The Lecture Diary starts from the next page.

Additionally, I wrote some additional programs and notes during the course, including:

- a library for LK/LJ formulas and proofs.
- a REPL for writing proofs interactively.
- a tool for exporting proofs in TeX format via the bussproofs package. Some examples here.
- a tool for constructing and exporting reduction trees from a sequent in TeX format. Examples at the bottom of the same document.
- a library for parsing and evaluating PRFs.
- example of PRF definitions that can be parsed and tested automatically.

All the materials are collected under the following repository.

Regards,
Alejandro

STRUCTURAL

	left	right
1.1) weakening	$\dfrac{\Gamma \to \Delta}{D, \Gamma \to \Delta}$	$\dfrac{\Gamma \to \Delta}{\Gamma \to \Delta, D}$
1.2) contraction	$\dfrac{D, D, \Gamma \to \Delta}{D, \Gamma \to \Delta}$	$\dfrac{\Gamma \to \Delta, D, D}{\Gamma \to \Delta, D}$
1.3) exchange	$\dfrac{\Gamma, C, D, \Pi \to \Delta}{\Gamma, D, C, \Pi \to \Delta}$	$\dfrac{\Gamma \to \Delta, C, D, \Lambda}{\Gamma \to \Delta, D, C, \Lambda}$

1.4) cut

$$\frac{\Gamma \to \Delta, D \qquad D, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}$$

LOGICAL

	left	right
2.1) $\neg$	$\dfrac{\Gamma \to \Delta, D}{\neg D, \Gamma \to \Delta}$	$\dfrac{D, \Gamma \to \Delta}{\Gamma \to \Delta, \neg D}$
2.2) $\wedge$	$\dfrac{C, \Gamma \to \Delta}{C \wedge D, \Gamma \to \Delta}$ $\quad$ $\dfrac{D, \Gamma \to \Delta}{C \wedge D, \Gamma \to \Delta}$	$\dfrac{\Gamma \to \Delta, C \qquad \Gamma \to \Delta, D}{\Gamma \to \Delta, C \wedge D}$
2.3) $\vee$	$\dfrac{C, \Gamma \to \Delta \qquad D, \Gamma \to \Delta}{C \vee D, \Gamma \to \Delta}$	$\dfrac{\Gamma \to \Delta, C}{\Gamma \to \Delta, C \vee D}$ $\quad$ $\dfrac{\Gamma \to \Delta, D}{\Gamma \to \Delta, C \vee D}$
2.4) $\supset$	$\dfrac{\Gamma \to \Delta, C \qquad D, \Pi \to \Lambda}{C \supset D, \Gamma, \Pi \to \Delta, \Lambda}$	$\dfrac{C, \Gamma \to \Delta, D}{\Gamma \to \Delta, C \supset D}$
2.5) $\forall$	$\dfrac{F(t), \Gamma \to \Delta}{\forall x F(x), \Gamma \to \Delta}$	$\dfrac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \forall x F(x)}$
2.6) $\exists$	$\dfrac{F(a), \Gamma \to \Delta}{\exists x F(x), \Gamma \to \Delta}$	$\dfrac{\Gamma \to \Delta, F(t)}{\Gamma \to \Delta, \exists x F(x)}$

2.5

1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A \wedge \neg B, A \to}}{A, \neg A \wedge \neg B \to}}{A \to \neg(\neg A \wedge \neg B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\neg A \wedge \neg B, B \to}}{B, \neg A \wedge \neg B \to}}{B \to \neg(\neg A \wedge \neg B)}}{A \vee B \to \neg(\neg A \wedge \neg B)}}{\to (A \vee B) \supset \neg(\neg A \wedge \neg B)}$$

[¬L, ∧L]
[XL]
[¬R]
[∨L]
[⊃R]

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A \vee B}}{\to A \vee B, \neg A} \qquad \dfrac{\dfrac{B \to B}{B \to A \vee B}}{\to A \vee B, \neg B}}{\to A \vee B, \neg A \wedge \neg B}}{\neg(\neg A \wedge \neg B) \to A \vee B}}{\to \neg(\neg A \wedge \neg B) \supset A \vee B}$$

[∧R]
[¬L]
[⊃R]

$\Leftarrow$

2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to \neg A, A}}{\to \neg A \vee B, A} \qquad \dfrac{B \to B}{B \to \neg A \vee B}}{A \supset B \to \neg A \vee B, \neg A \vee B}}{A \supset B \to \neg A \vee B}}{\to (A \supset B) \supset (\neg A \vee B)}$$

[∨R]
[⊃L]
[CR]
[⊃R]

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg A \to}}{A, \neg A \to B}}{\neg A \to A \supset B} \qquad \dfrac{\dfrac{\dfrac{B \to B}{A, B \to B}}{}}{B \to A \supset B}}{\neg A \vee B \to A \supset B}}{\to (\neg A \vee B) \supset (A \supset B)}$$

[⊃R]
[∨L]
[⊃R]

$\Leftarrow$

3)

$$
\begin{array}{rl}
F(a) \to F(a) & [\text{axiom}] \\ \hline
\neg F(a), F(a) \to & [\neg L] \\ \hline
\forall y \neg F(y), F(a) \to & [\forall L] \\ \hline
F(a) \to \neg \forall y \neg F(y) & [\neg R] \\ \hline
\exists x F(x) \to \neg \forall y \neg F(y) & [\exists L]
\end{array}
\quad \Rightarrow
$$

$$
\begin{array}{rl}
F(a) \to \exists x F(x) & [\text{axiom}, \exists R] \\ \hline
\to \exists x F(x), \neg F(a) & [\neg R] \\ \hline
\to \exists x F(x), \forall y \neg F(y) & [\forall R] \\ \hline
\neg \forall y \neg F(y) \to \exists x F(x) & [\neg L] \\ \hline
\neg \forall y \neg F(y) \to \exists x F(x) & [\exists R]
\end{array}
\quad \Leftarrow
$$

4)

$$
\begin{array}{rl}
F(a) \to F(a) & [\text{axiom}] \\ \hline
\forall y F(y) \to F(a) & [\forall L] \\ \hline
\neg F(a) \to \neg \forall y F(y) & [\neg L, \neg R, X] \\ \hline
\exists x \neg F(x) \to \neg \forall y F(y) & [\exists L]
\end{array}
\quad \Leftarrow
$$

$$
\begin{array}{rl}
F(a) \to F(a) & [\text{axiom}] \\ \hline
\to F(a), \neg F(a) & [\neg R] \\ \hline
\to F(a), \exists x \neg F(x) & [\exists R] \\ \hline
\to \forall y F(y), \exists x \neg F(x) & [\forall R, XR] \\ \hline
\neg \forall y F(y) \to \exists x \neg F(x) & [\neg L]
\end{array}
\quad \Rightarrow
$$

5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to \neg A, A}}{\to \neg A \vee \neg B, A} \qquad \dfrac{\dfrac{B \to B}{\to \neg B, B}}{\to \neg A \vee \neg B, B}}{\to \neg A \vee \neg B, A \wedge B}}{\neg(A \wedge B) \to \neg A \vee \neg B}} \quad \Rightarrow$$

$$\dfrac{\dfrac{\dfrac{A \to A}{A \wedge B \to A}}{\neg A \to \neg(A \wedge B)} \qquad \dfrac{\dfrac{B \to B}{A \wedge B \to B}}{\neg B \to \neg(A \wedge B)}}{\neg A \vee \neg B \to \neg(A \wedge B)} \quad \Leftarrow$$

2.6

1)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B(a) \to B(a)}{A \supset B(a), A \to B(a)}\ [\supset L] \leftarrow \text{detachment}}{A \supset B(a), A \to \exists x B(x)}\ [\exists R]}{\exists x (A \supset B(x)), A \to \exists x B(x)}\ [\exists L]}{\exists x (A \supset B(x)) \to A \supset \exists x B(x)}\ [\supset R] \quad \Rightarrow$$

$\Leftarrow$

$$\dfrac{\dfrac{[\text{ex } 2.5.2]}{A \supset \exists x B(x) \to \neg A \vee \exists x B(x)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to B(a)}}{A, \neg A \to B(a)}}{\neg A \to A \supset B(a)}}{\neg A \to \exists x (A \supset B(x))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B(a) \to B(a)}{A, B(a) \to B(a)}}{B(a) \to A \supset B(a)}}{B(a) \to \exists x (A \supset B(x))}}{\exists x B(x) \to \exists x (A \supset B(x))}}{\neg A \vee \exists x B(x) \to \exists x (A \supset B(x))}}{A \supset \exists x B(x) \to \exists x (A \supset B(x))}\ [\text{cut}]$$

2)

$$\begin{array}{cl}
A(a),\ A(a)\supset B \to B & [\text{detachment}] \\ \hline
\forall x A(x),\ A(a)\supset B \to B & [\forall L] \\ \hline
A(a)\supset B \to \forall x A(x)\supset B & [\supset R] \\ \hline
\exists x(A(x)\supset B) \to \forall x A(x)\supset B & [\exists L]
\end{array}
\quad \Rightarrow$$

$\Leftarrow$

$$\begin{array}{c}
[\text{ex } 2.5.2] \\ \hline \hline
\forall x A(x)\supset B \to \neg\forall x A(x)\vee B
\end{array}
\qquad
\begin{array}{c}
A(a)\to A(a) \\ \hline
A(a)\to A(a), B \\ \hline
\to A(a),\ A(a)\supset B \\ \hline
\to A(a),\ \exists x(A(x)\supset B) \\ \hline
\to \exists x(A(x)\supset B),\ \forall x A(x) \\ \hline
\neg\forall x A(x) \to \exists x(A(x)\supset B)
\end{array}
\qquad
\begin{array}{c}
B\to B \\ \hline
A(a), B\to B \\ \hline
B\to A(a)\supset B \\ \hline
B\to \exists x(A(x)\supset B)
\end{array}$$

$$\neg\forall x A(x)\vee B \to \exists x(A(x)\supset B)$$

$$\begin{array}{cl}
\hline
\forall x A(x)\supset B \to \exists x(A(x)\supset B) & [\text{Cut}]
\end{array}$$

3)

$$\begin{array}{cl}
A(a),\ A(a)\supset B(a) \to B(a) & [\text{detachment}] \\ \hline
\forall x A(x),\ A(a)\supset B(a) \to \exists x B(x) & [\forall L, \exists R] \\ \hline
A(a)\supset B(a) \to \forall x A(x)\supset \exists x B(x) & [\supset R] \\ \hline
\exists x(A(x)\supset B(x)) \to \forall x A(x)\supset \exists x B(x) & [\exists L]
\end{array}
\quad \Rightarrow$$

$\Leftarrow$

$$\begin{array}{c}
[\text{ex } 2.5.2] \\ \hline \hline
\forall x A(x)\supset \exists x B(x) \to \neg\forall x A(x)\vee \exists x B(x)
\end{array}
\qquad
\begin{array}{c}
A(a)\to A(a) \\ \hline
A(a)\to A(a), B(a) \\ \hline
\to A(a),\ A(a)\supset B(a) \\ \hline
\to A(a),\ \exists x(A(x)\supset B(x)) \\ \hline
\to \exists x(A(x)\supset B(x)),\ \forall x A(x) \\ \hline
\neg\forall x A(x) \to \exists x(A(x)\supset B(x))
\end{array}
\qquad
\begin{array}{c}
B(a)\to B(a) \\ \hline
A(a), B(a)\to B(a) \\ \hline
B(a)\to A(a)\supset B(a) \\ \hline
B(a)\to \exists x(A(x)\supset B(x)) \\ \hline
\exists x B(x)\to \exists x(A(x)\supset B(x))
\end{array}$$

$$\neg\forall x A(x)\vee \exists x B(x) \to \exists x(A(x)\supset B(x))$$

$$\begin{array}{cl}
\hline
\forall x A(x)\supset \exists x B(x) \to \exists x(A(x)\supset B(x)) & [\text{Cut}]
\end{array}$$

4)

$$\begin{array}{c}
[\text{ex } 2.5.2] \\ \hline \hline
\neg A\supset B \to \neg\neg A\vee B
\end{array}
\qquad
\begin{array}{c}
A\to A \\ \hline
A\to \neg B\supset A \\ \hline
\neg\neg A\to \neg B\supset A
\end{array}
\qquad
\begin{array}{c}
B\to B \\ \hline
B\to B, A \\ \hline
B\to \neg B\supset A
\end{array}$$

$$\neg\neg A\vee B \to \neg B\supset A$$

$$\begin{array}{cl}
\hline
\neg A\supset B \to \neg B\supset A & [\text{cut}]
\end{array}$$

