -\frac{1}{2}e^{-2B}(-2A'e^{2A}) = A'e^{2(A-B)}$.
   - $\Gamma^r{}_{t\phi} = 0$; $\Gamma^r{}_{t\theta} = 0$; $\Gamma^r{}_{\theta\phi} = 0$.
   - $\boxed{\Gamma^r{}_{\theta\theta}} = -\frac{1}{2}g^{rr}g_{\theta\theta,r} = -\frac{1}{2}e^{-2B}2r = -re^{-2B}$.
   - $\boxed{\Gamma^r{}_{\phi\phi}} = \frac{1}{2}g^{rr}g_{\phi\phi,r} = -e^{-2B}rsin^2\theta$.

3. For $\mu = \theta$ at (5):

   - $\Gamma^\theta{}_{\theta\theta} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\theta} = 0$.
   - $\Gamma^\theta{}_{\theta t} = 0$; $\Gamma^\theta{}_{\theta\phi} = 0$.
   - $\Gamma^\theta{}_{tt} = -\frac{1}{2}g^{\theta\theta}g_{tt,\theta} = 0$; $\Gamma^\theta{}_{rr} = 0$.
   - $\boxed{\Gamma^\theta{}_{\theta r}} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,r} = \frac{1}{2}r^{-2}2r = \frac{1}{r}$.
   - $\boxed{\Gamma^\theta{}_{\phi\phi}} = -\frac{1}{2}g^{\theta\theta}g_{\phi\phi,\theta} = -\frac{1}{2}r^{-2}r^2 2sin\theta cos\theta = -sin\theta cos\theta$.
   - $\Gamma^\theta{}_{tr} = 0$; $\Gamma^\theta{}_{t\phi} = 0$; $\Gamma^\theta{}_{r\phi} = 0$.

4. For $\mu = \phi$ at (5):

   - $\Gamma^\phi{}_{\phi\phi} = 0$.
   - $\Gamma^\phi{}_{\phi t} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,t} = 0$.
   - $\boxed{\Gamma^\phi{}_{\phi r}} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,r} = \frac{1}{2}\frac{1}{r^2sin^2\theta}2rsin^2\theta = \frac{1}{r}$.
   - $\boxed{\Gamma^\phi{}_{\phi\theta}} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\theta} = \frac{1}{2}\frac{1}{r^2sin^2\theta}r^2 2sin\theta cos\theta = cotg\theta$.
   - $\Gamma^\phi{}_{tt} = -\frac{1}{2}g^{\phi\phi}g_{tt,\phi} = 0$; $\Gamma^\phi{}_{rr} = 0$; $\Gamma^\phi{}_{\theta\theta} = 0$.
   - $\Gamma^\phi{}_{tr} = 0$; $\Gamma^\phi{}_{t\theta} = 0$; $\Gamma^\phi{}_{r\theta} = 0$.

Then, we could substitute in (4) to obtain its non-zero components:

$$R_{tt} = \frac{(-A'B'r + A''r + rA'^2 + 2A')e^{2A} + re^{2B}(\dot{A}\dot{B} - \ddot{B} - \dot{B}^2)}{re^{2B}}, \tag{7}$$

$$R_{tr} = \frac{\dot{B}}{r}, \tag{8}$$

$$R_{rr} = \frac{(2B' + A'B'r - rA'^2 - A''r)e^{2A} + re^{2B}(\ddot{B} + \dot{B}^2 - \dot{A}\dot{B})}{re^{2A}}, \tag{9}$$

$$R_{\theta\theta} = -\frac{1 - e^{2B} - rB' + A'r}{e^{2B}}, \tag{10}$$

$$R_{\phi\phi} = sin^2\theta R_{\theta\theta}. \tag{11}$$

Now, particularising for the vacuum ($R_{\mu\nu} = 0$), from (8), it is evident that $\dot{B} = 0$, so $B(t,r) = B(r)$. This result allows us to simplify R_{tt} and R_{rr}, so let's see what we get from the following calculation:

$$R_{tt}e^{-2(A-B)} + R_{rr} = [e^{2(A-B)}(A'' + A'^2 - A'B' + 2A'r^{-1})]e^{-2(A-B)} + [-A'' - A'^2 + A'B' + 2B'r^{-1}] =$$
$$= 2(A' + B')r^{-1} = 0 \Longrightarrow A' = -B' \Longrightarrow A(t,r) = -B(r) + f(t), \tag{12}$$

where $f(t)$ is a time-dependent function. With the general forms of A and B, we can substitute in (10) (which gives the same information as (11) in vacuum):

$$1 - e^{-2B}(rA' - rB' + 1) = 0 = 1 - e^{-2f(t)}e^{2A}(1 + 2rA') = 1 - e^{-2f(t)}[re^{2A}]' \Longrightarrow 1 = e^{-2f(t)}[re^{2A}]'. \tag{13}$$

