

2.5.1)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{(\neg A \wedge \neg B), A \rightarrow}{A \rightarrow \neg(\neg A \wedge \neg B)} [\neg R]}{A \rightarrow \neg(\neg A \wedge \neg B)} [\wedge L] \quad \frac{\frac{B \rightarrow B}{\neg B, B \rightarrow} [\neg L] \quad \frac{(\neg A \wedge \neg B), B \rightarrow}{B \rightarrow \neg(\neg A \wedge \neg B)} [\neg R]}{B \rightarrow \neg(\neg A \wedge \neg B)} [\wedge L] \\
\frac{(A \vee B) \rightarrow \neg(\neg A \wedge \neg B)}{\rightarrow ((A \vee B) \supset \neg(\neg A \wedge \neg B))} [\supset R] \quad \frac{\frac{A \rightarrow A}{A \rightarrow (A \vee B)} [\vee R] \quad \frac{B \rightarrow B}{B \rightarrow (A \vee B)} [\vee R]}{\rightarrow (A \vee B), \neg A} [\neg R] \quad \frac{\rightarrow (A \vee B), \neg B}{\rightarrow (A \vee B), \neg B} [\neg R] \\
\frac{\rightarrow (A \vee B), (\neg A \wedge \neg B)}{\neg(\neg A \wedge \neg B) \rightarrow (A \vee B)} [\neg L] \quad \frac{\neg(\neg A \wedge \neg B) \rightarrow (A \vee B)}{\rightarrow (\neg(\neg A \wedge \neg B) \supset (A \vee B))} [\supset R] \\
\frac{\rightarrow ((A \vee B) \supset \neg(\neg A \wedge \neg B))}{\rightarrow ((A \vee B) \equiv \neg(\neg A \wedge \neg B))} [\wedge R]
\end{array}$$

2.5.2)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R] \quad \frac{\rightarrow A, (\neg A \vee B)}{\rightarrow (\neg A \vee B), A} [\vee R]}{\rightarrow A, (\neg A \vee B)} [\wedge R] \quad \frac{B \rightarrow B}{B \rightarrow (\neg A \vee B)} [\vee R] \\
\frac{(A \supset B) \rightarrow (\neg A \vee B), (\neg A \vee B)}{(A \supset B) \rightarrow (\neg A \vee B)} [\wedge R] \quad \frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{\neg A, A \rightarrow B}{A, \neg A \rightarrow B} [\wedge R] \quad \frac{B \rightarrow B}{A, B \rightarrow B} [\wedge R] \\
\frac{(A \supset B) \rightarrow (\neg A \vee B)}{\rightarrow ((A \supset B) \supset (\neg A \vee B))} [\supset R] \quad \frac{\neg A \rightarrow (A \supset B)}{B \rightarrow (A \supset B)} [\supset R] \quad \frac{(\neg A \vee B) \rightarrow (A \supset B)}{\rightarrow ((\neg A \vee B) \supset (A \supset B))} [\supset R] \\
\frac{\rightarrow ((A \supset B) \supset (\neg A \vee B))}{\rightarrow ((A \supset B) \equiv (\neg A \vee B))} [\wedge R]
\end{array}$$

2.5.3)

$$\begin{array}{c}
\frac{F(a) \rightarrow F(a)}{\neg F(a), F(a) \rightarrow} [\neg L] \quad \frac{\neg F(a), F(a) \rightarrow}{\forall y \neg F(y), F(a) \rightarrow} [\forall L] \quad \frac{F(a) \rightarrow \neg \forall y \neg F(y)}{\exists x F(x) \rightarrow \neg \forall y \neg F(y)} [\exists L] \\
\frac{\exists x F(x) \rightarrow \neg \forall y \neg F(y)}{\rightarrow (\exists x F(x) \supset \neg \forall y \neg F(y))} [\supset R] \quad \frac{F(a) \rightarrow F(a)}{\rightarrow F(a), \neg F(a)} [\neg R] \quad \frac{\rightarrow F(a), \neg F(a)}{\rightarrow \neg F(a), F(a)} [\wedge R] \\
\frac{\rightarrow \neg F(a), F(a)}{\rightarrow \neg F(a), \exists x F(x)} [\exists R] \quad \frac{\rightarrow \neg F(a), \exists x F(x)}{\rightarrow \exists x F(x), \neg F(a)} [\wedge R] \quad \frac{\rightarrow \exists x F(x), \neg F(a)}{\rightarrow \exists x F(x), \forall y \neg F(y)} [\forall R] \\
\frac{\rightarrow \exists x F(x), \forall y \neg F(y)}{\neg \forall y \neg F(y) \rightarrow \exists x F(x)} [\neg L] \quad \frac{\neg \forall y \neg F(y) \rightarrow \exists x F(x)}{\rightarrow (\neg \forall y \neg F(y) \supset \exists x F(x))} [\supset R] \\
\frac{\rightarrow (\exists x F(x) \supset \neg \forall y \neg F(y))}{\rightarrow (\exists x F(x) \equiv \neg \forall y \neg F(y))} [\wedge R]
\end{array}$$

2.5.4)

$$\begin{array}{c}
\frac{F(a) \rightarrow F(a)}{\rightarrow F(a), \neg F(a)} [\neg R] \quad \frac{\rightarrow F(a), \neg F(a)}{\rightarrow F(a), \exists x \neg F(x)} [\exists R] \quad \frac{\rightarrow F(a), \exists x \neg F(x)}{\rightarrow \exists x \neg F(x), F(a)} [\wedge R] \quad \frac{\rightarrow \exists x \neg F(x), F(a)}{\rightarrow \exists x \neg F(x), \forall y F(y)} [\forall R] \\
\frac{\rightarrow \exists x \neg F(x), \forall y F(y)}{\neg \forall y F(y) \rightarrow \exists x \neg F(x)} [\neg L] \quad \frac{\neg \forall y F(y) \rightarrow \exists x \neg F(x)}{\rightarrow (\neg \forall y F(y) \supset \exists x \neg F(x))} [\supset R] \quad \frac{F(a) \rightarrow F(a)}{\forall y F(y) \rightarrow F(a)} [\forall L] \quad \frac{\forall y F(y) \rightarrow F(a)}{\neg F(a), \forall y F(y) \rightarrow} [\neg L] \\
\frac{\neg F(a), \forall y F(y) \rightarrow}{\forall y F(y), \neg F(a) \rightarrow} [\wedge R] \quad \frac{\forall y F(y), \neg F(a) \rightarrow}{\neg F(a) \rightarrow \neg \forall y F(y)} [\neg R] \quad \frac{\neg F(a) \rightarrow \neg \forall y F(y)}{\exists x \neg F(x) \rightarrow \neg \forall y F(y)} [\exists L] \\
\frac{\exists x \neg F(x) \rightarrow \neg \forall y F(y)}{\rightarrow (\exists x \neg F(x) \supset \neg \forall y F(y))} [\supset R] \quad \frac{\rightarrow (\exists x \neg F(x) \supset \neg \forall y F(y))}{\rightarrow (\neg \forall y F(y) \equiv \exists x \neg F(x))} [\wedge R]
\end{array}$$

2.5.5)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R] \quad \frac{B \rightarrow B}{\rightarrow B, \neg B} [\neg R]}{\frac{\rightarrow A, (\neg A \vee \neg B)}{\rightarrow (\neg A \vee \neg B), A} [\vee R] \quad \frac{\rightarrow B, (\neg A \vee \neg B)}{\rightarrow (\neg A \vee \neg B), B} [\vee R]} [\vee R] \\
\frac{\rightarrow (\neg A \vee \neg B), A}{\rightarrow (\neg A \vee \neg B), (A \wedge B)} [\wedge R] \quad \frac{\rightarrow (\neg A \vee \neg B), B}{\rightarrow (\neg A \vee \neg B), (A \wedge B)} [\wedge R] \\
\frac{\rightarrow (\neg A \vee \neg B), (A \wedge B)}{\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)} [\neg L] \\
\frac{\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)}{\rightarrow (\neg(A \wedge B) \supset (\neg A \vee \neg B))} [\supset R] \\
\frac{\rightarrow (\neg(A \wedge B) \supset (\neg A \vee \neg B))}{\rightarrow (\neg(A \wedge B) \equiv (\neg A \vee \neg B))} [\wedge R]
\end{array}$$