5)

$$
\begin{prooftree}
\AxiomC{[ex 2.5.2]}
\doubleLine
\UnaryInfC{$\neg A \supset \neg B \rightarrow \neg\neg A \vee \neg B$}
\AxiomC{$A \rightarrow A$}
\UnaryInfC{$B, A \rightarrow A$}
\UnaryInfC{$A \rightarrow B \supset A$}
\UnaryInfC{$\neg\neg A \rightarrow B \supset A$}
\AxiomC{$B \rightarrow B$}
\UnaryInfC{$\neg B, B \rightarrow$}
\UnaryInfC{$\neg B, B \rightarrow A$}
\UnaryInfC{$B, \neg B \rightarrow A$}
\UnaryInfC{$\neg B \rightarrow B \supset A$}
\BinaryInfC{$\neg\neg A \vee \neg B \rightarrow B \supset A$}
\RightLabel{[cut]}
\BinaryInfC{$\neg A \supset \neg B \rightarrow B \supset A$}
\end{prooftree}
$$

2.7

$$
\begin{prooftree}
\AxiomC{$A(a) \rightarrow A(a)$}
\UnaryInfC{$A(a) \rightarrow A(a), B$}
\UnaryInfC{$\rightarrow A(a), A(a) \supset B$}
\UnaryInfC{$\rightarrow A(a), \exists x (A(x) \supset B)$}
\UnaryInfC{$\rightarrow \exists x (A(x) \supset B), \forall x A(x)$}
\AxiomC{$B \rightarrow B$}
\UnaryInfC{$A(a), B \rightarrow B$}
\UnaryInfC{$B \rightarrow A(a) \supset B$}
\UnaryInfC{$B \rightarrow \exists x (A(x) \supset B)$}
\RightLabel{[$\supset$L]}
\BinaryInfC{$\forall x A(x) \supset B \rightarrow \exists x (A(x) \supset B), \exists x (A(x) \supset B)$}
\RightLabel{[contr.]}
\UnaryInfC{$\forall x A(x) \supset B \rightarrow \exists x (A(x) \supset B)$}
\end{prooftree}
$$

3.9

1)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg A \to}}{A, \neg A \to B}}{\neg A \to A \supset B} \qquad \dfrac{\dfrac{B \to B}{A, B \to B}}{B \to A \supset B}}{\neg A \vee B \to A \supset B}$$

2)

$$\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\neg F(a), F(a) \to}}{\forall y \neg F(y), F(a) \to}}{\forall y \neg F(y), \exists x F(x) \to}}{\exists x F(x) \to \neg \forall y \neg F(y)}$$

[axiom]
[¬L]
[∀L]
[XL, ∃L, XL]
[¬R]

3)

$$\dfrac{A \to A}{A \wedge B \to A}$$

4)

$$\dfrac{A \to A}{A \to A \vee B}$$

5)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \wedge B \to A}}{A \wedge B, \neg A \to}}{\neg A \to \neg(A \wedge B)} \qquad \dfrac{\dfrac{\dfrac{B \to B}{A \wedge B \to B}}{A \wedge B, \neg B \to}}{\neg B \to \neg(A \wedge B)}}{\neg A \vee \neg B \to \neg(A \wedge B)}$$

6)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A \vee B}}{A, \neg(A \vee B) \to}}{\neg(A \vee B) \to \neg A} \qquad \dfrac{\dfrac{\dfrac{B \to B}{B \to A \vee B}}{B, \neg(A \vee B) \to}}{\neg(A \vee B) \to \neg B}}{\neg(A \vee B) \to \neg A \wedge \neg B} \quad \Rightarrow$$

$$\left.\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg A \to}}{A, \neg A \wedge \neg B \to} \qquad \dfrac{\dfrac{B \to B}{B, \neg B \to}}{B, \neg A \wedge \neg B \to}}{A \vee B, \neg A \wedge \neg B \to}}{\neg A \wedge \neg B \to \neg (A \vee B)}\right\} \Leftarrow$$

7)

$$\left.\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \to A \qquad A, B \to B}{A, B \to A \wedge B}}{A, B \to (A \wedge B) \vee C} \qquad \dfrac{C \to C}{C, B \to (A \wedge B) \vee C}}{A \vee C, B \to (A \wedge B) \vee C}}{B, A \vee C \to (A \wedge B) \vee C} \qquad C, A \vee C \to (A \wedge B) \vee C}{B \vee C, A \vee C \to (A \wedge B) \vee C}}{(A \vee C) \wedge (B \vee C), (A \vee C) \wedge (B \vee C) \to (A \wedge B) \vee C}}{(A \vee C) \wedge (B \vee C) \to (A \wedge B) \vee C}\right\} \Rightarrow$$

$$\left.\dfrac{\dfrac{\dfrac{A \to A}{A \wedge B \to A \vee C} \qquad \dfrac{C \to C}{C \to A \vee C}}{(A \wedge B) \vee C \to A \vee C} \qquad \dfrac{\dfrac{B \to B}{A \wedge B \to B \vee C} \qquad \dfrac{C \to C}{C \to B \vee C}}{(A \wedge B) \vee C \to B \vee C}}{(A \wedge B) \vee C \to (A \vee C) \wedge (B \vee C)}\right\} \Leftarrow$$

8)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a) \quad [\text{axiom}]}{\neg F(a), F(a) \to} \; [\neg L]}{F(a), \neg F(a) \to} \; [XL]}{\forall x\, F(x), \neg F(a) \to} \; [\forall L]}{\neg F(a) \to \neg \forall x\, F(x)} \; [\neg R]}{\exists x\, \neg F(x) \to \neg \forall x\, F(x)} \; [\exists L]$$

9)

$$\Rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{F(a)\wedge G(a) \to F(a)}}{\forall x(F(x)\wedge G(x)) \to F(a)}}{\forall x(F(x)\wedge G(x)) \to \forall x F(x)} \qquad \dfrac{\dfrac{\dfrac{G(a) \to G(a)}{F(a)\wedge G(a) \to G(a)}}{\forall x(F(x)\wedge G(x)) \to G(a)}}{\forall x(F(x)\wedge G(x)) \to \forall x G(x)}}{\forall x\,(F(x)\wedge G(x)) \to \forall x F(x) \wedge \forall x G(x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a)\to F(a)}{\forall x F(x) \to F(a)}}{\forall x F(x) \wedge \forall x G(x) \to F(a)} \qquad \dfrac{\dfrac{G(a) \to G(a)}{\forall x G(x) \to G(a)}}{\forall x F(x) \wedge \forall x G(x) \to G(a)}}{\forall x F(x) \wedge \forall x G(x) \to F(a) \wedge G(a)}}{\forall x F(x) \wedge \forall x G(x) \to \forall x\,(F(x)\wedge G(x))}} \quad \Leftarrow$$

10)

$$\dfrac{\dfrac{\dfrac{A \to A \qquad \dfrac{B \to B}{\neg B,\ B \to}}{A,\ B,\ A \supset \neg B \to}}{B,\ A\supset \neg B \to \neg A}}{A \supset \neg B \to B \supset \neg A} \qquad [\supset L]$$

11)

$$\dfrac{\dfrac{\dfrac{A,\ A\supset B(a) \to B(a)}{A,\ A\supset B(a) \to \exists x B(x)}}{A\supset B(a) \to A \supset \exists x\, B(x)}}{\exists x\,(A\supset B(x)) \to A \supset \exists x\, B(x)}$$

[detachment]
[∃R]
[⊃R]
[∃L]

12)

$$\dfrac{\dfrac{\dfrac{A(a),\ A(a)\supset B \to B}{\forall x\, A(x),\ A(a)\supset B \to B}}{A(a)\supset B \to \forall x\, A(x) \supset B}}{\exists x\,(A(x)\supset B) \to \forall x\, A(x) \supset B}$$

[detachment]
[∀L]
[⊃R]
[∃L]

13)

$$
\begin{array}{ll}
A(a), A(a)\supset B(a) \to B(a) & [\text{detachment}] \\ \hline
A(a), A(a)\supset B(a) \to \exists x\, B(x) & [\exists R] \\ \hline
\forall x\, A(x), A(a)\supset B(a) \to \exists x\, B(x) & [\forall L] \\ \hline
A(a)\supset B(a) \to \forall x\, A(x) \supset \exists x\, B(x) & [\supset R] \\ \hline
\exists x (A(x)\supset B(x)) \to \forall x\, A(x) \supset \exists x\, B(x) & [\exists L]
\end{array}
$$

3.10.

1)

$$
\begin{array}{ll}
A, A\supset B \to B & [\text{detachment}] \\ \hline
A\supset B, A \to B & [XL] \\ \hline
\neg B, A\supset B, A \to & [\neg L] \\ \hline
A\supset B, \neg B, A \to & [XL] \\ \hline
\neg B, A \to \neg(A\supset B) & [\neg R] \\ \hline
\neg\neg(A\supset B), \neg B, A \to & [\neg L] \\ \hline
\neg B, \neg\neg(A\supset B), A \to & [XL] \\ \hline
\neg\neg(A\supset B), A \to \neg\neg B & [\neg R]
\end{array}
$$

2)

$$
\begin{array}{ll}
\neg\neg(A\supset B), A \to \neg\neg B \qquad B \to B & [3.10.1] \\ \hline
\neg\neg B\supset B, A, \neg\neg(A\supset B) \to B & [\supset L] \\ \hline
A, \neg\neg B\supset B, \neg\neg(A\supset B) \to B & [XL] \\ \hline
\neg\neg B\supset B, \neg\neg(A\supset B) \to A\supset B & [\supset R]
\end{array}
$$

3)

$$
\begin{array}{l}
A \to A \\ \hline
\neg A, A \to \\ \hline
A \to \neg\neg A \\ \hline
\neg\neg\neg A, A \to \\ \hline
A, \neg\neg\neg A \to \\ \hline
\neg\neg\neg A \to \neg A \\ \hline
\to \neg\neg\neg A \supset \neg A
\end{array}
\qquad
\begin{array}{l}
\neg A \to \neg A \\ \hline
\neg\neg A, \neg A \to \\ \hline
\neg A \to \neg\neg\neg A \\ \hline
\to \neg A \supset \neg\neg\neg A
\end{array}
$$

$$
\begin{array}{ll}
\hline
\to \neg\neg\neg A \equiv \neg A & [\wedge R]
\end{array}
$$

3.11 By induction on the number of logical symbols in A:

* **case** A is atomic: then $\neg\neg A \to A$ is an axiom of LJ'.

* **case** A is $\neg B$: $\neg\neg B \to B$ is provable by the induction hypothesis. But $\neg\neg A \to A$ is $\neg\neg\neg B \to \neg B$ which is provable in LJ by exercise 3.10.3.

* **case** A is $B \wedge C$: then $\neg\neg B \to B$ and $\neg\neg C \to C$ are provable in LJ' by the induction hypothesis. From them we can obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B \wedge C \to B}}{\neg\neg(B\wedge C) \to \neg\neg B} \quad [\neg\neg B \to B]}{\neg\neg(B \wedge C) \to B} \qquad \dfrac{\dfrac{\dfrac{C \to C}{B \wedge C \to C}}{\neg\neg(B\wedge C) \to \neg\neg C} \quad [\neg\neg C \to C]}{\neg\neg(B \wedge C) \to C}}{\neg\neg(B \wedge C) \to B \wedge C}$$

* **case** A is $B \supset C$: then $\neg\neg B \to B$ and $\neg\neg C \to C$ are provable in LJ' by the induction hypothesis.

$$\dfrac{\dfrac{[\neg\neg C \to C]}{\to \neg\neg C \supset C} \qquad \neg\neg C \supset C,\ \neg\neg(B \supset C) \to B \supset C}{\neg\neg(B \supset C) \to B \supset C}$$

* **case** A is $\forall x B(x)$: then $\neg\neg B(a) \to B(a)$ is provable in LJ' by the induction hypothesis.

$$\dfrac{\dfrac{\dfrac{\dfrac{B(a) \to B(a)}{\forall x B(x) \to B(a)}}{\neg\neg \forall x B(x) \to \neg\neg B(a)} \qquad [\neg\neg B(a) \to B(a)]}{\neg\neg \forall x B(x) \to B(a)}}{\neg\neg \forall x B(x) \to \forall x B(x)}$$

3.12

$$A^* \triangleq \begin{cases} \neg\neg A & : A \text{ is atomic} \\ \neg B^* & : A \text{ is } \neg B \\ B^* \wedge C^* & : A \text{ is } B \wedge C \\ \neg(\neg B^* \wedge \neg C^*) & : A \text{ is } B \vee C \\ B^* \supset C^* & : A \text{ is } B \supset C \\ \forall x\, B^*(x) & : A \text{ is } \forall x\, B(x) \\ \neg \forall x \neg B^*(x) & : A \text{ is } \exists x\, B(x) \end{cases}$$

Claim: A is LK-provable iff A^* is LJ-provable

1) For any A, $A \equiv A^*$ is LK-provable: since we can always obtain $A \equiv A^*$ from $A \to A^*$ and $A^* \to A$, we only need to prove that $A \to A^*$ and $A^* \to A$ are LK-provable for any A. We proceed by induction on the number of logical symbols in A:

