Black Holes: Problems II

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In this work, $c = G = 1$, a fact that must be considered if we want the results at the international system of units.

1 An astronaut falling into a black hole

1.1 Pain time

We have an astronaut that is $\triangle r = 2\ m$ tall, that is falling into a black hole, where the differential acceleration between head and feet is equal to Earth's gravity $g_E = 9.81\ m/s^2$.

As the acceleration (variation of the infalling velocity with the proper time) can be computed as:

$$\frac{d\dot{r}}{d\tau} = \frac{d\dot{r}}{dr}\frac{dr}{d\tau} = \frac{d\dot{r}}{dr}\dot{r} = \frac{m}{r^2}, \tag{1}$$

the relative acceleration between the head and feet of an independant observer will be $dg = (2m/r^3)dr$ (taking into account that, for a particle starting at rest at infinite distance, the radial infall equation is $\dot{r}^2 = v_{loc}^2 = 2m/r$).

Following this thought, the proper distance measured by the falling observer, $dr(1-2m/r)^{-1/2}$, is contracted by a factor $1/\gamma = 1/(1-v_{loc}^2)^{1/2}$, so the proper length in the freely falling frame is:

$$\gamma^{-1} dr(1-2m/r)^{-1/2} = dr. \tag{2}$$

Then, the tidal acceleration on a body of extension $\triangle r$ is $dg \approx (2m/r^3)\triangle r$.

Then, the time of the fall in which the astronaut suffers by tidal forces is:

$$\tau = -\int_{r_p}^{0} \left(\frac{r}{2m}\right)^{1/2} dr = \frac{2r_p^{3/2}}{3\sqrt{2m}}, \tag{3}$$

where r_p is the distance at which the tidal acceleration on the astronaut equals Eath's gravity:

$$\frac{dg}{\triangle x} \approx \frac{g_E}{\triangle x} \approx \frac{2m}{r_p^3} \longrightarrow r_p^{3/2} \approx \left(\frac{2m\triangle x}{g_E}\right)^{1/2}. \tag{4}$$

Substituting this last expression at 3:

$$\tau \approx \frac{2}{3\sqrt{2m}}\frac{\sqrt{2m\triangle x}}{\sqrt{g_E}} = \frac{2}{3}\sqrt{\frac{\triangle x}{g_E}} = 0.30\ s. \tag{5}$$

At least, we have a good news for our poor astronaut: he will suffer, but it will be quick.

This result its interesting because is independant of the black hole mass, such that, if $m = M_\odot$, the painful time will be $0.30\ s$ (answering the last question).

1.2 BH mass

To obtain the black hole mass necessary to extend free fall without pain for a year, we know that we are working with a supermassive black hole. Then, our astronaut will suffer, as we obtained in the last subsection, only the last $0.3\ s$, that is a really small value respect to 1 year. In other words, this time will be the one that the astronaut pass inside the black hole until it reaches the singularity (more or less).

Analogously, this time has a similar expression to 3, but now:

$$\tau = -\int_{2m}^{0} \left(\frac{r}{2m}\right)^{1/2} dr = \frac{2(2m)^{3/2}}{3\sqrt{2m}} = \frac{4m}{3}. \tag{6}$$

Taking $G = 6.67 \cdot 10^{-11}\ m^3 kg^{-1} s^{-2}$, $c = 3 \cdot 10^8\ m/s$, and $M_\odot = 2 \cdot 10^{30}\ kg$, the black hole mass is:

$$m = \frac{3\tau}{4}\frac{c^3}{G} = 4.79 \cdot 10^{12} M_\odot, \tag{7}$$

that is of the same order of magnitude that an entire galaxy. Indeed, we are talking about a super-massive black hole.

2 Energy loss of a particle

If we consider that the particle (that has a unit mass) was at rest before it started approaching the black hole (so it has an energy $E = -1$), the maximun amount of energy that this particle may lose as it spirals into a Kerr BH (assuming that it were equal to its maximun binding energy) is:

$$1 - \tilde{E}, \tag{8}$$

where $\tilde{E}$ appears due to the fact that the particle spirals through a succesion of almost circular equatorial orbits. This value will be maximum if:

$$\frac{a}{m} = 1 = \mp\frac{4\sqrt{2}(1-\tilde{E}^2)^{1/2} - 2\tilde{E}}{3\sqrt{3}(1-\tilde{E}^2)}, \tag{9}$$

and also if $\tilde{E}$ has the miminum value, associated to co-rotation, with a value of $\tilde{E} = \sqrt{1/3}$. Then, the energy loss will be:

$$1 - \sqrt{\frac{1}{3}} = 0.423 = 42.3\%. \tag{10}$$

This value is almost two order of magnitude higher than the energy released at nuclear reactions (0.7%), so this process just described its more energetic by far.

3 Relation between mass and spin parameter

For circular orbits:

$$\tilde{E} = \frac{r^2 - 2mr \pm a\sqrt{mr}}{r(r^2 - 3mr \pm 2a\sqrt{mr})^{1/2}}. \tag{11}$$

If we require stability, $1 - \tilde{E}^2 = 2m/3r$. With this two expressions, its very difficult to clear a, so we have used *Mathematica* to stay alive after the exercise. Then, eliminating r, what we obtained is:

$$a = \frac{2}{9}\left[\frac{\sqrt{3}\tilde{E}m}{\sqrt{1 - 2\tilde{E}^2 + \tilde{E}^4}} \pm 2\sqrt{6}\frac{m}{\sqrt{(1-\tilde{E})(1+\tilde{E})}}\right] = \frac{2}{9}\frac{\sqrt{3}m\tilde{E} \pm 2\sqrt{6}m\sqrt{1-\tilde{E}^2}}{(1-\tilde{E}^2)}. \tag{12}$$

Finally, the expression withput r is:

$$\frac{a}{m} = \frac{2\tilde{E} \pm 4\sqrt{2}(1-\tilde{E}^2)^{1/2}}{3\sqrt{3}(1-\tilde{E}^2)}. \tag{13}$$