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Electron Capture I: Resonance Capture from Hydrogen Atoms by Fast Protons

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(A fuller discussion of this question of radii is given in Bhatia *et al.* 1952.) However, in this case this will not happen since the final nucleus in the (d, p) reaction corresponds to the initial nucleus in the present reaction, and one always starts with a stable nucleus.

Finally, Wolfgang and Libby (1952) have measured the total cross section of the ^{9}Be(d, t)^{8}Be reaction and have found it to be approximately constant at about 2×10^{-25} cm^2 over the range of energy from 3 to 8 Mev. It is difficult to estimate this theoretically, but a rough estimate can be made of 10^{-25} cm^2 at 8 Mev. In the limit of high energies the cross section varies as $1/E_d$. These results are in as good agreement as can be expected with the experimental data.

In conclusion it may be pointed out that the above analysis will apply also to the (d, ^{3}He) reaction by changing all quantities relating to the triton to the corresponding ones for the ^{3}He nucleus. The parameters in Irving's wave function remain the same since these give the correct binding energy for the ^{3}He nucleus when the coulomb energy is taken into account.

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Electron Capture
I: Resonance Capture from Hydrogen Atoms by Fast Protons

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ABSTRACT. Born's approximation is used to calculate the cross section associated with resonance charge exchange between protons and hydrogen atoms, it being pointed out that in earlier treatments of this problem unjustified simplifying assumptions were made. An energy range of up to 250 kev is covered. The predicted energy variation of the cross section is in harmony with the results of Ribe's recent laboratory work on collisions of protons with hydrogen molecules.

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§ 1. INTRODUCTION

Electron capture through energetic collisions between positive ions and neutral atoms or molecules, besides being of interest in itself, is important in aurorae and in certain other natural phenomena. It is therefore desirable to carry out quantal calculations on the process. The simplest possible case

$$H^+ + H(1s) \rightarrow H(1s) + H^+ \qquad \ldots\ldots(1)$$

was studied many years ago by Oppenheimer (1928) and by Brinkmann and Kramers (1930) who gave a closed formula for the cross section which could readily be generalized to more complicated collisions. Unfortunately, examination of their treatment shows that the interaction potential employed is incorrect and that as a consequence so also are both the absolute magnitude and the energy variation of the deduced cross sections. The present paper is devoted to a reinvestigation of the problem. Attention will be restricted to the collision represented by (1), but the method is applicable to other cases and will be extended to them later.

§ 2. ANALYSIS

2.1

At high energies the cross section Q associated with the rearrangement collision

(nucleus A + electron e) + nucleus B → nucleus A + (nucleus B + electron e),

is given by Born's approximation (cf. Mott and Massey 1949) to be

$$Q = \frac{8\pi^3 M^2}{h^4}\frac{v_f}{v_i}\int_{-1}^{+1} |\mathscr{M}|^2\, d(\cos\theta) \qquad \ldots\ldots(2)$$

in which M is the reduced mass of the system, v_i and v_f are the magnitudes of the initial and final velocities of relative motion, θ is the angle between unit vectors $\mathbf{n}_i$ and $\mathbf{n}_f$ parallel to these velocities, and $\mathscr{M}$ is the relevant transition integral. Denoting the position vector of the centre of mass of (A + e) relative to B by $\boldsymbol{\rho}$, that of the centre of mass of (B + e) relative to A by $\boldsymbol{\sigma}$, and that of e relative to A and B by $\mathbf{r}$ and $\mathbf{s}$ respectively,

$$|\mathscr{M}| = |\textstyle\iint \exp[i(k_i\mathbf{n}_i\cdot\boldsymbol{\rho} - k_f\mathbf{n}_f\cdot\boldsymbol{\sigma})]\phi_i(\mathbf{r})\phi_f^*(\mathbf{s})V(\mathbf{r},\mathbf{s})\,d\mathbf{s}\,d\boldsymbol{\sigma}| \qquad \ldots\ldots(3)$$

where

$$k_i = \frac{2\pi v_i}{h}\frac{(M_A+m)M_B}{M_A+M_B+m}, \quad k_f = \frac{2\pi v_f}{h}\frac{(M_B+m)M_A}{M_A+M_B+m}, \qquad \ldots\ldots(4)$$

ϕ_i is the initial and ϕ_f the final wave function of the electron, and $V(\mathbf{r},\mathbf{s})$ is the interaction potential between A and (B + e) which is

$$\frac{Z_A e^2}{r} - \frac{Z_A Z_B e^2}{|\mathbf{r}-\mathbf{s}|} \qquad \ldots\ldots(5)$$