Integrating over r:

$$r + C = e^{-2f(t)}e^{2A}r \Longrightarrow e^{2(A-f(t))} = e^{-2B} = 1 + \frac{C}{r}, \tag{14}$$

with C a constant, and from where $e^{2A} = (1 + C/r)e^{2f(t)}$. Finally, we have what we need to obtain (6) as a function of r and $f(t)$:

$$ds^2 = -\left(1 + \frac{C}{r}\right)e^{2f(t)}dt^2 + \left(1 + \frac{C}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{15}$$

It is possible to absorb the function $f(t)$ by redefining $d\tilde{t} = e^{f(t)}dt$, so the metric of any spherically symmetric vacuum is given by:

$$ds^2 = -\left(1 + \frac{C}{r}\right)dt^2 + \left(1 + \frac{C}{r}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{16}$$

which is static, and its Schwarzschild metric, proving the Birkhoff's theorem.

3 Exercise 3

Consider Schwarzschild in incoming Eddington-Finkelstein coordinates $(r, \tilde{V}, \theta, \phi)$, whose line-element is:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\tilde{V}^2 + 2d\tilde{V}dr + r^2d\Omega^2, \tag{17}$$

and the surface $S(r) = r - 2M$. Let's find the components of the normal vector $l = \tilde{f}(x)(g^{\mu\nu}\partial_v S)\partial/(\partial x^\mu)$ to that family of hypersurfaces, $S(r)$, which are given by:

$$l^\mu = \tilde{f}(x)(g^{\mu\nu}\partial_\nu S). \tag{18}$$

In IEF coordinates, $\tilde{V} = t + r + 2Mlog((r - 2M)/2M)$ and $R = r$ (just to distinguish from the original coordinates), so:

$$g_{rr} = g(\partial_r, \partial_r) = g\left(\frac{\partial\tilde{V}}{\partial r}\partial_{\tilde{V}} + \partial_R, \partial_r\right) = g\left(\frac{1}{1 - 2M/r}\partial_{\tilde{V}} + \partial_R, \frac{1}{1 - 2M/r}\partial_{\tilde{V}} + \partial_R\right) =$$
$$= \frac{g_{\tilde{V}\tilde{V}}}{(1 - 2M/r)^2} + \frac{2g_{\tilde{V}r}}{1 - 2M/r} + g_{RR} = -\frac{1}{1 - 2M/r} + \frac{2}{1 - 2M/r} = \frac{1}{1 - 2M/r}. \tag{19}$$

With this in mind, the inverse metric is:

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - 2M/r & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2sin^2\theta \end{pmatrix}, \tag{20}$$

which apparently may contradict the usual definition of $ds^2 = g_\mu dx^\mu dx^\nu$, but it does not. Since $S(r)$ can only be derived with respect to r such that $\partial_r S = 1$, the non-zero terms of (18) are:

$$l^{\tilde{V}} = \tilde{f}(x)(g^{\tilde{V}\nu}\partial_\nu S) = \tilde{f}(x)g^{\tilde{V}r} = \tilde{f}(x), \tag{21}$$

$$l^r = \tilde{f}(x)(g^{r\nu}\partial_\nu S) = \tilde{f}(x)g^{rr} = \tilde{f}(x)\left(1 - \frac{2M}{r}\right). \tag{22}$$

Lastly, to show that $S(r)$ is a null hypersurface in $r = 2M$, let us calculate:

$$l^2 = l^\alpha l_\alpha = g^{\mu\nu}\partial_\mu S \partial_\nu S \tilde{f}^2 = g^{rr}\tilde{f}^2 = \left(1 - \frac{2M}{r}\right)\tilde{f}^2. \tag{23}$$

This quantity verifies at $r = 2M$ that $l^2 = 0$, showing that it is a null hypersurface at that value of r.