*** case** A is atomic: then A^* is $\neg\neg A$ and hence

$$\dfrac{\dfrac{A \to A}{\neg A, A \to}}{A \to \neg\neg A} \qquad \dfrac{\dfrac{A \to A}{\to A, \neg A}}{\neg\neg A \to A}$$

*** case** A is $\neg B$: then $B \to B^*$ and $B^* \to B$ are LK-provable by the induction hypothesis and from them

$$\dfrac{\dfrac{\dfrac{[B \to B^*]}{\neg B^*, B \to}}{B, \neg B^* \to}}{\neg B^* \to \neg B} \qquad \dfrac{\dfrac{\dfrac{[B^* \to B]}{\neg B, B^* \to}}{B^*, \neg B \to}}{\neg B \to \neg B^*}$$

*** case** A is $B \wedge C$: then $B \to B^*$, $B^* \to B$, $C \to C^*$ and $C^* \to C$ are LK-provable by the induction hypothesis. From them

$$\dfrac{\dfrac{[B \to B^*]}{B \wedge C \to B^*} \quad \dfrac{[C \to C^*]}{B \wedge C \to C^*}}{B \wedge C \to B^* \wedge C^*} \qquad \dfrac{\dfrac{[B^* \to B]}{B^* \wedge C^* \to B} \quad \dfrac{[C^* \to C]}{B^* \wedge C^* \to C}}{B^* \wedge C^* \to B \wedge C}$$

case A is $B \vee C$: then $B \to B^*$, $B^* \to B$, $C \to C^*$ and $C^* \to C$ by the induction hypothesis and we have

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \to B^*]}{\neg B^*, B \to}}{\neg B^* \wedge \neg C^*, B \to}}{B \to \neg(\neg B^* \wedge \neg C^*)} \qquad \dfrac{\dfrac{\dfrac{[C \to C^*]}{\neg C^*, C \to}}{\neg B^* \wedge \neg C^*, C \to}}{C \to \neg(\neg B^* \wedge \neg C^*)}}{B \vee C \to \neg(\neg B^* \wedge \neg C^*)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[B^* \to B]}{B^* \to B \vee C}}{\to B \vee C, \neg B^*} \qquad \dfrac{\dfrac{[C^* \to C]}{C^* \to B \vee C}}{\to B \vee C, \neg C^*}}{\to B \vee C, \neg B^* \wedge \neg C^*}}{\neg(\neg B^* \wedge \neg C^*) \to B \vee C}$$

case A is $B \supset C$: then $B \to B^*$, $B^* \to B$, $C \to C^*$ and $C^* \to C$ are LK-provable by the induction hypothesis

$$\dfrac{\dfrac{[B^* \to B] \qquad [C \to C^*]}{B^*, B \supset C \to C^*}}{B \supset C \to B^* \supset C^*} \qquad\qquad \dfrac{\dfrac{[B \to B^*] \qquad [C^* \to C]}{B, B^* \supset C^* \to C}}{B^* \supset C^* \to B \supset C}$$

case A is $\forall x B(x)$: then $B(a) \to B^*(a)$ and $B^*(a) \to B(a)$ are LK-provable by the induction hypothesis, so

$$\dfrac{\dfrac{[B(a) \to B^*(a)]}{\forall x B(x) \to B^*(a)}}{\forall x B(x) \to \forall x B^*(x)} \qquad\qquad \dfrac{\dfrac{[B^*(a) \to B(a)]}{\forall x B^*(x) \to B(a)}}{\forall x B^*(x) \to \forall x B(x)}$$

case A is $\exists x B(x)$: then $B(a) \to B^*(a)$ and $B^*(a) \to B(a)$ are LK-provable by the induction hypothesis, so

$$\dfrac{\dfrac{\dfrac{\dfrac{[B(a) \to B^*(a)]}{\neg B^*(a), B(a) \to}}{\forall x \neg B^*(x), B(a) \to}}{B(a) \to \neg \forall x \neg B^*(x)}}{\exists x B(x) \to \neg \forall x \neg B^*(x)} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{[B^*(a) \to B(a)]}{B^*(a) \to \exists x B(x)}}{\to \exists x B(x), \neg B^*(a)}}{\to \exists x B(x), \forall x \neg B^*(x)}}{\neg \forall x \neg B^*(x) \to \exists x B(x)}$$

2) For the forward direction, assume that

$$A_1, \ldots, A_m \to B_1, \ldots, B_n.$$

From part 1), we know that $A_i^* \to A_i$ and $B_i \to B_i^*$ are LK-provable. Then, notice that

$$\dfrac{\dfrac{\dfrac{A_i^* \to A_i \qquad [A_i, \ldots, A_m, A_1^*, \ldots, A_{i-1}^* \to B_1, \ldots, B_n]}{A_i^*, A_{i+1}, \ldots, A_m, A_1^*, \ldots, A_{i-1}^* \to B_1, \ldots, B_n}\,[\text{Cut}]}{}}{A_{i+1}, \ldots, A_m, A_1^*, \ldots, A_i^* \to B_1, \ldots, B_n}\,\left[\begin{array}{l}\text{several}\\ \text{exchanges}\end{array}\right]$$

for all $1 \leq i \leq m$. Starting with the premise and and applying the above step repeatedly we obtain

(a) $$A_1^*, \ldots, A_m^* \to B_1, \ldots, B_n.$$

Next, notice a symmetric idea for the succedent

$$\dfrac{\dfrac{\dfrac{[A_1^*, \ldots, A_m^*, \neg B_{i+1}^*, \ldots, \neg B_n^* \to B_1, \ldots, B_i] \qquad B_i \to B_i^*}{A_1^*, \ldots, A_m^*, \neg B_{i+1}^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}, B_i^*}}{\neg B_i^*, A_1^*, \ldots, A_m^*, \neg B_{i+1}^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}}}{A_1^*, \ldots, A_m^*, \neg B_i^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}} \qquad (1 \leq i \leq n)$$

and hence starting with (a) and applying the above step repeatedly we obtain the desired

$$A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to$$

This concludes the proof for the forward direction.

For the backward direction, assume that

$$A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to .$$

From part 1), we know that $A_i \to A_i^*$ and $B_i^* \to B_i$ are LK-provable. We can additionally prove $\neg\neg B_i^* \to B_i$ from $B_i^* \to B_i$, the axiom $B_i^* \to B_i^*$ and the cut rule. Then, notice that

$$\dfrac{\dfrac{\dfrac{[A_1^*, \ldots, A_m^*, \neg B_i^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}]}{\neg B_i^*, A_1^*, \ldots, A_m^*, \neg B_{i+1}^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}}}{A_1^*, \ldots, A_m^*, \neg B_{i+1}^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}, \neg\neg B_i^*} \qquad \dfrac{B_i^* \to B_i^*}{\neg\neg B_i^* \to B_i}}{A_1^*, \ldots, A_m^*, \neg B_{i+1}^*, \ldots, \neg B_n^* \to B_1, \ldots, B_{i-1}, B_i}$$

for $1 \leq i \leq n$. Starting from the premise and applying the above step repeatedly we obtain

(b) $A_1^*, \ldots, A_m^* \to B_1, \ldots, B_n$

Next, we note the following for the antecedent

$$\dfrac{A_i \to A_i^* \qquad \dfrac{A_{i+1}, \ldots, A_m, A_1^*, \ldots, A_i^* \to B_1, \ldots, B_n}{A_i^*, A_{i+1}, \ldots, A_m, A_1^*, \ldots, A_{i-1}^* \to B_1, \ldots, B_n}\,[\text{exchanges}]}{A_i, \ldots, A_m, A_1^*, \ldots, A_{i-1}^* \to B_1, \ldots, B_n}\,[\text{cut}]$$

for $1 \leq i \leq m$. Starting from (b) and applying the above step repeatedly, we obtain

$$A_1, \ldots, A_m \to B_1, \ldots, B_n.$$

This completes the backward direction and the proof.

3) The forward implication is trivial

$$\dfrac{\dfrac{A^* \to A^*}{\neg A^*, A^* \to}}{A^* \to \neg\neg A^*}.$$

For the backward implication, we first notice that it is intuitively provable on the basis of exercise 3.11 and the definition of A^*. Namely, A^* is defined in terms of subformulas which are also B^* for some B, do not contain $\vee$ or $\exists$, and for which the constructive proof of exercise 3.11 uses axioms of the form $\neg\neg A^* \to A^*$ which are LJ-provable for A atomic. Formally, we can prove it by induction on the number of logical symbols in A:

* case A is atomic: then A^* is $\neg\neg A$ and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}}{A, \neg A \to}}{\neg A \to \neg A}}{\neg\neg A, \neg A \to}}{\neg A \to \neg\neg\neg A}}{\neg\neg\neg\neg A, \neg A \to}}{\neg A, \neg\neg\neg\neg A \to}}{\neg\neg\neg\neg A \to \neg\neg A}}{\neg\neg A^* \to A^*}$$

*** case** A is $\neg B$: then A^* is $\neg B^*$ and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^* \to B^*}{\neg B^*, B^* \to}}{B^* \to \neg\neg B^*}}{\neg\neg\neg B^*, B^* \to}}{B^*, \neg\neg\neg B^* \to}}{\neg\neg\neg B^* \to \neg B^*}$$

$$\neg\neg A^* \to A^*$$

*** case** A is $B \wedge C$: then A^* is $B^* \wedge C^*$. We have $\neg\neg B^* \to B^*$ and $\neg\neg C^* \to C^*$ LJ-provable by the induction hypothesis and

$$\dfrac{\dfrac{\dfrac{\dfrac{B^* \to B^*}{B^* \wedge C^* \to B^*}}{\neg\neg(B^* \wedge C^*) \to \neg\neg B^*} \quad [\neg\neg B^* \to B^*]}{\neg\neg(B^* \wedge C^*) \to B^*} \qquad \dfrac{\dfrac{\dfrac{C^* \to C^*}{B^* \wedge C^* \to C^*}}{\neg\neg(B^* \wedge C^*) \to \neg\neg C^*} \quad [\neg\neg C^* \to C^*]}{\neg\neg(B^* \wedge C^*) \to C^*}}{\neg\neg(B^* \wedge C^*) \to B^* \wedge C^*}$$

*** case** A is $B \vee C$: then A^* is $\neg(\neg B^* \wedge \neg C^*)$. We have $\neg\neg B^* \to B^*$ and $\neg\neg C^* \to C^*$ LJ-provable by the induction hypothesis and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg B^* \wedge \neg C^* \to \neg B^* \wedge \neg C^*}{\neg(\neg B^* \wedge \neg C^*), \neg B^* \wedge \neg C^* \to}}{\neg B^* \wedge \neg C^* \to \neg\neg(\neg B^* \wedge \neg C^*)}}{\neg\neg\neg(\neg B^* \wedge \neg C^*), \neg B^* \wedge \neg C^* \to}}{\neg B^* \wedge \neg C^*, \neg\neg\neg(\neg B^* \wedge \neg C^*) \to}}{\neg\neg\neg(\neg B^* \wedge \neg C^*) \to \neg(\neg B^* \wedge \neg C^*)}$$

(generally, $\overbrace{\neg \dots \neg}^{m} A \to \overbrace{\neg \dots \neg}^{n} A$ is LJ-provable if m and n have the same parity, i.e. both odd or both even, by repeated use of $[\neg L]$ and $[\neg R]$)

* case A is $B \supset C$: then A^* is $B^* \supset C^*$. We have $\neg\neg B^* \to B^*$ and $\neg\neg C^* \to C^*$ LJ-provable by the induction hypothesis and

$$\dfrac{\dfrac{\dfrac{\dfrac{[\text{ex } 3.10.1]}{\neg\neg(B^* \supset C^*), B^* \to \neg\neg C^*} \qquad [\neg\neg C^* \to C^*]}{\neg\neg(B^* \supset C^*), B^* \to C^*}\,[\text{Cut}]}{B^*, \neg\neg(B^* \supset C^*) \to C^*}\,[XL]}{\neg\neg(B^* \supset C^*) \to B^* \supset C^*}\,[\supset R]$$

* case A is $\forall x B(x)$: then A^* is $\forall x B^*(x)$. We have $\neg\neg B^*(a) \to B^*(a)$ LJ-provable by the induction hypothesis and

$$\dfrac{\dfrac{\dfrac{\dfrac{B^*(a) \to B^*(a)}{\forall x B^*(x) \to B^*(a)}}{\neg\neg\forall x B^*(x) \to \neg\neg B^*(a)} \qquad [\neg\neg B^*(a) \to B^*(a)]}{\neg\neg\forall x B^*(x) \to B^*(a)}\,[\text{Cut}]}{\neg\neg\forall x B^*(x) \to \forall x B^*(x)}\,[\forall R]$$

* case A is $\exists x B(x)$: then A^* is $\neg\forall x \neg B^*(x)$. We have $\neg\neg B^*(a) \to B^*(a)$ LJ-provable by the induction hypothesis and

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x \neg B^*(x) \to \forall x \neg B^*(x)}{\forall x \neg B^*(x) \to \neg\neg\forall x \neg B^*(x)}}{\neg\neg\neg\forall x \neg B^*(x), \forall x \neg B^*(x) \to}}{\forall x \neg B^*(x), \neg\neg\neg\forall x \neg B^*(x) \to}}{\neg\neg\neg\forall x \neg B^*(x) \to \neg\forall x \neg B^*(x)}$$

This completes the proof.

