2.5.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), A \to}\,[\wedge\mathrm{L}]}{A \to \neg(\neg A \wedge \neg B)}\,[\neg\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{B \to B}{\neg B, B \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), B \to}\,[\wedge\mathrm{L}]}{B \to \neg(\neg A \wedge \neg B)}\,[\neg\mathrm{R}]}{(A \vee B) \to \neg(\neg A \wedge \neg B)}\,[\vee\mathrm{L}]}{\to ((A \vee B) \supset \neg(\neg A \wedge \neg B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to (A \vee B)}\,[\vee\mathrm{R}]}{\to (A \vee B), \neg A}\,[\neg\mathrm{R}] \quad \dfrac{\dfrac{B \to B}{B \to (A \vee B)}\,[\vee\mathrm{R}]}{\to (A \vee B), \neg B}\,[\neg\mathrm{R}]}{\to (A \vee B), (\neg A \wedge \neg B)}\,[\wedge\mathrm{R}]}{\neg(\neg A \wedge \neg B) \to (A \vee B)}\,[\neg\mathrm{L}]}{\to (\neg(\neg A \wedge \neg B) \supset (A \vee B))}\,[\supset\mathrm{R}]}{\to ((A \vee B) \equiv \neg(\neg A \wedge \neg B))}\,[\wedge\mathrm{R}]$$

2.5.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\mathrm{R}]}{\to A, (\neg A \vee B)}\,[\vee\mathrm{R}]}{\to (\neg A \vee B), A}\,[\mathrm{XR}] \quad \dfrac{B \to B}{B \to (\neg A \vee B)}\,[\vee\mathrm{R}]}{(A \supset B) \to (\neg A \vee B), (\neg A \vee B)}\,[\supset\mathrm{L}]}{(A \supset B) \to (\neg A \vee B)}\,[\mathrm{CR}]}{\to ((A \supset B) \supset (\neg A \vee B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{\neg A, A \to B}\,[\mathrm{WR}]}{A, \neg A \to B}\,[\mathrm{XL}]}{\neg A \to (A \supset B)}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{B \to B}{A, B \to B}\,[\mathrm{WL}]}{B \to (A \supset B)}\,[\supset\mathrm{R}]}{(\neg A \vee B) \to (A \supset B)}\,[\vee\mathrm{L}]}{\to ((\neg A \vee B) \supset (A \supset B))}\,[\supset\mathrm{R}]}{\to ((A \supset B) \equiv (\neg A \vee B))}\,[\wedge\mathrm{R}]$$

2.5.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\neg F(a), F(a) \to}\,[\neg\mathrm{L}]}{\forall y \neg F(y), F(a) \to}\,[\forall\mathrm{L}]}{F(a) \to \neg\forall y \neg F(y)}\,[\neg\mathrm{R}]}{\exists x F(x) \to \neg\forall y \neg F(y)}\,[\exists\mathrm{L}]}{\to (\exists x F(x) \supset \neg\forall y \neg F(y))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\to F(a), \neg F(a)}\,[\neg\mathrm{R}]}{\to \neg F(a), F(a)}\,[\mathrm{XR}]}{\to \neg F(a), \exists x F(x)}\,[\exists\mathrm{R}]}{\to \exists x F(x), \neg F(a)}\,[\mathrm{XR}]}{\to \exists x F(x), \forall y \neg F(y)}\,[\forall\mathrm{R}]}{\neg\forall y \neg F(y) \to \exists x F(x)}\,[\neg\mathrm{L}]}{\to (\neg\forall y \neg F(y) \supset \exists x F(x))}\,[\supset\mathrm{R}]}{\to (\exists x F(x) \equiv \neg\forall y \neg F(y))}\,[\wedge\mathrm{R}]$$

2.5.4)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\to F(a), \neg F(a)}\,[\neg\mathrm{R}]}{\to F(a), \exists x \neg F(x)}\,[\exists\mathrm{R}]}{\to \exists x \neg F(x), F(a)}\,[\mathrm{XR}]}{\to \exists x \neg F(x), \forall y F(y)}\,[\forall\mathrm{R}]}{\neg\forall y F(y) \to \exists x \neg F(x)}\,[\neg\mathrm{L}]}{\to (\neg\forall y F(y) \supset \exists x \neg F(x))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\forall y F(y) \to F(a)}\,[\forall\mathrm{L}]}{\neg F(a), \forall y F(y) \to}\,[\neg\mathrm{L}]}{\forall y F(y), \neg F(a) \to}\,[\mathrm{XL}]}{\neg F(a) \to \neg\forall y F(y)}\,[\neg\mathrm{R}]}{\exists x \neg F(x) \to \neg\forall y F(y)}\,[\exists\mathrm{L}]}{\to (\exists x \neg F(x) \supset \neg\forall y F(y))}\,[\supset\mathrm{R}]}{\to (\neg\forall y F(y) \equiv \exists x \neg F(x))}\,[\wedge\mathrm{R}]$$

2.5.5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\mathrm{R}]}{\to A, (\neg A \vee \neg B)}\,[\vee\mathrm{R}]}{\to (\neg A \vee \neg B), A}\,[\mathrm{XR}] \quad \dfrac{\dfrac{\dfrac{B \to B}{\to B, \neg B}\,[\neg\mathrm{R}]}{\to B, (\neg A \vee \neg B)}\,[\vee\mathrm{R}]}{\to (\neg A \vee \neg B), B}\,[\mathrm{XR}]}{\to (\neg A \vee \neg B), (A \wedge B)}\,[\wedge\mathrm{R}]}{\to \neg(A \wedge B) \to (\neg A \vee \neg B)}\,[\neg\mathrm{L}]}{\to (\neg(A \wedge B) \supset (\neg A \vee \neg B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\mathrm{L}]}{\to A, \neg(A \wedge B)}\,[\neg\mathrm{R}]}{\to \neg(A \wedge B), A}\,[\mathrm{XR}]}{\neg A \to \neg(A \wedge B)}\,[\neg\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{B \to B}{(A \wedge B) \to B}\,[\wedge\mathrm{L}]}{\to B, \neg(A \wedge B)}\,[\neg\mathrm{R}]}{\to \neg(A \wedge B), B}\,[\mathrm{XR}]}{\neg B \to \neg(A \wedge B)}\,[\neg\mathrm{L}]}{(\neg A \vee \neg B) \to \neg(A \wedge B)}\,[\vee\mathrm{L}]}{\to ((\neg A \vee \neg B) \supset \neg(A \wedge B))}\,[\supset\mathrm{R}]}{\to (\neg(A \wedge B) \equiv (\neg A \vee \neg B))}\,[\wedge\mathrm{R}]$$

2.6.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A, B(a)}\,[\mathrm{WR}]}{A \to B(a), A}\,[\mathrm{XR}] \qquad B(a) \to B(a)}{(A \supset B(a)), A \to B(a), B(a)}\,[\supset\mathrm{L}]}{(A \supset B(a)), A \to B(a)}\,[\mathrm{CR}]}{(A \supset B(a)), A \to \exists x B(x)}\,[\exists\mathrm{R}]}{A, (A \supset B(a)) \to \exists x B(x)}\,[\mathrm{XL}]}{(A \supset B(a)) \to (A \supset \exists x B(x))}\,[\supset\mathrm{R}]}{\exists x(A \supset B(x)) \to (A \supset \exists x B(x))}\,[\exists\mathrm{L}]}{\to (\exists x(A \supset B(x)) \supset (A \supset \exists x B(x)))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{A \to A \qquad \exists x B(x) \to \exists x B(x)}{(A \supset \exists x B(x)) \to (\neg A \vee \exists x B(x))}\,[2.5.2.\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{\neg A, A \to B(a)}\,[\mathrm{WR}]}{A, \neg A \to B(a)}\,[\mathrm{XL}]}{\neg A \to (A \supset B(a))}\,[\supset\mathrm{R}]}{\neg A \to \exists x(A \supset B(x))}\,[\exists\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{B(a) \to B(a)}{A, B(a) \to B(a)}\,[\mathrm{WL}]}{B(a) \to (A \supset B(a))}\,[\supset\mathrm{R}]}{B(a) \to \exists x(A \supset B(x))}\,[\exists\mathrm{R}]}{\exists x B(x) \to \exists x(A \supset B(x))}\,[\exists\mathrm{L}]}{(\neg A \vee \exists x B(x)) \to \exists x(A \supset B(x))}\,[\vee\mathrm{L}]}{(A \supset \exists x B(x)) \to \exists x(A \supset B(x))}\,[\mathrm{Cut}]}{\to ((A \supset \exists x B(x)) \supset \exists x(A \supset B(x)))}\,[\supset\mathrm{R}]}{\to (\exists x(A \supset B(x)) \equiv (A \supset \exists x B(x)))}\,[\wedge\mathrm{R}]$$