2.6.1)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{A \rightarrow A, B(a)} [\text{WR}] \quad \frac{A \rightarrow A, B(a)}{A \rightarrow B(a), A} [\text{XR}]}{\frac{A \rightarrow B(a), A}{B(a) \rightarrow B(a)} [\supset L]} \\
\frac{B(a) \rightarrow B(a)}{(A \supset B(a)), A \rightarrow B(a), B(a)} [\supset L] \\
\frac{(A \supset B(a)), A \rightarrow B(a)}{(A \supset B(a)), A \rightarrow \exists x B(x)} [\text{CR}] \\
\frac{(A \supset B(a)), A \rightarrow \exists x B(x)}{A, (A \supset B(a)) \rightarrow \exists x B(x)} [\exists R] \\
\frac{A, (A \supset B(a)) \rightarrow \exists x B(x)}{(A \supset B(a)) \rightarrow (A \supset \exists x B(x))} [\text{XL}] \\
\frac{(A \supset B(a)) \rightarrow (A \supset \exists x B(x))}{\exists x(A \supset B(x)) \rightarrow (A \supset \exists x B(x))} [\supset R] \\
\frac{\exists x(A \supset B(x)) \rightarrow (A \supset \exists x B(x))}{\rightarrow (\exists x(A \supset B(x)) \supset (A \supset \exists x B(x)))} [\exists L] \\
\frac{\rightarrow (\exists x(A \supset B(x)) \supset (A \supset \exists x B(x)))}{\rightarrow (\exists x(A \supset B(x)) \equiv (A \supset \exists x B(x)))} [\supset R]
\end{array}$$

2.6.2)

$$\begin{array}{c}
\frac{\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B} [\text{WR}] \quad \frac{A(a) \rightarrow A(a), B}{A(a) \rightarrow B, A(a)} [\text{XR}]}{\frac{A(a) \rightarrow B, A(a)}{B \rightarrow B} [\supset L]} \\
\frac{B \rightarrow B}{(A(a) \supset B), A(a) \rightarrow B, B} [\supset L] \\
\frac{(A(a) \supset B), A(a) \rightarrow B, B}{(A(a) \supset B), A(a) \rightarrow B} [\text{CR}] \\
\frac{(A(a) \supset B), A(a) \rightarrow B}{A(a), (A(a) \supset B) \rightarrow B} [\text{XL}] \\
\frac{A(a), (A(a) \supset B) \rightarrow B}{\forall x A(x), (A(a) \supset B) \rightarrow B} [\forall L] \\
\frac{\forall x A(x), (A(a) \supset B) \rightarrow B}{(A(a) \supset B) \rightarrow (\forall x A(x) \supset B)} [\supset R] \\
\frac{(A(a) \supset B) \rightarrow (\forall x A(x) \supset B)}{\exists x(A(x) \supset B) \rightarrow (\forall x A(x) \supset B)} [\exists L] \\
\frac{\exists x(A(x) \supset B) \rightarrow (\forall x A(x) \supset B)}{\rightarrow (\exists x(A(x) \supset B) \supset (\forall x A(x) \supset B))} [\supset R] \\
\frac{\rightarrow (\exists x(A(x) \supset B) \supset (\forall x A(x) \supset B))}{\rightarrow (\exists x(A(x) \supset B) \equiv (\forall x A(x) \supset B))} [\wedge R]
\end{array}$$

2.6.3)

$$\begin{array}{c}
\frac{\frac{A(a) \rightarrow A(a) \quad B(a) \rightarrow B(a)}{A(a), (A(a) \supset B(a)) \rightarrow B(a)} [\text{detachment}]}{\frac{\forall x A(x), (A(a) \supset B(a)) \rightarrow B(a)}{\forall x A(x), (A(a) \supset B(a)) \rightarrow \exists x B(x)} [\forall L]} \\
\frac{\frac{\forall x A(x), (A(a) \supset B(a)) \rightarrow \exists x B(x)}{(A(a) \supset B(a)) \rightarrow (\forall x A(x) \supset \exists x B(x))} [\exists R]}{\frac{\exists x(A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x))}{\rightarrow (\exists x(A(x) \supset B(x)) \supset (\forall x A(x) \supset \exists x B(x)))} [\supset R]} \\
\frac{\frac{\forall x A(x) \rightarrow \forall x A(x) \quad \exists x B(x) \rightarrow \exists x B(x)}{(\forall x A(x) \supset \exists x B(x)) \rightarrow (\neg \forall x A(x) \vee \exists x B(x))} [2.5.2.L]}{\frac{(\neg \forall x A(x) \vee \exists x B(x)) \rightarrow \exists x(A(x) \supset B(x))}{\rightarrow ((\forall x A(x) \supset \exists x B(x)) \supset \exists x(A(x) \supset B(x)))} [\text{Cut}]} \\
\frac{\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B(a)} [\text{WR}]}{\frac{\rightarrow A(a), (A(a) \supset B(a))}{\rightarrow A(a), \exists x(A(x) \supset B(x))} [\supset R]} \\
\frac{\frac{\rightarrow A(a), \exists x(A(x) \supset B(x))}{\rightarrow \exists x(A(x) \supset B(x)), A(a)} [\exists R]}{\frac{\rightarrow \exists x(A(x) \supset B(x)), \forall x A(x)}{\neg \forall x A(x) \rightarrow \exists x(A(x) \supset B(x))} [\forall R]} \\
\frac{\frac{B(a) \rightarrow B(a)}{A(a), B(a) \rightarrow B(a)} [\text{WL}]}{\frac{A(a), B(a) \rightarrow B(a)}{B(a) \rightarrow (A(a) \supset B(a))} [\supset R]} \\
\frac{\frac{B(a) \rightarrow (A(a) \supset B(a))}{B(a) \rightarrow \exists x(A(x) \supset B(x))} [\exists R]}{\frac{B(a) \rightarrow \exists x(A(x) \supset B(x))}{\exists x B(x) \rightarrow \exists x(A(x) \supset B(x))} [\exists L]} \\
\frac{\frac{\rightarrow \exists x(A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x))}{\rightarrow (\exists x(A(x) \supset B(x)) \supset (\forall x A(x) \supset \exists x B(x)))} [\supset R]}{\frac{\rightarrow ((\forall x A(x) \supset \exists x B(x)) \supset \exists x(A(x) \supset B(x)))}{\rightarrow (\exists x(A(x) \supset B(x)) \equiv (\forall x A(x) \supset \exists x B(x)))} [\wedge R]}
\end{array}$$

2.6.4)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\neg B, A \rightarrow A} [\text{WL}]}{A \rightarrow (\neg B \supset A)} [\supset R]} \\
\frac{A \rightarrow (\neg B \supset A)}{\rightarrow (\neg B \supset A), \neg A} [\neg R]} \\
\frac{\rightarrow (\neg B \supset A), \neg A}{\neg \neg A \rightarrow (\neg B \supset A)} [\neg L]} \\
\frac{\neg A \rightarrow \neg A \quad B \rightarrow B}{(\neg A \supset B) \rightarrow (\neg \neg A \vee B)} [2.5.2.L]} \\
\frac{\neg \neg A \rightarrow (\neg B \supset A)}{(\neg \neg A \vee B) \rightarrow (\neg B \supset A)} [\text{Cut}] \\
\frac{B \rightarrow B}{B \rightarrow B, A} [\text{WR}] \\
\frac{B \rightarrow B, A}{B \rightarrow A, B} [\text{XR}] \\
\frac{B \rightarrow A, B}{\neg B, B \rightarrow A} [\neg L]} \\
\frac{\neg B, B \rightarrow A}{B \rightarrow (\neg B \supset A)} [\supset R]} \\
\frac{B \rightarrow (\neg B \supset A)}{B \rightarrow (\neg B \supset A)} [\vee L]}
\end{array}$$

