5. Dynamic analysis techniques

Even though dynamic analyses of railway vehicles today usually are performed by means of computer simulations, all calculations are based on mathematical models of vehicles. Modelling of carbody, bogie, wheelset and some suspension elements has been discussed in Chapter 3, modelling of track irregularities and track flexibility has been discussed in Chapter 2. Geometrical and mechanical properties of the wheel-rail contact are described in the following chapters.

In this chapter simple models describing the vertical dynamics of a typical railway vehicle are introduced and the most common types of dynamic analyses are discussed with help of these models. Simple models are also very useful to check simulation results. It can be difficult to decide whether results of simulations with a model with 50 or 100 degrees of freedom are reasonable or not. More detailed descriptions of simulation of railway vehicle dynamics can be found in [36], [61] and [84]. See also Chapter 13.

5.1 Simple models for analysis

Defining a model it is important to think about the phenomena which shall be studied. In studying the dynamic behaviour of railway vehicles, it is often sufficient to represent reality in a frequency range between 0 and 150 Hz. Studying only carbody dynamics normally frequencies up to 20 Hz have to be considered. Higher frequencies are filtered out in the primary and secondary suspension.

5.1.1 One-dimensional model with 1 DOF

The most simple model to investigate the vertical dynamics of a railway vehicle is shown in Figure 5-1. The model is one-dimensional. It only consists of an unsuspended (unsprung) mass and a suspended (sprung) mass.

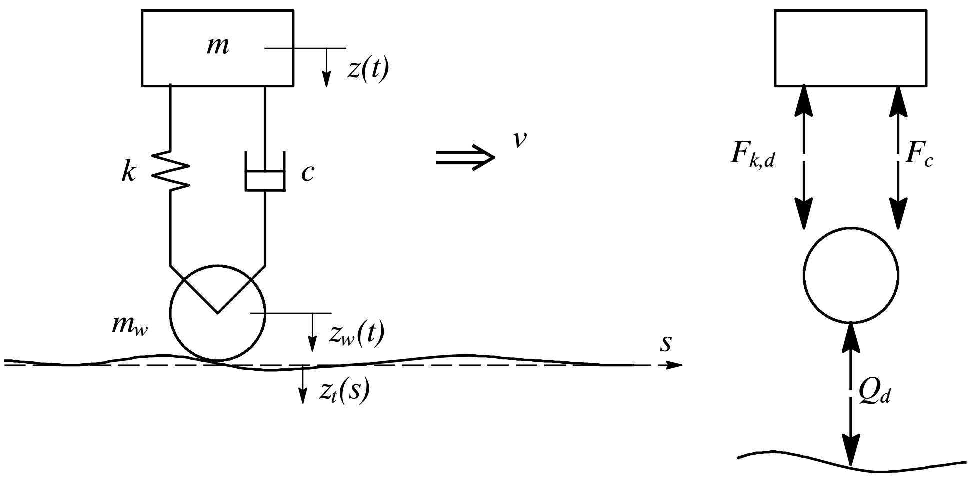

Figure 5-1 ***One-dimensional model with one DOF* (z).** *Masses* m *and* m_w, *viscous damping* c, *stiffness* k, speed v, *displacements* z(t) *and* $z_w(t)$ *and track irregularities* $z_t(s)$. *Damping force* F_c, *dynamic spring force* $F_{k,d}$ *and dynamic contact force* Q_d.

The model is associated with the following assumptions:

- The vehicle is running on tangent track (without superelevation or gradient).
- The vehicle is symmetric with respect to a longitudinal vertical plane.
- Only vertical track irregularities act on the vehicle.
- Wheel and rail are totally stiff (rigid).
- Springs and dampers have linear characteristics.
- The length of the vehicle is much shorter than the dominating wavelength in the irregularities. This means that pitch motion can be neglected.
- The vehicle runs at constant speed.
- Wheel and rail do not loose contact.
- The vehicle has perfectly circular wheels.

The assumption that wheel and rail do not loose contact with each other results in that the vertical displacement of the wheelset is the same as for the track irregularity. Therefore $z_w(t)$ is not an independent degree of freedom, thus the model consists of only one degree of freedom. The position $s(t)$ of the vehicle at each time t can be described as

$$s(t) = s_0 + vt \qquad (5\text{-}1)$$

The displacement, velocity and acceleration of the wheel can be expressed by

$$z_w = z_t[s(t)] \qquad (5\text{-}2a)$$

$$\dot{z}_w = \frac{d}{dt}z_t[s(t)] = \frac{dz_t}{ds}\frac{ds}{dt} = z_t' \, v \qquad (5\text{-}2b)$$

$$\ddot{z}_w = \frac{d^2}{dt^2}z_t[s(t)] = \frac{d}{dt}\dot{z}_w = \frac{d}{dt}(z_t' \, v) = z_t'' \, v^2 \qquad (5\text{-}2c)$$

Both the spring force and the contact force have a static and a dynamic part:

$$F_k = F_{k,o} + F_{k,d} = mg + F_{k,d} \qquad (5\text{-}3a)$$

$$Q = Q_o + Q_d = (m + m_w)g + Q_d \qquad (5\text{-}3b)$$

For simplicity in Figure 5-1 only the dynamic parts are taken into acount. The displacement of the suspended mass is therefore only the dynamic displacement relative to the static equilibrium. Accordingly the equilibrium equations for the two masses become

$$m\ddot{z} = -F_{k,d} - F_c \qquad (5\text{-}4a)$$

$$m_w\ddot{z}_w = F_{k,d} + F_c - Q_d \qquad (5\text{-}4b)$$

With the assumption of linear characteristics for spring and damper we get

$$F_{k,d} = k(z - z_w) \tag{5-5a}$$

$$F_c = c(\dot{z} - \dot{z}_w) \tag{5-5b}$$

Substituting Equation (5-5) into Equation (5-4) yields

$$m\ddot{z} + c(\dot{z} - \dot{z}_w) + k(z - z_w) = 0 \tag{5-6a}$$

$$m_w\ddot{z}_w - c(\dot{z} - \dot{z}_w) - k(z - z_w) = -Q_d \tag{5-6b}$$

Finally by substituting Eq. (5-2a,b) into Eq. (5-6a) we get the equation of motion for the suspended mass

$$m\ddot{z} + c\dot{z} + kz = c\dot{z}_w + kz_w = cz_t'v + kz_t \tag{5-7a}$$

while substituting Q_d from Eq. (5-6b) into Equation (5-3b) gives the contact force

$$\begin{aligned} Q &= Q_o + Q_d = (m + m_w)g - m\ddot{z} - m_w\ddot{z}_w = \\ &(m + m_w)g - m\ddot{z} - m_w z_t'' v^2 \end{aligned} \tag{5-7b}$$

In conclusion the model consists of one DOF, and thus one equation of motion. The equation of motion is an ordinary second order differential equation. The right hand side can be interpreted as force representing the track excitation. The Q-force which often is of interest (cf. Chapter 9) can be calculated with help of Equation (5-7b) if $z(t)$ is known. As can be seen in (5-7b) the dynamic part of Q is directly related to the inertia forces of the two masses. For a well suspended mass the acceleration $\ddot{z}$ is much lower than the acceleration $\ddot{z}_w$ of the unsuspended wheelset. Even though m is significantly higher than m_w the wheelset mass usually is responsible for the largest part of the inertia force and in turn the largest part of the dynamic contact force.

Therefore it is important to keep the unsuspended mass as low as possible, especially for high-speed trains. From Equation (5-7b) it can be seen as well that the acceleration $\ddot{z}_w$ is equal to v^2 divided by the curvature of the track irregularity z_s'' . This can be compared with the centrifugal acceleration in a circular curve v^2/R (cf. Chapter 4).

5.1.2 One-dimensional model with 2 DOF

Almost all railway vehicles consist of carbody, bogie frames and wheelsets. Therefore an improvement of the model above is to introduce as second suspended mass, i.e. to introduce a second vertical degree of freedom. Such a model is shown in Figure 5-2.