2.6.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to A(a), B}\,[\mathrm{WR}]}{A(a) \to B, A(a)}\,[\mathrm{XR}] \qquad B \to B}{(A(a) \supset B), A(a) \to B, B}\,[\supset\mathrm{L}]}{(A(a) \supset B), A(a) \to B}\,[\mathrm{CR}]}{A(a), (A(a) \supset B) \to B}\,[\mathrm{XL}]}{\forall x A(x), (A(a) \supset B) \to B}\,[\forall\mathrm{L}]}{(A(a) \supset B) \to (\forall x A(x) \supset B)}\,[\supset\mathrm{R}]}{\exists x(A(x) \supset B) \to (\forall x A(x) \supset B)}\,[\exists\mathrm{L}]}{\to (\exists x(A(x) \supset B) \supset (\forall x A(x) \supset B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\forall x A(x) \to \forall x A(x) \qquad B \to B}{(\forall x A(x) \supset B) \to (\neg \forall x A(x) \vee B)}\,[2.5.2.\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to A(a), B}\,[\mathrm{WR}]}{\to A(a), (A(a) \supset B)}\,[\supset\mathrm{R}]}{\to A(a), \exists x(A(x) \supset B)}\,[\exists\mathrm{R}]}{\to \exists x(A(x) \supset B), A(a)}\,[\mathrm{XR}]}{\to \exists x(A(x) \supset B), \forall x A(x)}\,[\forall\mathrm{R}]}{\neg \forall x A(x) \to \exists x(A(x) \supset B)}\,[\neg\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{B \to B}{A(a), B \to B}\,[\mathrm{WL}]}{B \to (A(a) \supset B)}\,[\supset\mathrm{R}]}{B \to \exists x(A(x) \supset B)}\,[\exists\mathrm{R}]}{(\neg \forall x A(x) \vee B) \to \exists x(A(x) \supset B)}\,[\vee\mathrm{L}]}{(\forall x A(x) \supset B) \to \exists x(A(x) \supset B)}\,[\mathrm{Cut}]}{\to ((\forall x A(x) \supset B) \supset \exists x(A(x) \supset B))}\,[\supset\mathrm{R}]}{\to (\exists x(A(x) \supset B) \equiv (\forall x A(x) \supset B))}\,[\wedge\mathrm{R}]$$

2.6.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\to A(a)\qquad B(a)\to B(a)}{A(a),(A(a)\supset B(a))\to B(a)}\,[\text{detachment}]}{\forall xA(x),(A(a)\supset B(a))\to B(a)}\,[\forall\text{L}]}{\forall xA(x),(A(a)\supset B(a))\to \exists xB(x)}\,[\exists\text{R}]}{(A(a)\supset B(a))\to(\forall xA(x)\supset\exists xB(x))}\,[\supset\text{R}]}{\exists x(A(x)\supset B(x))\to(\forall xA(x)\supset\exists xB(x))}\,[\exists\text{L}]}{\to(\exists x(A(x)\supset B(x))\supset(\forall xA(x)\supset\exists xB(x)))}\,[\supset\text{R}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\to A(a)}{A(a)\to A(a),B(a)}\,[\text{WR}]}{\to A(a),(A(a)\supset B(a))}\,[\supset\text{R}]}{\to A(a),\exists x(A(x)\supset B(x))}\,[\exists\text{R}]}{\to\exists x(A(x)\supset B(x)),A(a)}\,[\text{XR}]}{\to\exists x(A(x)\supset B(x)),\forall xA(x)}\,[\forall\text{R}]}{\neg\forall xA(x)\to\exists x(A(x)\supset B(x))}\,[\neg\text{L}]\qquad \dfrac{\dfrac{\dfrac{B(a)\to B(a)}{A(a),B(a)\to B(a)}\,[\text{WL}]}{B(a)\to(A(a)\supset B(a))}\,[\supset\text{R}]}{B(a)\to\exists x(A(x)\supset B(x))}\,[\exists\text{R}]}{\exists xB(x)\to\exists x(A(x)\supset B(x))}\,[\exists\text{L}]$$

$$\dfrac{\dfrac{\dfrac{\forall xA(x)\to\forall xA(x)\qquad\exists xB(x)\to\exists xB(x)}{(\forall xA(x)\supset\exists xB(x))\to(\neg\forall xA(x)\vee\exists xB(x))}\,[2.5.2.\text{L}]\qquad(\neg\forall xA(x)\vee\exists xB(x))\to\exists x(A(x)\supset B(x))}{(\forall xA(x)\supset\exists xB(x))\to\exists x(A(x)\supset B(x))}\,[\text{Cut}]}{\to((\forall xA(x)\supset\exists xB(x))\supset\exists x(A(x)\supset B(x)))}\,[\supset\text{R}]$$

$$\to(\exists x(A(x)\supset B(x))\equiv(\forall xA(x)\supset\exists xB(x)))\,[\wedge\text{R}]$$

2.6.4)

$$\dfrac{\dfrac{\dfrac{\neg A\to\neg A\qquad B\to B}{(\neg A\supset B)\to(\neg\neg A\vee B)}\,[2.5.2.\text{L}]\qquad\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg B,A\to A}\,[\text{WL}]}{A\to(\neg B\supset A)}\,[\supset\text{R}]}{\to(\neg B\supset A),\neg A}\,[\neg\text{R}]}{\neg\neg A\to(\neg B\supset A)}\,[\neg\text{L}]\qquad\dfrac{\dfrac{\dfrac{B\to B}{B\to B,A}\,[\text{WR}]}{B\to A,B}\,[\text{XR}]}{\neg B,B\to A}\,[\neg\text{L}]}{B\to(\neg B\supset A)}\,[\supset\text{R}]}{(\neg\neg A\vee B)\to(\neg B\supset A)}\,[\vee\text{L}]}{(\neg A\supset B)\to(\neg B\supset A)}\,[\text{Cut}]$$

2.6.5)

$$\dfrac{\dfrac{\dfrac{\neg A\to\neg A\qquad\neg B\to\neg B}{(\neg A\supset\neg B)\to(\neg\neg A\vee\neg B)}\,[2.5.2.\text{L}]\qquad\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{B,A\to A}\,[\text{WL}]}{A\to(B\supset A)}\,[\supset\text{R}]}{\to(B\supset A),\neg A}\,[\neg\text{R}]}{\neg\neg A\to(B\supset A)}\,[\neg\text{L}]\qquad\dfrac{\dfrac{\dfrac{\dfrac{B\to B}{B\to B,A}\,[\text{WR}]}{B\to A,B}\,[\text{XR}]}{\neg B,B\to A}\,[\neg\text{L}]}{B,\neg B\to A}\,[\text{XL}]}{\neg B\to(B\supset A)}\,[\supset\text{R}]}{(\neg\neg A\vee\neg B)\to(B\supset A)}\,[\vee\text{L}]}{(\neg A\supset\neg B)\to(B\supset A)}\,[\text{Cut}]$$