$Z_A e$ and $Z_B e$ being the nuclear charges. Previous workers have taken only the first term of $V(\mathbf{r},\mathbf{s})$ into account, asserting that orthogonality would make the contribution from the second zero if proper wave functions were used. This simplification is without justification. In the standard collision theory treatment (cf. Mott and Massey 1949) the complete wave function $\Psi(\boldsymbol{\sigma},\mathbf{s})$ is expanded in the form

$$\sum_m F_m(\boldsymbol{\sigma})\phi_m(\mathbf{s}) \qquad \ldots\ldots(6)$$

and substituted in the Schrödinger equation yielding

$$(\nabla^2 + k_f^2)F_f(\boldsymbol{\sigma}) = \frac{8\pi^2 M}{h^2}\int \Psi(\boldsymbol{\sigma},\mathbf{s})V(\mathbf{r},\mathbf{s})\phi_f^*(\mathbf{s})\,d\mathbf{s} \qquad \ldots\ldots(7)$$

from which (2), with $\mathscr{M}$ defined as in (3), follows at once on making the usual approximation that

$$\Psi(\boldsymbol{\sigma},\mathbf{s}) \simeq \exp(ik_i\mathbf{n}_i\cdot\boldsymbol{\rho})\phi_i(\mathbf{r}). \qquad \ldots\ldots(8)\dagger$$

† In the case of identical nuclei, the wave function should, of course, be made antisymmetrical, but this is an unnecessary complication in an investigation concerned only with exchange scattering.

Clearly the contribution from the controversial term of $V(\mathbf{r},\mathbf{s})$ would *not* vanish even if, in place of (8), the exact expansion (6) were used. It would instead give on the right-hand side of (7) the function

$$\frac{8\pi^2 M}{h^2}\left\{-\frac{Z_A Z_B e^2}{|\mathbf{r}-\mathbf{s}|}\right\} F_f(\boldsymbol{\sigma}) \qquad \ldots\ldots(9)$$

which is certainly not negligible, being in fact equivalent to a coulomb potential on the left-hand side. The corresponding function resulting from the adoption of (8) is very different since the $\phi_f(\mathbf{s})$ content of $\exp(ik_1\mathbf{n}_1 . \boldsymbol{\rho})\phi_1(\mathbf{r})$ is far from equal to $F_f(\boldsymbol{\sigma})$. This is, to be sure, most grievous but it is not a reason for ignoring the $-Z_A Z_B e^2/|\mathbf{r}-\mathbf{s}|$ term of $V(\mathbf{r},\mathbf{s})$ altogether. The effect of $Z_A e^2/r$, the remaining term of $V(\mathbf{r},\mathbf{s})$, is also likely to be poorly represented. However, the similarity of the two terms suggests that the errors may partially cancel, especially if Z_A and Z_B are unity. In consequence approximation (8) may be not unsatisfactory. The situation is analogous to that arising in the more familiar case of excitation by electrons for which the interaction potential is of essentially the same form as that under discussion. Some early investigators thought that owing to orthogonality one of the two terms should be omitted. It is now known that such a procedure is incorrect and leads to serious overestimates of the exchange cross sections, particularly at low impact energies (cf. Bates, Fundaminsky, Leech and Massey 1950).

2.2

To evaluate $\mathscr{M}$ it is convenient (following Brinkmann and Kramers) to change the integration variables, and introduce the vectors

$$\boldsymbol{\alpha} = k_f\mathbf{n}_f + k_1\mathbf{n}_1 M_A/(M_A+m) \qquad \ldots\ldots(10)$$

$$\boldsymbol{\beta} = -k_1\mathbf{n}_1 - k_f\mathbf{n}_f M_B/(M_B+m) \qquad \ldots\ldots(11)$$

so that (3) may be written,

$$|\mathscr{M}| = \left|\iint \exp\,[i(\boldsymbol{\alpha}.\mathbf{r}+\boldsymbol{\beta}.\mathbf{s})]\phi_1(\mathbf{r})\phi_f^*(\mathbf{s})\left\{\frac{Z_A e^2}{r} - \frac{Z_A Z_B e^2}{|\mathbf{r}-\mathbf{s}|}\right\} d\mathbf{r}\, d\mathbf{s}\right|. \qquad \ldots\ldots(12)$$