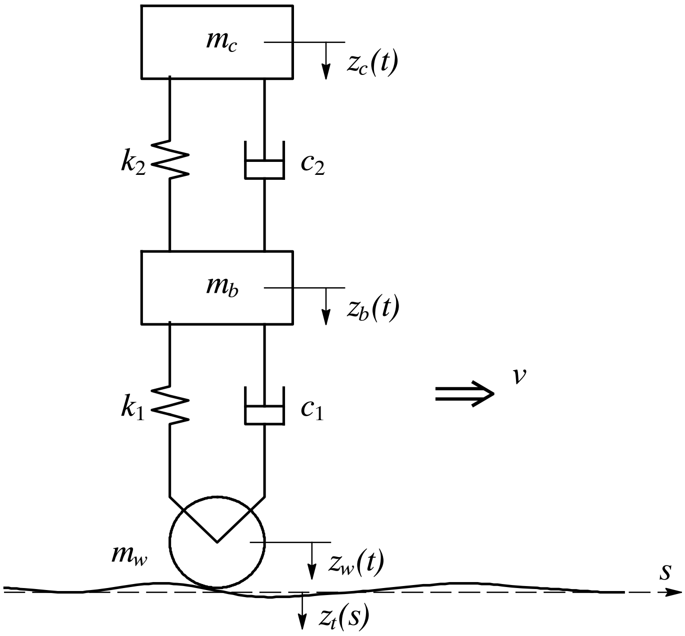

Figure 5-2 ***One-dimensional model with two degrees of freedom (z_c, z_b).*** *Carbody mass m_c, bogieframe mass m_b and wheelset mass m_w. Primary suspension with stiffness k_1 and damping c_1. Secondary suspension with stiffness k_2 and damping c_2. Speed v. Displacements $z_c(t)$, $z_b(t)$, $z_w(t)$ and track irregularity $z_t(s)$.*

The three force equations of the system with respect to the static equilibrium can be written as

$$m_c\ddot{z}_c + c_2(\dot{z}_c - \dot{z}_b) + k_2(z_c - z_b) = 0 \qquad (5\text{-}8a)$$

$$m_b\ddot{z}_b - c_2(\dot{z}_c - \dot{z}_b) - k_2(z_c - z_b) + c_1(\dot{z}_b - \dot{z}_w) + k_1(z_b - z_w) = 0 \qquad (5\text{-}8b)$$

$$m_w\ddot{z}_w - c_1(\dot{z}_b - \dot{z}_w) - k_1(z_b - z_w) = -Q_{dyn} \qquad (5\text{-}8c)$$

Using Equation (5-2) one gets the two equations of motion in matrix form as

$$\begin{bmatrix} m_c & 0 \\ 0 & m_b \end{bmatrix}\begin{Bmatrix} \ddot{z}_c \\ \ddot{z}_b \end{Bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix}\begin{Bmatrix} \dot{z}_c \\ \dot{z}_b \end{Bmatrix} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}\begin{Bmatrix} z_c \\ z_b \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ c_1\dot{z}_w + k_1 z_w \end{Bmatrix} = \begin{Bmatrix} 0 \\ c_1 z_t' v + k_1 z_t \end{Bmatrix} \qquad (5\text{-}9)$$

or in short form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F} \qquad (5\text{-}10)$$

and an expression for the dynamic wheel-rail contact force

$$Q_{dyn} = -m_c\ddot{z}_c - m_b\ddot{z}_b - m_w z_t'' v^2 \tag{5-11}$$

The total contact force can be calculated to

$$Q = Q_o + Q_{dyn} = (m_c + m_b + m_w)g + Q_{dyn} \tag{5-12}$$

5.1.3 Two-dimensional model with two DOF

Now we go back to a model with only one suspension level but instead include the length dimension of the vehicle. The model then looks like in Figure 5-3. This model should be able to represent the vertical dynamics for a two-axle freight wagon or a bogie with two wheelsets but without carbody.

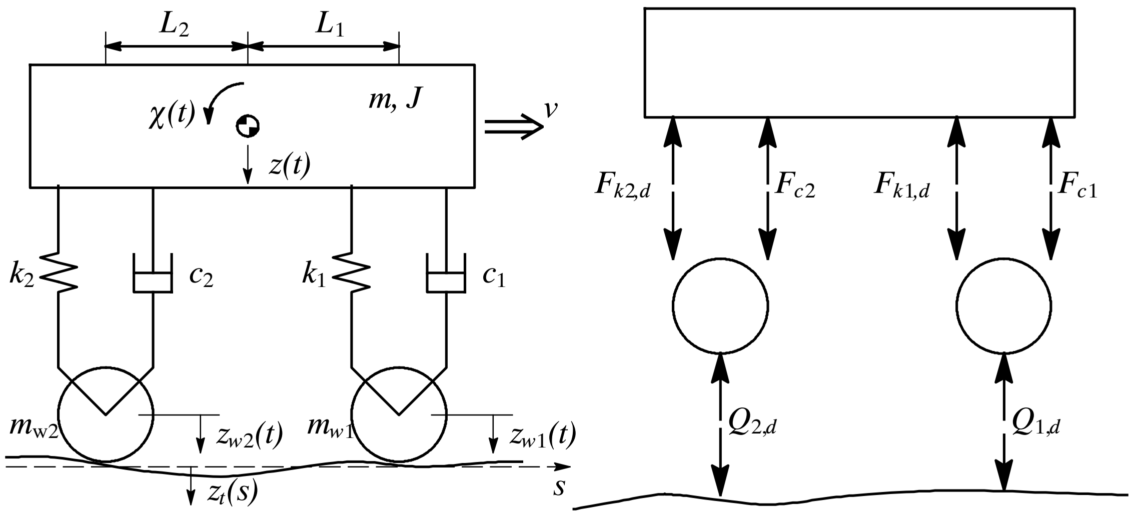

Figure 5-3 ***Two-dimensional model with two DOF* (z,χ).** *Masses* m, m_{w1} *and* m_{w2}. *Mass inertia moment J. Damping* c_1 *and* c_2*; stiffnesses* k_1 *and* k_2*. Longitudinal distance from centre of gravity to wheelset* L_1 *och* L_2*. Speed v. Displacements* z(t), $z_{w1}(t)$ *och* $z_{w2}(t)$ *and track irregularities* $z_t(s)$*. Rotation, or pitch,* $\chi(t)$*. Damping forces* F_{c1} *and* F_{c2}*; dynamic suspension forces* $F_{k1,d}$ *and* $F_{k2,d}$*; dynamic contakt forces* $Q_{1,d}$ *och* $Q_{2,d}$.

The force and moment equations for the suspended mass write as

$$m\ddot{z} = -F_{k1,d} - F_{c1} - F_{k2,d} - F_{c2} \tag{5-13a}$$

$$J\ddot{\chi} = F_{k1,d}L_1 + F_{c1}L_1 - F_{k2,d}L_2 - F_{c2}L_2 \tag{5-13b}$$

while the force equations for the two wheelsets become

$$m_{w1}\ddot{z}_{w1} = F_{k1,d} + F_{c1} - Q_{1,d} \tag{5-13c}$$

$$m_{w2}\ddot{z}_{w2} = F_{k2,d} + F_{c2} - Q_{2,d} \tag{5-13d}$$

With linear charasteristics for springs and dampers we can write

$$F_{k1,d} = k_1(z - \chi L_1 - z_{w1}), \qquad F_{k2,d} = k_2(z + \chi L_2 - z_{w2}) \tag{5-14a,b}$$

$$F_{c1} = c_1\left(\dot{z} - \dot{\chi} L_1 - \dot{z}_{w1}\right), \qquad F_{c2} = c_2\left(\dot{z} + \dot{\chi} L_2 - \dot{z}_{w2}\right) \tag{5-14c,d}$$

Substituting Equation (5-14) into Equation (5-13) gives

$$\begin{aligned} & m\ddot{z} + c_1\left(\dot{z} - \dot{\chi} L_1 - \dot{z}_{w1}\right) + c_2\left(\dot{z} + \dot{\chi} L_2 - \dot{z}_{w2}\right) + \\ & + k_1(z - \chi L_1 - z_{w1}) + k_2(z + \chi L_2 - z_{w2}) = 0 \end{aligned} \tag{5-15a}$$

$$\begin{aligned} & J\ddot{\chi} - c_1 L_1\left(\dot{z} - \dot{\chi} L_1 - \dot{z}_{w1}\right) + c_2 L_2\left(\dot{z} + \dot{\chi} L_2 - \dot{z}_{w2}\right) - \\ & - k_1 L_1(z - \chi L_1 - z_{w1}) + k_2 L_2(z + \chi L_2 - z_{w2}) = 0 \end{aligned} \tag{5-15b}$$

$$m_{w1}\ddot{z}_{w1} - c_1\left(\dot{z} - \dot{\chi} L_1 - \dot{z}_{p1}\right) - k_1(z - \chi L_1 - z_{w1}) = -Q_{1,d} \tag{5-15c}$$

$$m_{w2}\ddot{z}_{w2} - c_2\left(\dot{z} + \dot{\chi} L_2 - \dot{z}_{w2}\right) - k_2(z + \chi L_2 - z_{w2}) = -Q_{2,d} \tag{5-15d}$$