4) We use induction on the number n of inferences, as suggested in the prescription.

* case $n=0$ and S is an initial sequent $A \to A$, then

$$A^* \to A^*$$
$$\neg A^*, A^* \to$$
$$A^*, \neg A^* \to$$

* case the last inference is Weakening left, then

$$\dfrac{\begin{matrix}\vdots \\ A_1, \ldots, A_m \to B_1, \ldots, B_n\end{matrix}}{D, A_1, \ldots, A_m \to B_1, \ldots, B_n}\,[WL]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\begin{matrix}\vdots \\ A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to\end{matrix}}{D^*, A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to}\,[WL]$$

* case the last inference is Weakening right, then

$$\dfrac{\begin{matrix}\vdots \\ A_1, \ldots, A_m \to B_1, \ldots, B_n\end{matrix}}{A_1, \ldots, A_m \to B_1, \ldots, B_n, D}\,[WR]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to\end{matrix}}{\neg D^*, A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to}\,[WL]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^*, \neg D^* \to}\,[\text{several exchanges}]$$

*** case** the last inference is Contraction left, then

$$\dfrac{\begin{matrix}\vdots\\ D, D, A_1, \dots, A_m \to B_1, \dots, B_n\end{matrix}}{D, A_1, \dots, A_m \to B_1, \dots, B_n}\,[CL]$$

and from the induction hypothesis the following is LJ-provable

$$\dfrac{\begin{matrix}\vdots\\ D^*, D^*, A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to\end{matrix}}{D^*, A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[CL]$$

*** case** the last inference is Contraction right, then

$$\dfrac{\begin{matrix}\vdots\\ A_1, \dots, A_m \to B_1, \dots, B_n, D, D\end{matrix}}{A_1, \dots, A_m \to B_1, \dots, B_n, D}\,[CR]$$

and from the induction hypothesis the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^*, \neg D^*, \neg D^* \to\end{matrix}}{\neg D^*, \neg D^*, A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[\text{several exchanges}]}{\neg D^*, A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[CL]}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^*, \neg D^* \to}\,[\text{several exchanges}]$$

*** case** the last inference is Exchange left, then

$$\dfrac{\begin{matrix}\vdots\\ A_1, \dots, A_i, A_{i+1}, \dots, A_m \to B_1, \dots B_n\end{matrix}}{A_1, \dots, A_{i+1}, A_i, \dots, A_m \to B_1, \dots, B_n}\,[XL]$$

and from the induction hypothesis the following is LJ-provable

$$\dfrac{\begin{matrix}\vdots\\ A_1^*, \dots, A_i^*, A_{i+1}^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to\end{matrix}}{A_1^*, \dots, A_{i+1}^*, A_i^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[XL]$$

* case the last inference is Exchange right, then

$$\dfrac{A_1,\dots,A_m \to B_1,\dots,B_i,B_{i+1},\dots,B_n}{A_1,\dots,A_m \to B_1,\dots,B_{i+1},B_i,\dots,B_n}\,[XR]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_i^*,\neg B_{i+1}^*,\dots,\neg B_n^* \to}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_{i+1}^*,\neg B_i^*,\dots,\neg B_n^* \to}\,[XL]$$

* case the last inference is Cut, then we have

$$[Cut]\ \dfrac{A_1,\dots,A_m \to B_1,\dots,B_n,F \qquad F,C_1,\dots,C_p \to D_1,\dots,D_q}{A_1,\dots,A_m,C_1,\dots,C_p \to B_1,\dots,B_m,D_1,\dots,D_q}$$

where F is the cut formula. From the induction hypothesis the following is LJ-provable

$$(a)\quad \dfrac{\dfrac{\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg F^* \to}{\neg F^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to}\,[\text{several exchanges}]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to \neg\neg F^*}\,[\neg R] \qquad \dfrac{}{\neg\neg F^* \to F^*}\,[\text{ex } 3.12.3]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to F^*}\,[Cut]$$

$$\dfrac{\dfrac{F^*,C_1^*,\dots,C_p^*,\neg D_1^*,\dots,\neg D_q^* \to}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,C_1^*,\dots,C_p^*,\neg D_1^*,\dots,\neg D_q^* \to}\,[cut(a)]}{A_1^*,\dots,A_m^*,C_1^*,\dots,C_p^*,\neg B_1^*,\dots,\neg B_n^*,\neg D_1^*,\dots,\neg D_q^* \to}\,[exch.]$$

* case the last inference is $\neg$:left, then

$$\dfrac{A_1,\dots,A_m \to B_1,\dots,B_n,D}{\neg D,A_1,\dots,A_m \to B_1,\dots,B_n}\,[\neg L]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\begin{array}{c}\vdots\\ A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^*,\neg D^* \to\end{array}}{\neg D^*, A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^* \to}\;[\text{several exchanges}]$$

* case the last inference is $\neg$: right, then

$$\dfrac{\begin{array}{c}\vdots\\ D, A_1,\ldots,A_m \to B_1,\ldots,B_n\end{array}}{A_1,\ldots,A_m \to B_1,\ldots,B_n,\neg D}\;[\neg R]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ D^*, A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^* \to\end{array}}{A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^* \to \neg D^*}\;[\neg R]}{\neg\neg D^*, A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^* \to}\;[\neg L]}{A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^*,\neg\neg D^* \to}\;[\text{several exchanges}]$$

* case the last inference is $\wedge$: left, then

$$\dfrac{\begin{array}{c}\vdots\\ C, A_1,\ldots,A_m \to B_1,\ldots,B_n\end{array}}{C\wedge D, A_1,\ldots,A_m \to B_1,\ldots,B_n}\;[\wedge L]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\begin{array}{c}\vdots\\ C^*, A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^* \to\end{array}}{C^*\wedge D^*, A_1^*,\ldots,A_m^*,\neg B_1^*,\ldots,\neg B_n^* \to}\;[\wedge L]$$

The left-conjunct variant is treated similarly.

* case the last inference is $\wedge$: right, then

$$\dfrac{A_1,\dots,A_m \to B_1,\dots,B_n, C \qquad A_1,\dots,A_m \to B_1,\dots,B_n, D}{A_1,\dots,A_m \to B_1,\dots,B_n, C\wedge D}\,[\wedge R]$$

and from the induction hypotheses the following is LJ-provable

$$\text{(a)}\quad \dfrac{\dfrac{\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg C^* \to}{\neg C^*, A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to}\,[\text{exch}]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to \neg\neg C^*}\,[\neg R] \qquad \dfrac{[\text{ex } 3.12.3]}{\neg\neg C^* \to C^*}}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to C^*}\,[\text{cut}]$$

$$\text{(b)}\quad \dfrac{\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg D^* \to}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to \neg\neg D^*}\,\left[\begin{array}{l}\text{same as}\\ \text{above}\end{array}\right] \qquad \dfrac{[\text{ex } 3.12.3]}{\neg\neg D^* \to D^*}}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to D^*}\,[\text{cut}]$$

and finally using $\wedge$:right with (a) and (b) we obtain

$$\dfrac{\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to C^*\wedge D^*}{\neg(C^*\wedge D^*), A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to}\,[\neg L]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg(C^*\wedge D^*) \to}\,\left[\begin{array}{l}\text{several}\\ \text{exchanges}\end{array}\right]$$

* case the last inference is $\vee$: left, then

$$\dfrac{C, A_1,\dots,A_m \to B_1,\dots,B_n \qquad D, A_1,\dots,A_m \to B_1,\dots,B_n}{C\vee D, A_1,\dots,A_m \to B_1,\dots,B_n}\,[\vee L]$$

and from the induction hypotheses the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{\dfrac{C^*, A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow \neg C^*}[\neg R] \quad \dfrac{D^*, A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow \neg D^*}[\neg R]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow \neg C^* \wedge \neg D^*}[\wedge R]}{\neg(\neg C^* \wedge \neg D^*), A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow}[\neg L]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^*, \neg(\neg C^* \wedge \neg D^*) \rightarrow}[\text{several exch.}]$$

* case the last inference is $\vee$: right, then

$$\dfrac{A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n, C}{A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n, C \vee D}[\vee R]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_m^*, \neg C^* \rightarrow}{\neg C^*, A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_m^* \rightarrow}[\text{several exchanges}]}{\neg C^* \wedge \neg D^*, A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_m^* \rightarrow}[\wedge L]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_m^* \rightarrow \neg(\neg C^* \wedge \neg D^*)}[\neg R]}{\neg\neg(\neg C^* \wedge \neg D^*), A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_m^* \rightarrow}[\neg L]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_m^*, \neg\neg(\neg C^* \wedge \neg D^*) \rightarrow}[\text{several exchanges}]$$

The left-disjunct variant is treated similarly.

* case the last inference is $\supset$: left, then

$$\dfrac{A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n, F \qquad G, C_1, \ldots, C_p \rightarrow D_1, \ldots, D_q}{F \supset G, A_1, \ldots, A_m, C_1, \ldots, C_p \rightarrow B_1, \ldots, B_n, D_1, \ldots, D_q}[\supset L]$$

and from the induction hypotheses, the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg F^* \to}{\neg F^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to}\,[\text{exch}]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to \neg\neg F^*}\,[\neg R] \qquad \dfrac{[\text{ex } 3.12.3]}{\neg\neg F^* \to F^*}}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to F^*}\,[\text{Cut}]$$

$$[\supset L]\;\dfrac{\dfrac{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to F^* \qquad G^*,C_1^*,\dots,C_p^*,\neg D_1^*,\dots,\neg D_q^* \to}{F^*\supset G^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,C_1^*,\dots,C_p^*,\neg D_1^*,\dots,\neg D_q^* \to}}{F^*\supset G^*,A_1^*,\dots,A_m^*,C_1^*,\dots,C_p^*,\neg B_1^*,\dots,\neg B_n^*,\neg D_1^*,\dots,\neg D_q^* \to}\;[\text{exch}]$$

* case the last inference is $\supset$:right, then

$$\dfrac{C,A_1,\dots,A_m \to B_1,\dots,B_n,D}{A_1,\dots,A_m \to B_1,\dots,B_n,C\supset D}\,[\supset R]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg D^* \to}{\neg D^*,C^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to}\,[\text{exch}]}{C^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to \neg\neg D^*}\,[\neg R] \qquad \dfrac{[\text{ex } 3.12.3]}{\neg\neg D^* \to D^*}}{C^*,A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to D^*}\,[\text{Cut}]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to C^*\supset D^*}\,[\supset R]}{\neg(C^*\supset D^*),A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^* \to}\,[\neg L]}{A_1^*,\dots,A_m^*,\neg B_1^*,\dots,\neg B_n^*,\neg(C^*\supset D^*) \to}\,[\text{exch}]$$

* case the last inference is $\forall$:left, then

$$\dfrac{F(t),A_1,\dots,A_m \to B_1,\dots,B_n}{\forall x F(x),A_1,\dots,A_m \to B_1,\dots,B_n}\,[\forall L]$$

and from the induction hypothesis, the following is LJ-provable

$$\dfrac{F^*(t), A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}{\forall x\, F^*(x), A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[\forall L]$$

* case the last inference is $\forall$:right, then

$$\dfrac{A_1, \dots, A_m \to B_1, \dots, B_n, F(a)}{A_1, \dots, A_m \to B_1, \dots, B_n, \forall x\, F(x)}\,[\forall R]$$

where a does not appear in A_i or B_i. From the induction hypothesis, the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^*, \neg F^*(a) \to}{\neg F^*(a), A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[\text{exch}]}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to \neg\neg F^*(a)}\,[\neg R] \qquad \dfrac{[\text{ex } 3.12.3]}{\neg\neg F^*(a) \to F^*(a)}}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to F^*(a)}\,[\text{cut}]}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to \forall x\, F^*(x)}\,[\forall R]}{\neg \forall x\, F^*(x), A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[\neg L]}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^*, \neg\forall x\, F^*(x) \to}\,[\text{exch}]$$

* case the last inference is $\exists$:left, then

$$\dfrac{F(a), A_1, \dots, A_m \to B_1, \dots, B_n}{\exists x\, F(x), A_1, \dots, A_m \to B_1, \dots, B_n}\,[\exists L]$$

where a does not appear in A_i or B_i. From the induction hypothesis, the following is LJ-provable

$$\dfrac{\dfrac{\dfrac{F^*(a), A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to \neg F^*(a)}\,[\neg R]}{A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to \forall x \neg F^*(x)}\,[\forall R]}{\neg\forall x \neg F^*(x), A_1^*, \dots, A_m^*, \neg B_1^*, \dots, \neg B_n^* \to}\,[\neg L]$$

* case the last inference is $\exists$: right, then

$$\frac{A_1, \ldots, A_m \to B_1, \ldots, B_n, F(t)}{A_1, \ldots, A_m \to B_1, \ldots, B_n, \exists x F(x)} [\exists R]$$

and from the induction hypothesis the following is LJ-provable

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^*, \neg F^*(t) \to}{\neg F^*(t), A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to} [exch]}{\forall x \neg F^*(x), A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to} [\forall L]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to \neg \forall x \neg F^*(x)} [\neg R]}{\neg\neg \forall x \neg F^*(x), A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \to} [\neg L]}{A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^*, \neg\neg \forall x \neg F^*(x) \to} [exch]$$

□

Hence, we see that if S is LK-provable, then S' is LJ-provable, from result 4). On the other hand, if S' is LJ-provable, it is also LK-provable since LK is an extension of LJ, and using result 2), we obtain that S is LK-provable.

