

Chapter 2

NOTATION

- a, b, c, d**: syntactical variables over terms.
- A, B, C, D**: syntactical variables over formulas.
- e**: syntactical variables over constant symbols.
- f, g**: syntactical variables over function symbols.
- i, j**: syntactical variables over names.
- p, q**: syntactical variables over predicate symbols.
- u, v**: syntactical variables over expressions.
- x, y, z, w**: syntactical variables over (individual) variables.

DEFINITIONS

- A *first-order language* has as symbols:
 - a) the *variables*: $x, y, z, w, x', y', z', w', x'', y'', z'', w'', \dots$
 - b) for each n , the n -ary *function symbols* and the n -ary *predicate symbols*.
 - c) the symbols \neg, \vee and \exists .
- A *term* is defined inductively as:
 - i) \mathbf{x} is a term;
 - ii) if \mathbf{f} is n -ary, then $\mathbf{fa}_1 \dots \mathbf{a}_n$ is a term.
- A *formula* is defined inductively as:
 - i) if \mathbf{p} is n -ary, then an atomic formula $\mathbf{pa}_1 \dots \mathbf{a}_n$ is a formula;
 - ii) $\neg \mathbf{A}$ is a formula;
 - iii) $\vee \mathbf{AB}$ is a formula;
 - iv) $\exists \mathbf{xA}$ is a formula.
- A *designator* is an expression which is either a term or a formula.
- A *structure* \mathcal{A} for a first-order language L consist of:
 - i) A nonempty set $|\mathcal{A}|$, the *universe* and its *individuals*.
 - ii) For each n -ary function symbol \mathbf{f} of L , an n -ary function $\mathbf{f}_{\mathcal{A}} : |\mathcal{A}|^n \rightarrow |\mathcal{A}|$. (In particular, for each constant \mathbf{e} of L , $\mathbf{e}_{\mathcal{A}}$ is an individual of \mathcal{A} .)
 - iii) For each n -ary predicate symbol \mathbf{p} of L other than $=$, an n -ary predicate $\mathbf{p}_{\mathcal{A}}$ in $|\mathcal{A}|$.Also, $\mathcal{A}(\mathbf{a})$ designates an individual and $\mathcal{A}(\mathbf{A})$ designates a truth value.
- A formula \mathbf{A} is *valid* in a structure \mathcal{A} if $\mathcal{A}(\mathbf{A}') = \top$ for every \mathcal{A} -instance \mathbf{A}' of \mathbf{A} . In particular, a closed formula \mathbf{A} is valid in \mathcal{A} iff $\mathcal{A}(\mathbf{A}) = \top$.
- A formula \mathbf{A} is *logically valid* if it's valid in every structure.
- A formula \mathbf{A} is a *consequence* of a set Γ of formulas if the validity of \mathbf{A} follows from the validity of the formulas in Γ .
- A formula \mathbf{A} is a *logical consequence* of a set Γ of formulas if \mathbf{A} is valid in every structure for L in which all of the formulas in Γ are valid.
- A *first-order theory* is a formal system T such that
 - i) the language of T is a first-order language;
 - ii) the axioms of T are the logical axioms of $L(T)$ and certain further axioms, the *nonlogical axioms*;
 - iii) the rules of T are Expansion, Contraction, Associative, Cut and \exists -Introduction.
- A *model* of a theory T , is a structure for $L(T)$ in which all the nonlogical axioms of T are valid.
- A formula \mathbf{A} is *valid* in a theory T if it is valid in every model of T .

LOGICAL AXIOMS

Propositional: $\neg A \vee A$

Substitution: $A_x[a] \rightarrow \exists x A$

Identity: $x = x$

Equality: $x_1 = y_1 \rightarrow \dots \rightarrow x_n = y_n \rightarrow f x_1 \dots x_n = f y_1 \dots y_n$

$x_1 = y_1 \rightarrow \dots \rightarrow x_n = y_n \rightarrow p x_1 \dots x_n \rightarrow p y_1 \dots y_n$

RULES OF INFERENCE

Expansion. Infer $B \vee A$ from A .

Contraction. Infer A from $A \vee A$.

Associative. Infer $(A \vee B) \vee C$ from $A \vee (B \vee C)$.

Cut. Infer $B \vee C$ from $A \vee B$ and $\neg A \vee C$.

\exists -Introduction. If x is not free in B , infer $\exists x A \rightarrow B$ from $A \rightarrow B$.

RESULTS

§2.4

Lemma 1. If $u_1, \dots, u_n, u'_1, \dots, u'_n$ are designators and $u_1 \dots u_n$ and $u'_1 \dots u'_n$ are compatible, then u_i is u'_i for $i = 1, \dots, n$.

Formation Theorem. Every designator can be written in the form $u v_1 \dots v_n$, where u is a symbol of index n and v_1, \dots, v_n are designators, in one and only one way.