The equations of motion (5-15a,b) can be written as

$$\begin{aligned} & \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\chi} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 L_1 + c_2 L_2 \\ -c_1 L_1 + c_2 L_2 & c_1 L_1^2 + c_2 L_2^2 \end{bmatrix} \begin{Bmatrix} \dot{z} \\ \dot{\chi} \end{Bmatrix} + \\ & + \begin{bmatrix} k_1 + k_2 & -k_1 L_1 + k_2 L_2 \\ -k_1 L_1 + k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{Bmatrix} z \\ \chi \end{Bmatrix} = \begin{Bmatrix} c_1\dot{z}_{w1} + c_2\dot{z}_{w2} + k_1 z_{w1} + k_2 z_{w2} \\ -c_1\dot{z}_{w1}L_1 + c_2\dot{z}_{w2}L_2 - k_1 z_{w1} L_1 + k_2 z_{w2} L_2 \end{Bmatrix} \end{aligned} \tag{5-16a,b}$$

while the expressions for the contact forces become, cf. Eqs. (5-3) and (5-15c,d),

$$Q_1 = \left(\frac{mL_2}{L_1 + L_2} + m_{w1}\right)g - m_{w1}\ddot{z}_{w1} + c_1\left(\dot{z} - \dot{\chi} L_1 - \dot{z}_{w1}\right) + k_1(z - \chi L_1 - z_{w1}) \tag{5-16c}$$

$$Q_2 = \left(\frac{mL_1}{L_1 + L_2} + m_{w2}\right)g - m_{w2}\ddot{z}_{w2} + c_2\left(\dot{z} + \dot{\chi} L_2 - \dot{z}_{w2}\right) + k_2(z + \chi L_2 - z_{w2}) \tag{5-16d}$$

where the static parts of the contact forces can be calculated with a static equilibrium.

In matrix form the equations of motion (5-16a,b) can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F} \tag{5-17}$$

where

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \tag{5-17a}$$

$$\boldsymbol{C} = \begin{bmatrix} c_1 + c_2 & -c_1L_1 + c_2L_2 \\ -c_1L_1 + c_2L_2 & c_1L_1^2 + c_2L_2^2 \end{bmatrix} \tag{5-17b}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_1L_1 + k_2L_2 \\ -k_1L_1 + k_2L_2 & k_1L_1^2 + k_2L_2^2 \end{bmatrix} \tag{5-17c}$$

$$\boldsymbol{F} = \begin{Bmatrix} c_1\dot{z}_{w1} + c_2\dot{z}_{w2} + k_1z_{w1} + k_2z_{w2} \\ -c_1\dot{z}_{w1}L_1 + c_2\dot{z}_{w2}L_2 - k_1z_{w1}L_1 + k_2z_{w2}L_2 \end{Bmatrix} \tag{5-17d}$$

$$\ddot{\boldsymbol{x}} = \begin{Bmatrix} \ddot{z} \\ \ddot{\chi} \end{Bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{Bmatrix} \dot{z} \\ \dot{\chi} \end{Bmatrix}, \quad \boldsymbol{x} = \begin{Bmatrix} z \\ \chi \end{Bmatrix} \tag{5-17e,f,g}$$

Like before z_{w1} and z_{w2}, and their derivatives, in Equations (5-16) and (5-17d) can be replaced by the expresions in Eq. (5-2).

For the special case $c_1 = c_2 = c$, $k_1 = k_2 = k$ and $L_1 = L_2 = L$, i.e. a symmetric vehicle, the matrices $\boldsymbol{C}$, $\boldsymbol{K}$ and $\boldsymbol{F}$ are simplified to

$$\boldsymbol{C} = \begin{bmatrix} 2c & 0 \\ 0 & 2cL^2 \end{bmatrix} \tag{5-18a}$$

$$\boldsymbol{K} = \begin{bmatrix} 2k & 0 \\ 0 & 2kL^2 \end{bmatrix} \tag{5-18b}$$

$$\boldsymbol{F} = \begin{Bmatrix} c\left(\dot{z}_{w1} + \dot{z}_{w2}\right) + k\left(z_{w1} + z_{w2}\right) \\ cL\left(-\dot{z}_{w1} + \dot{z}_{w2}\right) + kL\left(-z_{w1} + z_{w2}\right) \end{Bmatrix} \tag{5-18c}$$

All matrices become diagonal matrices and thus the two equations of motion get uncoupled.

5.2 Calculation of eigenvalues and eigenmodes

Calculation of eigenvalues (eigenfrequencies, relative damping) and eigenmodes (eigenvectors) gives valuable information about the principle dynamic properties of a vehicle. It is important to know the eigenfrequencies of the vehicle in order to avoid resonances. For example there is the risk that the frequency of the hunting motion of the wheelsets or bogies coincides with the yaw eigenfrequency of the carbody. Eigenmodes give a good indication about the type of the dominant motions in the vehicle.

A disadvantage of this method of analysis is that the model has to be linearized before the calculation of eigenvalues and eigenmodes. In strongly nonlinear models this might lead to major errors in the results. It is important to be aware of the displacement amplitudes around the working points for suspensions and wheel-rail contact used for the linearisation.

The solution for the free vibration of the system in Figure 5-2 is calculated from the homogeneous part of the equations of motion (5-10)

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \qquad \textit{(5-19a)}$$

The solution of this equation system is assumed to be

$$\boldsymbol{x}(t) = \gamma e^{\lambda t} \qquad \textit{(5-19b)}$$

where λ and γ are the eigenvalues and eigenvectors, respectively, of the system. For a system with n degrees of freedom one gets $2n$ eigenvalues and $2n$ eigenvectors.

Substituting Equation (5-19b) into (5-19a) gives

$$\left(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K}\right)\gamma = \boldsymbol{0} \qquad \textit{(5-19c)}$$

because of $e^{\lambda t} > 0$. For a non-trivial solution the following condition has to be satisfied:

$$\det\left(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K}\right) = 0 \qquad \textit{(5-19d)}$$

The eigenvalue λ can be calculated from this equation. Substituting the eigenvalues into (5-19c) gives the eigenvectors. (5-19d) is quadratic in λ. For a system with n DOF we get $2n$ eigenvalues and thus $2n$ eigenvectors.

For systems with many DOF there are in principle no numerical algorithms which solve Eqs. (5-19c) and (5-19d). Therefore often the second order differential equations are rewritten into twice as many first order differential equations. Eq. (5-19a) can then be written as

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{K} & -\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix} \qquad \textit{(5-20)}$$

where $\boldsymbol{I}$ is a n x n unit matrix. Inverting the matrix on the left hand side of the equation gives

$$\begin{Bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix} \qquad \textit{(5-21)}$$

or

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B} \qquad (5\text{-}22)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \qquad (5\text{-}22a)$$

$$\boldsymbol{B} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{F} \end{Bmatrix} \qquad (5\text{-}22b)$$

$$\boldsymbol{u} = \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix}, \qquad \dot{\boldsymbol{u}} = \frac{d}{dt}\boldsymbol{u} = \frac{d}{dt}\begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix} = \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{Bmatrix} \qquad (5\text{-}22c,d)$$

The so called system matrix $\boldsymbol{A}$ has the dimension *2n* x *2n.* There are several algorithms existing to solve Eq. (5-22). With Eq. (5-22) and $\boldsymbol{B}$=$\boldsymbol{0}$ we get

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} \qquad (5\text{-}23a)$$

With the same as above, i.e.

$$\boldsymbol{u}(t) = \begin{Bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{Bmatrix} = \begin{Bmatrix} \gamma \\ \lambda\gamma \end{Bmatrix} e^{\lambda t} = \boldsymbol{a}e^{\lambda t} \qquad (5\text{-}23b)$$

we get

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{a} = \boldsymbol{0} \qquad (5\text{-}23c)$$

with a non-trivial solution that has to fulfill

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0 \qquad (5\text{-}23d)$$

5.2.1 One-dimensional model with one DOF

If the right hand side (track excitation) is set to zero we get the following equation of motion

$$m\ddot{z} + c\dot{z} + kz = 0 \qquad (5\text{-}24a)$$

Because the model only has one DOF Eq. (5-24a) can be solved easily without neglecting damping or transforming into two first order equations (cf. below). With the same guessed solution like in (5-19b) we get

$$z(t) = \gamma e^{\lambda t} \qquad (5\text{-}24b)$$

where γ is a constant. Substitution into Eq. (5-24a) yields

$$\left(m\lambda^2 + c\lambda + k\right)\gamma = 0 \tag{5-24c}$$

which in turn gives for the non-trivial solution

$$\det\left(m\lambda^2 + c\lambda + k\right) = m\lambda^2 + c\lambda + k = 0 \tag{5-24d}$$