2.6.5)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{B, A \rightarrow A} [\text{WL}]}{A \rightarrow (B \supset A)} [\supset R]} \\
\frac{A \rightarrow (B \supset A)}{\rightarrow (B \supset A), \neg A} [\neg R]} \\
\frac{\rightarrow (B \supset A), \neg A}{\neg \neg A \rightarrow (B \supset A)} [\neg L]} \\
\frac{\neg A \rightarrow \neg A \quad \neg B \rightarrow \neg B}{(\neg A \supset \neg B) \rightarrow (\neg \neg A \vee \neg B)} [2.5.2.L]} \\
\frac{\neg \neg A \rightarrow (B \supset A)}{(\neg \neg A \vee \neg B) \rightarrow (B \supset A)} [\text{Cut}] \\
\frac{B \rightarrow B}{B \rightarrow B, A} [\text{WR}] \\
\frac{B \rightarrow B, A}{B \rightarrow A, B} [\text{XR}] \\
\frac{B \rightarrow A, B}{\neg B, B \rightarrow A} [\neg L]} \\
\frac{\neg B, B \rightarrow A}{B, \neg B \rightarrow A} [\text{XL}] \\
\frac{B, \neg B \rightarrow A}{\neg B \rightarrow (B \supset A)} [\supset R]} \\
\frac{\neg B \rightarrow (B \supset A)}{\neg B \rightarrow (B \supset A)} [\vee L]}
\end{array}$$

2.7)

$$\begin{array}{c}
\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B} [\text{WR}] \\
\frac{A(a) \rightarrow A(a), B}{\rightarrow A(a), (A(a) \supset B)} [\supset R] \\
\frac{\rightarrow A(a), (A(a) \supset B)}{\rightarrow A(a), \exists x(A(x) \supset B)} [\exists R] \\
\frac{\rightarrow A(a), \exists x(A(x) \supset B)}{\rightarrow \exists x(A(x) \supset B), A(a)} [\text{XR}] \\
\frac{\rightarrow \exists x(A(x) \supset B), \forall x A(x)}{(\forall x A(x) \supset B) \rightarrow \exists x(A(x) \supset B), \exists x(A(x) \supset B)} [\forall R] \\
\frac{(\forall x A(x) \supset B) \rightarrow \exists x(A(x) \supset B), \exists x(A(x) \supset B)}{(\forall x A(x) \supset B) \rightarrow \exists x(A(x) \supset B)} [\text{CR}] \\
\frac{B \rightarrow B}{A(a), B \rightarrow B} [\text{WL}] \\
\frac{A(a), B \rightarrow B}{B \rightarrow (A(a) \supset B)} [\supset R] \\
\frac{B \rightarrow (A(a) \supset B)}{B \rightarrow \exists x(A(x) \supset B)} [\exists R] \\
\frac{B \rightarrow \exists x(A(x) \supset B)}{B \rightarrow \exists x(A(x) \supset B)} [\supset L]
\end{array}$$

3.9.1)

$$\frac{\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{\neg A, A \rightarrow B}{A, \neg A \rightarrow B} [XL] \quad \frac{B \rightarrow B}{A, B \rightarrow B} [WL]}{\neg A \rightarrow (A \supset B)} [\supset R] \quad \frac{B \rightarrow (A \supset B)}{(\neg A \vee B) \rightarrow (A \supset B)} [\vee L]$$

3.9.2)

$$\frac{\frac{\frac{F(a) \rightarrow F(a)}{\neg F(a), F(a) \rightarrow} [\neg L] \quad \frac{\neg F(a), F(a) \rightarrow}{\forall y \neg F(y), F(a) \rightarrow} [\forall L] \quad \frac{\forall y \neg F(y), F(a) \rightarrow}{F(a) \rightarrow \neg \forall y \neg F(y)} [\neg R] \quad \frac{F(a) \rightarrow \neg \forall y \neg F(y)}{\exists x F(x) \rightarrow \neg \forall y \neg F(y)} [\exists L]$$

3.9.3)

$$\frac{A \rightarrow A}{(A \wedge B) \rightarrow A} [\wedge L]$$

3.9.4)

$$\frac{A \rightarrow A}{A \rightarrow (A \vee B)} [\vee R]$$

3.9.5)

$$\frac{\frac{\frac{A \rightarrow A}{(A \wedge B) \rightarrow A} [\wedge L] \quad \frac{\neg A, (A \wedge B) \rightarrow}{(A \wedge B), \neg A \rightarrow} [XL] \quad \frac{(A \wedge B), \neg A \rightarrow}{\neg A \rightarrow \neg(A \wedge B)} [\neg R]}{\neg A \rightarrow \neg(A \wedge B)} [\neg R] \quad \frac{\frac{B \rightarrow B}{(A \wedge B) \rightarrow B} [\wedge L] \quad \frac{\neg B, (A \wedge B) \rightarrow}{(A \wedge B), \neg B \rightarrow} [XL] \quad \frac{(A \wedge B), \neg B \rightarrow}{\neg B \rightarrow \neg(A \wedge B)} [\neg R]}{(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)} [\vee L]$$

3.9.6)

$$\frac{\frac{\frac{A \rightarrow A}{A \rightarrow (A \vee B)} [\vee R] \quad \frac{\neg(A \vee B), A \rightarrow}{A, \neg(A \vee B) \rightarrow} [XL] \quad \frac{A, \neg(A \vee B) \rightarrow}{\neg(A \vee B) \rightarrow \neg A} [\neg R]}{\neg(A \vee B) \rightarrow \neg(A \wedge \neg B)} [\wedge R] \quad \frac{\frac{B \rightarrow B}{B \rightarrow (A \vee B)} [\vee R] \quad \frac{\neg(A \vee B), B \rightarrow}{B, \neg(A \vee B) \rightarrow} [XL] \quad \frac{B, \neg(A \vee B) \rightarrow}{\neg(A \vee B) \rightarrow \neg B} [\neg R]}{\neg(A \vee B) \rightarrow \neg(A \wedge \neg B)} [\wedge R] \quad \frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{\neg A, A \rightarrow}{(\neg A \wedge \neg B), A \rightarrow} [\wedge L] \quad \frac{(\neg A \wedge \neg B), A \rightarrow}{A, (\neg A \wedge \neg B) \rightarrow} [XL] \quad \frac{\frac{B \rightarrow B}{\neg B, B \rightarrow} [\neg L] \quad \frac{\neg B, B \rightarrow}{(\neg A \wedge \neg B), B \rightarrow} [\wedge L] \quad \frac{(\neg A \wedge \neg B), B \rightarrow}{B, (\neg A \wedge \neg B) \rightarrow} [XL]}{(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)} [\neg R] \quad \frac{\neg(A \vee B) \rightarrow \neg(A \wedge \neg B)}{\rightarrow ((\neg A \wedge \neg B) \supset \neg(A \vee B))} [\supset R] \quad \frac{\rightarrow ((\neg A \wedge \neg B) \supset \neg(A \vee B))}{\rightarrow (\neg(A \vee B) \equiv (\neg A \wedge \neg B))} [\wedge R]$$

