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Modificación a funciones $Y^k(a,b,r)$

1. Las funciones $Y^k(a,b,r)$

Por definición,

$$
\begin{aligned}
\frac{Y_{ab}^k(r)}{r} &= \int_0^\infty \frac{r_<^k}{r_>^{k+1}} P_a(t)P_b(t)\,dt && (1)\\
&= \int_0^r \frac{t^k}{r^{k+1}} P_a(t)P_b(t)\,dt + \int_r^\infty \frac{r^k}{t^{k+1}} P_a(t)P_b(t)\,dt && (2)\\
&= \frac{1}{r}\left[\int_0^r \left(\frac{t}{r}\right)^k P_a(t)P_b(t)\,dt + \int_r^\infty \left(\frac{r}{t}\right)^{k+1} P_a(t)P_b(t)\,dt\right] && (3)\\
Y_{ab}^k(r) &= \frac{1}{r^k}\int_0^r t^k P_a(t)P_b(t)\,dt + r^{k+1}\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt\,. && (4)
\end{aligned}
$$

Definiendo,

$$
Z_{ab}^k(r) = \frac{1}{r^k}\int_0^r t^k P_a(t)P_b(t)\,dt\,. \tag{5}
$$

Finalmente,

$$
Y_{ab}^k(r) = Z_{ab}^k(r) + r^{k+1}\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt\,. \tag{6}
$$

1.1. Ecuaciones diferenciales de $Y^k(a,b,r)$

Derivando $Z_{ab}^k(r)$, tenemos

$$
\begin{aligned}
\frac{d}{dr} Z_{ab}^k(r) &= \frac{d}{dr}\left[\frac{1}{r^k}\int_0^r t^k P_a(t)P_b(t)\,dt\right] && (7)\\
&= -\frac{k}{r^{k+1}}\int_0^r t^k P_a(t)P_b(t)\,dt + \frac{1}{r^k}\left[r^k P_a(r)P_b(r)\right] && (8)\\
&= P_a(r)P_b(r) - \frac{k}{r^{k+1}}\int_0^r t^k P_a(t)P_b(t)\,dt && (9)\\
&= P_a(r)P_b(r) - \frac{k}{r} Z_{ab}^k(r)\,. && (10)
\end{aligned}
$$

En donde se usa el hecho que

$$
\frac{d}{dr}\int_0^r f(t)\,dt = f(r)\,. \tag{11}
$$

Derivando $Y^k_{ab}(r)$, tenemos

$$\frac{d}{dr}Y^k_{ab}(r) = \frac{d}{dr}\left[Z^k_{ab}(r) + r^{k+1}\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt\right] \tag{12}$$
$$= \frac{d}{dr}Z^k_{ab}(r) + (k+1)r^k\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt - r^{k+1}\left[\frac{1}{r^{k+1}}P_a(r)P_b(r)\right] \tag{13}$$
$$= \left[P_a(r)P_b(r) - \frac{k}{r}Z^k_{ab}(r)\right] + (k+1)r^k\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt - P_a(r)P_b(r) \tag{14}$$
$$= -\frac{k}{r}Z^k_{ab}(r) + (k+1)r^k\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt \tag{15}$$
$$= -\frac{k}{r}Z^k_{ab}(r)\left[-\frac{k+1}{r}Z^k_{ab}(r) + \frac{k+1}{r}Z^k_{ab}(r)\right] + (k+1)r^k\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt \tag{16}$$
$$= -\frac{2k+1}{r}Z^k_{ab}(r) + \frac{k+1}{r}Z^k_{ab}(r) + \frac{k+1}{r}r^{k+1}\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt \tag{17}$$
$$= -\frac{2k+1}{r}Z^k_{ab}(r) + \frac{k+1}{r}\left[Z^k_{ab}(r) + r^{k+1}\int_r^\infty \frac{P_a(t)P_b(t)}{t^{k+1}}\,dt\right] \tag{18}$$
$$= -\frac{2k+1}{r}Z^k_{ab}(r) + \frac{k+1}{r}Y^k_{ab}(r)\,. \tag{19}$$

Siendo $r = e^\rho$,

$$\frac{df^k}{dr} = \frac{df^k}{d\rho}\frac{d\rho}{dr} \quad \to \quad \frac{df^k}{d\rho} = \frac{df^k}{dr}\frac{dr}{d\rho}\,, \tag{20}$$

y $\frac{dr}{d\rho} = r$, entonces

$$\frac{df^k}{d\rho} = r\frac{df^k}{dr}\,. \tag{21}$$

Además, siendo $P_i(r) = \sqrt{r}\,\overline{P}_i(\rho)$

$$\frac{d}{d\rho}Z^k_{ab} = r^2\,\overline{P}_a(\rho)\overline{P}_b(\rho) - kZ^k_{ab}(\rho)\,, \tag{22}$$
$$\frac{d}{d\rho}Y^k_{ab} = -(2k+1)Z^k_{ab}(\rho) + (k+1)Y^k_{ab}(\rho)\,. \tag{23}$$

1.2. Factor de integración

Siendo f_1 el factor de integración de Z^k_{ab}, entonces

$$\frac{d\,f_1\,Z^k_{ab}}{d\rho} = \frac{df_1}{d\rho}Z^k_{ab} + f_1\frac{d}{d\rho}Z^k_{ab} \tag{24}$$
$$\tag{25}$$