This second order polynomial in λ has two eigenvalues

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{5-25}$$

Moreorless all railway vehicles have a (vertical) damping which is *undercritical*, i.e. $c < c_c = 2\sqrt{mk}$ where c_c is the critical damping. This means that a single excitation is damped out with an oscillating motion. Overcritical damping, $c > c_c$, usually leads to a rather bumpy running behaviour. For Eq. (5-25) undercritical damping means that $(c/2m)^2 < k/m$ and thus we can write

$$\lambda_{1,2} = -\zeta\omega_o \pm i\omega_d \tag{5-26}$$

where $i^2 = -1$ and where

$$\omega_o = \sqrt{\frac{k}{m}} \tag{5-26a}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{c}{2m\omega_o} \tag{5-26b}$$

$$\omega_d = \omega_o\sqrt{1 - \zeta^2} \tag{5-26c}$$

are the undamped eigenfrequency, relative damping ($\zeta < 1$) and the damped eigenfrequency of the system. The two eigenvalues in Eq. (5-26) are thus complex conjugated and result in the same eigenfrequency of the system (only one DOF). The eigenvalues can be shown in a complex coordinate system like in Figure 5-4.

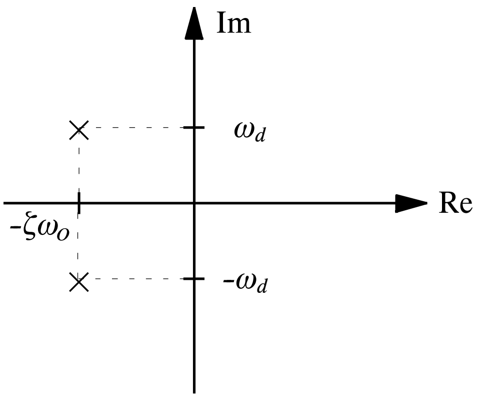

Figure 5-4 **Eigenvalues for model with one DOF.** *(See Figure 5-1).*

For relative damping $\zeta \geq 0$ the eigenvalues are on the left side of the imaginary axis. The linear system can in this case be regarded as *stable.* Note that the distance from the ”crosses” to the origin in Figure 5-4 is equal to ω_o.

The eigenvector γ is for this 1-DOF system an arbitrary number, because with eigenvalues according to Eq. (5-26) all values for γ fulfill Equation (5-24c).

Example: A carbody with mass m = 38200 kg, damping c = 160 kNs/m and suspension stiffness k = 2160 kN/m has with Eq. (5-26a) the undamped eigenfrequency $\omega_o = \sqrt{2160000/38200} = 7.52$ rad/s, and thus in Hz $f_o = 7.52/2\pi = 1.20$ Hz. With Eq. (5-26b) we get for the relative damping $\zeta = c/(2m\omega_o) = 160000/(2 \cdot 38200 \cdot 7.52) = 0.278$, i.e. $\zeta < 1$. According to (5-26c) the damped eigenfrequency becomes $\omega_d = 7.52\sqrt{1 - 0.278^2} = 7.22$ rad/s, and in Hz $f_d = 7.22/2\pi = 1.15$ Hz. With Eq. (5-26) finally the eigenvalues are $\lambda_{1,2} = -2.09 \pm 7.22i$.

In the following it is shown that transformation into two first order equations like in Eq. (5-23d) of course yields the same eigenvalues like Eq. (5-26). The system matrix is here given by, cf. Eq. (5-22a),

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \tag{5-27}$$

Substituting into Eq. (5-23d) yields

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det\begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{bmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \tag{5-28}$$

which after multiplying with m gives Equation (5-24d).

We will now look into the solution of (5-24a) in more detail. With (5-24b) and the eigenvlaues in (5-26) we have to extend the solution to

$$z(t) = \gamma_1 e^{(-\zeta\omega_o + i\omega_d)t} + \gamma_2 e^{(-\zeta\omega_o - i\omega_d)t} = e^{-\zeta\omega_o t}\left(\gamma_1 e^{i\omega_d t} + \gamma_2 e^{-i\omega_d t}\right) \tag{5-29}$$

where γ_1 and γ_2 are constants. With Euler's formula

$$e^{iy} = \cos y + i \sin y \tag{5-30}$$

Eq. (5-29) can be rewritten to

$$z(t) = e^{-\zeta\omega_o t}\left(C_1 \sin\omega_d t + C_2 \cos\omega_d t\right) \tag{5-31}$$

where C_1 and C_2 are constants given by the initial conditions $z(0)$ and $\dot{z}(0)$. It can be shown that the solution is

$$z(t) = e^{-\zeta\omega_o t}\left(\frac{\dot{z}(0) + z(0)\zeta\omega_o}{\omega_d}\sin\omega_d t + z(0)\cos\omega_d t\right) \tag{5-32}$$

Figure 5-5 illustrates how the solution could look like.

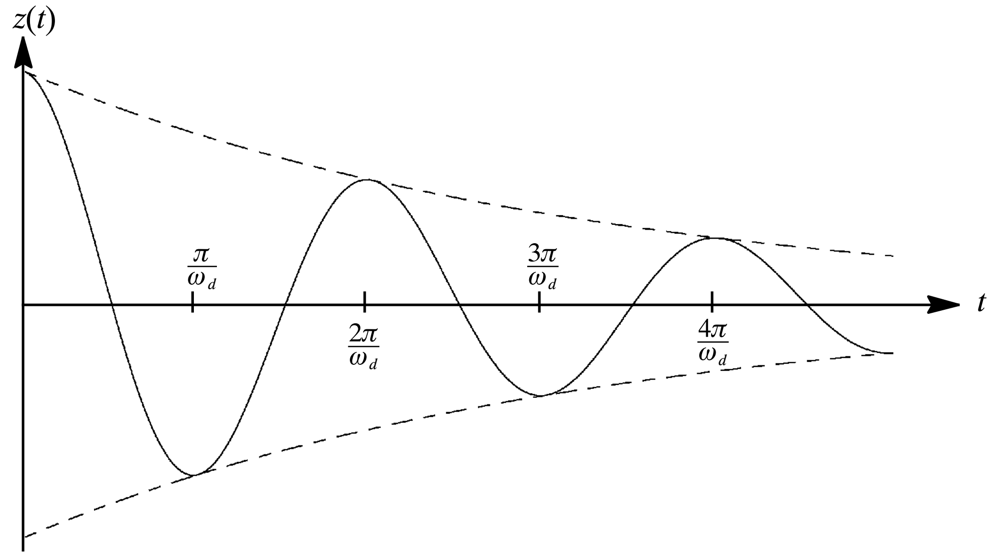

Figure 5-5 Free vibration for system with undercritical damping.

5.2.2 One-dimensional model with two DOF

For the model from Figure 5-2 with 2 DOF we get the following equation of motion for free vibrations

$$\begin{bmatrix} m_c & 0 \\ 0 & m_b \end{bmatrix}\begin{Bmatrix} \ddot{z}_c \\ \ddot{z}_b \end{Bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix}\begin{Bmatrix} \dot{z}_c \\ \dot{z}_b \end{Bmatrix} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}\begin{Bmatrix} z_c \\ z_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5-33a}$$

With the solution

$$\begin{Bmatrix} z_c(t) \\ z_b(t) \end{Bmatrix} = \begin{Bmatrix} \gamma_c \\ \gamma_b \end{Bmatrix} e^{\lambda t} \tag{5-33b}$$

we get

$$\begin{bmatrix} m_c\lambda^2 + c_2\lambda + k_2 & -c_2\lambda - k_2 \\ -c_2\lambda - k_2 & m_b\lambda^2 + (c_1 + c_2)\lambda + (k_1 + k_2) \end{bmatrix}\begin{Bmatrix} \gamma_c \\ \gamma_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5-33c}$$

The non-trivial solution is given by

$$\det\begin{bmatrix} m_c\lambda^2 + c_2\lambda + k_2 & -c_2\lambda - k_2 \\ -c_2\lambda - k_2 & m_b\lambda^2 + (c_1 + c_2)\lambda + (k_1 + k_2) \end{bmatrix} = 0 \tag{5-34}$$

and we get the characteristic equation

$$m_c m_b\lambda^4 + (m_c c_1 + m_c c_2 + m_b c_2)\lambda^3 + (m_c k_1 + m_c k_2 + m_b k_2 + c_1 c_2)\lambda^2 +$$

$$+ (c_1 k_2 + c_2 k_1)\lambda + k_1 k_2 = 0 \tag{5-35}$$

Since it is almost impossible to calculate the symbolic solution of this equation by hand, only the solution of the undamped case, i e $c_1 = c_2 = 0$, will be given

$$m_c m_b \lambda^4 + (m_c k_1 + m_c k_2 + m_b k_2)\lambda^2 + k_1 k_2 = 0 \tag{5-36}$$