3.9.7)

$$\begin{array}{c}
\frac{A \rightarrow A}{B, A \rightarrow A} [\text{WL}] \quad \frac{B \rightarrow B}{A, B \rightarrow B} [\text{WL}] \quad \frac{C \rightarrow C}{B, C \rightarrow C} [\text{WL}] \\
\frac{B, A \rightarrow A}{A, B \rightarrow A} [\text{XL}] \quad \frac{A, B \rightarrow B}{A, B \rightarrow B} [\text{XR}] \quad \frac{C, B \rightarrow C}{C, B \rightarrow C} [\text{XL}] \\
\frac{A, B \rightarrow (A \wedge B)}{A, B \rightarrow ((A \wedge B) \vee C)} [\vee\text{R}] \quad \frac{C, B \rightarrow C}{C, B \rightarrow ((A \wedge B) \vee C)} [\vee\text{R}] \quad \frac{C \rightarrow C}{C \rightarrow ((A \wedge B) \vee C)} [\vee\text{R}] \\
\frac{(A \vee C), B \rightarrow ((A \wedge B) \vee C)}{B, (A \vee C) \rightarrow ((A \wedge B) \vee C)} [\text{XL}] \quad \frac{(A \vee C), C \rightarrow ((A \wedge B) \vee C)}{C, (A \vee C) \rightarrow ((A \wedge B) \vee C)} [\text{XL}] \\
\frac{(B \vee C), (A \vee C) \rightarrow ((A \wedge B) \vee C)}{(A \vee C), (B \vee C) \rightarrow ((A \wedge B) \vee C)} [\text{XL}] \\
\frac{((A \vee C) \wedge (B \vee C)), (B \vee C) \rightarrow ((A \wedge B) \vee C)}{(B \vee C), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)} [\wedge\text{L}] \\
\frac{(B \vee C), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}{((A \vee C) \wedge (B \vee C)), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)} [\text{XL}] \\
\frac{((A \vee C) \wedge (B \vee C)), ((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}{((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)} [\wedge\text{L}] \\
\frac{((A \vee C) \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)}{\rightarrow (((A \vee C) \wedge (B \vee C)) \supset ((A \wedge B) \vee C))} [\supset\text{R}] \\
\frac{\rightarrow (((A \vee C) \wedge (B \vee C)) \supset ((A \wedge B) \vee C))}{\rightarrow (((A \vee C) \wedge (B \vee C)) \equiv ((A \wedge B) \vee C))} [\text{CL}]
\end{array}$$

3.9.8)

$$\begin{array}{c}
\frac{F(a) \rightarrow F(a)}{\neg F(a), F(a) \rightarrow} [\neg\text{L}] \\
\frac{\neg F(a), F(a) \rightarrow}{F(a), \neg F(a) \rightarrow} [\text{XL}] \\
\frac{F(a), \neg F(a) \rightarrow}{\forall x F(x), \neg F(a) \rightarrow} [\forall\text{L}] \\
\frac{\forall x F(x), \neg F(a) \rightarrow}{\neg F(a) \rightarrow \neg \forall x F(x)} [\neg\text{R}] \\
\frac{\neg F(a) \rightarrow \neg \forall x F(x)}{\exists x \neg F(x) \rightarrow \neg \forall x F(x)} [\exists\text{L}]
\end{array}$$

3.9.9)

$$\begin{array}{c}
\frac{F(a) \rightarrow F(a)}{(F(a) \wedge G(a)) \rightarrow F(a)} [\wedge\text{L}] \quad \frac{G(a) \rightarrow G(a)}{(F(a) \wedge G(a)) \rightarrow G(a)} [\wedge\text{L}] \quad \frac{F(a) \rightarrow F(a)}{\forall x F(x) \rightarrow F(a)} [\forall\text{L}] \quad \frac{G(a) \rightarrow G(a)}{\forall x G(x) \rightarrow G(a)} [\forall\text{L}] \\
\frac{(F(a) \wedge G(a)) \rightarrow F(a)}{\forall x (F(x) \wedge G(x)) \rightarrow F(a)} [\forall\text{L}] \quad \frac{(F(a) \wedge G(a)) \rightarrow G(a)}{\forall x (F(x) \wedge G(x)) \rightarrow G(a)} [\forall\text{L}] \quad \frac{(\forall x F(x) \wedge \forall x G(x)) \rightarrow F(a)}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow F(a)} [\wedge\text{L}] \quad \frac{(\forall x F(x) \wedge \forall x G(x)) \rightarrow G(a)}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow G(a)} [\wedge\text{L}] \\
\frac{\forall x (F(x) \wedge G(x)) \rightarrow \forall x F(x)}{\forall x (F(x) \wedge G(x)) \rightarrow \forall x F(x)} [\forall\text{R}] \quad \frac{\forall x (F(x) \wedge G(x)) \rightarrow \forall x G(x)}{\forall x (F(x) \wedge G(x)) \rightarrow \forall x G(x)} [\forall\text{R}] \quad \frac{(\forall x F(x) \wedge \forall x G(x)) \rightarrow (F(a) \wedge G(a))}{(\forall x F(x) \wedge \forall x G(x)) \rightarrow \forall x (F(x) \wedge G(x))} [\forall\text{R}] \\
\frac{\rightarrow (\forall x (F(x) \wedge G(x)) \supset (\forall x F(x) \wedge \forall x G(x)))}{\rightarrow (\forall x (F(x) \wedge G(x)) \supset (\forall x F(x) \wedge \forall x G(x)))} [\supset\text{R}] \quad \frac{\rightarrow ((\forall x F(x) \wedge \forall x G(x)) \supset \forall x (F(x) \wedge G(x)))}{\rightarrow ((\forall x F(x) \wedge \forall x G(x)) \supset \forall x (F(x) \wedge G(x)))} [\supset\text{R}] \\
\frac{\rightarrow (\forall x (F(x) \wedge G(x)) \equiv (\forall x F(x) \wedge \forall x G(x)))}{\rightarrow (\forall x (F(x) \wedge G(x)) \equiv (\forall x F(x) \wedge \forall x G(x)))} [\wedge\text{R}]
\end{array}$$

3.9.10)

$$\begin{array}{c}
\frac{B \rightarrow B}{\neg B, B \rightarrow} [\neg\text{L}] \\
\frac{A \rightarrow A}{(A \supset \neg B), A, B \rightarrow} [\supset\text{L}] \\
\frac{(A \supset \neg B), A, B \rightarrow}{A, (A \supset \neg B), B \rightarrow} [\text{XL}] \\
\frac{A, (A \supset \neg B), B \rightarrow}{A, B, (A \supset \neg B) \rightarrow} [\text{XL}] \\
\frac{A, B, (A \supset \neg B) \rightarrow}{B, (A \supset \neg B) \rightarrow \neg A} [\neg\text{R}] \\
\frac{B, (A \supset \neg B) \rightarrow \neg A}{(A \supset \neg B) \rightarrow (B \supset \neg A)} [\supset\text{R}]
\end{array}$$

3.9.11)

$$\frac{\frac{\frac{A \rightarrow A \quad B(a) \rightarrow B(a)}{A, (A \supset B(a)) \rightarrow B(a)} [\text{detachment}]}{A, (A \supset B(a)) \rightarrow \exists x B(x)} [\exists R]}{\frac{(A \supset B(a)) \rightarrow (A \supset \exists x B(x))}{\exists x (A \supset B(x)) \rightarrow (A \supset \exists x B(x))} [\supset R]} [\exists L]$$

3.9.12)

$$\frac{\frac{\frac{A(a) \rightarrow A(a) \quad B \rightarrow B}{A(a), (A(a) \supset B) \rightarrow B} [\text{detachment}]}{\forall x A(x), (A(a) \supset B) \rightarrow B} [\forall L]}{\frac{(A(a) \supset B) \rightarrow (\forall x A(x) \supset B)}{\exists x (A(x) \supset B) \rightarrow (\forall x A(x) \supset B)} [\supset R]} [\exists L]$$