If each of the nuclear changes is Z, and if ϕ_1 and ϕ_f are 1s wave functions (12) becomes

$$|\mathscr{M}| = \left|\frac{Za^3e^2}{\pi}\iint \exp\,[i(\boldsymbol{\alpha}.\mathbf{r}+\boldsymbol{\beta}.\mathbf{s}) - a(r+s)]\left\{\frac{1}{r} - \frac{Z}{|\mathbf{r}-\mathbf{s}|}\right\} d\mathbf{r}\, d\mathbf{s}\right| \qquad \ldots\ldots(13)$$

where $a = Z/a_0$, a_0 being the radius of the lowest Bohr orbit of hydrogen. Moreover, it may easily be shown that K, the modulus of both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and δ, the angle between these two vectors, are given with sufficient accuracy by the simple equations

$$K^2 = \frac{2M^2p^2}{m^2a_0^2}(1+\cos\theta) + \frac{p^2}{4a_0^2}, \quad \cos\delta = \frac{p^2}{2K^2a_0^2} - 1 \qquad \ldots\ldots(14)$$

with $p = 2\pi mva_0/h$, the subscript on v being omitted as unnecessary since the collision is elastic.

Integrals similar to (13) have been discussed by Massey and Mohr (1931) in connection with another problem. The contribution from the first term of the interaction potential may be obtained by elementary methods; that from the second term may be obtained by choosing the z-axis parallel to either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, expanding $1/|\mathbf{r}-\mathbf{s}|$ in the standard Legendre polynomial series, and using the formula

$$(-i)^n\int_{-1}^{+1}\exp(iz\mu)P_n(\mu)\,d\mu=(2\pi/z)^{1/2}J_{n+1/2}(z)=2j_n(z) \quad \ldots\ldots(15)$$

(cf. Watson 1944). Writing $t=K/a$ it is found that

$$|\mathscr{M}|=\left|\frac{4\pi Ze^2}{a^2}\left\{f(t)-\sum_n(-1)^nP_n(\cos\delta)g_n(t)\right\}\right|, \quad \ldots\ldots(16)$$

where $f(t)=8/(1+t^2)^3$

$$g_n(t)=4a^5Z\int_0^\infty e^{-ar}j_n(Kr)$$

$$\times\left[r^{-(n-1)}\int_0^r e^{-as}s^{n+2}j_n(Ks)\,ds+r^{n+2}\int_r^\infty e^{-as}s^{-(n-1)}j_n(Ks)\,ds\right]dr$$

$$=8a^5Z\int_0^\infty e^{-ar}j_n(Kr)i_n(Kr)r^{(-n-1)}\,dr,$$

with $i_n(Kr)=\int_0^r e^{-as}s^{n+2}j_n(Ks)\,ds=-K^{n-1}\frac{\partial}{\partial K}\{K^{-(n-1)}i_{n-1}(Kr)\}$.

Simple analysis shows that

$$g_n(t)=\frac{Z}{t^2(1+t^2)^3}[A_n(t)-B_n(t)\log(1+t^2)-tC_n\tan^{-1}t],$$

$A_n(t)$ and $B_n(t)$ being polynomials and C_n being a constant. The first six members of the series (some of which have also been derived by Massey and Mohr) are as follows:

$A_0(t)=t^4+5t^2$
$B_0(t)=0$
$C_0\ =0$

$A_1(t)=3t^4+21t^2-2$
$B_1(t)=6t^2-12-2t^{-2}$
$C_1\ =32$

$A_2(t)=5t^4+85t^2+24+12t^{-2}$
$B_2(t)=30t^2+0+30t^{-2}+12t^{-4}$
$C_2\ =96$

$A_3(t)=7t^4+259t^2+438+498t^{-2}+180t^{-4}$
$B_3(t)=84t^2+252+672t^{-2}+588t^{-4}+180t^{-6}$
$C_3\ =192$

$A_4(t)=9t^4+615t^2+2500+\frac{1}{3}(15710)t^{-2}+4520t^{-4}+1400t^{-6}$
$B_4(t)=180t^2+1260+4800t^{-2}+7380t^{-4}+5220t^{-6}+1400t^{-8}$
$C_4\ =320$

$A_5(t)=11t^4+1232t^2+9024+30245t^{-2}+45840t^{-4}+32550t^{-6}+8820t^{-8}$
$B_5(t)=330t^2+3960+21450t^{-2}+50820t^{-4}+61380t^{-6}+36960t^{-8}+8820t^{-10}$
$C_5\ =480.$

Finally, substitution of (16) in (2) yields

$$Q(p^2)=\left[\frac{4}{p^2}\int_{p^2/4Z^2}^\infty\left|f(t)-\sum_n(-1)^nP_n\left(\frac{p^2}{2Z^2t^2}-1\right)g_n(t)\right|^2d(t^2)\right]\pi a_0^2. \quad \ldots(17)$$

The adoption of the infinite upper limit is justified by (14) which shows that when θ is zero K, and hence t, is extremely large. It may be noted that p^2, the variable in which Q is expressed, is m/M_A times the energy of relative motion of the incident nucleus measured in rydbergs; thus if the incident nucleus is a proton a p^2 of unity corresponds to an energy of $24{\cdot}9_7$ kev.