2.7)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\to A(a)}{A(a)\to A(a),B}\,[\text{WR}]}{\to A(a),(A(a)\supset B)}\,[\supset\text{R}]}{\to A(a),\exists x(A(x)\supset B)}\,[\exists\text{R}]}{\to\exists x(A(x)\supset B),A(a)}\,[\text{XR}]}{\to\exists x(A(x)\supset B),\forall xA(x)}\,[\forall\text{R}]\qquad\dfrac{\dfrac{\dfrac{B\to B}{A(a),B\to B}\,[\text{WL}]}{B\to(A(a)\supset B)}\,[\supset\text{R}]}{B\to\exists x(A(x)\supset B)}\,[\exists\text{R}]}{(\forall xA(x)\supset B)\to\exists x(A(x)\supset B),\exists x(A(x)\supset B)}\,[\supset\text{L}]$$

$$\dfrac{(\forall xA(x)\supset B)\to\exists x(A(x)\supset B),\exists x(A(x)\supset B)}{(\forall xA(x)\supset B)\to\exists x(A(x)\supset B)}\,[\text{CR}]$$

3.9.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{\neg A, A \to B}\,[\mathrm{WR}]}{A, \neg A \to B}\,[\mathrm{XL}]}{\neg A \to (A \supset B)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{B \to B}{A, B \to B}\,[\mathrm{WL}]}{B \to (A \supset B)}\,[\supset\mathrm{R}]}{(\neg A \vee B) \to (A \supset B)}\,[\vee\mathrm{L}]$$

3.9.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\neg F(a), F(a) \to}\,[\neg\mathrm{L}]}{\forall y \neg F(y), F(a) \to}\,[\forall\mathrm{L}]}{F(a) \to \neg\forall y \neg F(y)}\,[\neg\mathrm{R}]}{\exists x F(x) \to \neg\forall y \neg F(y)}\,[\exists\mathrm{L}]$$

3.9.3)

$$\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\mathrm{L}]$$

3.9.4)

$$\dfrac{A \to A}{A \to (A \vee B)}\,[\vee\mathrm{R}]$$

3.9.5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\mathrm{L}]}{\neg A, (A \wedge B) \to}\,[\neg\mathrm{L}]}{(A \wedge B), \neg A \to}\,[\mathrm{XL}]}{\neg A \to \neg(A \wedge B)}\,[\neg\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{(A \wedge B) \to B}\,[\wedge\mathrm{L}]}{\neg B, (A \wedge B) \to}\,[\neg\mathrm{L}]}{(A \wedge B), \neg B \to}\,[\mathrm{XL}]}{\neg B \to \neg(A \wedge B)}\,[\neg\mathrm{R}]}{(\neg A \vee \neg B) \to \neg(A \wedge B)}\,[\vee\mathrm{L}]$$

3.9.6)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to (A \vee B)}\,[\vee\mathrm{R}]}{\neg(A \vee B), A \to}\,[\neg\mathrm{L}]}{A, \neg(A \vee B) \to}\,[\mathrm{XL}]}{\neg(A \vee B) \to \neg A}\,[\neg\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B \to (A \vee B)}\,[\vee\mathrm{R}]}{\neg(A \vee B), B \to}\,[\neg\mathrm{L}]}{B, \neg(A \vee B) \to}\,[\mathrm{XL}]}{\neg(A \vee B) \to \neg B}\,[\neg\mathrm{R}]}{\neg(A \vee B) \to (\neg A \wedge \neg B)}\,[\wedge\mathrm{R}]}{\to (\neg(A \vee B) \supset (\neg A \wedge \neg B))}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), A \to}\,[\wedge\mathrm{L}]}{A, (\neg A \wedge \neg B) \to}\,[\mathrm{XL}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\neg B, B \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), B \to}\,[\wedge\mathrm{L}]}{B, (\neg A \wedge \neg B) \to}\,[\mathrm{XL}]}{(A \vee B), (\neg A \wedge \neg B) \to}\,[\vee\mathrm{L}]}{(\neg A \wedge \neg B) \to \neg(A \vee B)}\,[\neg\mathrm{R}]}{\to ((\neg A \wedge \neg B) \supset \neg(A \vee B))}\,[\supset\mathrm{R}]}{\to (\neg(A \vee B) \equiv (\neg A \wedge \neg B))}\,[\wedge\mathrm{R}]$$

3.9.7)

$$\dfrac{A \rightarrow A}{B, A \rightarrow A}\,[\text{WL}] \qquad \dfrac{B, A \rightarrow A}{A, B \rightarrow A}\,[\text{XL}] \qquad \dfrac{B \rightarrow B}{A, B \rightarrow B}\,[\text{WL}]$$

$$\dfrac{A, B \rightarrow A \quad A, B \rightarrow B}{A, B \rightarrow (A \wedge B)}\,[\wedge\text{R}] \qquad \dfrac{A, B \rightarrow (A \wedge B)}{A, B \rightarrow ((A \wedge B) \vee C)}\,[\vee\text{R}]$$

$$\dfrac{C \rightarrow C}{B, C \rightarrow C}\,[\text{WL}] \qquad \dfrac{B, C \rightarrow C}{C, B \rightarrow C}\,[\text{XL}] \qquad \dfrac{C, B \rightarrow C}{C, B \rightarrow ((A \wedge B) \vee C)}\,[\vee\text{R}]$$

$$\dfrac{A, B \rightarrow ((A \wedge B) \vee C) \quad C, B \rightarrow ((A \wedge B) \vee C)}{(A \vee C), B \rightarrow ((A \wedge B) \vee C)}\,[\vee\text{L}] \qquad \dfrac{(A \vee C), B \rightarrow ((A \wedge B) \vee C)}{B, (A \vee C) \rightarrow ((A \wedge B) \vee C)}\,[\text{XL}]$$

$$\dfrac{C \rightarrow C}{C \rightarrow ((A \wedge B) \vee C)}\,[\vee\text{R}] \qquad \dfrac{C \rightarrow ((A \wedge B) \vee C)}{(A \vee C), C \rightarrow ((A \wedge B) \vee C)}\,[\text{WL}] \qquad \dfrac{(A \vee C), C \rightarrow ((A \wedge B) \vee C)}{C, (A \vee C) \rightarrow ((A \wedge B) \vee C)}\,[\text{XL}]$$

$$\dfrac{B, (A \vee C) \rightarrow ((A \wedge B) \vee C) \quad C, (A \vee C) \rightarrow ((A \wedge B) \vee C)}{(B \vee C), (A \vee C) \rightarrow ((A \wedge B) \vee C)}\,[\vee\text{L}]$$

$$\dfrac{(B \vee C), (A \vee C) \rightarrow ((A \wedge B) \vee C)}{(A \vee C), (B \vee C) \rightarrow ((A \wedge B) \vee C)}\,[\text{XL}]$$

$$\dfrac{(A \vee C), (B \vee C) \rightarrow ((A \wedge B) \vee C)}{((A \vee C) \wedge (B \vee C)), (B \vee C) \rightarrow ((A \wedge B) \vee C)}\,[\wedge\text{L}]$$

$$\dfrac{((A \vee C) \wedge (B \vee C)), (B \vee C) \rightarrow ((A \wedge B) \vee C)}{(B \vee C), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}\,[\text{XL}]$$

$$\dfrac{(B \vee C), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}{((A \vee C) \wedge (B \vee C)), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}\,[\wedge\text{L}]$$

$$\dfrac{((A \vee C) \wedge (B \vee C)), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}{((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}\,[\text{CL}]$$

$$\dfrac{((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}{\rightarrow (((A \vee C) \wedge (B \vee C)) \supset ((A \wedge B) \vee C))}\,[\supset\text{R}]$$