3.9.13)

$$\frac{\frac{\frac{\frac{A(a) \rightarrow A(a) \quad B(a) \rightarrow B(a)}{A(a), (A(a) \supset B(a)) \rightarrow B(a)} [\text{detachment}]}{A(a), (A(a) \supset B(a)) \rightarrow \exists x B(x)} [\exists R]}{\forall x A(x), (A(a) \supset B(a)) \rightarrow \exists x B(x)} [\forall L]}{\frac{(A(a) \supset B(a)) \rightarrow (\forall x A(x) \supset \exists x B(x))}{\exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x))} [\supset R]} [\exists L]$$

3.10.1)

$$\frac{\frac{\frac{A \rightarrow A \quad B \rightarrow B}{A, (A \supset B) \rightarrow B} [\text{detachment}]}{(A \supset B), A \rightarrow B} [\text{XL}]}{\frac{\neg B, (A \supset B), A \rightarrow B}{(A \supset B), \neg B, A \rightarrow B} [\neg L]} [\text{XL}]}{\frac{\neg B, A \rightarrow \neg(A \supset B)}{\neg \neg(A \supset B), \neg B, A \rightarrow B} [\neg R]} [\neg L]} [\text{XL}]}{\frac{\neg B, \neg \neg(A \supset B), A \rightarrow B}{\neg \neg(A \supset B), A \rightarrow \neg \neg B} [\neg R]} [\neg R]$$

3.10.2)

$$\frac{\frac{\frac{A \rightarrow A \quad B \rightarrow B}{\neg \neg(A \supset B), A \rightarrow \neg \neg B} [3.10.1]}{(\neg \neg B \supset B), \neg \neg(A \supset B), A \rightarrow B} [\supset L]}{\frac{(\neg \neg B \supset B), A, \neg \neg(A \supset B) \rightarrow B}{A, (\neg \neg B \supset B), \neg \neg(A \supset B) \rightarrow B} [\text{XL}]} [\text{XL}]}{\frac{(\neg \neg B \supset B), \neg \neg(A \supset B) \rightarrow (A \supset B)}{(\neg \neg B \supset B), \neg \neg(A \supset B) \rightarrow (A \supset B)} [\supset R]}$$

3.10.3)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L]}{\frac{A \rightarrow \neg\neg A}{\neg\neg A, A \rightarrow} [\neg R]} \\
\frac{}{\frac{A, \neg\neg A \rightarrow}[\text{XL}]} \\
\frac{}{\frac{\neg\neg A \rightarrow \neg A}{\neg\neg A, \neg A \rightarrow} [\neg L]} \\
\frac{}{\frac{\neg A \rightarrow \neg\neg A}{\neg A \rightarrow \neg\neg A} [\neg R]} \\
\frac{}{\rightarrow (\neg\neg A \supset \neg A)} [\sup R]
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L]}{\frac{A, \neg A \rightarrow}[\text{XL}]} \\
\frac{}{\frac{\neg A \rightarrow \neg A}{\neg\neg A, \neg A \rightarrow} [\neg L]} \\
\frac{}{\frac{\neg A \rightarrow \neg\neg A}{\neg A \rightarrow \neg\neg A} [\neg R]} \\
\frac{}{\rightarrow (\neg A \supset \neg\neg A)} [\sup R]
\end{array}$$

3.11. \wedge)

$$\begin{array}{c}
\frac{B \rightarrow B}{(B \wedge C) \rightarrow B} [\wedge L] \\
\frac{}{\neg B, (B \wedge C) \rightarrow} [\neg L] \\
\frac{}{(B \wedge C), \neg B \rightarrow} [XL] \\
\frac{}{\neg B \rightarrow \neg(B \wedge C)} [\neg R] \\
\frac{}{\neg\neg(B \wedge C), \neg B \rightarrow} [\neg L] \\
\frac{}{\neg B, \neg\neg(B \wedge C) \rightarrow} [XL] \\
\frac{}{\neg\neg(B \wedge C) \rightarrow \neg\neg B} [\neg R] \\
\frac{}{\neg\neg(B \wedge C) \rightarrow B} [\neg\neg B \rightarrow B] \\
\hline
\neg\neg(B \wedge C) \rightarrow B
\end{array}
\quad
\begin{array}{c}
\frac{C \rightarrow C}{(B \wedge C) \rightarrow C} [\wedge L] \\
\frac{}{\neg C, (B \wedge C) \rightarrow} [\neg L] \\
\frac{}{(B \wedge C), \neg C \rightarrow} [XL] \\
\frac{}{\neg C \rightarrow \neg(B \wedge C)} [\neg R] \\
\frac{}{\neg\neg(B \wedge C), \neg C \rightarrow} [\neg L] \\
\frac{}{\neg C, \neg\neg(B \wedge C) \rightarrow} [XL] \\
\frac{}{\neg\neg(B \wedge C) \rightarrow \neg\neg C} [\neg R] \\
\frac{}{\neg\neg(B \wedge C) \rightarrow \neg\neg C} [\neg\neg C \rightarrow C] \\
\hline
\neg\neg(B \wedge C) \rightarrow \neg\neg C
\end{array}
\quad
\begin{array}{c}
\frac{}{\neg\neg(B \wedge C) \rightarrow (B \wedge C)} [\wedge R]
\end{array}$$

3.11.⊂)

$$\frac{\frac{[\neg\neg C \rightarrow C]}{\rightarrow (\neg\neg C \supset C)} [\supset R] \quad \frac{B \rightarrow B \quad C \rightarrow C}{(\neg\neg C \supset C), \neg\neg(B \supset C) \rightarrow (B \supset C)} [3.10.2]}{\neg\neg(B \supset C) \rightarrow (B \supset C)} [\text{Cut}]$$

3.11. \forall)

$$\frac{\frac{\frac{B(a) \rightarrow B(a)}{\forall x B(x) \rightarrow B(a)} [\forall L]}{\neg B(a), \forall x B(x) \rightarrow} [\neg L]}{\forall x B(x), \neg B(a) \rightarrow} [XL]}{\neg B(a) \rightarrow \neg \forall x B(x)} [\neg R]}{\neg \neg \forall x B(x), \neg B(a) \rightarrow} [\neg L]}{\neg B(a), \neg \neg \forall x B(x) \rightarrow} [XL]}{\neg \neg \forall x B(x) \rightarrow \neg \neg B(a)} [\neg R]}{\frac{\neg \neg \forall x B(x) \rightarrow B(a)}{\neg \neg \forall x B(x) \rightarrow \forall x B(x)} [\forall R]} [\neg \neg B(a) \rightarrow B(a)]} [\text{Cut}]$$

3.12.1.atomic)

$$\frac{\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R]}{\frac{A \rightarrow \neg \neg A}{\rightarrow (A \supset \neg \neg A)} [\supset R]} \quad \frac{\frac{\frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R] \quad \frac{A \rightarrow A}{\neg \neg A \rightarrow A} [\neg L]}{\frac{\neg \neg A \rightarrow A}{\rightarrow (\neg \neg A \supset A)} [\supset R]} [\wedge R]}{\rightarrow (A \equiv \neg \neg A)} [\wedge R]$$

3.12.1.¬)

$$\frac{\frac{\frac{[B^* \rightarrow B]}{\neg B, B^* \rightarrow} [\neg L] \quad \frac{B^* \rightarrow B}{B^*, \neg B \rightarrow} [\neg R]}{\frac{\neg B \rightarrow \neg B^*}{\rightarrow (\neg B \supset \neg B^*)} [\supset R]} \quad \frac{\frac{\frac{[B \rightarrow B^*]}{\neg B^*, B \rightarrow} [\neg L] \quad \frac{B \rightarrow B^*}{B, \neg B^* \rightarrow} [\neg R]}{\frac{\neg B^* \rightarrow \neg B}{\rightarrow (\neg B^* \supset \neg B)} [\supset R]} [\wedge R]}{\rightarrow (\neg B \equiv \neg B^*)} [\wedge R]$$