If the g_n terms are omitted the integration in (17) can be carried out analytically giving

$$Q(p^2) = \left[\frac{2^{18}Z^{10}}{5p^2}(p^2+4Z^2)^{-5}\right]\pi a_0^2, \qquad \ldots\ldots(18)$$

which is, of course, identical with the formula obtained by Brinkmann and Kramers.

It is instructive also to consider the effect of ignoring the momentum transfer occurring when the electron is removed from the stationary to the incident nucleus. If m is neglected in (10) and (11) it can be seen that in this case the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are antiparallel. Hence $P_n(\cos\delta)$ is $(-1)^n$, and the summation in the expression for Q can be replaced by

$$\sum_n g_n(t) = Z\left\{\frac{1}{(1+t^2)} + \frac{4}{3(1+t^2)^2} + \frac{8}{3(1+t^2)^3}\right\}, \qquad \ldots\ldots(19)$$

as may be proved by the consideration of

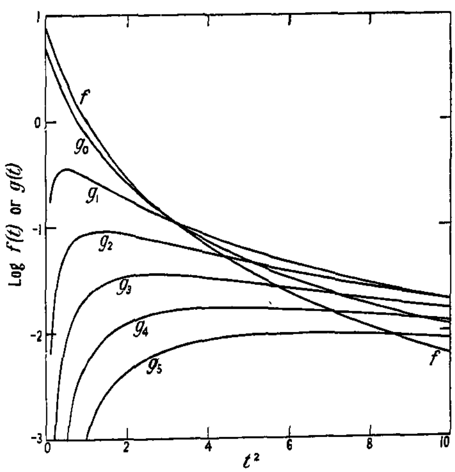
$$\iint \frac{1}{|\mathbf{r}-\mathbf{s}|}\exp\left[iK\mathbf{n}.(\mathbf{r}-\mathbf{s}) - a(r+s)\right] d\mathbf{r}\, d\mathbf{s}, \qquad \ldots\ldots(20)$$

which can be evaluated easily (by, for example, using spheroidal coordinates). Since in addition the lower limit of the integral involved in it becomes zero, (17) thus reduces to $Q(p^2) = \{4/135p^2\}[1728 - 2592Z + 1067Z^2]\pi a_0^2$ or, if Z is unity, to

$$Q(p^2) = [6{\cdot}01/p^2]\pi a_0^2. \qquad \ldots\ldots(21)$$

§ 3. RESULTS

The functions $f(t)$ and $g_n(t)$ ($n=0$ to 5, $t^2=0$ to 100) were computed from the expressions given in the previous section. Great cancellation occurs when t is small but fortunately a useful check on the work can be obtained from (19) since for such t the g_n series converges very rapidly. The figure shows the form of the



Log of functions $f(t)$, $g_0(t)$, $g_1(t)$, $g_2(t)$, $g_3(t)$, $g_4(t)$ and $g_5(t)$ plotted against t^2.

functions in the earlier part of the range. A logarithmic scale is used. It will be noted that the g_n's decrease comparatively slowly, becoming much larger than f at quite moderate values of t^2, and further that one of them, g_0, is comparable with

f even at the origin. However, owing to the oscillatory character of the multiplying Legendre polynomials their influence on the magnitude of Q is not as great as might be supposed at first.

Formula (17) was evaluated by numerical methods with Z unity and with p^2 between 0 and 10 (which covers energies up to some 250 kev in the case of protons). Account was taken of only those members of the g_n series that are illustrated. The omission of the higher members probably makes the calculated cross sections slightly smaller or larger than they should be according to whether p^2 is below or above about 2, but from some calculations on the effect of including g_6 and g_7 (which were estimated by extrapolation) it seems most unlikely that an error of as much as 10 % could be introduced.