$$\dfrac{A \rightarrow A}{(A \wedge B) \rightarrow A}\,[\wedge\text{L}] \qquad \dfrac{(A \wedge B) \rightarrow A}{(A \wedge B) \rightarrow (A \vee C)}\,[\vee\text{R}] \qquad \dfrac{C \rightarrow C}{C \rightarrow (A \vee C)}\,[\vee\text{R}]$$

$$\dfrac{(A \wedge B) \rightarrow (A \vee C) \quad C \rightarrow (A \vee C)}{((A \wedge B) \vee C) \rightarrow (A \vee C)}\,[\vee\text{L}]$$

$$\dfrac{B \rightarrow B}{(A \wedge B) \rightarrow B}\,[\wedge\text{L}] \qquad \dfrac{(A \wedge B) \rightarrow B}{(A \wedge B) \rightarrow (B \vee C)}\,[\vee \qquad ((A \wedge B) \vee C)$$

$$\dfrac{((A \wedge B) \vee C) \rightarrow ((A \vee C) \wedge (B \vee C))}{\rightarrow (((A \wedge B) \vee C) \supset ((A \vee C) \wedge (B \vee C)))}\,[\supset\text{R}]$$

$$\dfrac{\rightarrow (((A \vee C) \wedge (B \vee C)) \supset ((A \wedge B) \vee C)) \quad \rightarrow (((A \wedge B) \vee C) \supset ((A \vee C) \wedge (B \vee C)))}{\rightarrow (((A \vee C) \wedge (B \vee C)) \equiv ((A \wedge B) \vee C))}\,[\wedge\text{R}]$$

3.9.8)

$$\dfrac{F(a) \rightarrow F(a)}{\neg F(a), F(a) \rightarrow}\,[\neg\text{L}] \qquad \dfrac{\neg F(a), F(a) \rightarrow}{F(a), \neg F(a) \rightarrow}\,[\text{XL}] \qquad \dfrac{F(a), \neg F(a) \rightarrow}{\forall x F(x), \neg F(a) \rightarrow}\,[\forall\text{L}]$$

$$\dfrac{\forall x F(x), \neg F(a) \rightarrow}{\neg F(a) \rightarrow \neg \forall x F(x)}\,[\neg\text{R}] \qquad \dfrac{\neg F(a) \rightarrow \neg \forall x F(x)}{\exists x \neg F(x) \rightarrow \neg \forall x F(x)}\,[\exists\text{L}]$$

3.9.9)

$$\dfrac{F(a) \rightarrow F(a)}{(F(a) \wedge G(a)) \rightarrow F(a)}\,[\wedge\text{L}] \qquad \dfrac{(F(a) \wedge G(a)) \rightarrow F(a)}{\forall x(F(x) \wedge G(x)) \rightarrow F(a)}\,[\forall\text{L}] \qquad \dfrac{\forall x(F(x) \wedge G(x)) \rightarrow F(a)}{\forall x(F(x) \wedge G(x)) \rightarrow \forall x F(x)}\,[\forall\text{R}]$$

$$\dfrac{G(a) \rightarrow G(a)}{(F(a) \wedge G(a)) \rightarrow G(a)}\,[\wedge\text{L}] \qquad \dfrac{(F(a) \wedge G(a)) \rightarrow G(a)}{\forall x(F(x) \wedge G(x)) \rightarrow G(a)}\,[\forall\text{L}] \qquad \dfrac{\forall x(F(x) \wedge G(x)) \rightarrow G(a)}{\forall x(F(x) \wedge G(x)) \rightarrow \forall x G(x)}\,[\forall\text{R}]$$

$$\dfrac{\forall x(F(x) \wedge G(x)) \rightarrow \forall x F(x) \quad \forall x(F(x) \wedge G(x)) \rightarrow \forall x G(x)}{\forall x(F(x) \wedge G(x)) \rightarrow (\forall x F(x) \wedge \forall x G(x))}\,[\wedge\text{R}] \qquad \dfrac{\forall x(F(x) \wedge G(x)) \rightarrow (\forall x F(x) \wedge \forall x G(x))}{\rightarrow (\forall x(F(x) \wedge G(x)) \supset (\forall x F(x) \wedge \forall x G(x)))}\,[\supset\text{R}]$$

$$\dfrac{F(a) \rightarrow F(a)}{\forall x F(x) \rightarrow F(a)}\,[\forall\text{L}] \qquad \dfrac{\forall x F(x) \rightarrow F(a)}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow F(a)}\,[\wedge\text{L}]$$

$$\dfrac{G(a) \rightarrow G(a)}{\forall x G(x) \rightarrow G(a)}\,[\forall\text{L}] \qquad \dfrac{\forall x G(x) \rightarrow G(a)}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow G(a)}\,[\wedge\text{L}]$$

$$\dfrac{(\forall x F(x) \wedge \forall x G(x)) \rightarrow F(a) \quad (\forall x F(x) \wedge \forall x G(x)) \rightarrow G(a)}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow (F(a) \wedge G(a))}\,[\wedge\text{R}] \qquad \dfrac{(\forall x F(x) \wedge \forall x G(x)) \rightarrow (F(a) \wedge G(a))}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow \forall x(F(x) \wedge G(x))}\,[\forall\text{R}]$$

$$\dfrac{(\forall x F(x) \wedge \forall x G(x)) \rightarrow \forall x(F(x) \wedge G(x))}{\rightarrow ((\forall x F(x) \wedge \forall x G(x)) \supset \forall x(F(x) \wedge G(x)))}\,[\supset\text{R}]$$

$$\dfrac{\rightarrow (\forall x(F(x) \wedge G(x)) \supset (\forall x F(x) \wedge \forall x G(x))) \quad \rightarrow ((\forall x F(x) \wedge \forall x G(x)) \supset \forall x(F(x) \wedge G(x)))}{\rightarrow (\forall x(F(x) \wedge G(x)) \equiv (\forall x F(x) \wedge \forall x G(x)))}\,[\wedge\text{R}]$$

3.9.10)

$$\dfrac{B \rightarrow B}{\neg B, B \rightarrow}\,[\neg\text{L}] \qquad \dfrac{A \rightarrow A \quad \neg B, B \rightarrow}{(A \supset \neg B), A, B \rightarrow}\,[\supset\text{L}]$$

$$\dfrac{(A \supset \neg B), A, B \rightarrow}{A, (A \supset \neg B), B \rightarrow}\,[\text{XL}] \qquad \dfrac{A, (A \supset \neg B), B \rightarrow}{A, B, (A \supset \neg B) \rightarrow}\,[\text{XL}]$$

$$\dfrac{A, B, (A \supset \neg B) \rightarrow}{B, (A \supset \neg B) \rightarrow \neg A}\,[\neg\text{R}] \qquad \dfrac{B, (A \supset \neg B) \rightarrow \neg A}{(A \supset \neg B) \rightarrow (B \supset \neg A)}\,[\supset\text{R}]$$

3.9.11)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B(a) \to B(a)}{}\,[\text{detachment}]}{A, (A \supset B(a)) \to B(a)}\,[\exists\text{R}]}{A, (A \supset B(a)) \to \exists x B(x)}\,[\supset\text{R}]}{(A \supset B(a)) \to (A \supset \exists x B(x))}\,[\exists\text{L}]}{\exists x(A \supset B(x)) \to (A \supset \exists x B(x))}$$

3.9.12)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a) \qquad B \to B}{}\,[\text{detachment}]}{A(a), (A(a) \supset B) \to B}\,[\forall\text{L}]}{\forall x A(x), (A(a) \supset B) \to B}\,[\supset\text{R}]}{(A(a) \supset B) \to (\forall x A(x) \supset B)}\,[\exists\text{L}]}{\exists x(A(x) \supset B) \to (\forall x A(x) \supset B)}$$