3.12.1.∧)

$$\frac{\frac{\frac{[B \rightarrow B^*]}{(B \wedge C) \rightarrow B^*} [\wedge L] \quad \frac{[C \rightarrow C^*]}{(B \wedge C) \rightarrow C^*} [\wedge L]}{\frac{(B \wedge C) \rightarrow (B^* \wedge C^*)}{\rightarrow ((B \wedge C) \supset (B^* \wedge C^*))} [\supset R]} \quad \frac{\frac{\frac{[B^* \rightarrow B]}{(B^* \wedge C^*) \rightarrow B} [\wedge L] \quad \frac{[C^* \rightarrow C]}{(B^* \wedge C^*) \rightarrow C} [\wedge L]}{\frac{(B^* \wedge C^*) \rightarrow (B \wedge C)}{\rightarrow ((B^* \wedge C^*) \supset (B \wedge C))} [\supset R]} [\wedge R]}{\rightarrow ((B \wedge C) \equiv (B^* \wedge C^*))} [\wedge R]$$

3.12.1.∨)

$$\frac{\frac{\frac{\frac{[B \rightarrow B^*]}{\neg B^*, B \rightarrow} [\neg L] \quad \frac{[C \rightarrow C^*]}{\neg C^*, C \rightarrow} [\neg L]}{(\neg B^* \wedge \neg C^*), B \rightarrow} [\wedge L] \quad \frac{(\neg B^* \wedge \neg C^*), C \rightarrow} {C \rightarrow \neg(\neg B^* \wedge \neg C^*)} [\neg R]}{\frac{(B \vee C) \rightarrow \neg(\neg B^* \wedge \neg C^*)}{\rightarrow ((B \vee C) \supset \neg(\neg B^* \wedge \neg C^*))} [\supset R]} [\vee L] \quad \frac{\frac{\frac{[B^* \rightarrow B]}{B^* \rightarrow (B \vee C)} [\vee R] \quad \frac{[C^* \rightarrow C]}{C^* \rightarrow (B \vee C)} [\vee R]}{\rightarrow (B \vee C), \neg B^*} [\neg R] \quad \frac{\rightarrow (B \vee C), \neg C^*}{\rightarrow (B \vee C), \neg C^*} [\neg R]}{\frac{\rightarrow (B \vee C), (\neg B^* \wedge \neg C^*)}{\neg(\neg B^* \wedge \neg C^*) \rightarrow (B \vee C)} [\neg L] \quad \frac{\neg(\neg B^* \wedge \neg C^*) \rightarrow (B \vee C)}{\rightarrow (\neg(\neg B^* \wedge \neg C^*) \supset (B \vee C))} [\supset R]} [\wedge R]}{\rightarrow ((B \vee C) \equiv \neg(\neg B^* \wedge \neg C^*))} [\wedge R]$$

3.12.1.⊃)

$$\frac{\frac{\frac{[B^* \rightarrow B]}{(B \supset C), B^* \rightarrow C^*} [\supset L] \quad \frac{[C \rightarrow C^*]}{B^*, (B \supset C) \rightarrow C^*} [\supset R]}{\frac{B^*, (B \supset C) \rightarrow C^*}{(B \supset C) \rightarrow (B^* \supset C^*)} [\supset R]} [\supset L] \quad \frac{\frac{\frac{[B \rightarrow B^*]}{(B^* \supset C^*), B \rightarrow C} [\supset L] \quad \frac{[C^* \rightarrow C]}{B, (B^* \supset C^*) \rightarrow C} [\supset R]}{\frac{B^* \supset C^*, B \rightarrow C}{(B^* \supset C^*) \rightarrow (B \supset C)} [\supset R]} [\supset L]}{\frac{\rightarrow ((B \supset C) \supset (B^* \supset C^*))}{\rightarrow ((B \supset C) \equiv (B^* \supset C^*))} [\supset R]} [\wedge R]$$

3.12.1.∀)

$$\frac{\frac{\frac{[B(a) \rightarrow B^*(a)]}{\forall x B(x) \rightarrow B^*(a)} [\forall L]}{\forall x B(x) \rightarrow \forall x B^*(x)} [\forall R]}{\rightarrow (\forall x B(x) \supset \forall x B^*(x))} [\supset R] \quad \frac{\frac{\frac{[B^*(a) \rightarrow B(a)]}{\forall x B^*(x) \rightarrow B(a)} [\forall L]}{\forall x B^*(x) \rightarrow \forall x B(x)} [\forall R]}{\rightarrow (\forall x B^*(x) \supset \forall x B(x))} [\supset R] \quad \frac{}{\rightarrow (\forall x B(x) \equiv \forall x B^*(x))} [\wedge R]$$

3.12.1.∃)

$$\frac{\frac{\frac{[B(a) \rightarrow B^*(a)]}{\neg B^*(a), B(a) \rightarrow} [\neg L]}{\forall x \neg B^*(x), B(a) \rightarrow} [\forall L]}{B(a) \rightarrow \neg \forall x \neg B^*(x)} [\neg R] \quad \frac{\frac{[B^*(a) \rightarrow B(a)]}{B^*(a) \rightarrow \exists x B(x)} [\exists R]}{\rightarrow \exists x B(x), \neg B^*(a)} [\neg R] \quad \frac{\frac{\frac{B(a) \rightarrow \neg \forall x \neg B^*(x)}{\exists x B(x) \rightarrow \neg \forall x \neg B^*(x)} [\exists L]}{\rightarrow (\exists x B(x) \supset \neg \forall x \neg B^*(x))} [\supset R] \quad \frac{\frac{\frac{\neg \forall x \neg B^*(x) \rightarrow \exists x B(x)}{\rightarrow \exists x B(x), \forall x \neg B^*(x)} [\forall R]}{\neg \forall x \neg B^*(x) \rightarrow \exists x B(x)} [\neg L] \quad \frac{}{\rightarrow (\exists x B(x) \equiv \neg \forall x \neg B^*(x))} [\wedge R]$$

3.12.3.atomic)

$$\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L]}{A, \neg A \rightarrow} [XL] \quad \frac{\frac{\neg A \rightarrow \neg A}{\neg \neg A, \neg A \rightarrow} [\neg L]}{\neg A \rightarrow \neg \neg A} [\neg R] \quad \frac{\frac{\neg \neg A, \neg A \rightarrow}{\neg A \rightarrow \neg \neg \neg A} [\neg R]}{\neg \neg \neg A, \neg A \rightarrow} [\neg L] \quad \frac{\frac{\neg \neg \neg A, \neg A \rightarrow}{\neg A, \neg \neg \neg A \rightarrow} [XL]}{\neg \neg \neg A \rightarrow \neg \neg A} [\neg R]$$

3.12.3.¬)

$$\frac{\frac{B^* \rightarrow B^*}{\neg B^*, B^* \rightarrow} [\neg L]}{B^* \rightarrow \neg \neg B^*} [\neg R] \quad \frac{\frac{\neg \neg \neg B^*, B^* \rightarrow}{B^*, \neg \neg \neg B^* \rightarrow} [XL]}{\neg \neg \neg B^* \rightarrow \neg B^*} [\neg R]$$