Lemma 2. Every occurrence of a symbol in a designator u begins an occurrence of a designator in u .

Occurrence Theorem. Let u be a symbol of index n , and let v_1, \dots, v_n be designators. Then any occurrence of a designator v in $u v_1 \dots v_n$ is either all of $u v_1 \dots v_n$ or a part of one of the v_i .

§2.5

Lemma. Let \mathcal{A} be a structure for L ; \mathbf{a} a variable-free term in $L(\mathcal{A})$; \mathbf{i} the name of $\mathcal{A}(\mathbf{a})$. If \mathbf{b} is a term of $L(\mathcal{A})$ in which no variable except \mathbf{x} occurs, then $\mathcal{A}(\mathbf{b}_x[\mathbf{a}]) = \mathcal{A}(\mathbf{b}_x[\mathbf{i}])$. If \mathbf{A} is a formula of $L(\mathcal{A})$ in which no variable except \mathbf{x} is free, then $\mathcal{A}(\mathbf{A}_x[\mathbf{a}]) = \mathcal{A}(\mathbf{A}_x[\mathbf{i}])$.

Validity Theorem. If T is a theory, then every theorem of T is valid in T .

EXERCISES

1.

(a) Let $F(a_1, \dots, a_n)$ be any truth function. We can construct another function

$$F'(a_1, \dots, a_n) = H_{d,m}(H_{c,n}(a_1^1, \dots, a_n^1), \dots, H_{c,n}(a_1^m, \dots, a_n^m))$$

where the a_1^i, \dots, a_n^i are all the tuples of truth values such that $F(a_1^i, \dots, a_n^i) = \top$. Thus, $a_j^i = a_j$ or $a_j^i = H_{\neg}(a_j)$, for some values of i and j . Now, we can see that F and F' are the same function, since any truth assignment a'_1, \dots, a'_n that satisfies (falsifies) F , also satisfies (falsifies) F' , respectively. This is called *Disjunctive Normal Form (DNF)*.

We can also construct a similar function

$$\begin{aligned} F''(a_1, \dots, a_n) &= H_{c,m}(H_{\neg}(H_{c,n}(a_1^1, \dots, a_n^1)), \dots, H_{\neg}(H_{c,n}(a_1^m, \dots, a_n^m))) \\ &= H_{c,m}(H_{d,n}(H_{\neg}(a_1^1), \dots, H_{\neg}(a_n^1)), \dots, H_{d,n}(H_{\neg}(a_1^m), \dots, H_{\neg}(a_n^m))) \end{aligned}$$

where the a_1^i, \dots, a_n^i are all the tuples of truth values such that $F(a_1^i, \dots, a_n^i) = \perp$. It can be seen by a reasoning similar to above, that F and F'' are the same function. This is called *Conjunctive Normal Form (CNF)*.

(b) It can be seen that

$$\begin{aligned} H_{c,n} &= H_{\wedge}(a_1, H_{\wedge}(a_2, \dots)) \\ H_{d,n} &= H_{\vee}(a_1, H_{\vee}(a_2, \dots)). \end{aligned}$$

This means we can define any truth function F in terms of H_{\neg} , H_{\vee} and H_{\wedge} , due to (a). Additionally, we can convert each instance of $H_{\wedge}(a, b)$ into $H_{\neg}(H_{\vee}(H_{\neg}(a), H_{\neg}(b)))$. Thus, every truth function is definable in terms of H_{\neg} and H_{\vee} .

(c) Since $H_{\vee}(a, b)$ can be defined as $H_{\rightarrow}(H_{\neg}(a), b)$, every truth function is definable in terms of H_{\neg} and H_{\rightarrow} , due to (b).

(d) Since $H_{\vee}(a, b)$ can be defined as $H_{\neg}(H_{\wedge}(H_{\neg}(a), H_{\neg}(b)))$, every truth function is definable in terms of H_{\neg} and H_{\wedge} , due to (b).

(e) Consider the following identities, which can be easily verified e.g. via their truth tables

$$\begin{aligned} H_{\vee}(a, a) &= a, & H_{\vee}(a, \top) &= \top \\ H_{\wedge}(a, a) &= a, & H_{\wedge}(a, \top) &= a \\ H_{\rightarrow}(a, a) &= \top, & H_{\rightarrow}(a, \top) &= \top, & H_{\rightarrow}(\top, a) &= a \\ H_{\leftrightarrow}(a, a) &= \top, & H_{\leftrightarrow}(a, \top) &= a, & H_{\leftrightarrow}(\top, a) &= a. \end{aligned}$$

Thus, any formula consisting of only those connectives and the free variable a can be inductively reduced to either a or \top and can never define H_{\neg} . Those connectives can only define monotone functions while negation is not monotone. Note that allowing constants in the expression would allow to define negation as e.g. $H_{\neg}(a) = H_{\rightarrow}(a, \perp)$.