Finally, by taking S to be $\to A$, we can see that if A is LK-provable, then S' is LJ-provable and thus

$$\cfrac{\cfrac{\neg A^* \to}{\to \neg\neg A^*} \qquad \cfrac{[ex\ 3.12.3]}{\neg\neg A^* \to A^*}}{\to A^*}$$

and A^* is LJ-provable. In the other direction, if A^* is LJ-provable, then $\neg A^* \to$ is also LJ-provable and thus $\to A$ is LK-provable. This concludes the proof.

□

Lemma 5.2

Replacing a cut rule by a mix rule:

$$\dfrac{\Gamma \to \Delta, D \qquad D, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}\,[\text{cut}]
\qquad\qquad
\begin{array}{c}
\dfrac{\Gamma \to \Delta, D \qquad D, \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\,(D) \\
\hline
\text{some weakenings} \\
\hline
\text{some exchanges} \\
\hline
\Gamma, \Pi \to \Delta, \Lambda
\end{array}$$

Replacing a mix rule by a cut rule:

$$\dfrac{\Gamma \to \Delta \qquad \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\,(D)
\qquad\qquad
\dfrac{
\begin{array}{c}
\Gamma \to \Delta \\
\hline
\text{some exchanges} \\
\hline
\Gamma \to \Delta^*, D, \dots, D \\
\hline
\text{some contractions} \\
\hline
\Gamma \to \Delta^*, D
\end{array}
\qquad
\begin{array}{c}
\Pi \to \Lambda \\
\hline
\text{some exchanges} \\
\hline
D, \dots, D, \Pi^* \to \Lambda \\
\hline
\text{some contractions} \\
\hline
D, \Pi^* \to \Lambda
\end{array}
}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\,[\text{cut}]$$

Proposition 7.2

The first claim can be proven by induction on the number of logical symbols in A. For zero symbols (namely, A is atomic), this is just an equality axiom of LKe.

For the second claim, notice that

$$s = t,\ s = s \to t = s$$

is an equality axiom where $R(a)$ is $a = s$. From this we obtain

$$\dfrac{\rightarrow s=s \qquad \dfrac{s=t,\ s=s \rightarrow t=s}{s=s,\ s=t \rightarrow t=s}\,[XL]}{s=t \rightarrow t=s}\,[cut]$$

Finally, the following is an equality axiom where $R(a)$ is $a = s_3$

$$S_2 = S_1,\ R(s_2) \rightarrow R(s_1)$$
$$S_2 = S_1,\ S_2 = S_3 \rightarrow S_1 = S_3$$

and using the previous result

$$\dfrac{S_1 = S_2 \rightarrow S_2 = S_1 \qquad S_2 = S_1,\ S_2 = S_3 \rightarrow S_1 = S_3}{S_1 = S_2,\ S_2 = S_3 \rightarrow S_1 = S_3}\,[cut]$$

$\square$

Theorem 8.2

Soundness (also called "Validity Theorem" in [Shoenfield, p23]): by induction on the number of inferences. Namely, we show that the axioms are valid and the inference rules preserve validity.

* $n=0$ inferences: axioms are of the form $A \rightarrow A$.
  Given an interpretation τ:
  - if τ satisfies A, then $A \rightarrow A$ is valid since a formula in its succedent is satisfied.
  - if τ falsifies A, then $A \rightarrow A$ is valid since a formula in its antecedent is falsified.

Hence, axioms are valid since they are satisfied in every interpretation.

* $n > 0$ inferences:

Note that for a sequent $\Gamma \to \Delta$ obtained by a structural rule from a sequent $\Gamma' \to \Delta'$, we have that every formula A which occurs in Γ' also occurs in Γ, and every formula B which occurs in Δ' also occurs in Δ. Hence, if $\Gamma' \to \Delta'$ is satisfied by a given interpretation τ then:

- there is a formula $A \in \Gamma'$ falsified by τ. But this formula also appears in Γ and thus $\Gamma \to \Delta$ is also satisfied by τ;
- or there is a formula $B \in \Delta'$ satisfied by τ. But this formula also appears in Δ and thus $\Gamma \to \Delta$ is also satisfied by τ.

Thus, structural rules preserve validity.

Now we cover the logical rules:

* case negation left

$$\frac{\Gamma \to \Delta, D}{\neg D, \Gamma \to \Delta}\ [\neg L]$$

Assume $\Gamma \to \Delta, D$ is satisfied by interpretation τ. Then we have 3 cases:

i) τ falsifies a formula in Γ. Then τ falsifies the same formula in $\neg D, \Gamma$.

ii) τ satisfies a formula in Δ. Then τ satisfies the same formula in Δ in the lower sequent.

iii) τ satisfies D. Then τ falsifies $\neg D$ since $\Phi(\neg D) = \neg \Phi(D)$. So τ falsifies a formula in $\neg D, \Gamma$.

It follows that $\neg D, \Gamma \to \Delta$ is also satisfied by τ.

* case negation right: symmetric to previous.

* case conjunction left: [ex 8.2.1]

$$\frac{C, \Gamma \to \Delta}{C \wedge D, \Gamma \to \Delta}[\wedge L]$$

Assume $C, \Gamma \to \Delta$ is satisfied by interpretation τ. Then we have 3 cases:

i) τ falsifies C. Then τ also falsifies $C \wedge D$ since $\Phi(C \wedge D) = \Phi(C) \wedge \Phi(D)$ and thus τ falsifies a formula in $C \wedge D, \Gamma$.

ii) τ falsifies a formula in Γ. Then τ falsifies the same formula in $C \wedge D, \Gamma$.

iii) τ satisfies a formula in Δ. Then τ satisfies the same formula in Δ in the lower sequent.

It follows that $C \wedge D, \Gamma \to \Delta$ is also satisfied by τ. The case for $D \vee C, \Gamma \to \Delta$ is treated similarly.

* case conjunction right: [ex 8.2.1]

$$\frac{\Gamma \to \Delta, C \qquad \Gamma \to \Delta, D}{\Gamma \to \Delta, C \wedge D}[\wedge R]$$

Assume $\Gamma \to \Delta, C$ and $\Gamma \to \Delta, D$ are satisfied by interpretation τ. We have 3 cases:

i) τ falsifies a formula in Γ. Then τ falsifies the same formula in Γ in the lower sequent.

ii) τ satisfies a formula in Δ. Then τ satisfies the same formula in $\Delta, C \wedge D$.

iii) τ satisfies both C and D. Then τ satisfies $C \wedge D$ since $\phi(C \wedge D) = \phi(C) \wedge \phi(D)$ and thus satisfies a formula in $\Delta, C \wedge D$.

It follows that $\Gamma \to \Delta, C \wedge D$ is also satisfied by τ.

* case disjunction left: [ex 8.2.1]

$$\frac{C, \Gamma \to \Delta \qquad D, \Gamma \to \Delta}{C \vee D, \Gamma \to \Delta} [\vee L]$$

Assume $C, \Gamma \to \Delta$ and $D, \Gamma \to \Delta$ are satisfied by interpretation τ. We have 3 cases:

i) τ satisfies a formula in Δ. Then τ satisfies the same formula in Δ in the lower segment.

ii) τ falsifies a formula in Γ. Then τ falsifies the same formula in $C \vee D, \Gamma$.

iii) τ falsifies both C and D (note that it is necessary that it falsifies both since the other cases cover the other possibilities and we assume both $C, \Gamma \to \Delta$ and $D, \Gamma \to \Delta$ are satisfied). Then it also falsifies $C \vee D$ since $\phi(C \vee D) = \phi(C) \vee \phi(D)$ and thus falsifies a formula in $C \vee D, \Gamma \to \Delta$.

It follows that $C \vee D, \Gamma \to \Delta$ is also satisfied by τ.

* case disjunction right: [ex 8.2.1]

$$\frac{\Gamma \to \Delta, C}{\Gamma \to \Delta, C \vee D} [\vee R]$$

Assume $\Gamma \to \Delta, C$ is satisfied by interpretation τ. We have cases:

i) τ falsifies a formula in Γ. Then τ falsifies the same formula in Γ in the lower segment.

ii) τ satisfies a formula in Δ. Then τ satisfies the same formula in $\Delta, C \vee D$.

iii) τ satisfies C. Then τ also satisfies $C \vee D$ since $\phi(C \vee D) = \phi(C) \vee \phi(D)$ and thus satisfies a formula in $\Delta, C \vee D$.

It follows that $\Gamma \to \Delta, C \vee D$ is also satisfied by τ.

*** case** implication left: **[ex 8.2.1]**

$$\frac{\Gamma \to \Delta, C \qquad D, \Pi \to \Lambda}{C \supset D, \Gamma, \Pi \to \Delta, \Lambda} [\supset L]$$

Assume $\Gamma \to \Delta, C$ and $D, \Pi \to \Lambda$ are satisfied by interpretation τ. We have 5 cases:

i) τ falsifies a formula in Γ. Then it falsifies the same formula in $C \supset D, \Gamma, \Pi$.

ii) τ falsifies a formula in Π. Then it falsifies the same formula in $C \supset D, \Gamma, \Pi$.

iii) τ satisfies a formula in Δ. Then it satisfies the same formula in Δ, Λ.

iv) τ satisfies a formula in Λ. Then it satisfies the same formula in Δ, Λ.

v) τ satisfies C and falsifies D, which is the only remaining possibility. Then τ falsifies $C \supset D$ since $\Phi(C \supset D) = \neg\Phi(C) \vee \Phi(D)$ and thus falsifies a formula in $C \supset D, \Gamma, \Pi$.

It follows that $C \supset D, \Gamma, \Pi \to \Delta, \Lambda$ is also satisfied by τ.

*** case** implication right: **[ex 8.2.1]**

$$\frac{C, \Gamma \to \Delta, D}{\Gamma \to \Delta, C \supset D} [\supset R]$$

Assume that $C, \Gamma \to \Delta, D$ is satisfied by interpretation τ. We have 4 cases:

i) τ falsifies a formula in Γ. Then τ falsifies the same formula in Γ in the lower sequent.

ii) τ satisfies a formula in Δ. Then τ satisfies the same formula in $\Delta, C \supset D$.

iii) τ falsifies C. Then τ satisfies $C \supset D$ since $\Phi(C \supset D) = \neg\Phi(C) \vee \Phi(D)$ and thus satisfies a formula in $\Delta, C \supset D$.

iv) τ satisfies D. Then τ also satisfies $C \supset D$ since $\phi(C \supset D) = \neg\phi(C) \vee \phi(D)$. Thus τ satisfies a formula in $\Delta, C \supset D$.

It follows that $\Gamma \to \Delta, C \supset D$ is also satisfied by τ.

* case forall left: [ex 8.2.2]

$$\frac{F(t), \Gamma \to \Delta}{\forall x\, F(x), \Gamma \to \Delta}\ [\forall L]$$

Assume $F(t), \Gamma \to \Delta$ is satisfied by interpretation τ. The cases where τ falsifies a formula in Γ or satisfies a formula in Δ are treated similarly. The remaining case is when τ falsifies $F(t)$.

Let's denote τ by $(\langle D, \Phi\rangle, \Phi_0)$. Recall that τ satisfies $\forall x\, F(x)$ iff for every Φ_0' such that Φ_0 and Φ_0' agree, except possibly in x, $(\langle D, \Phi\rangle, \Phi_0')$ satisfies $F(x)$.

But we can construct an interpretation $(\langle D, \Phi\rangle, \Phi_0')$ where Φ_0' is identical with Φ_0, except in x, for which we define $\Phi_0'(x) = \Phi(t)$. That is, Φ_0' maps x to the individual which is the meaning of t under τ. This new interpretation falsifies $F(x)$, and thus τ falsifies $\forall x\, F(x)$.

It follows that τ falsifies a formula in $\forall x\, F(x), \Gamma$ and hence $\forall x\, F(x), \Gamma \to \Delta$ is also satisfied by τ.