3.12.3.∧)

$$\begin{array}{c}
\frac{\frac{B^* \rightarrow B^*}{(B^* \wedge C^*) \rightarrow B^*} [\wedge L]}{\neg B^*, (B^* \wedge C^*) \rightarrow} [\neg L] \\
\frac{}{(B^* \wedge C^*), \neg B^* \rightarrow} [XL] \\
\frac{}{\neg B^* \rightarrow \neg(B^* \wedge C^*)} [\neg R] \\
\frac{}{\neg \neg(B^* \wedge C^*), \neg B^* \rightarrow} [\neg L] \\
\frac{}{\neg B^*, \neg \neg(B^* \wedge C^*) \rightarrow} [XL] \\
\frac{}{\neg \neg(B^* \wedge C^*) \rightarrow \neg \neg B^*} [\neg R] \\
\frac{}{\neg \neg(B^* \wedge C^*) \rightarrow B^*} [\text{Cut}]
\end{array}
\quad
\begin{array}{c}
\frac{\frac{C^* \rightarrow C^*}{(B^* \wedge C^*) \rightarrow C^*} [\wedge L]}{\neg C^*, (B^* \wedge C^*) \rightarrow} [\neg L] \\
\frac{}{(B^* \wedge C^*), \neg C^* \rightarrow} [XL] \\
\frac{}{\neg C^* \rightarrow \neg(B^* \wedge C^*)} [\neg R] \\
\frac{}{\neg \neg(B^* \wedge C^*), \neg C^* \rightarrow} [\neg L] \\
\frac{}{\neg C^*, \neg \neg(B^* \wedge C^*) \rightarrow} [XL] \\
\frac{}{\neg \neg(B^* \wedge C^*) \rightarrow \neg \neg C^*} [\neg R] \\
\frac{}{\neg \neg(B^* \wedge C^*) \rightarrow C^*} [\text{Cut}]
\end{array}$$

$$\frac{\neg \neg(B^* \wedge C^*) \rightarrow B^* \quad \neg \neg(B^* \wedge C^*) \rightarrow C^*}{\neg \neg(B^* \wedge C^*) \rightarrow (B^* \wedge C^*)} [\wedge R]$$

3.12.3.∨)

$$\begin{array}{c}
\frac{(\neg B^* \wedge \neg C^*) \rightarrow (\neg B^* \wedge \neg C^*)}{\neg(\neg B^* \wedge \neg C^*), (\neg B^* \wedge \neg C^*) \rightarrow} [\neg L] \\
\frac{}{(\neg B^* \wedge \neg C^*) \rightarrow \neg \neg(\neg B^* \wedge \neg C^*)} [\neg R] \\
\frac{}{\neg \neg \neg(\neg B^* \wedge \neg C^*), (\neg B^* \wedge \neg C^*) \rightarrow} [\neg L] \\
\frac{}{(\neg B^* \wedge \neg C^*), \neg \neg \neg(\neg B^* \wedge \neg C^*) \rightarrow} [XL] \\
\frac{}{\neg \neg \neg(\neg B^* \wedge \neg C^*) \rightarrow \neg(\neg B^* \wedge \neg C^*)} [\neg R]
\end{array}$$

3.12.3.⊃)

$$\frac{\frac{B^* \rightarrow B^* \quad C^* \rightarrow C^*}{\neg \neg(B^* \supset C^*), B^* \rightarrow \neg \neg C^*} [3.10.1]}{\neg \neg(B^* \supset C^*), B^* \rightarrow C^*} [\neg \neg C^* \rightarrow C^*] [\text{Cut}]$$

$$\frac{}{B^*, \neg \neg(B^* \supset C^*) \rightarrow C^*} [XL]$$

$$\frac{}{\neg \neg(B^* \supset C^*) \rightarrow (B^* \supset C^*)} [\supset R]$$

3.12.3.∀)

$$\begin{array}{c}
\frac{B^*(a) \rightarrow B^*(a)}{\forall x B^*(x) \rightarrow B^*(a)} [\forall L] \\
\frac{}{\neg B^*(a), \forall x B^*(x) \rightarrow} [\neg L] \\
\frac{}{\forall x B^*(x), \neg B^*(a) \rightarrow} [XL] \\
\frac{}{\neg B^*(a) \rightarrow \neg \forall x B^*(x)} [\neg R] \\
\frac{}{\neg \neg \forall x B^*(x), \neg B^*(a) \rightarrow} [\neg L] \\
\frac{}{\neg B^*(a), \neg \neg \forall x B^*(x) \rightarrow} [XL] \\
\frac{}{\neg \neg \forall x B^*(x) \rightarrow \neg \neg B^*(a)} [\neg R] \\
\frac{}{\neg \neg \forall x B^*(x) \rightarrow B^*(a)} [\neg \neg B^*(a) \rightarrow B^*(a)] [\text{Cut}]
\end{array}$$

$$\frac{}{\neg \neg \forall x B^*(x) \rightarrow \forall x B^*(x)} [\forall R]$$

3.12.3.3.)

$$\begin{array}{c}
 \frac{\forall x \neg B^*(x) \rightarrow \forall x \neg B^*(x)}{\neg \forall x \neg B^*(x), \forall x \neg B^*(x) \rightarrow} [\neg L] \\
 \frac{\forall x \neg B^*(x) \rightarrow \neg \neg \forall x \neg B^*(x)}{\neg \neg \neg \forall x \neg B^*(x), \forall x \neg B^*(x) \rightarrow} [\neg R] \\
 \frac{\neg \neg \neg \forall x \neg B^*(x), \forall x \neg B^*(x) \rightarrow}{\forall x \neg B^*(x), \neg \neg \neg \forall x \neg B^*(x) \rightarrow} [\neg L] \\
 \frac{\forall x \neg B^*(x), \neg \neg \neg \forall x \neg B^*(x) \rightarrow}{\neg \neg \neg \forall x \neg B^*(x) \rightarrow \neg \forall x \neg B^*(x)} [\neg R]
 \end{array}$$

Detachment Rule)

$$\frac{\frac{A \rightarrow A \quad B \rightarrow B}{(A \supset B), A \rightarrow B} [\supset L]}{A, (A \supset B) \rightarrow B} [\text{XL}]$$

Double Negation)

$$\frac{\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L]}{A \rightarrow \neg \neg A} [\neg R]}{\rightarrow (A \supset \neg \neg A)} [\supset R] \quad \frac{\frac{\frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R]}{\neg \neg A \rightarrow A} [\neg L]}{\rightarrow (\neg \neg A \supset A)} [\supset R]}{\rightarrow (A \equiv \neg \neg A)} [\wedge R]$$