3.9.13)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a) \qquad B(a) \to B(a)}{}\,[\text{detachment}]}{A(a), (A(a) \supset B(a)) \to B(a)}\,[\exists\text{R}]}{A(a), (A(a) \supset B(a)) \to \exists x B(x)}\,[\forall\text{L}]}{\forall x A(x), (A(a) \supset B(a)) \to \exists x B(x)}\,[\supset\text{R}]}{(A(a) \supset B(a)) \to (\forall x A(x) \supset \exists x B(x))}\,[\exists\text{L}]}{\exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x))}$$

3.10.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{}\,[\text{detachment}]}{A, (A \supset B) \to B}\,[\text{XL}]}{(A \supset B), A \to B}\,[\neg\text{L}]}{\neg B, (A \supset B), A \to}\,[\text{XL}]}{(A \supset B), \neg B, A \to}\,[\neg\text{R}]}{\neg B, A \to \neg(A \supset B)}\,[\neg\text{L}]}{\neg\neg(A \supset B), \neg B, A \to}\,[\text{XL}]}{\neg B, \neg\neg(A \supset B), A \to}\,[\neg\text{R}]}{\neg\neg(A \supset B), A \to \neg\neg B}$$

3.10.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{}\,[3.10.1]}{\neg\neg(A \supset B), A \to \neg\neg B} \qquad B \to B}{}\,[\supset\text{L}]}{(\neg\neg B \supset B), \neg\neg(A \supset B), A \to B}\,[\text{XL}]}{(\neg\neg B \supset B), A, \neg\neg(A \supset B) \to B}\,[\text{XL}]}{A, (\neg\neg B \supset B), \neg\neg(A \supset B) \to B}\,[\supset\text{R}]}{(\neg\neg B \supset B), \neg\neg(A \supset B) \to (A \supset B)}$$

3.10.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{A \to \neg\neg A}\,[\neg\mathrm{R}]}{\neg\neg\neg A, A \to}\,[\neg\mathrm{L}]}{A, \neg\neg\neg A \to}\,[\mathrm{XL}]}{\neg\neg\neg A \to \neg A}\,[\neg\mathrm{R}]}{\to (\neg\neg\neg A \supset \neg A)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{A, \neg A \to}\,[\mathrm{XL}]}{\neg A \to \neg A}\,[\neg\mathrm{R}]}{\neg\neg A, \neg A \to}\,[\neg\mathrm{L}]}{\neg A \to \neg\neg\neg A}\,[\neg\mathrm{R}]}{\to (\neg A \supset \neg\neg\neg A)}\,[\supset\mathrm{R}]}{\to (\neg\neg\neg A \equiv \neg A)}\,[\wedge\mathrm{R}]$$

3.11.$\wedge$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{(B \wedge C) \to B}\,[\wedge\mathrm{L}]}{\neg B, (B \wedge C) \to}\,[\neg\mathrm{L}]}{(B \wedge C), \neg B \to}\,[\mathrm{XL}]}{\neg B \to \neg(B \wedge C)}\,[\neg\mathrm{R}]}{\neg\neg(B \wedge C), \neg B \to}\,[\neg\mathrm{L}]}{\neg B, \neg\neg(B \wedge C) \to}\,[\mathrm{XL}]}{\neg\neg(B \wedge C) \to \neg\neg B}\,[\neg\mathrm{R}] \qquad [\neg\neg B \to B]}{\neg\neg(B \wedge C) \to B}\,[\mathrm{Cut}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to C}{(B \wedge C) \to C}\,[\wedge\mathrm{L}]}{\neg C, (B \wedge C) \to}\,[\neg\mathrm{L}]}{(B \wedge C), \neg C \to}\,[\mathrm{XL}]}{\neg C \to \neg(B \wedge C)}\,[\neg\mathrm{R}]}{\neg\neg(B \wedge C), \neg C \to}\,[\neg\mathrm{L}]}{\neg C, \neg\neg(B \wedge C) \to}\,[\mathrm{XL}]}{\neg\neg(B \wedge C) \to \neg\neg C}\,[\neg\mathrm{R}] \qquad [\neg\neg C \to C]}{\neg\neg(B \wedge C) \to C}\,[\mathrm{Cut}]}{\neg\neg(B \wedge C) \to (B \wedge C)}\,[\wedge\mathrm{R}]$$

3.11.$\subset$)

$$\dfrac{\dfrac{[\neg\neg C \to C]}{\to (\neg\neg C \supset C)}\,[\supset\mathrm{R}] \qquad \dfrac{B \to B \qquad C \to C}{(\neg\neg C \supset C), \neg\neg(B \supset C) \to (B \supset C)}\,[3.10.2]}{\neg\neg(B \supset C) \to (B \supset C)}\,[\mathrm{Cut}]$$

3.11.$\forall$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B(a) \to B(a)}{\forall x B(x) \to B(a)}\,[\forall\mathrm{L}]}{\neg B(a), \forall x B(x) \to}\,[\neg\mathrm{L}]}{\forall x B(x), \neg B(a) \to}\,[\mathrm{XL}]}{\neg B(a) \to \neg\forall x B(x)}\,[\neg\mathrm{R}]}{\neg\neg\forall x B(x), \neg B(a) \to}\,[\neg\mathrm{L}]}{\neg B(a), \neg\neg\forall x B(x) \to}\,[\mathrm{XL}]}{\neg\neg\forall x B(x) \to \neg\neg B(a)}\,[\neg\mathrm{R}] \qquad [\neg\neg B(a) \to B(a)]}{\neg\neg\forall x B(x) \to B(a)}\,[\mathrm{Cut}]}{\neg\neg\forall x B(x) \to \forall x B(x)}\,[\forall\mathrm{R}]$$

3.12.1.atomic)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{A \to \neg\neg A}\,[\neg\mathrm{R}]}{\to (A \supset \neg\neg A)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\mathrm{R}]}{\neg\neg A \to A}\,[\neg\mathrm{L}]}{\to (\neg\neg A \supset A)}\,[\supset\mathrm{R}]}{\to (A \equiv \neg\neg A)}\,[\wedge\mathrm{R}]$$

3.12.1.¬)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B^* \to B]}{\neg B, B^* \to}\,[\neg\mathrm{L}]}{B^*, \neg B \to}\,[\mathrm{XL}]}{\neg B \to \neg B^*}\,[\neg\mathrm{R}]}{\to (\neg B \supset \neg B^*)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{[B \to B^*]}{\neg B^*, B \to}\,[\neg\mathrm{L}]}{B, \neg B^* \to}\,[\mathrm{XL}]}{\neg B^* \to \neg B}\,[\neg\mathrm{R}]}{\to (\neg B^* \supset \neg B)}\,[\supset\mathrm{R}]}{\to (\neg B \equiv \neg B^*)}\,[\wedge\mathrm{R}]$$

3.12.1.∧)

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \to B^*]}{(B \wedge C) \to B^*}\,[\wedge\mathrm{L}] \quad \dfrac{[C \to C^*]}{(B \wedge C) \to C^*}\,[\wedge\mathrm{L}]}{(B \wedge C) \to (B^* \wedge C^*)}\,[\wedge\mathrm{R}]}{\to ((B \wedge C) \supset (B^* \wedge C^*))}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{[B^* \to B]}{(B^* \wedge C^*) \to B}\,[\wedge\mathrm{L}] \quad \dfrac{[C^* \to C]}{(B^* \wedge C^*) \to C}\,[\wedge\mathrm{L}]}{(B^* \wedge C^*) \to (B \wedge C)}\,[\wedge\mathrm{R}]}{\to ((B^* \wedge C^*) \supset (B \wedge C))}\,[\supset\mathrm{R}]}{\to ((B \wedge C) \equiv (B^* \wedge C^*))}\,[\wedge\mathrm{R}]$$