(Note: here the fact that if $\phi(s) = \phi(t)$, then $\phi(F(s)) = \phi(F(t))$, is tacitly used. Compare with the Lemma 2.5 [Shoenfield, p19], where the language is extended with individual names for this purpose.)

* case for all right: [ex 8.2.2]

$$\frac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \forall x\, F(x)} [\forall R]$$

where a does not appear in Γ or Δ.

Assume that $\Gamma \to \Delta, F(a)$ is valid in structure $\langle D, \phi \rangle$. That is, every interpretation $(\langle D, \phi \rangle, \phi_0)$ satisfies $\Gamma \to \Delta, F(a)$.

Now suppose that $\Gamma \to \Delta, \forall x\, F(x)$ is falsified by some interpretation $(\langle D, \phi \rangle, \phi_0')$. Then, necessarily

i) $(\langle D, \phi \rangle, \phi_0')$ satisfies all formulas in Γ.
ii) $(\langle D, \phi \rangle, \phi_0')$ falsifies all formulas in Δ.
iii) $(\langle D, \phi \rangle, \phi_0')$ falsifies $\forall x\, F(x)$. Or in other words, there is a ϕ_0'' which agrees with ϕ_0' except possibly in x, such that $(\langle D, \phi \rangle, \phi_0'')$ falsifies $F(x)$.

We can construct a new mapping ϕ_0^* which agrees with the ϕ_0'' above, except possibly in a, by setting $\phi_0^*(a) = \phi_0''(x)$. Since a does not occur in Γ or Δ, $(\langle D, \phi \rangle, \phi_0^*)$ still satisfies all formulas in Γ and falsifies all formulas in Δ. Additionally, $(\langle D, \phi \rangle, \phi_0^*)$ also falsifies $F(a)$ since $(\langle D, \phi \rangle, \phi_0'')$ falsifies $F(x)$. Thus, $(\langle D, \phi \rangle, \phi_0^*)$ also falsifies $\Gamma \to \Delta, F(a)$; a contradiction.

Hence, no such interpretation $(\langle D, \phi \rangle, \phi_0')$ exists and $\Gamma \to \Delta, \forall x\, F(x)$ is also valid in $\langle D, \phi \rangle$.

* case exists left: [ex 8.2.2]

$$\frac{F(a), \Gamma \to \Delta}{\exists x\, F(x), \Gamma \to \Delta} [\exists L]$$

where a does not appear in Γ or Δ.

As in previous case, assume that $F(a), \Gamma \to \Delta$ is valid in $\langle D, \phi \rangle$. That is, $F(a), \Gamma \to \Delta$ is satisfied by every interpretation $(\langle D, \phi\rangle, \phi_0)$.

Now suppose that $\exists x\, F(x), \Gamma \to \Delta$ is falsified by some interpretation $(\langle D, \phi\rangle, \phi_0')$. Then, necessarily

i) $(\langle D, \phi\rangle, \phi_0')$ satisfies $\exists x\, F(x)$. In other words, there is some ϕ_0'' such that ϕ_0' and ϕ_0'' agree except possibly in x, and $(\langle D, \phi\rangle, \phi_0'')$ satisfies $F(x)$.

ii) $(\langle D, \phi\rangle, \phi_0')$ satisfies all formulas in Γ.

iii) $(\langle D, \phi\rangle, \phi_0')$ falsifies all formulas in Δ

We can construct a new mapping ϕ_0^* which agrees with ϕ_0'', except possibly in a, by setting $\phi_0^*(a) = \phi_0''(x)$. Since a does not occur in Γ or Δ, $(\langle D, \phi\rangle, \phi_0^*)$ still satisfies all formulas in Γ and falsifies all formulas in Δ. Additionally, $(\langle D, \phi\rangle, \phi_0^*)$ also satisfies $F(a)$ since $(\langle D, \phi\rangle, \phi_0'')$ satisfies $F(x)$. Thus $(\langle D, \phi\rangle, \phi_0^*)$ falsifies $F(a), \Gamma \to \Delta$; a contradiction.

Hence no such interpretation $(\langle D, \phi\rangle, \phi_0')$ exists and $\exists x\, F(x), \Gamma \to \Delta$ is also valid in $\langle D, \phi\rangle$.

* case exists right: [ex 8.2.2]

$$\frac{\Gamma \to \Delta, F(t)}{\Gamma \to \Delta, \exists x F(x)} [\exists R]$$

Assume that $\Gamma \to \Delta, F(t)$ is satisfied by interpretation τ. The cases where τ falsifies a formula in Γ or satisfies a formula in Δ are treated similarly. The remaining case is when τ satisfies $F(t)$. Let's denote τ by $(\langle D, \phi \rangle, \phi_0)$. Recall that τ satisfies $\exists x F(x)$ iff there is a ϕ_0' which agrees with ϕ_0 except possibly in x and $(\langle D, \phi \rangle, \phi_0')$ satisfies $F(x)$. We can contruct such a ϕ_0' from ϕ_0, by setting $\phi_0'(x) = \phi(t)$. Then $(\langle D, \phi \rangle \phi_0')$ satisfies $F(x)$ and thus $(\langle D, \phi \rangle, \phi_0)$ satisfies $\exists x F(x)$.

Hence, τ satisfies a formula in $\Delta, \exists x F(x)$ and $\Gamma \to \Delta, \exists x F(x)$ is also satisfied by τ.

This concludes the proof. $\square$

Reduction Trees

$$(\ldots)$$

$K=12 \quad A(a,b), \exists y A(a,y), \forall x \exists y A(x,y) \to$

$K=12 \quad A(a,b), \exists y A(a,y), \forall x \exists y A(x,y) \to$

$K=10 \quad A(a,b), \exists y A(a,y), \forall x \exists y A(x,y) \to$

$K=8 \quad \exists y A(a,y), \forall x \exists y A(x,y) \to$

$\forall x \exists y A(x,y) \to$

$K \equiv 0$ $\qquad\qquad$ $A, A \supset B \to \neg A \vee B, \neg A, B, A$

$K \equiv 6$ $\quad$ $B, A \supset B \to \neg A \vee B, \neg A, B$ $\qquad$ $A \supset B \to \neg A \vee B, \neg A, B, A$

$K \equiv 5$ $\quad$ $A \supset B \to \neg A \vee B, \neg A, B$

S $\quad$ $A \supset B \to \neg A \vee B$

8.3.1

$K \equiv 1$ $\quad$ $F(a), \neg \exists x F(x) \to \forall y \neg F(y), \exists x F(x), \neg F(a), F(a)$

$K \equiv 11$ $\quad$ $\neg \exists x F(x) \to \forall y \neg F(y), \exists x F(x), \neg F(a), F(a)$

$K \equiv 9$ $\quad$ $\neg \exists x F(x) \to \forall y \neg F(y), \exists x F(x), \neg F(a)$

$K \equiv 0$ $\quad$ $\neg \exists x F(x) \to \forall y \neg F(y), \exists x F(x)$

S $\quad$ $\neg \exists x F(x) \to \forall y \neg F(y)$

8.3.2

$K=8$ $A(b), A(b)\supset B(b), A(a), \forall x A(x), \ldots \rightarrow \ldots, \exists x B(x), B(a), A(b)$

$K=6$ $A(b)\supset B(b), A(a), \forall x A(x), \ldots \rightarrow \ldots, \exists x B(x), B(a), A(b)$

$K=11$ $A(b), B(b), A(b)\supset B(b), A(a), \forall x A(x), \ldots \rightarrow \ldots, \exists x B(x), B(a), B(b)$

$K=8$ $A(b), B(b), A(b)\supset B(b), A(a), \forall x A(x), \ldots \rightarrow \ldots, \exists x B(x), B(a)$

$K=6$ $B(b), A(b)\supset B(b), A(a), \forall x A(x), \ldots \rightarrow \ldots, \exists x B(x), B(a)$

$K=11$ $A(b)\supset B(b), A(a), \forall x A(x), \exists x(A(x)\supset B(x)) \rightarrow \forall x A(x)\supset \exists x B(x), \exists x B(x), B(a)$

$K=10$ $A(b)\supset B(b), A(a), \forall x A(x), \exists x(A(x)\supset B(x)) \rightarrow \forall x A(x)\supset \exists x B(x), \exists x B(x)$

$K=8$ $A(a), \forall x A(x), \exists x(A(x)\supset B(x)) \rightarrow \forall x A(x)\supset \exists x B(x), \exists x B(x)$

$K=7$ $\forall x A(x), \exists x(A(x)\supset B(x)) \rightarrow \forall x A(x)\supset \exists x B(x), \exists x B(x)$

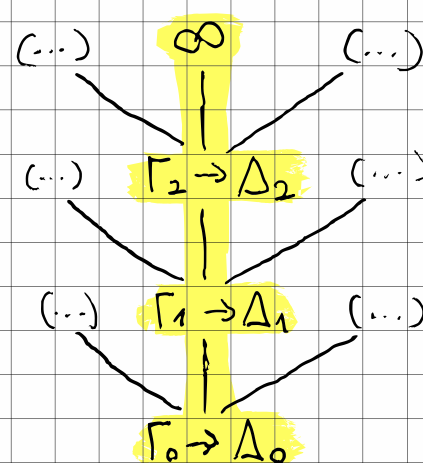
S $\exists x(A(x)\supset B(x)) \rightarrow \forall x A(x)\supset \exists x B(x)$

Given a sequent S and its reduction tree $T(S)$. Let's assume that $T(S)$ has an infinite branch $S_0, S_1, \ldots$ and let S_i be $\Gamma_i \to \Delta_i$, $\cup\Gamma$ the set of all formulas in Γ_i for same i and $\cup\Delta$ the set of all formulas in Δ_i for same i.

Note that $\cup\Gamma$ and $\cup\Delta$ have no formula in common. This is because:

i) If Γ_i and Δ_i had a formula in common for same i, the branch would have stopped at that sequent

ii) If Γ_i and Δ_j had a formula in common for $i<j$, it means that such formula was eliminated at same Γ_K with $i<K<j$. But this is clearly impossible since none of the reductions remove formulas from sequents.



Now we construct an interpretation that falsifies S, which is $\Gamma_0 \to \Delta_0$. More precisely, an interpretation that satisfies all formulas in $\cup\Gamma$ and falsifies all formulas in $\cup\Delta$. Let this interpretation be

$$\tau = (\langle D, \phi \rangle, \phi_0)$$

where

D the set of all free variables
$\phi_0(a) = a$
$\phi_0(x) =$ doesn't matter
ϕR = a subset of D^n such that

$$(a_1, \ldots, a_n) \in \phi R \quad \text{iff} \quad `R(a_1, \ldots, a_n)' \in \cup\Gamma$$

Now we prove that for any A in $\cup\Gamma$ or $\cup\Delta$, τ satisfies A if A in $\cup\Gamma$ or τ falsifies A if A in $\cup\Delta$, by induction on the number of logical symbols in A.

* case A is atomic: recall that there are no individual or function constants. Then A is of the form $R(a_1, \dots, a_n)$:

i) If A appears in $\cup\Gamma$, then $R(a_1, \dots, a_n)$ also appears in $\cup\Gamma$ and by the definition of Φ, $(a_1, \dots, a_n) \in \Phi R$. So τ satisfies A.
ii) If A appears in $\cup\Delta$, then $R(a_1, \dots, a_n)$ also appears in $\cup\Delta$ and by the definition of Φ, $(a_1, \dots, a_n) \notin \Phi R$. So τ falsifies A.

* case A is $\neg B$:
i) A occurs in Γ_i: then at some point, a $\neg$:left reduction will cause B to occur in Δ_j with $i<j$. By the induction hypothesis, τ falsifies B.
ii) A occurs in Δ_i: then at some point, a $\neg$:right reduction will cause B to occur in Γ_j with $i<j$. By the induction hypothesis, τ satisfies B.

* case A is $B \wedge C$:
i) A occurs in Γ_i: then at some point, a $\wedge$:left reduction will cause both B and C to appear in Γ_j, with $i<j$. By the induction hypothesis, τ satisfies B and C.
ii) A occurs in Δ_i: then at some point, a $\wedge$:right reduction will cause either B or C to occur in Δ_j with $i<j$. By the induction hypothesis, τ falsifies that B or C which appears in the branch. (note that every sequent produced by such reduction always contains either B or C)

* case A is $B \vee C$: [ex 8.3.3]

i) A occurs in Γ_i: then at some point a $\vee$:left reduction will cause either B or C to appear in Γ_j for some $j > i$. By the induction hypothesis, τ satisfies such B or C which appears in the branch.

ii) A occurs in Δ_i: then at some point a $\vee$:right reduction will cause both B and C to appear in Δ_j for some $j > i$. By the induction hypothesis, τ falsifies both B and C.