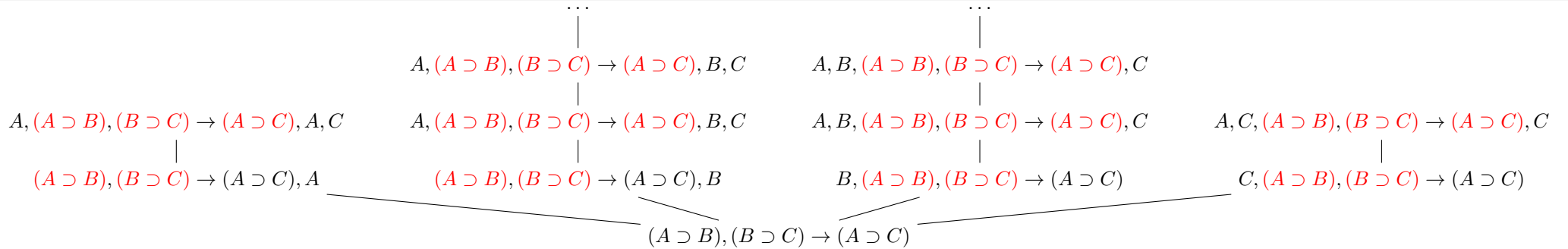
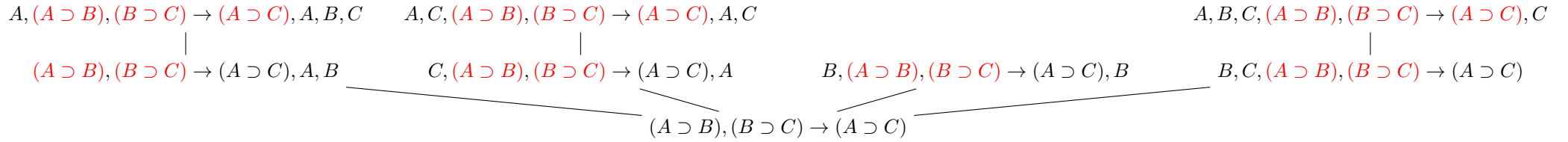
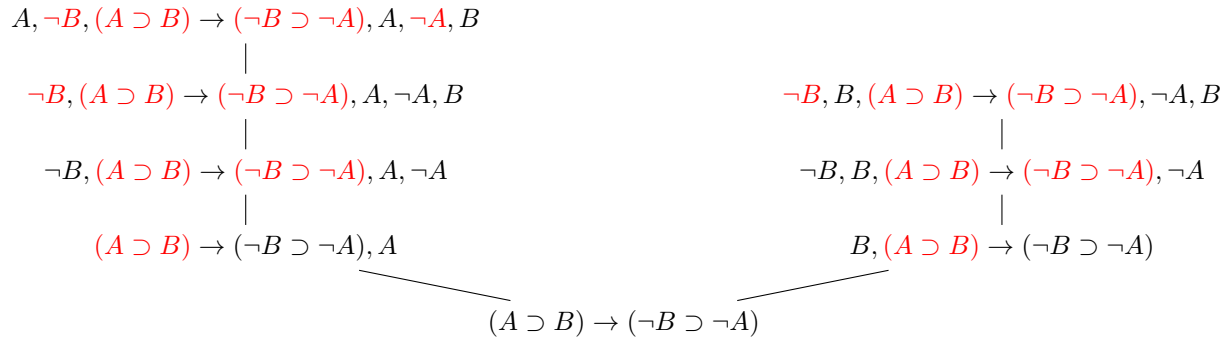
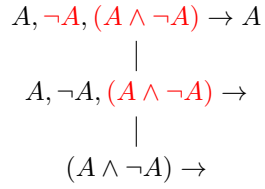
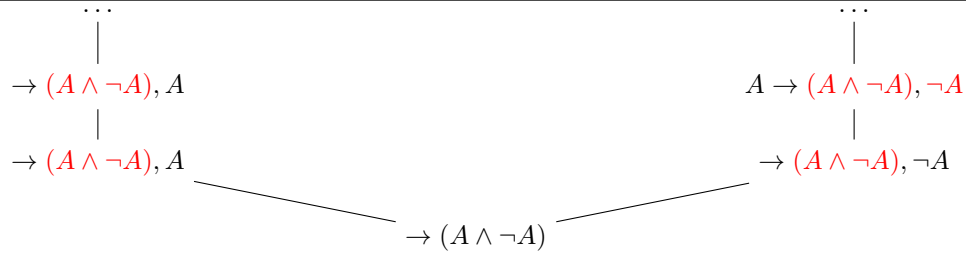
Contraction Redundancy)

$$\frac{\frac{\frac{D \rightarrow D}{\neg D, D \rightarrow} [\neg L]}{D \rightarrow \neg \neg D} [\neg R]}{D, \Gamma \rightarrow \Delta} \quad \frac{\frac{\frac{D, D, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg D, \neg D} [\neg R, \neg R]}{\Gamma \rightarrow \Delta, \neg D} [\text{CR}]}{\neg \neg D, \Gamma \rightarrow \Delta} [\neg L]}{D, \Gamma \rightarrow \Delta} [\text{Cut}]$$

Implication Transitivity)

$$\frac{\frac{\frac{A \rightarrow A \quad B \rightarrow B}{A, (A \supset B) \rightarrow B} [\text{detachment}]}{A, (A \supset B) \rightarrow B, C} [\text{WR}]}{A, (A \supset B) \rightarrow C, B} [\text{XR}] \quad \frac{\frac{\frac{C \rightarrow C}{(A \supset B), C \rightarrow C} [\text{WL}]}{C, (A \supset B) \rightarrow C} [\text{XL}]}{A, C, (A \supset B) \rightarrow C} [\text{WL}]}{C, A, (A \supset B) \rightarrow C} [\text{XL}]}{\frac{(B \supset C), A, (A \supset B), A, (A \supset B) \rightarrow C, C}{(B \supset C), A, (A \supset B), A, (A \supset B) \rightarrow C} [\text{CR}]}{\frac{A, (B \supset C), (A \supset B), A, (A \supset B) \rightarrow C}{A, (B \supset C), A, (A \supset B), (A \supset B) \rightarrow C} [\text{XL}]}{\frac{A, (B \supset C), A, (A \supset B), (A \supset B) \rightarrow C}{A, A, (B \supset C), (A \supset B), (A \supset B) \rightarrow C} [\text{XL}]}{\frac{A, A, (B \supset C), (A \supset B), (A \supset B) \rightarrow C}{A, (B \supset C), (A \supset B), (A \supset B) \rightarrow C} [\text{CL}]}{\frac{A, (B \supset C), (A \supset B), (A \supset B) \rightarrow C}{A, (A \supset B), (B \supset C), (A \supset B) \rightarrow C} [\text{XL}]}{\frac{(A \supset B), A, (B \supset C), (A \supset B) \rightarrow C}{(A \supset B), A, (A \supset B), (B \supset C) \rightarrow C} [\text{XL}]}{\frac{(A \supset B), A, (A \supset B), (B \supset C) \rightarrow C}{(A \supset B), (A \supset B), A, (B \supset C) \rightarrow C} [\text{XL}]}{\frac{(A \supset B), (A \supset B), A, (B \supset C) \rightarrow C}{(A \supset B), A, (B \supset C) \rightarrow C} [\text{CL}]}{\frac{A, (A \supset B), (B \supset C) \rightarrow C}{(A \supset B), (B \supset C) \rightarrow (A \supset C)} [\text{XL}]}{\frac{A, (A \supset B), (B \supset C) \rightarrow C}{(A \supset B), (B \supset C) \rightarrow (A \supset C)} [\supset R]}$$

Reduction Trees)



$$\begin{array}{c}
A, (A \supset B) \rightarrow (\neg A \vee B), \neg A, B, A \\
| \\
(A \supset B) \rightarrow (\neg A \vee B), \neg A, B, A \\
\swarrow \quad \searrow \\
(A \supset B) \rightarrow (\neg A \vee B), \neg A, B \quad B, (A \supset B) \rightarrow (\neg A \vee B), \neg A, B \\
| \\
(A \supset B) \rightarrow (\neg A \vee B)
\end{array}$$

8.3.1)

$$\begin{array}{c}
F(a), \neg \exists x F(x) \rightarrow \forall y \neg F(y), \exists x F(x), \neg F(a), F(a) \\
| \\
\neg \exists x F(x) \rightarrow \forall y \neg F(y), \exists x F(x), \neg F(a), F(a) \\
| \\
\neg \exists x F(x) \rightarrow \forall y \neg F(y), \exists x F(x), \neg F(a) \\
| \\
\neg \exists x F(x) \rightarrow \forall y \neg F(y), \exists x F(x) \\
| \\
\neg \exists x F(x) \rightarrow \forall y \neg F(y)
\end{array}$$

8.3.2)

$$\begin{array}{c}
A(b), (A(b) \supset B(b)), A(a), \forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b), A(b) \\
| \\
(A(b) \supset B(b)), A(a), \forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b), A(b) \quad B(b), (A(b) \supset B(b)), A(a), \forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b) \\
\swarrow \quad \searrow \\
(A(b) \supset B(b)), A(a), \forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x), B(b) \\
| \\
(A(b) \supset B(b)), A(a), \forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x) \\
| \\
A(a), \forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x) \\
| \\
\forall x A(x), \exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x)), \exists x B(x) \\
| \\
\exists x (A(x) \supset B(x)) \rightarrow (\forall x A(x) \supset \exists x B(x))
\end{array}$$