2.

(a) Note that $H_d(a, b) = H_{\wedge}(H_{\neg}(a), H_{\neg}(b))$. We can then define

$$\begin{aligned} H_{\neg}(a) &= H_d(a, a) \\ H_{\vee}(a, b) &= H_d(H_d(a, b), H_d(a, b)) \end{aligned}$$

and thus every truth function is definable in terms of H_d (using result from 1.1(b)).

(b) Note that $H_s(a, b) = H_{\neg}(H_{\wedge}(a, b))$. We can then define

$$\begin{aligned} H_{\neg}(a) &= H_s(a, a) \\ H_{\vee}(a, b) &= H_s(H_s(a, a), H_s(b, b)) \end{aligned}$$

and thus every truth function is definable in terms of H_s (using result from 1.1(b)).

(c) Let H be singular with $H(a_1, \dots, a_n) = H'(a_i)$. The syntax of every truth function $F(a_1, \dots, a_m)$ definable in terms of H can be inductively defined by

$$e ::= a_j | H(e_1, \dots, e_n)$$

where $1 \leq j \leq m$ and e_1, \dots, e_n are valid expressions.

We can then reduce every expression to an equivalent expression that involves a single a_j : as long as the expression has the form $H(e_1, \dots, e_n)$, we can replace it with $H'(e_i)$ and inductively reduce e_i . Thus, every truth function F definable in terms of H is singular and furthermore

$$F(a_1, \dots, a_m) = H'^k(a_j)$$

for some integers $k \geq 0$ and $1 \leq j \leq m$.

(d) Note that since any n -ary truth function is completely determined by its truth table, there are 2^{2^n} of them. So we know there are $2^{2^2} = 16$ binary truth functions. Let's analyze them:

- Consider the four binary truth functions H such that

$$H(a, a) = a.$$

It is easy to see that any function definable in terms of such H can be inductively reduced to a , in a similar fashion as before. Thus, none of these four functions can define every truth function (e.g. negation H_{\neg} cannot be defined).

- Consider the four binary truth functions H such that

$$H(a, a) = \perp.$$

For each of these four functions, we have

$$H(a, \perp) \in \{a, \perp\}, \quad H(\perp, a) \in \{a, \perp\}$$

and thus none of these four functions can define every truth function (e.g. negation H_{\neg} cannot be defined).

- Consider the four binary truth functions H such that

$$H(a, a) = \top.$$

This case is symmetric to the previous one. For each of these four functions, we have

$$H(a, \top) \in \{a, \top\}, \quad H(\top, a) \in \{a, \top\}$$

and thus none of these four functions can define every truth function (e.g. negation H_{\neg} cannot be defined).

- For the four remaining binary truth functions, we have

$$H(\top, \top) = \perp, \quad H(\perp, \perp) = \top.$$

Two of those functions

$$\begin{aligned} H_1(\top, \perp) &= \top, & H_1(\perp, \top) &= \perp \\ H_2(\top, \perp) &= \perp, & H_2(\perp, \top) &= \top \end{aligned}$$

are singular and thus cannot define functions such as H_{\vee} , due to the result from 2.2(c). The two remaining functions are H_d and H_s , presented in 2.2(a) and 2.2(b), respectively.

3. If \mathbf{v} is empty, then trivially neither \mathbf{u} or \mathbf{v}' are empty, and they are both designators.

Let's assume that \mathbf{v} is not empty and that the designator \mathbf{uv} has the form $\mathbf{tt}_1 \dots \mathbf{t}_n$. Since \mathbf{uv} and \mathbf{vv}' are designators, they both begin with a symbol: thus \mathbf{v} also begins with a symbol, since it is a non-empty prefix of \mathbf{vv}' . The occurrence of this symbol in \mathbf{uv} begins the occurrence of a designator \mathbf{u}' in \mathbf{uv} (by Lemma 2), which is compatible with \mathbf{v} . Moreover, the occurrence of \mathbf{u}' in \mathbf{uv} is either all of \mathbf{uv} or part of one of the \mathbf{t}_i (by the Occurrence Theorem). In the former case, it means that \mathbf{v} is a designator and \mathbf{u} and \mathbf{v}' are empty. On the other hand, if \mathbf{u}' is part of one of the \mathbf{t}_i , it means that \mathbf{vv}' begins with \mathbf{u}' , and thus \mathbf{u}' and \mathbf{v} are the same (by the Formation Theorem) and \mathbf{v}' is empty.

4. If a term is:

- i) a variable \mathbf{x}' , then the substitution result is \mathbf{x} itself, which is also a term.
- ii) a function application $\mathbf{fa}_1 \dots \mathbf{a}_n$, then \mathbf{a} is one of the \mathbf{a}_i and the substitution result is also a term, or \mathbf{a} is substituted in one of the terms \mathbf{a}_i , and it remains a term, by the induction hypothesis.