This equation is called the characteristic equation for the calculation of λ. This is a quadratic expression in λ^2 and the roots can be found as

$$\lambda_{1,2}^2 = -\frac{1}{2}\left[\frac{k_1}{m_b} + \frac{k_2}{m_c}\left(1 + \frac{m_c}{m_b}\right) \pm \sqrt{\left[\frac{k_1}{m_b} + \frac{k_2}{m_c}\left(1 + \frac{m_c}{m_b}\right)\right]^2 - \frac{4k_1 k_2}{m_b m_c}}\right] \tag{5-37}$$

Example: A carbody with mass m_c = 38200 kg is isolated against vibrations with a secondary suspension stiffness of k_2 = 2160 kN/m, a bogieframe mass m_b = 6000 kg and a primary stiffness k_1 = 11200 kN/m ($c_1 = c_2 = 0$). With Equation (5-37) the quadratic eigenvalues become
$\lambda_{1,2}^2 = -1141.6 \pm 1094.4$, which leads to
$\lambda_1^2 = -47.2$ and $\lambda_2^2 = -2236$.
The four eigenvalues are calculated to
$\lambda_{1,2} = \pm 6.87i$ and $\lambda_{3,4} = \pm 47.3i$,
giving the undamped natural frequencies
ω_1 = 6.87 rad/s respectively ω_2 = 47.3 rad/s, i.e. f_1 = 1.09 Hz respectively f_2 = 7.53 Hz.

The eigenvectors of the example above can be calculated to

$$\gamma_1 = \begin{Bmatrix} \gamma_{c1} \\ \gamma_{b1} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.165 \end{Bmatrix} \text{ respectively } \gamma_2 = \begin{Bmatrix} \gamma_{c2} \\ \gamma_{b2} \end{Bmatrix} = \begin{Bmatrix} -0.026 \\ 1 \end{Bmatrix} \tag{5-38a,b}$$

From the eigenvectors we get information about the amplitudes of the displacements and about the phase shift between the oscillations of the different masses. In an undamped system only 0°and 180° phase shift exist. In the first eigenmode mainly the carbody oscillates, while in the second eigenmode the bogie frame mass oscillates. In the first eigenmode the two masses oscillate in phase and in the second mode they oscillate with a phase shift of 180°. The two modes are shown in Figure 5-6.

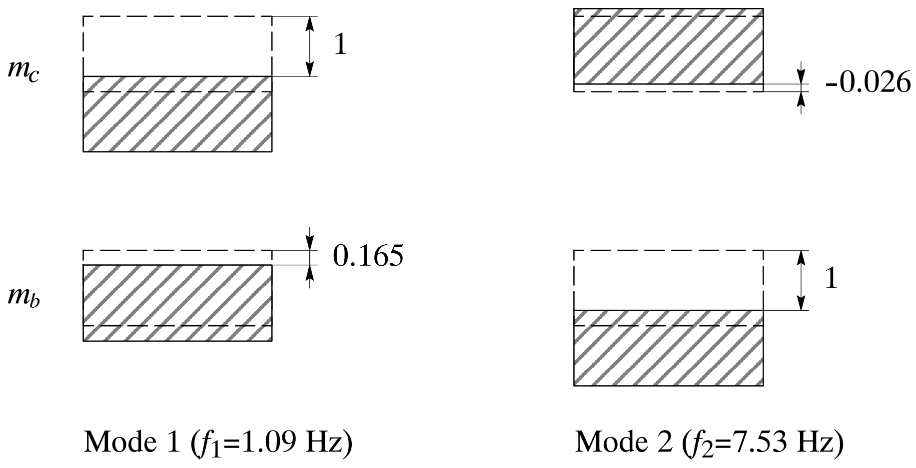

Figure 5-6 Eigenvectors for example with two degrees of freedom.

An example for a system with two degrees of freedom containing damping is discussed in Chapter 8.

5.3 Calculation of frequency response functions

In frequency response analysis the particular solution of the equations of motion is obtained that is due to the excitation of the system irrespective of the homogeneous solution. Frequency response functions describe the relationship between a harmonic excitation at different points in the system and the response of the system, also at different points of the system. The functions are in general complex-valued and contain information about amplitudes and phases with respect to the exciting force.

Also frequency response analysis is a method which demands linear or linearized equations of motion. Frequency response functions can for example give valuable information about how well the carbody is isolated against different excitation frequencies or wavelengths in the track irregularities. Peaks in the frequency response function indicate that the pertinent excitation frequencies can be critical. Often peaks arise near eigenfrequencies (natural frequencies) of the system.

Introduce a periodic force into the equation system (5-19a)

$$\boldsymbol{F}(t) = 2\,\mathrm{Re}(\overline{\boldsymbol{F}}e^{\bar{s}t}) \qquad (5\text{-}39)$$

In this case a linear system has to respond with a periodic oscillation

$$\boldsymbol{x}(t) = 2\,\mathrm{Re}(\overline{\boldsymbol{x}}e^{\bar{s}t}) \qquad (5\text{-}40)$$

Especially an undamped harmonic input is interesting to study. This means that $\bar{s} = i\omega$ where ω is the circular frequency of the excitation.

If this is introduced into Eq. (5-40) we get

$$\left(-\boldsymbol{M}\omega^2 + \boldsymbol{C}i\omega + \boldsymbol{K}\right)\overline{\boldsymbol{x}} = \overline{\boldsymbol{F}} \qquad (5\text{-}41)$$

If the system has n degrees of freedom n^2 frequency response functions can be calculated because every input $\overline{F}_k$ (k=1,n) gives an output $\bar{x}_j$ (j=1,n). The frequency response functions are therefore given as

$$H_{jk}(i\omega) = H_{F_k}^{x_j}(i\omega) = \frac{\bar{x}_j}{\bar{F}_k} \tag{5-42a}$$

To investigate vibration isolation in the carbody it is more interesting to evaluate its accelerations due to the excitation. The corresponding frequency response function can be stated as

$$H_{F_k}^{\ddot{x}_j}(i\omega) = \frac{\omega^2 \bar{x}_j}{\bar{F}_k} = \omega^2 H_{F_k}^{x_j}(i\omega) \tag{5-42b}$$

One-dimensional model with 1 DOF

The equation of motion is

$$m\ddot{z} + c\dot{z} + kz = c\dot{z}_w + kz_w \tag{5-43}$$

With what is said above in this section we get

$$z_w(t) = 2\,\mathrm{Re}\,(\bar{z}_w e^{i\omega t}) \tag{5-44}$$

and

$$z(t) = 2\,\mathrm{Re}\,(\bar{z} e^{i\omega t}) \tag{5-45}$$

Substituting Eqs. (5-44) and (5-45) into (5-43) yields

$$\left(-m\omega^2 + ci\omega + k\right)\bar{z} = (ci\omega + k)\bar{z}_w \tag{5-46}$$

and thus

$$H_{z_w}^{z}(i\omega) = \frac{ci\omega + k}{-m\omega^2 + ci\omega + k} \tag{5-47a}$$

$$H_{z_w}^{\ddot{z}}(i\omega) = -\omega^2 H_{z_w}^{z}(i\omega) = -\omega^2 \frac{ci\omega + k}{-m\omega^2 + ci\omega + k} \tag{5-47b}$$

With $k/m = \omega_o^2$ and $c/m = 2\zeta\omega_o$ from Eq. (5-26a,b), Eq. (5-47) can be rewritten as

$$H_{z_w}^{z}(\beta, \zeta) = \frac{1 + i2\zeta\beta}{(1 - \beta^2) + i2\zeta\beta} \tag{5-48a}$$

$$\frac{H_{z_w}^{\ddot{z}}}{-\omega_o^2}(\beta, \zeta) = \beta^2 \frac{1 + i2\zeta\beta}{(1 - \beta^2) + i2\zeta\beta} \tag{5-48b}$$

where the frequency response function for accelerations has been normalised with $-\omega_o{}^2$ and where

$$\beta = \frac{\omega}{\omega_o} \tag{5-49}$$

i.e. the ratio between excitation and undamped system frequency. The frequency response functions in Eq. (5-48) are thus only depending on the dimesionless variables β and ζ.