* case A is $B \supset C$:

i) A occurs in Γ_i: at some point, a $\supset$:left reduction will be applied. In each of the resulting sequents, either B will appear in the succedent or C will appear in the antecedent; so by the induction hypothesis, τ will falsify B or satisfy C, respectively.

ii) A occurs in Δ_i: then at some point a $\supset$:right reduction will cause B to occur in Γ_j and C to occur in Δ_j, for some $j > i$. By the induction hypothesis, τ satisfies B and falsifies C.

* case A is $\forall x A(x)$:

i) A occurs in Γ_i: note that, since the branch is infinite and the $\forall$:left does not have a restriction on the formula to which it applies in terms of previous reductions, it will be applied to $\forall x A(x)$ an infinite number of times, each time using a new free variable. As a consequence, $A(a)$ appears in $\cup\Gamma$ for every free variable a. By the induction hypothesis, τ satisfies $A(a)$ for every free variable a. Finally, note that this is a sufficient condition for τ to satisfy $\forall x A(x)$, since the domain D consists of all free variables and $\phi_0(a)=a$.

ii) A occurs in Δ_i: at some point, a $\forall$:right reduction will cause $A(a)$ to appear in Δ_j with $i<j$, and a is a free variable not available at stage j. By the induction hypothesis, τ falsifies $A(a)$. But since the domain is D, consisting of all free variables and, $\phi_0(a)=a$, note that the interpretation $(\langle D, \Phi\rangle, \Phi_0')$, where ϕ_0' agrees with ϕ_0 except for $\phi_0'(x)=a$, falsifies $A(x)$. Hence, τ falsifies $\forall x A(x)$.

*** case** A is $\exists x A(x)$: symmetric to previous.

Summary of the argument:

(1)	S provable	$\to$	S valid	[Soundness]
(2)	not S valid	$\to$	not S provable	[Contrapos. (1)]
(3)	$T(S)$ finite	$\to$	S provable	[Lemma 8.3]
(4)	not $T(S)$ finite	$\to$	not S valid	[Lemma 8.3]
(5)	not $T(S)$ finite	$\to$	not S provable	[(4), (2)]
(6)	$T(S)$ finite	$\equiv$	S provable	[(3), (5)]
(7)	not S provable	$\to$	not S valid	[(6), (4)]
(8)	S provable	$\equiv$	S valid	[(1), (7)]

$\square$

Notes:
- the proof utilizes a "semantics-by-syntax" trick, similar to Henkin's. Check other proofs as well.
- handle constant and function symbols!

Cut-Elimination in LKe (Lemma 5.4)

*** case 1:** $r = 2$ ($\mathrm{rank}_\ell(P) = 1$, $\mathrm{rank}_r(P) = 1$)

1.1) S_1 is an initial sequent

$$\dfrac{A \to A \qquad \Pi \to \Lambda}{A, \Pi^* \to \Lambda}\ (A)$$

it can be obtained, instead as follows

$$\dfrac{\dfrac{\Pi \to \Lambda}{A, \dots, A, \Pi^* \to \Lambda}\ \text{[some exchanges]}}{A, \Pi^* \to \Lambda}\ \text{[some contractions]}$$

1.2) S_2 is an initial sequent: symmetric to 1.1).

1.3) S_1 is obtained from a structural inference. Since $\mathrm{rank}_\ell(P) = 1$, only a weakening: left could introduce the mix formula:

$$\dfrac{[WL]\ \dfrac{\Gamma \to \Delta^*}{\Gamma \to \Delta^*, A} \qquad \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\ (A)$$

and it can be obtained instead as follows

$$\dfrac{\dfrac{\Gamma \to \Delta^*}{\Pi^*, \Gamma \to \Delta^*, \Lambda}\ \text{[some weakenings]}}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\ \text{[some exchanges]}$$

1.4) S_2 is obtained from a structural inference: symmetric to 1.3).

1.5) S_1 and S_2 are obtained from logical inferences: since $rank_l(P) = rank_r(P) = 1$, the mix formula must be the principal formula for both inferences. Each case is listed next:

* case A is $B \land C$:

$$\dfrac{\dfrac{\Gamma \to \Delta, B \qquad \Gamma \to \Delta, C}{\Gamma \to \Delta, B\land C}\,[\land R] \qquad \dfrac{B, \Pi \to \Lambda}{B\land C, \Pi \to \Lambda}\,[\land L]}{\Gamma, \Pi \to \Delta, \Lambda}\,(B\land C) \quad (r=1, g)$$

can be replaced with

$$\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, B \qquad B, \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\,(B) \quad (r=1, g-1)}{B, \dots, B, \Gamma, \Pi^* \to \Delta^*, \Lambda, B, \dots, B}\,[\text{several weakenings}]}{\Gamma, \Pi \to \Delta, \Lambda}\,[\text{several exchanges}]$$

* case A is $B \lor C$: symmetric to previous case.

* case A is $\neg B$:

$$\dfrac{\dfrac{B, \Gamma \to \Delta}{\Gamma \to \Delta, \neg B}\,[\neg R] \qquad \dfrac{\Pi \to \Lambda, B}{\neg B, \Pi \to \Lambda}\,[\neg L]}{\Gamma, \Pi \to \Delta, \Lambda}\,(\neg B) \quad (r=1, g)$$

can be replaced with

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \to \Lambda, B \qquad B, \Gamma \to \Delta}{\Pi, \Gamma^* \to \Lambda^*, \Delta}\,(B) \quad (r=1, g-1)}{B, \dots, B, \Pi, \Gamma^* \to \Lambda^*, \Delta, B, \dots, B}\,[\text{some weakenings}]}{\Pi, \Gamma \to \Lambda, \Delta}\,[\text{some exchanges}]}{\Gamma, \Pi \to \Delta, \Lambda}\,[\text{some more exchanges}]$$

* **case** A is $B \supset C$:

$$\dfrac{\dfrac{B, \Gamma \to \Delta, C}{\Gamma \to \Delta, B \supset C}\,[\supset R] \qquad \dfrac{\Pi_1 \to \Lambda_1, B \qquad C, \Pi_2 \to \Lambda_2}{B \supset C, \Pi_1, \Pi_2 \to \Lambda_1, \Lambda_2}\,[\supset L]}{\Gamma, \Pi_1, \Pi_2 \to \Delta, \Lambda_1, \Lambda_2}\,(B \supset C)_{(r=1,\, g)}$$

can be replaced with

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \to \Lambda_1, B \qquad B, \Gamma \to \Delta, C}{\Pi_1, \Gamma^* \to \Lambda_1^*, \Delta, C}\,(B)\,(r=1,\, g-1) \qquad C, \Pi_2 \to \Lambda_2}{\Pi_1, \Gamma^*, \Pi_2^\# \to \Lambda_1^{*\#}, \Delta^\#, \Lambda_2}\,(C)\,(r=1,\, g-1)}{B, \ldots, B, C, \ldots, C, \Pi_1, \Gamma^*, \Pi_2^\# \to \Lambda_1^{*\#}, \Delta^\#, \Lambda_2, B, \ldots, B, C, \ldots, C}\,[\text{weaks.}]}{\Pi_1, \Gamma, \Pi_2 \to \Lambda_1, \Delta, \Lambda_2}\,[\text{exchs.}]}{\Gamma, \Pi_1, \Pi_2 \to \Delta, \Lambda_1, \Lambda_2}\,[\text{some exchanges}]$$

where cedents with $*$ do not contain B and cedents with $\#$ do not contain C. Note that the new mixes have lesser grade than the original mix.

* **case** A is $\forall x B(x)$:

$$\dfrac{\dfrac{\Gamma \to \Delta, B(a)}{\Gamma \to \Delta, \forall x B(x)}\,[\forall R] \qquad \dfrac{B(t), \Pi \to \Lambda}{\forall x B(x), \Pi \to \Lambda}\,[\forall L]}{\Gamma, \Pi \to \Delta, \Lambda}\,(\forall x B(x)) \quad (r=1,\, g)$$

using Lemma 2.12 we can obtain a proof of $\Gamma \to \Delta, B(t)$ from the proof of $\Gamma \to \Delta, B(a)$ and

$$\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, B(t) \qquad B(t), \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\,(B(t)) \quad (r=1,\, g-1)}{B(t), \ldots, B(t), \Gamma, \Pi^* \to \Delta^*, \Lambda, B(t), \ldots, B(t)}\,[\text{some weakenings}]}{\Gamma, \Pi \to \Delta, \Lambda}\,[\text{some exchanges}]$$

* **case** A is $\exists x B(x)$: symmetric to previous.

1.6) S_1 is an equality axiom [ex 7.6]

$$r_1 = s_1, \ldots, r_n = s_n, R(r_1, \ldots, r_n) \to R(s_1, \ldots, s_n)$$

and since S_2 cannot be an initial sequent, or have been obtained from a structural inference, or have been obtained from a logical inference (since R is atomic and the principal formula in a logical inference has an outermost logical symbol), it must also be an equality axiom of the form

$$s_1 = t_1, \ldots, s_n = t_n, R(s_1, \ldots, s_n) \to R(t_1, \ldots, t_n)$$

and the endsequent of the mix is

$$r_1 = s_1, \ldots, r_n = s_n, R(r_1, \ldots, r_n), s_1 = t_1, \ldots, s_n = t_n \to R(t_1, \ldots, t_n).$$

We can obtain the same endsequent as follows

$$\dfrac{\dfrac{r_1 = s_1, s_1 = t_1 \to r_1 = t_1 \qquad r_1 = t_1, \ldots, r_n = t_n, R(r_1, \ldots, r_n) \to R(t_1, \ldots, t_n)}{\dfrac{r_1 = s_1, s_1 = t_1, \ldots, r_n = t_n, R(r_1, \ldots, r_n) \to R(t_1, \ldots, t_n)}{r_2 = t_2, \ldots, r_n = t_n, r_1 = s_1, s_1 = t_1, R(r_1, \ldots, r_n) \to R(t_1, \ldots, t_n)}\ \text{several exchanges}}}{}$$

cuts with $r_i = s_i, s_i = t_i \to r_i = t_i$ for $2 \leq i \leq n$ and several exchanges

$$\dfrac{\overline{r_1 = s_1, s_1 = t_1, \ldots, r_n = s_n, s_n = t_n, R(r_1, \ldots, r_n) \to R(t_1, \ldots, t_n)}}{r_1 = s_1, \ldots, r_n = s_n, R(r_1, \ldots, r_n), s_1 = t_1, \ldots, s_n = t_n \to R(t_1, \ldots, t_n)}\ \text{several exchanges}$$

, where all the introduced cuts are inessential.

* case 2: $r > 2$ (omitted. See [Takeuti] p25)

Peano Arithmetic (extends LKe)

Nonlogical constants: 0
Nonlogical symbols: $=\ '\ +\ \cdot$
Nonlogical axioms:

$$\left.\begin{array}{l} \to s = s \\ s_1 = t_1, \ldots, s_n = t_n \to f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \\ s_1 = t_1, \ldots, s_n = t_n, R(s_1, \ldots, s_n) \to R(t_1, \ldots, t_n) \end{array}\right\}$$ From LKe. These characterize equality.

$$\left.\begin{array}{l} s' = t' \to s = t \\ s' = 0 \to \end{array}\right\}$$ These characterize the successor function.

$$\left.\begin{array}{l} \to s + 0 = s \\ \to s + t' = (s+t)' \end{array}\right\}$$ These characterize addition.

$$\left.\begin{array}{l} \to s \cdot 0 = 0 \\ \to s \cdot t' = s \cdot t + s \end{array}\right\}$$ These characterize multiplication.

Rules:

$$\frac{F(a), \Gamma \to \Delta, F(a')}{F(0), \Gamma \to \Delta, F(s)}\,[\mathrm{ind}]$$ with a not in $F(0)$, Γ or Δ and s an arbitrary term.

PA 1

$$\dfrac{\to 0''+0=0'' \qquad \overbrace{0''+0}^{s}=\overbrace{0''}^{t} \to \overbrace{(0''+0)'}^{f(s)}=\overbrace{0'''}^{f(t)}\ [\text{eq. axiom}]}{\to (0''+0)'=0'''}\ [\text{cut}]$$

then

$$\overbrace{0''+0'}^{r}=\overbrace{(0''+0)'}^{s},\ \overbrace{(0''+0)'}^{s}=\overbrace{0'''}^{t} \to \overbrace{0''+0'}^{r}=\overbrace{0'''}^{t} \quad [\text{Prop. 7.2}]$$

and after cut with $\to 0''+0'=(0''+0)'$ (a nonlogical axiom) and the previous result $\to (0''+0)'=0'''$ we finally obtain

$$\to 0''+0'=0'''$$
$$\stackrel{\Delta}{\to} \bar{2}+\bar{1}=\bar{3}$$

More generally, and again using Proposition 7.2

$$\dfrac{[\to t=\bar{m}] \qquad \dfrac{\dfrac{[\to s=t] \qquad \dfrac{\dfrac{[\to s=\bar{n}] \qquad s=\bar{n}\to\bar{n}=s}{\to \bar{n}=s} \qquad \bar{n}=s,\ s=t\to\bar{n}=t}{s=t\to\bar{n}=t}}{\to\bar{n}=t} \qquad \bar{n}=t,\ t=\bar{m}\to\bar{n}=\bar{m}}{t=\bar{m}\to\bar{n}=\bar{m}}}{\to\bar{n}=\bar{m}}$$

so equal terms can be substituted for their numerals.