4 Exercise 4

In Boyer-Linquist coordinates, the Kerr-Newman metric describing a spinning black hole (a) with electric and magnetic (monopole) charge (Q and P, respectively) is:

$$ds^2 = -\frac{\triangle - a^2 sin^2\theta}{\Sigma}dt^2 - 2asin^2\theta\frac{r^2 + a^2 - \triangle}{\Sigma}dtd\phi + \frac{(r^2+a^2)^2 - \triangle a^2 sin^2\theta}{\Sigma}sin^2\theta d\phi^2 + \frac{\Sigma}{\triangle}dr^2 + \Sigma d\theta^2, \tag{24}$$

where:

$$\Sigma = r^2 + a^2 cos^2\theta, \tag{25}$$

$$\triangle = r^2 - 2Mr + a^2 + e^2, \tag{26}$$

$$a = J/M, \tag{27}$$

$$e = \sqrt{Q^2 + P^2}. \tag{28}$$

On the one hand, let us find the radii of the two horizons that appear when the metric becomes singular. This happens if $\Sigma = 0$ and/or $\triangle = 0$:

$$\Sigma = 0 = r^2 + a^2 cos^2\theta \Longrightarrow r = 0 \ and \ \theta = \pi/2, \tag{29}$$

giving us the typical singularities of the black holes metric ($r = 0$ is the physical one); and:

$$\triangle = 0 = r^2 - 2Mr + a^2 + e^2 \Longrightarrow r_\pm = M \pm \frac{1}{2}\sqrt{4M^2 - 4(a^2 + e^2)} = M \pm \sqrt{M - (a^2 + Q^2 + P^2)}, \tag{30}$$

which are the two radii we were looking for. If $M^2 < a^2 + Qr + P^2$, there are complex solutions, but it is thought that this spacetime is not physical; if $M^2 \geq a^2 + Qr + P^2$, having a ring singularity, and at $r = r_\pm$ there are coordinate singularities. At $r = r_+$, we have the event horizon, and at $r = r_-$, the Cauchy horizon.

On the other hand, if we take $a = 0$ at (24):

$$ds^2 = -\frac{r^2 - 2Mr + e^2}{r^2}dt^2 + \frac{r^4}{r^2}sin^2\theta d\phi^2 + \frac{r^2}{r^2 - 2Mr + e^2}dr^2 + r^2 d\theta^2, \tag{31}$$

$$ds^2 = -\frac{r^2 - 2Mr + e^2}{r^2}dt^2 + \frac{r^2}{r^2 - 2Mr + e^2}dr^2 + r^2 d\Omega^2. \tag{32}$$

There are three particular cases here:

- If $e = 0$, translated into $Q = 0$ and $P = 0$, we recover the Schwarzschild metric.
- If $P = 0$, translated into $e^2 = Q^2$, we recover the Reissner-Nordström solution (that generalizes Schwarzschild).
- If $M = 0$ and $e = 0$, we recover the Minkowski metric.

These are non-rotating black holes by construction (remember that here we impose that $a = 0$).

5 Exercise 5

Let us study a head-on collision of two non-rotating black holes with masses M_1 and M_2, which forms another black hole with mass M_3. Our goal is to calculate the upper limit of the efficiency of gravitational wave emission in the process just described.

Let be the efficiency:

$$\eta = \frac{M_1 + M_2 - M_3}{M_1 + M_2} = 1 - \frac{M_3}{M_1 + M_2}. \tag{33}$$

From Hawking's area theorem, $A_3 \geq A_1 + A_2$. If we assume that the black holes are stationary (M_1 and M_2 at the beginning, and M_3 long after the collision), the area is $A = 4\pi r^2$, with $r = 2M$ being the Schwarzschild radius (M_3 only verify the equality at late times). Then, substituting at Hawking's area theorem:

$$16\pi M_3^2 \geq A_3 \geq 16\pi(M_1^2 + M_2^2) \Longrightarrow M_3 \geq \sqrt{M_1^2 + M_2^2}. \tag{34}$$

Taking this expression into (33):

$$\eta \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} = 1 - \frac{\sqrt{M_1^2 + M_2^2}}{\sqrt{(M_1 + M_2)^2}} = 1 - \frac{1}{\sqrt{1 + \frac{2M_1 M_2}{M_1^2 + M_2^2}}}, \tag{35}$$

such that it will take the maximum value if $M_1 = M_2$, so the upper limit of the efficiency is:

$$\eta \leq 1 - \frac{1}{\sqrt{2}} \approx 30\%. \tag{36}$$

This result is telling us that a lot of energy can be released from a collision of two black holes in the form of gravitational waves, which cannot be seen as easily as electromagnetic waves. This will cause the final black hole to have a mass less than the sum of the two initial black holes: there has been a loss of mass due to the emission of gravitational waves, so the higher the efficiency, the lower the mass of the resulting black hole.

For example, by definition of the efficiency (33), if $M_3 = M_1 + M_2$, $\eta = 0$, and it will not be a loss of mass, and the system will not emit gravitational waves. This is obvious: the initial and final masses are the same.