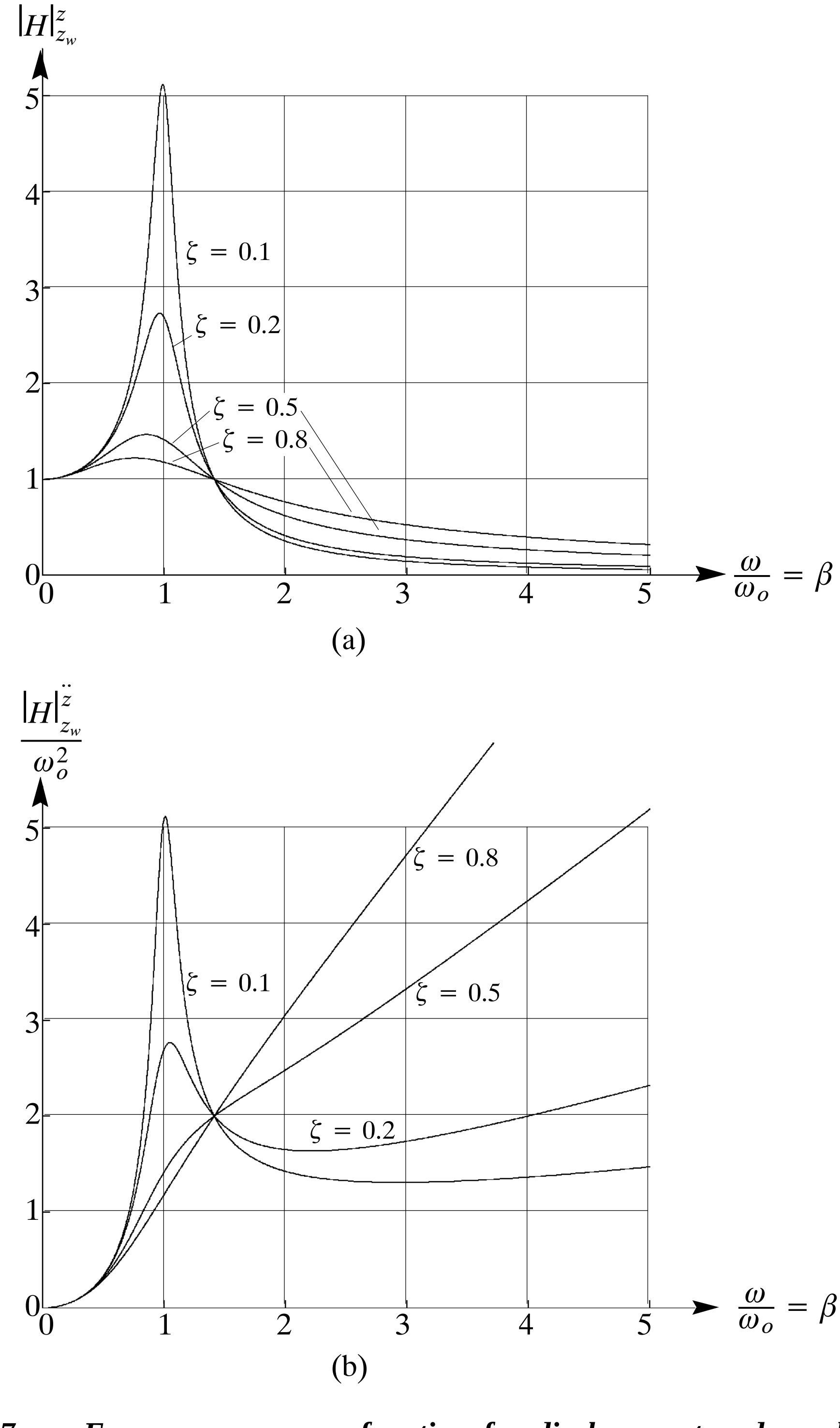

Figure 5-7 ***Frequency response function for displacement and acceleration of track excited 1 DOF system.***
(a) Displacement.
(b) Acceleration (normalised).

Amplification of displacement respectively acceleration is given by the absolute value of the frequency response functions, i.e.

$$|H_{z_w}^{z}(\beta,\zeta)| = \frac{\sqrt{1+(2\zeta\beta)^2}}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}} \qquad \text{(5-50a)}$$

$$\left|\frac{H_{z_w}^{\ddot{z}}}{-\omega_o^2}(\beta,\zeta)\right| = \beta^2\frac{\sqrt{1+(2\zeta\beta)^2}}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}} \qquad \text{(5-50b)}$$

Figure 5-7 illustrates these two expressions.

From the figure the following can be concluded:

- Low damping yields high amplification at resonance ($\beta = 1$).
- High relative damping gives large accelerations, also for e.g. $\beta > 3$.

An important task in vehicle dynamics is therefore to find adequate damping.

One-dimensional model with 2 DOF

If we introduce a bogie mass this acts like another filter for high frequency excitations. The frequency response functions for the model with 2 DOF in Figure 5-2 illustrates this.

The equations of motion for this model are

$$\begin{bmatrix} m_c & 0 \\ 0 & m_b \end{bmatrix}\begin{Bmatrix} \ddot{z}_c \\ \ddot{z}_b \end{Bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_1+c_2 \end{bmatrix}\begin{Bmatrix} \dot{z}_c \\ \dot{z}_b \end{Bmatrix} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} z_c \\ z_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ c_1\dot{z}_w + k_1 z_w \end{Bmatrix} \qquad \text{(5-51)}$$

With the solution

$$\begin{Bmatrix} 0 \\ z_w \end{Bmatrix} = 2\,\mathrm{Re}\,\left(\begin{Bmatrix} 0 \\ \bar{z}_w \end{Bmatrix} e^{i\omega t}\right) \qquad \text{(5-52)}$$

and

$$\begin{Bmatrix} z_c \\ z_b \end{Bmatrix} = 2\,\mathrm{Re}\,\left(\begin{Bmatrix} \bar{z}_c \\ \bar{z}_b \end{Bmatrix} e^{i\omega t}\right) \qquad \text{(5-53)}$$

and after substitution into Eq. (5-51) we get (cf. Equation (5-33c)),

$$\begin{bmatrix} -m_c\omega^2 + c_2 i\omega + k_2 & -c_2 i\omega - k_2 \\ -c_2 i\omega - k_2 & -m_b\omega^2 + (c_1+c_2)i\omega + (k_1+k_2) \end{bmatrix}\begin{Bmatrix} \bar{z}_c \\ \bar{z}_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ c_1 i\omega + k_1 \end{Bmatrix}\bar{z}_w \qquad \text{(5-54)}$$

We are looking for the frequency response function between z_w and z_c. With (5-42) it can be written

$$H_{z_w}^{z_c}(i\omega) = \frac{\bar{z}_c}{\bar{z}_w} \tag{5-55}$$

With Cramer's rule it can be shown that

$$H_{z_w}^{z_c}(i\omega) = \frac{\bar{z}_c}{\bar{z}_w} = \frac{\det\begin{bmatrix} 0 & -c_2 i\omega - k_2 \\ c_1 i\omega + k_1 & -m_b\omega^2 + (c_1 + c_2)i\omega + (k_1 + k_2) \end{bmatrix}}{\det\begin{bmatrix} -m_c\omega^2 + c_2 i\omega + k_2 & -c_2 i\omega - k_2 \\ -c_2 i\omega - k_2 & -m_b\omega^2 + (c_1 + c_2)i\omega + (k_1 + k_2) \end{bmatrix}} \tag{5-56}$$

which yields

$$H_{z_w}^{z_c}(i\omega) = \frac{c_1c_2(i\omega)^2 + (c_1k_2 + c_2k_1)i\omega + k_1k_2}{m_cm_b\omega^4 + (m_cc_1 + m_cc_2 + m_bc_2)(i\omega)^3 + (m_ck_1 + m_ck_2 + m_bk_2 + c_1c_2)(i\omega)^2 + (c_1k_2 + c_2k_1)i\omega + k_1k_2} \tag{5-57}$$

Frequency response functions between z_w and $\ddot{z}_c$ are like before

$$H_{z_w}^{\ddot{z}_c}(i\omega) = \frac{\omega^2\bar{z}_c}{\bar{z}_w} = \omega^2 H_{z_w}^{z_c}(i\omega) \tag{5-58}$$

In principle frequency response functions like in Eqs. (5-48a,b) can be derived. The expressions, however, become rather complicated. Instead an example with specific values is given.