$\rightarrow \bar{m} + \bar{n} = \overline{m+n}$ is provable for any $m, n \in \mathbb{N}$

First notice that

$$\left.\begin{aligned} &\rightarrow \bar{m} + \bar{n} = (\bar{m} + \overline{n-1})' \\ &\rightarrow \bar{m} + \overline{n-1} = (\bar{m} + \overline{n-2})' \\ &\qquad (\ldots) \\ &\rightarrow \bar{m} + \bar{1} = (\bar{m} + \bar{0})' \\ &\rightarrow \bar{m} + \bar{0} = \bar{m} \end{aligned}\right\} \text{addition axioms}$$

and starting from the bottom using cuts and the following

$$\bar{m} + \bar{k} = \bar{m}^{(k)}, \ \bar{m} + \overline{k+1} = (\bar{m} + \bar{k})' \rightarrow \bar{m} + \overline{k+1} = \bar{m}^{(k+1)}$$

we obtain $\rightarrow \bar{m} + \bar{n} = \bar{m}^{(n)}$, where $\bar{m}^{(n)}$ is of course $\overline{m+n}$.

A similar approach works for multiplication.

Lemma 9.6

(1) By induction on the length of s:

* **case** s is 0: $\rightarrow 0=0$ is an axiom of LK_e.
* **case** s is $r+t$: then by the induction hypothesis, there are unique (but not necessarily distinct) numerals $\bar{m}$ and $\bar{n}$ such that $\rightarrow r=\bar{m}$ and $\rightarrow t=\bar{n}$ are provable. Then

$$\dfrac{\rightarrow t=\bar{n} \qquad \dfrac{\rightarrow r=\bar{m} \qquad r=\bar{m},\, t=\bar{n} \rightarrow r+t=\bar{m}+\bar{n}}{t=\bar{n} \rightarrow r+t=\bar{m}+\bar{n}}\,[\text{cut}]}{\rightarrow r+t=\bar{m}+\bar{n}}\,[\text{cut}]$$

where $\bar{m}+\bar{n}$ (which is $\overline{m+n}$) is distinct from $\bar{m}$ and $\bar{n}$.

* **case** s is $r \cdot t$: symmetric to previous case.

* **case** s is r': then by the induction hypothesis there is a unique numeral $\bar{m}$ such that $\rightarrow r=\bar{m}$ is provable and then

$$\dfrac{\rightarrow r=\bar{m} \qquad r=\bar{m} \rightarrow r'=\bar{m}'}{\rightarrow r'=\bar{m}'}\,[\text{cut}]$$

where $\bar{m}'$ (which is $\overline{m+1}$) is distinct from $\bar{m}$.

The uniqueness can also be shown by contradiction. Suppose that $\rightarrow s=\bar{m}$ and $\rightarrow s=\bar{n}$ are provable. Then

$$\dfrac{\rightarrow s=\bar{n} \qquad \dfrac{\dfrac{\rightarrow s=\bar{m} \qquad s=\bar{m} \rightarrow \bar{m}=s}{\rightarrow \bar{m}=s} \qquad \bar{m}=s,\, s=\bar{n} \rightarrow \bar{m}=\bar{n}}{s=\bar{n} \rightarrow \bar{m}=\bar{n}}}{\rightarrow \bar{m}=\bar{n}}$$

So the numerals must coincide. $\square$

(2) PA is complete with respect to term equality: Since each closed term has a unique numeral, it suffices to show that either $\rightarrow \overline{m}=\overline{n}$ or $\overline{m}=\overline{n}\rightarrow$ is provable, where $\overline{m}$ and $\overline{n}$ are the numerals of s and t, respectively.

$$\dfrac{\dfrac{\dfrac{\dfrac{\rightarrow 0=0 \qquad 0=0 \rightarrow 0'=0'}{\rightarrow 0'=0'}\,[\text{cut}] \qquad 0'=0' \rightarrow 0''=0''}{\rightarrow 0''=0''}\,[\text{cut}] \qquad \cdots}{(\ldots)}}{\rightarrow \overline{m}=\overline{m}}$$

So $\overline{m}=\overline{m}$ is provable for all $m \in \mathbb{N}$.

Now consider $\overline{m}=\overline{n}\rightarrow$ with $m,n \in \mathbb{N}$ and $m>n$. (we can always bring it to this form due to Proposition 7.2). Let $d=m-n$ and recall that

$$\overline{d-1}' = \overline{0} \rightarrow$$

is an axiom and $\overline{d-1}'$ is simply $\overline{d}$.
Then, we can proceed as follows

$$\dfrac{\overline{d+2}=\overline{2} \rightarrow \overline{d+1}=\overline{1} \qquad \dfrac{\overline{d+1}=\overline{1} \rightarrow \overline{d}=\overline{0} \qquad \overline{d}=\overline{0}\rightarrow}{\overline{d+1}=\overline{1}\rightarrow}\,[\text{cut}]}{(\ldots)}\,[\text{cut}]$$

until we obtain

$$\overline{d+k} = \overline{k} \rightarrow$$

with $k=n$, which is

$$\overline{m} = \overline{n} \rightarrow$$

$\square$

Primitive Recursion (see [Kleene], §43)

(I) $\phi(x) = x'$

(II) $\phi(x_1, \dots, x_n) = q$

(III) $\phi(x_1, \dots, x_n) = x_i$

(IV) $\phi(x_1, \dots, x_n) = \psi(\chi_1(x_1, \dots, x_n), \dots, \chi_m(x_1, \dots, x_n))$

(Va) $\begin{cases} \phi(0) = q \\ \phi(y') = \chi(y, \phi(y)) \end{cases}$

(Vb) $\begin{cases} \phi(0, x_2, \dots, x_n) = \psi(x_2, \dots, x_n) \\ \phi(y', x_2, \dots, x_n) = \chi(y, \phi(y, x_2, \dots, x_n), x_2, \dots, x_n) \end{cases}$

(I) successor function
(II) constant functions
(III) identity/projection functions
(IV) substitution
(V) induction

Addition

(I) $S(x) = x'$
(III) $U_1^1(x) = x$
(III) $U_2^3(x, y, z) = y$
(IV) $\psi(b, c, a) = S(U_2^3(b, c, a))$
(Vb) $\begin{cases} \phi(0, a) = U_1^1(a) \\ \phi(b', a) = \psi(b, \phi(b, a), a) \end{cases}$
(III) $U_1^2(a, b) = a$
(III) $U_2^2(a, b) = b$
(IV) $\phi_1(a, b) = \phi(U_2^2(a, b), U_1^2(a, b))$

$\Big\}$ not strictly needed due to commutativity

Multiplication

[ex 10.2.1]

(II) $C_0^1(a) = 0$
(III) $U_2^3(a,b,c) = b$
(III) $U_3^3(a,b,c) = c$
(IV) $\psi(b,c,a) = add(U_2^3(b,c,a), U_3^3(b,c,a))$
(VIb) $\begin{cases} \phi(0,a) = C_0^1(a) \\ \phi(b',a) = \psi(b, \phi(b,a), a) \end{cases}$
(III) $U_1^2(a,b) = a$
(III) $U_2^2(a,b) = b$
(IV) $\phi_1(a,b) = \phi(U_2^2(a,b), U_1^2(a,b))$

} again, not strictly needed due to commutativity

Exponentiation

(II) $C_1^1(a) = 1$
(IV) $\psi(b,c,a) = mult(U_2^3(b,c,a), U_3^3(b,c,a))$
(VIb) $\begin{cases} \phi(0,a) = C_1^1(a) \\ \phi(b',a) = \psi(b, \phi(b,a), a) \end{cases}$
(IV) $\phi_1(a,b) = \phi(U_2^2(a,b), U_1^2(a,b))$

Predecessor (clamped)

(VIa) $\begin{cases} \phi(0) = 0 \\ \phi(b') = U_1^2(b, \phi(b)) \end{cases}$

Subtraction (clamped)

(IV) $\chi(b,c,a) = pred(U_2^3(b,c,a))$
(VIb) $\begin{cases} \phi(0,a) = U_1^1(a) \\ \phi(b',a) = \chi(b, \phi(b,a), a) \end{cases}$
(IV) $\phi_1(a,b) = \phi(U_2^2(a,b), U_1^2(a,b))$

Min

(IV) $\chi(a,b) = sub(U_2^2(a,b), U_1^2(a,b))$
(IV) $min(a,b) = sub(U_2^2(a,b), \chi(a,b))$

Equality

[ex 10.4]

(III) $\chi(a,b) = sub(U^2_2(a,b), U^2_1(a,b))$

(IV) $\phi(a,b) = add(sub(a,b), \chi(a,b))$

(IV) $eq(a,b) = min(C^2_1(a,b), \phi(a,b))$

Less-than

(IV) $\chi(a,b) = S(U^2_1(a,b))$

(IV) $\phi(a,b) = sub(\chi(a,b), U^2_2(a,b))$

(IV) $lt(a,b) = min(C^2_1(a,b), \phi(a,b))$

Incompleteness

Sketch of the proof:

1. PA is intended to describe the natural numbers.
2. The components of the formal system (symbols, formulas, proofs) are countable and can be assigned unique natural numbers (a so-called "Gödel Numbering").
3. A class of number-theoretic functions and predicates (so-called "primitive recursive") is introduced.
4. For a given PR function $f(x_1, \ldots, x_n)$, a formula $\bar{f}(x_1, \ldots, x_n)$ can be constructed such that if $f(x_1, \ldots, x_n) = p$, then $\bar{f}(\bar{x}_1, \ldots, \bar{x}_n) = \bar{p}$ is provable. Otherwise it is refutable. The same can be done with PR predicates. The formula is said to "numeralwise express" the PR function or predicate, respectively. Intuitively, it means that PR functions or predicates can be expressed within the formal system.
5. Some familiar number-theoretic functions and predicates are shown to be primitive recursive. Crucially, the predicate $P(x, p)$ meaning "x is the Gödel number of a formula A and p is the Gödel number of a proof of A" is primitive recursive.

6. To make the former point more precise and continue the idea, consider the following:

* let $\overline{m}: \mathbb{N} \to \text{Term}$ denote the numeral of a natural number
* let $\overline{f}: \text{Function} \to \text{Formula}$ denote the formula which numeralwise expresses the function f.
* let $\overline{P}: \text{Predicate} \to \text{Formula}$ denote the formula which numeralwise expresses the predicate P.
* let $\ulcorner A \urcorner: \text{Formula} \to \mathbb{N}$ denote the Gödel number of formula A (and similar for proofs).
* let $P(a, x)$ be the predicate such that
  $a: \mathbb{N}$ is $\ulcorner A(y) \urcorner$ for some formula $A(y)$
  $x: \mathbb{N}$ is $\ulcorner$proof of $A_y[\overline{a}]\urcorner$
  This predicate can be shown to be primitive recursive.

7. Now consider the formula $\forall x \neg \overline{P}(a, x)$ with free variable a. This formula says "the formula $A(y)$ for which a is the Gödel number, is unprovable."
8. Finally, consider the closed formula

$$\forall x \neg \overline{P}(\overline{\ulcorner \forall x \neg \overline{P}(a, x) \urcorner}, x)$$

which essentially says "I'm unprovable". Under the normally desirable premise that only true formulas are provable, this formula asserts its own unprovabity, thus showing that the underlyng system is incomplete, in that there is a true formula which is not provable. (note that "true" here means "true in the intended interpretation").

ex. Gödel 1

$$
\begin{array}{lccccccccc}
A \text{ is} & \exists & x & (& x & + & \bar{0} & = & \bar{0} &) \\
 & | & | & | & | & | & | & | & | & | \\
 & 3 & 4 & 6 & 4 & 8 & 2 & 1 & 2 & 7
\end{array}
$$

$$\ulcorner A \urcorner \text{ is } 2^3 \cdot 3^4 \cdot 5^6 \cdot 7^4 \cdot 11^8 \cdot 13^2 \cdot 17^1 \cdot 19^2 \cdot 23^7$$

$$= 18{,}402{,}056{,}222{,}200{,}818{,}868{,}210{,}954{,}318{,}875{,}000$$

$$
\begin{array}{lccccc}
n = 4261\,409\,460 = & 2^2 \cdot & 3^3 \cdot & 5^1 \cdot & 7^2 \cdot & 11^5 \\
 & | & | & | & | & | \\
 & \bar{0} & \exists & = & \bar{0} & '
\end{array}
$$

which is not a valid formula.