The cross sections finally obtained from formula (17) are displayed in the table together with the corresponding cross sections obtained from formulae (18) and (21). For convenience, the formulae numbers will be used as identifying subscripts. It can be seen that Q_{18} is considerably greater than Q_{17} at low energies

Cross Section associated with Resonance Charge Exchange between Protons and Hydrogen Atoms

Energy parameter*	Log of cross section (in units of πa_0^2)†		
p^2	formula (18)	formula (21)	formula (17)‡
(0·0)	$(1{\cdot}7_1 - \log p^2)$	$(0{\cdot}7_8 - \log p^2)$	$(0{\cdot}7_8 - \log p^2)$
0·4	$1{\cdot}9_0$	$1{\cdot}1_8$	$0{\cdot}9_8$
0·8	$1{\cdot}4_1$	$0{\cdot}8_8$	$0{\cdot}5_2$
1·2	$1{\cdot}0_6$	$0{\cdot}7_0$	$0{\cdot}2_0$
1·6	$0{\cdot}7_7$	$0{\cdot}5_8$	$-0{\cdot}0_6$
2·0	$0{\cdot}5_3$	$0{\cdot}4_8$	$-0{\cdot}2_8$
3·0	$0{\cdot}0_2$	$0\ 3_0$	$-0{\cdot}7_4$
4·0	$-0{\cdot}4_0$	$0{\cdot}1_8$	$-1{\cdot}1_2$
5·0	$-0\ 7_5$	$0\ 0_8$	$-1{\cdot}4_4$
6 0	$-1{\cdot}0_6$	$0\ 0_0$	$-1{\cdot}7_3$
8·0	$-1{\cdot}5_8$	$-0{\cdot}1_2$	$-2{\cdot}2_0$
10·0	$-2\ 0_1$	$-0\ 2_2$	$-2{\cdot}6_0$

* The energy of the incident proton is $(24{\cdot}9_7 p^2)$ kev.

† πa_0^2 equals $8{\cdot}8 \times 10^{-17}$ cm².

‡ Formula (17) is the best approximation at present available. The results obtained from formulae (18) and (21) are presented purely for the sake of comparison.

but falls off more rapidly as the energy is increased. The omission of the g_n's clearly provides a rather poor approximation. As would be expected Q_{21} equals Q_{17} when the energy is zero and falls off much more slowly so that at even moderate energies the difference between the two cross sections is very pronounced. The striking reduction of the probability of electron exchange due to the momentum transfer incurred is, of course, well known (cf. Mott and Massey 1949).

The reliability of Q_{17} itself remains to be considered. As the basic Born approximation is designed specifically for the high energy region it cannot be expected to be accurate in the low energy region. However, the range of its validity is not easy to predict. Results down to the zero energy limit are presented in the table in the hope that they may provide information on the matter when proper comparison data become available. At, and for some distance above, this limit, the cross sections given are certainly too large since in deriving them it was assumed that only transitions *into* the final state occur whereas *backward* transitions must be important. A theoretical investigation of this is in progress. The accuracy attained at moderate and high energies should be quite good though, as mentioned in § 2.1, assumption (8) is open to criticism in that what is relevant is not merely the general form, but a refined property, of the wave function (just as in the case of exchange excitation by electron impact, cf. Bates, Fundaminsky, Leech and Massey 1950).

No experiments have been carried out on proton–hydrogen atom collisions, but several workers have studied $H^+ + H_2 \rightarrow H' + H_2^+$. Unlike (1) the process is not of the resonance type so the results obtained in the low energy region are of little significance in the present connection. The latest measurements in the moderate and high energy region are those of Ribe (1951). Over the range covered (for p^2 between about 1·5 and 6) the observed variation with energy is, within the stated experimental error, identical with that given by the theory. The absolute magnitudes of Ribe's cross sections (per hydrogen atom) are, however, about 1·5 times those predicted. Part of the discrepancy may be due to the fact that the measurements relate to the *total* exchange cross sections and thus include the contribution from capture into excited states. Some calculations on this contribution are planned. The cross sections may, of course, be influenced by the molecular binding but it would scarcely be anticipated that they are thereby increased.

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The Transition Effect for Cosmic-Ray Bursts at Small Thicknesses of Lead

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ABSTRACT. The transition curve for cosmic-ray bursts under lead at sea level is analysed to find the contributions from stars, extensive showers, electromagnetic interactions of μ-mesons, and single high-energy electrons and photons. When this is done, a substantial fraction of the bursts under lead thicknesses less than about 2·5 cm remains unaccounted for. It is suggested that they are produced by 'narrow air showers' consisting predominantly of photons of about $2{\cdot}5 \times 10^8$ ev and having a sideways spread of less than 0·3 m.

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