Example: A carbody with mass m_C = 38200 kg is isolated against vibrations with a secondary stiffness k_2 = 2160 kN/m, a secondary damper c_2 = 160 kNs/m, a bogieframe mass m_b = 6000 kg, a primary stiffness k_1 = 11200 kN/m and a primary damper c_1 = 240 kNs/m. Figure 5-8 shows the frequency response function for acceleration in the carbody with and without primary suspension.

The frequency response function for the system with two degrees of freedom has two peaks, almost at the natural frequencies of the system (cf. Section 5.2). Comparing the frequency response function for the two degrees and one degree of freedom systems, it can be observed that the high frequency part of the acceleration is efficiently filtered out by the primary suspension.

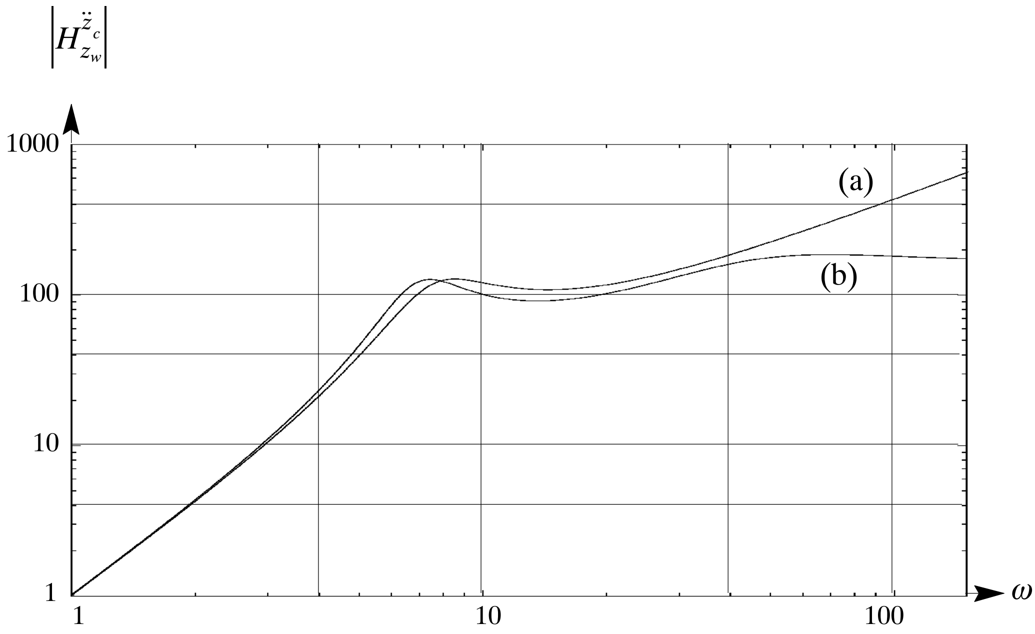

Figure 5-8 ***Frequency response function for acceleration in a system with and without primary suspension.***
(a) Without primary suspension (one degree of freedom).
(b) With primary suspension (two degrees of freedom).

5.4 Power spectral analysis

In Section 2.3 it is mentioned that track irregularities can be represented by power spectral densities (PSD). In power spectral analysis - which also is a linear method of analysis - power spectra of the vehicle response are calculated by multiplying power spectra of track irregularities with the frequency response functions between the point of excitation and the point in the vehicle to be analyzed. Power spectra are often calculated with help of the fourier transform of a signal. If C_K is the fourier coefficient of a discrete fourier spectrum and $\Delta\omega$ the frequency interval between two coefficients, power spectral density is calculated as [97]

$$S(\omega_K) = \frac{C_K^2}{2\Delta\omega}, \quad K = 1,\ldots,N. \tag{5-59}$$

In this way the spectral density gets independent of the frequency interval in contrast to the fourier transform, cf. Figure 5-9.

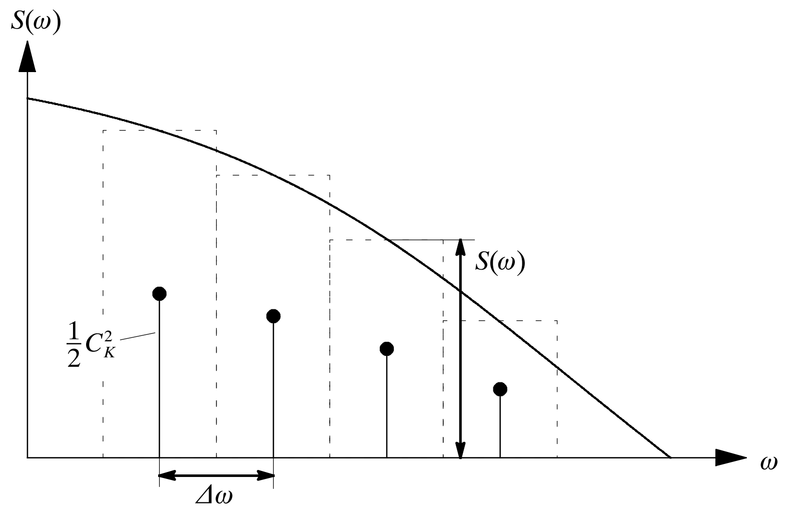

Figure 5-9 ***Calculation of spectral density from a discrete fourier transform (linear scale on horizontal axis).***

The power spectrum is the square of the fourier transform, this means that it is a real-valued expression. However, the phase information of the signal is lost. The spectral density of the vehicle response is found as follows

$$S_z(\omega) = |H|^2 \cdot S_{zw}(\omega) \qquad (5\text{-}60)$$

One has to be aware of the fact that measured power spectra of track irregularities usually are given with spatial circular frequency. Before multiplying them with the frequency response function they have to be divided by the speed

$$S_{zw}(\omega) = \frac{1}{v} \cdot S_{zw}(\omega_{spatial}) \left[\frac{\mathrm{s} \cdot \mathrm{m}^2 \cdot \mathrm{m}}{\mathrm{m} \cdot \mathrm{rad}}\right] \qquad (5\text{-}61)$$

The spectral density of accelerations can be calculated similar to the section before

$$S_{\ddot{z}} = |\omega^2 H|^2 \cdot S_{zw} = \omega^4 |H|^2 \cdot S_{zw} \qquad (5\text{-}62)$$

In Figure 5-10 power spectra of carbody accelerations for the example in the sections before are shown for a vertical excitation with the following spectral density

$$S_{zw}(\omega_{spatial}) = \frac{4.028 \cdot 10^{-7}}{0.288 \cdot 10^{-3} + 0.68\omega^2_{spatial} + \omega^4_{spatial}} \qquad (5\text{-}63)$$

The power spectrum is valid for a standard track in the network of the German railways [9]. For poor track the nominator has to be set to $1.0785 \cdot 10^{-6}$.