3.12.1.∨)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B \to B^*]}{\neg B^*, B \to}\,[\neg\mathrm{L}]}{(\neg B^* \wedge \neg C^*), B \to}\,[\wedge\mathrm{L}]}{B \to \neg(\neg B^* \wedge \neg C^*)}\,[\neg\mathrm{R}] \quad \dfrac{\dfrac{[C \to C^*]}{\neg C^*, C \to}\,[\neg\mathrm{L}]}{(\neg B^* \wedge \neg C^*), C \to}\,[\wedge\mathrm{L}]}{C \to \neg(\neg B^* \wedge \neg C^*)}\,[\neg\mathrm{R}]}{(B \vee C) \to \neg(\neg B^* \wedge \neg C^*)}\,[\vee\mathrm{L}]}{\to ((B \vee C) \supset \neg(\neg B^* \wedge \neg C^*))}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B^* \to B]}{B^* \to (B \vee C)}\,[\vee\mathrm{R}]}{\to (B \vee C), \neg B^*}\,[\neg\mathrm{R}] \quad \dfrac{[C^* \to C]}{C^* \to (B \vee C)}\,[\vee\mathrm{R}]}{\to (B \vee C), \neg C^*}\,[\neg\mathrm{R}]}{\to (B \vee C), (\neg B^* \wedge \neg C^*)}\,[\wedge\mathrm{R}]}{\neg(\neg B^* \wedge \neg C^*) \to (B \vee C)}\,[\neg\mathrm{L}]}{\to (\neg(\neg B^* \wedge \neg C^*) \supset (B \vee C))}\,[\supset\mathrm{R}]}{\to ((B \vee C) \equiv \neg(\neg B^* \wedge \neg C^*))}\,[\wedge\mathrm{R}]$$

3.12.1.⊂)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B^* \to B] \quad [C \to C^*]}{(B \supset C), B^* \to C^*}\,[\supset\mathrm{L}]}{B^*, (B \supset C) \to C^*}\,[\mathrm{XL}]}{(B \supset C) \to (B^* \supset C^*)}\,[\supset\mathrm{R}]}{\to ((B \supset C) \supset (B^* \supset C^*))}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{[B \to B^*] \quad [C^* \to C]}{(B^* \supset C^*), B \to C}\,[\supset\mathrm{L}]}{B, (B^* \supset C^*) \to C}\,[\mathrm{XL}]}{(B^* \supset C^*) \to (B \supset C)}\,[\supset\mathrm{R}]}{\to ((B^* \supset C^*) \supset (B \supset C))}\,[\supset\mathrm{R}]}{\to ((B \supset C) \equiv (B^* \supset C^*))}\,[\wedge\mathrm{R}]$$

3.12.1.∀)

$$\dfrac{\dfrac{\dfrac{\dfrac{[B(a) \to B^*(a)]}{\forall x B(x) \to B^*(a)}\,[\forall \mathrm{L}]}{\forall x B(x) \to \forall x B^*(x)}\,[\forall \mathrm{R}]}{\to (\forall x B(x) \supset \forall x B^*(x))}\,[\supset \mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{[B^*(a) \to B(a)]}{\forall x B^*(x) \to B(a)}\,[\forall \mathrm{L}]}{\forall x B^*(x) \to \forall x B(x)}\,[\forall \mathrm{R}]}{\to (\forall x B^*(x) \supset \forall x B(x))}\,[\supset \mathrm{R}]}{\to (\forall x B(x) \equiv \forall x B^*(x))}\,[\wedge \mathrm{R}]$$

3.12.1.∃)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B(a) \to B^*(a)]}{\neg B^*(a), B(a) \to}\,[\neg \mathrm{L}]}{\forall x \neg B^*(x), B(a) \to}\,[\forall \mathrm{L}]}{B(a) \to \neg \forall x \neg B^*(x)}\,[\neg \mathrm{R}]}{\exists x B(x) \to \neg \forall x \neg B^*(x)}\,[\exists \mathrm{L}]}{\to (\exists x B(x) \supset \neg \forall x \neg B^*(x))}\,[\supset \mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B^*(a) \to B(a)]}{B^*(a) \to \exists x B(x)}\,[\exists \mathrm{R}]}{\to \exists x B(x), \neg B^*(a)}\,[\neg \mathrm{R}]}{\to \exists x B(x), \forall x \neg B^*(x)}\,[\forall \mathrm{R}]}{\neg \forall x \neg B^*(x) \to \exists x B(x)}\,[\neg \mathrm{L}]}{\to (\neg \forall x \neg B^*(x) \supset \exists x B(x))}\,[\supset \mathrm{R}]}{\to (\exists x B(x) \equiv \neg \forall x \neg B^*(x))}\,[\wedge \mathrm{R}]$$

3.12.3.atomic)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg \mathrm{L}]}{A, \neg A \to}\,[\mathrm{XL}]}{\neg A \to \neg A}\,[\neg \mathrm{R}]}{\neg \neg A, \neg A \to}\,[\neg \mathrm{L}]}{\neg A \to \neg \neg \neg A}\,[\neg \mathrm{R}]}{\neg \neg \neg \neg A, \neg A \to}\,[\neg \mathrm{L}]}{\neg A, \neg \neg \neg \neg A \to}\,[\mathrm{XL}]}{\neg \neg \neg \neg A \to \neg \neg A}\,[\neg \mathrm{R}]$$

3.12.3.¬)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^* \to B^*}{\neg B^*, B^* \to}\,[\neg \mathrm{L}]}{B^* \to \neg \neg B^*}\,[\neg \mathrm{R}]}{\neg \neg \neg B^*, B^* \to}\,[\neg \mathrm{L}]}{B^*, \neg \neg \neg B^* \to}\,[\mathrm{XL}]}{\neg \neg \neg B^* \to \neg B^*}\,[\neg \mathrm{R}]$$

3.12.3.∧)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^* \to B^*}{(B^* \wedge C^*) \to B^*}\,[\wedge\mathrm{L}]}{\neg B^*, (B^* \wedge C^*) \to}\,[\neg\mathrm{L}]}{(B^* \wedge C^*), \neg B^* \to}\,[\mathrm{XL}]}{\neg B^* \to \neg(B^* \wedge C^*)}\,[\neg\mathrm{R}]}{\neg\neg(B^* \wedge C^*), \neg B^* \to}\,[\neg\mathrm{L}]}{\neg B^*, \neg\neg(B^* \wedge C^*) \to}\,[\mathrm{XL}]}{\neg\neg(B^* \wedge C^*) \to \neg\neg B^*}\,[\neg\mathrm{R}] \qquad [\neg\neg B^* \to B^*]}{\neg\neg(B^* \wedge C^*) \to B^*}\,[\mathrm{Cut}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C^* \to C^*}{(B^* \wedge C^*) \to C^*}\,[\wedge\mathrm{L}]}{\neg C^*, (B^* \wedge C^*) \to}\,[\neg\mathrm{L}]}{(B^* \wedge C^*), \neg C^* \to}\,[\mathrm{XL}]}{\neg C^* \to \neg(B^* \wedge C^*)}\,[\neg\mathrm{R}]}{\neg\neg(B^* \wedge C^*), \neg C^* \to}\,[\neg\mathrm{L}]}{\neg C^*, \neg\neg(B^* \wedge C^*) \to}\,[\mathrm{XL}]}{\neg\neg(B^* \wedge C^*) \to \neg\neg C^*}\,[\neg\mathrm{R}] \qquad [\neg\neg C^* \to C^*]}{\neg\neg(B^* \wedge C^*) \to C^*}\,[\mathrm{Cut}]}{\neg\neg(B^* \wedge C^*) \to (B^* \wedge C^*)}\,[\wedge\mathrm{R}]$$

3.12.3.∨)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\neg B^* \wedge \neg C^*) \to (\neg B^* \wedge \neg C^*)}{\neg(\neg B^* \wedge \neg C^*), (\neg B^* \wedge \neg C^*) \to}\,[\neg\mathrm{L}]}{(\neg B^* \wedge \neg C^*) \to \neg\neg(\neg B^* \wedge \neg C^*)}\,[\neg\mathrm{R}]}{\neg\neg\neg(\neg B^* \wedge \neg C^*), (\neg B^* \wedge \neg C^*) \to}\,[\neg\mathrm{L}]}{(\neg B^* \wedge \neg C^*), \neg\neg\neg(\neg B^* \wedge \neg C^*) \to}\,[\mathrm{XL}]}{\neg\neg\neg(\neg B^* \wedge \neg C^*) \to \neg(\neg B^* \wedge \neg C^*)}\,[\neg\mathrm{R}]$$

3.12.3.⊂)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^* \to B^* \qquad C^* \to C^*}{\neg\neg(B^* \supset C^*), B^* \to \neg\neg C^*}\,[3.10.1] \qquad [\neg\neg C^* \to C^*]}{\neg\neg(B^* \supset C^*), B^* \to C^*}\,[\mathrm{Cut}]}{B^*, \neg\neg(B^* \supset C^*) \to C^*}\,[\mathrm{XL}]}{\neg\neg(B^* \supset C^*) \to (B^* \supset C^*)}\,[\supset\mathrm{R}]$$

3.12.3.∀)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^*(a) \to B^*(a)}{\forall x B^*(x) \to B^*(a)}\,[\forall\mathrm{L}]}{\neg B^*(a), \forall x B^*(x) \to}\,[\neg\mathrm{L}]}{\forall x B^*(x), \neg B^*(a) \to}\,[\mathrm{XL}]}{\neg B^*(a) \to \neg\forall x B^*(x)}\,[\neg\mathrm{R}]}{\neg\neg\forall x B^*(x), \neg B^*(a) \to}\,[\neg\mathrm{L}]}{\neg B^*(a), \neg\neg\forall x B^*(x) \to}\,[\mathrm{XL}]}{\neg\neg\forall x B^*(x) \to \neg\neg B^*(a)}\,[\neg\mathrm{R}] \qquad [\neg\neg B^*(a) \to B^*(a)]}{\neg\neg\forall x B^*(x) \to B^*(a)}\,[\mathrm{Cut}]}{\neg\neg\forall x B^*(x) \to \forall x B^*(x)}\,[\forall\mathrm{R}]$$

3.12.3.∃)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\neg B^*(x) \to \forall x\neg B^*(x)}{\neg\forall x\neg B^*(x), \forall x\neg B^*(x) \to}\,[\neg\mathrm{L}]}{\forall x\neg B^*(x) \to \neg\neg\forall x\neg B^*(x)}\,[\neg\mathrm{R}]}{\neg\neg\neg\forall x\neg B^*(x), \forall x\neg B^*(x) \to}\,[\neg\mathrm{L}]}{\forall x\neg B^*(x), \neg\neg\neg\forall x\neg B^*(x) \to}\,[\mathrm{XL}]}{\neg\neg\neg\forall x\neg B^*(x) \to \neg\forall x\neg B^*(x)}\,[\neg\mathrm{R}]$$

Detachment Rule)

$$\dfrac{\dfrac{A \to A \qquad B \to B}{(A \supset B), A \to B}\,[\supset\text{L}]}{A, (A \supset B) \to B}\,[\text{XL}]$$

Double Negation)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\text{L}]}{A \to \neg\neg A}\,[\neg\text{R}]}{\to (A \supset \neg\neg A)}\,[\supset\text{R}] \qquad \dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\text{R}]}{\neg\neg A \to A}\,[\neg\text{L}]}{\to (\neg\neg A \supset A)}\,[\supset\text{R}]}{\to (A \equiv \neg\neg A)}\,[\wedge\text{R}]$$

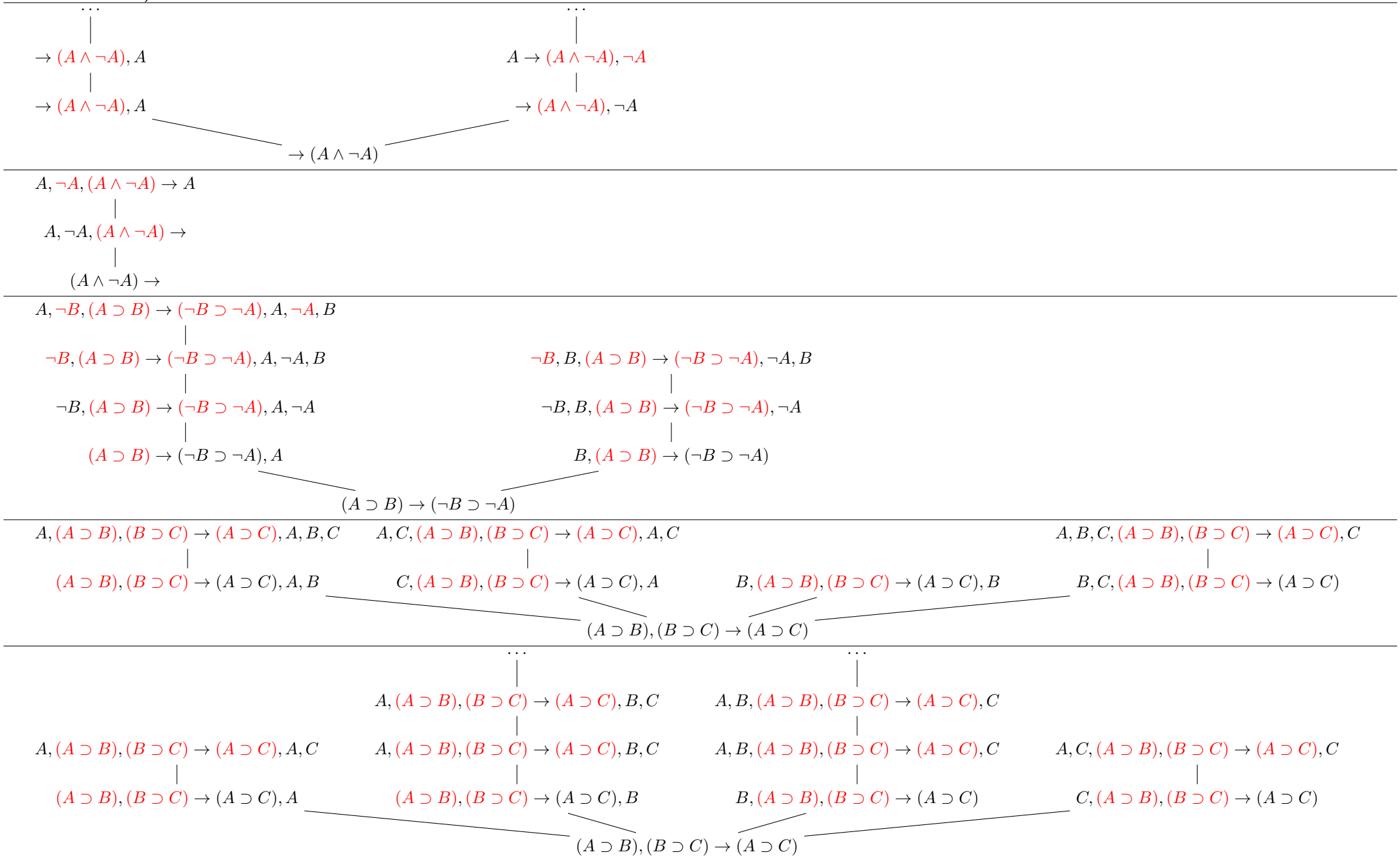
Contraction Redundancy)