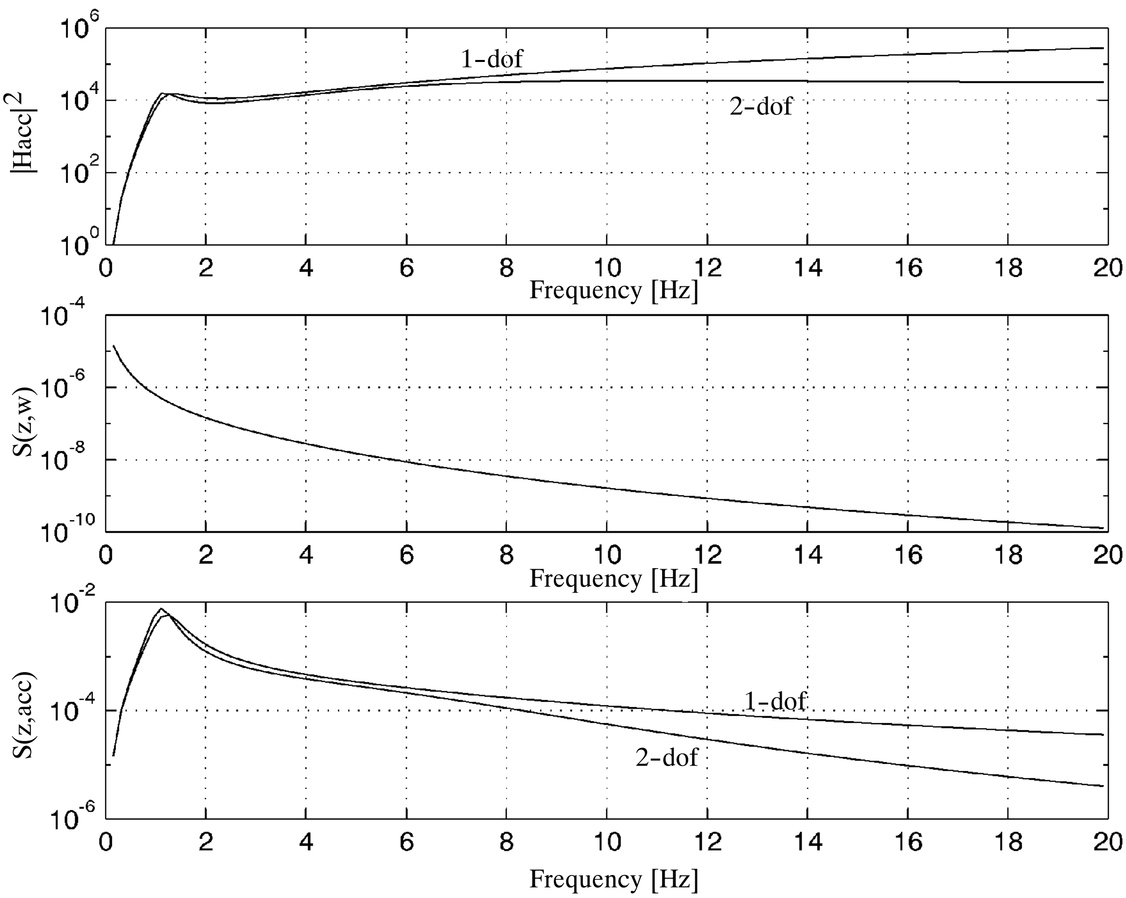

Figure 5-10 ***Power spectra of carbody acceleration for systems with one and two degrees of freedom.*** *Vehicle data as in the examples above.*

Usually a Gaussian distribution of track irregularities is assumed. In a linear system in this case also the vehicle response has a Gaussian distribution, and can be described by mean value and standard deviation. The standard deviation, b_z, is evaluated by integrating the power spectrum

$$b_z^2 = \int_0^{\infty} S_z(\omega)\, d\omega \tag{5-64}$$

In practice however it is not integrated between 0 and ∞ but as

$$b_z^2 \approx \int_{\omega_1}^{\omega_2} S_z(\omega)\, d\omega \tag{5-65}$$

Standard deviation is a quantity which is also used when analyzing measurements. Mean value $+3 \cdot$ standard deviation is sometimes assumed to be a statistical maximum value of a signal.

With the power spectrum of an acceleration like in Equation (5-62), it is easy to evaluate comfort values. If it is assumed that even a human being's sensibility for vibrations can be expressed by a frequency response function, the ”subjective standard deviation” of for example vertical carbody acceleration can be calculated as

$$b^2_{\ddot{z},carbody,subj} = \int_0^{\infty} \left|H_{human}(\omega)\right|^2 \cdot S_{\ddot{z},carbody}(\omega)\, d\omega \qquad (5\text{-}66)$$

More about evaluation of ride comfort can be found in Chapter 11.

5.5 Time step integration

Time step integration of the differential equations of motions is the most frequently used method of analysis today. The reason for this is that all non-linearities in the real vehicle - especially in the wheel-rail contact - can be taken into account in the model. Of course time step integration needs the longest computing time of all methods presented in this chapter. Usually 10-20 seconds of vehicle running, which means a travelled distance between 500-1000 m, depending on vehicle speed are simulated.

The equations of motion are often transformed to first order differential equations

$$\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u}, t), \quad \boldsymbol{u} = \{\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\}^T \qquad (5\text{-}67)$$

where function $\boldsymbol{f}$ describes spring and damper forces, contact forces, mass forces and track irregularities. A wide range of solvers exists for the numerical integration of the differential equations. It shall be mentioned here that the wheel-rail contact can cause problems for the integrator. One approximation is to regard the normal contact as rigid. In this case one gets an algebraic equation for the calculation of the contact force, which needs special algorithms.

Another possibility is to model the normal contact with a spring stiffness (Hertzian stiffness) of about $1.5 \cdot 10^9$ N/m. This introduces high eigenfrequencies in the system, one gets a so-called stiff equation system. Not all integrators are suited for stiff systems. Besides the time step has to be very short, at least shorter than $1/f_{max}$. With half the mass of a wheelset of about 800 kg the maximum time step becomes

$$\Delta t_{\max} = \frac{1}{f_{\max}} = \frac{2\pi}{\omega_{\max}} = \frac{2\pi}{\sqrt{1.5 \cdot 10^9/800}} \approx 0.005 \text{ s} \qquad (5\text{-}68)$$

Even 5 ms is often not small enough, but the lateral wheel-rail contact with the overcritical damped eigenvalues (compare Chapter 8) limits the maximum time step to about 0.1 - 1 ms.

Figure 5-11 shows some results from a time step integration with GENSYS [82]. The filter effect of the suspension can clearly be observed . The higher frequencies are filtered out gradually.

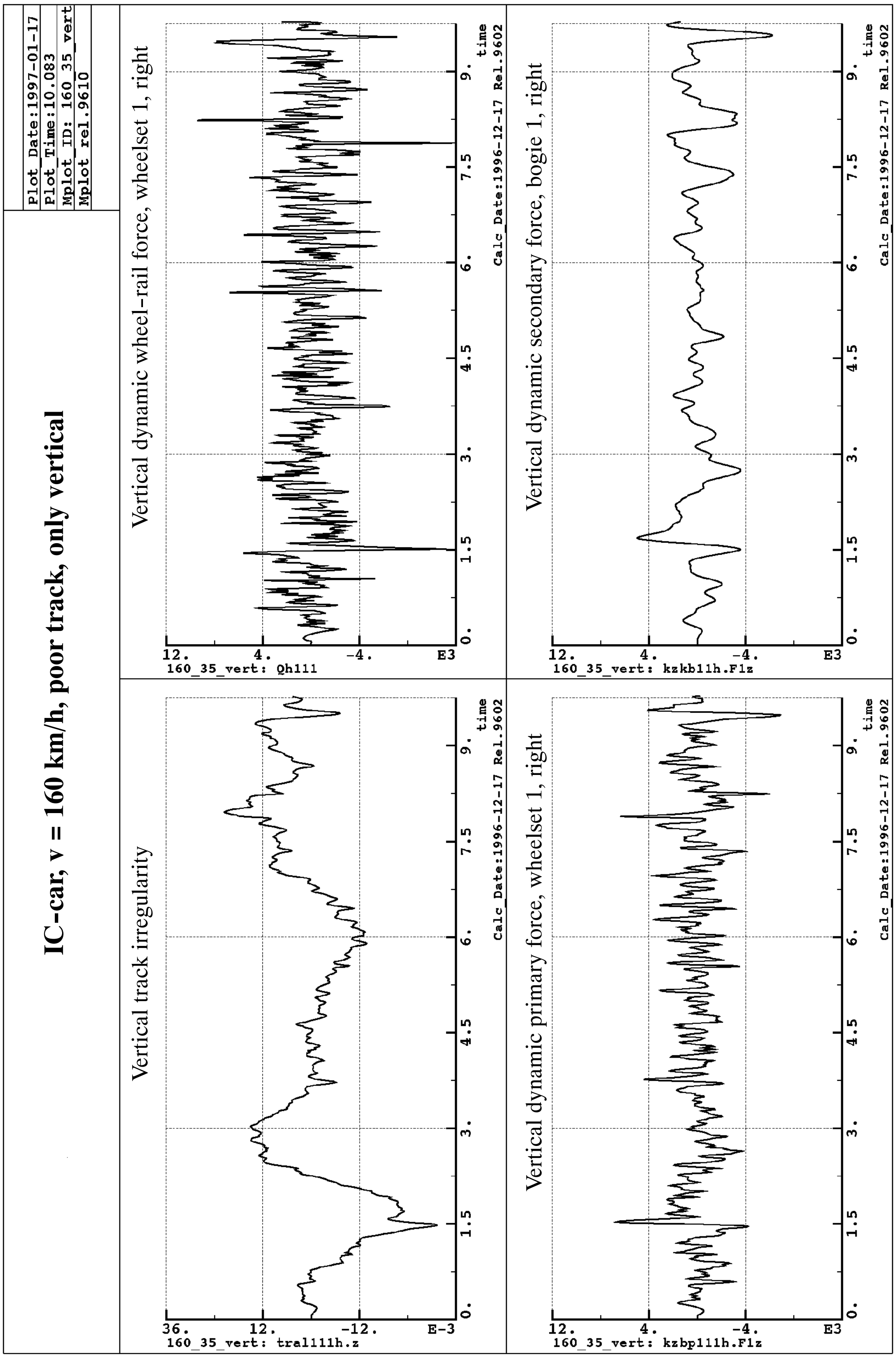

Figure 5-11 Results from time step integration.