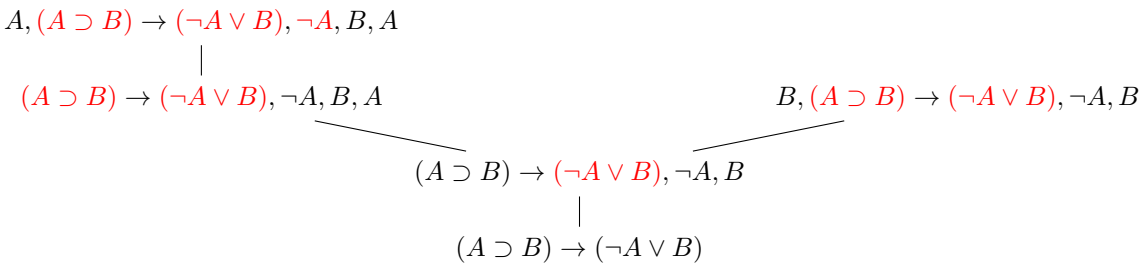
$$\dfrac{\dfrac{\dfrac{D \to D}{\neg D, D \to}\,[\neg\text{L}]}{D \to \neg\neg D}\,[\neg\text{R}] \qquad \dfrac{\dfrac{\dfrac{D, D, \Gamma \to \Delta}{\Gamma \to \Delta, \neg D, \neg D}\,[\neg\text{R}, \neg\text{R}]}{\Gamma \to \Delta, \neg D}\,[\text{CR}]}{\neg\neg D, \Gamma \to \Delta}\,[\neg\text{L}]}{D, \Gamma \to \Delta}\,[\text{Cut}]$$

Implication Transitivity)

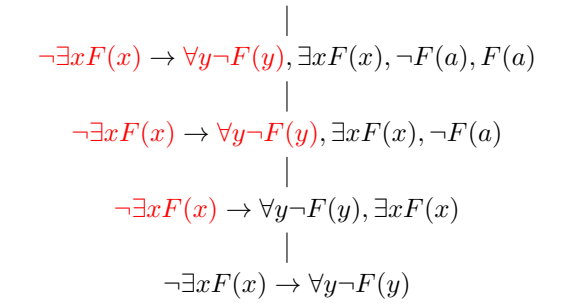
$$\begin{array}{cl}
\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{A, (A \supset B) \to B}\,[\text{detachment}]}{A, (A \supset B) \to B, C}\,[\text{WR}]}{A, (A \supset B) \to C, B}\,[\text{XR}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{C \to C}{(A \supset B), C \to C}\,[\text{WL}]}{C, (A \supset B) \to C}\,[\text{XL}]}{A, C, (A \supset B) \to C}\,[\text{WL}]}{C, A, (A \supset B) \to C}\,[\text{XL}]}{(B \supset C), A, (A \supset B), A, (A \supset B) \to C, C}\,[\supset\text{L}] & \\ \hline
(B \supset C), A, (A \supset B), A, (A \supset B) \to C & [\text{CR}] \\ \hline
A, (B \supset C), (A \supset B), A, (A \supset B) \to C & [\text{XL}] \\ \hline
A, (B \supset C), A, (A \supset B), (A \supset B) \to C & [\text{XL}] \\ \hline
A, A, (B \supset C), (A \supset B), (A \supset B) \to C & [\text{XL}] \\ \hline
A, (B \supset C), (A \supset B), (A \supset B) \to C & [\text{CL}] \\ \hline
A, (A \supset B), (B \supset C), (A \supset B) \to C & [\text{XL}] \\ \hline
(A \supset B), A, (B \supset C), (A \supset B) \to C & [\text{XL}] \\ \hline
(A \supset B), A, (A \supset B), (B \supset C) \to C & [\text{XL}] \\ \hline
(A \supset B), (A \supset B), A, (B \supset C) \to C & [\text{XL}] \\ \hline
(A \supset B), A, (B \supset C) \to C & [\text{CL}] \\ \hline
A, (A \supset B), (B \supset C) \to C & [\text{XL}] \\ \hline
(A \supset B), (B \supset C) \to (A \supset C) & [\supset\text{R}]
\end{array}$$

Reduction Trees)

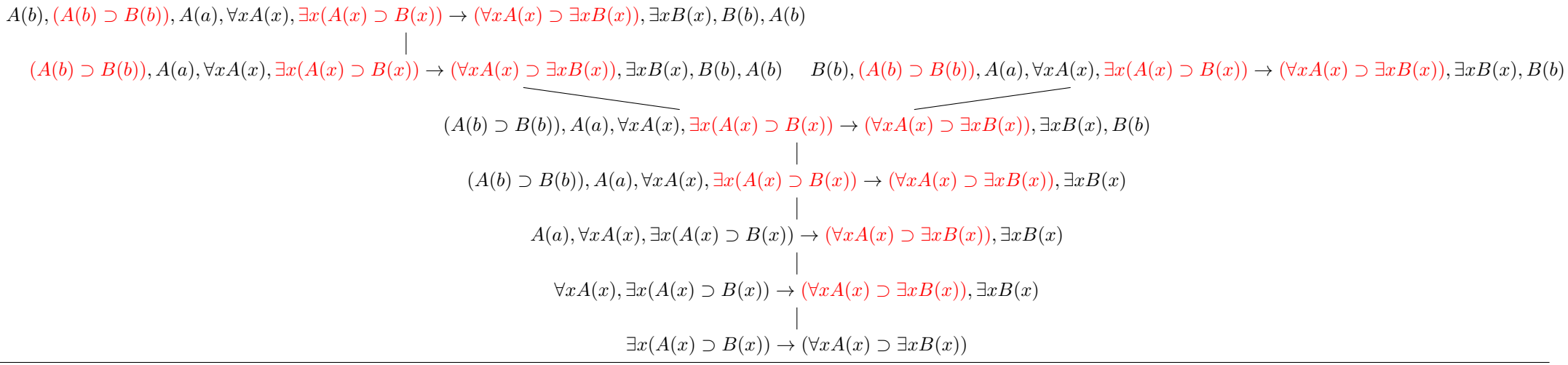



$A, (A \supset B) \to (\neg A \vee B), \neg A, B, A$
$(A \supset B) \to (\neg A \vee B), \neg A, B, A$
$B, (A \supset B) \to (\neg A \vee B), \neg A, B$
$(A \supset B) \to (\neg A \vee B), \neg A, B$
$(A \supset B) \to (\neg A \vee B)$

8.3.1)


$F(a), \neg\exists x F(x) \to \forall y \neg F(y), \exists x F(x), \neg F(a), F(a)$
$\neg\exists x F(x) \to \forall y \neg F(y), \exists x F(x), \neg F(a), F(a)$
$\neg\exists x F(x) \to \forall y \neg F(y), \exists x F(x), \neg F(a)$
$\neg\exists x F(x) \to \forall y \neg F(y), \exists x F(x)$
$\neg\exists x F(x) \to \forall y \neg F(y)$

8.3.2)


$A(b), (A(b) \supset B(b)), A(a), \forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b), A(b)$
$(A(b) \supset B(b)), A(a), \forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b), A(b)$
$B(b), (A(b) \supset B(b)), A(a), \forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b)$
$(A(b) \supset B(b)), A(a), \forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b)$
$(A(b) \supset B(b)), A(a), \forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x)$
$A(a), \forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x)$
$\forall x A(x), \exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x)), \exists x B(x)$
$\exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x))$

PA1)

$$\dfrac{\dfrac{\to 0'' + 0 = 0'' \qquad 0'' + 0 = 0'' \to (0'' + 0)' = 0'''}{\to (0'' + 0)' = 0'''}\,[\text{Cut}] \qquad \dfrac{\to 0'' + 0' = (0'' + 0)' \qquad 0'' + 0' = (0'' + 0)', (0'' + 0)' = 0''' \to 0'' + 0' = 0'''}{(0'' + 0)' = 0''' \to 0'' + 0' = 0'''}\,[\text{Cut}]}{\to 0'' + 0' = 0'''}\,[\text{Cut}]$$