If a formula is:

- i) an atomic formula $\mathbf{pa}_1 \dots \mathbf{a}_n$, then substituting \mathbf{a} in any of the \mathbf{a}_i results in a term, as previously shown. Thus it remains a formula.
- ii) $\neg \mathbf{A}$, then substituting \mathbf{a} in \mathbf{A} remains a formula by the induction hypothesis.
- iii) $\forall \mathbf{A}\mathbf{B}$, then substituting \mathbf{a} in \mathbf{A} or \mathbf{B} remains a formula by the induction hypothesis.
- iv) $\exists \mathbf{y}\mathbf{A}$, then substituting \mathbf{a} in \mathbf{A} remains a formula by the induction hypothesis.

5.

(a) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg \mathbf{A}) &= \perp; \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{B}); \\ f(\exists \mathbf{x}\mathbf{A}) &= \top. \end{aligned}$$

Let's prove that if \mathbf{A} is provable without propositional axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a substitution axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}'_x[\mathbf{a}] \vee \exists \mathbf{x}\mathbf{A}') = f(\exists \mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\neg \mathbf{A}' \vee \mathbf{C}') = f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x}\mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \rightarrow \mathbf{B}') = f(\neg \mathbf{A}' \vee \mathbf{B}') = f(\mathbf{B}') = f(\mathbf{A}) = \top$.

Thus, if \mathbf{A} is provable without propositional axioms, we have $f(\mathbf{A}) = \top$. But $f(\neg \neg(x = x) \vee \neg(x = x)) = f(\neg(x = x)) = \perp$ and so it is not provable without propositional axioms.

(b) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg \mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists \mathbf{x}\mathbf{A}) &= \perp. \end{aligned}$$

Let's prove that if \mathbf{A} is provable without substitution axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \perp$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x} \mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg \exists \mathbf{x} \mathbf{A}' \vee \mathbf{B}') = \neg f(\exists \mathbf{x} \mathbf{A}') \vee f(\mathbf{B}') = \top$.

Thus, if \mathbf{A} is provable without substitution axioms, we have $f(\mathbf{A}) = \top$. But $f(x = x \rightarrow \exists x(x = x)) = \neg f(x = x) \vee f(\exists x(x = x)) = \perp$ and so it is not provable without substitution axioms.

(c) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \perp, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg \mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists \mathbf{x} \mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if \mathbf{A} is provable without identity axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists \mathbf{x} \mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\mathbf{A}') = \top$ (see below for this case);
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \perp$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x} \mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg \exists \mathbf{x} \mathbf{A}' \vee \mathbf{B}') = \neg f(\mathbf{A}') \vee f(\mathbf{B}') = f(\neg \mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of \mathbf{A} :

- for \mathbf{A} atomic with form $\mathbf{pb}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{pb}_{1\mathbf{x}}[\mathbf{a}] \dots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \perp$ and $f(\mathbf{A}) = f(\mathbf{pb}_1 \dots \mathbf{b}_n) = \perp$.
- for \mathbf{A} with form $\neg \mathbf{A}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = \neg f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

- for \mathbf{A} with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}]) \vee f(\mathbf{B}'_x[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_x[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for \mathbf{A} with form $\exists \mathbf{y} \mathbf{A}'$: we have $f(\exists \mathbf{y} \mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}])$ and $f(\exists \mathbf{y} \mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if \mathbf{A} is provable without identity axioms, we have $f(\mathbf{A}) = \top$. But $f(x = x) = \perp$ and so it is not provable without identity axioms.

(d) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{e}_i = \mathbf{e}_j) &= \top \quad \text{iff } i \leq j; \\ f(\neg \mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists \mathbf{x} \mathbf{A}) &= \top \quad \text{iff } f(\mathbf{A}_x[\mathbf{e}_i]) = \top \text{ for some } i. \end{aligned}$$

Let's prove that if \mathbf{A} is provable without equality axioms then $f(\mathbf{A}') = \top$ for every formula obtained from \mathbf{A} by replacing each variable by some \mathbf{e}_i at all its free occurrences, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$ for every closed formula \mathbf{A}'' obtained from \mathbf{A}' by replacing each variable by some \mathbf{e}_i at all its free occurrences;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}'_x[\mathbf{a}] \vee \exists \mathbf{x} \mathbf{A}') = \neg f(\mathbf{A}'_x[\mathbf{a}]) \vee f(\exists \mathbf{x} \mathbf{A}')$. For every closed formula \mathbf{A}'' obtained from \mathbf{A}' by replacing each variable (except \mathbf{x}) by some \mathbf{e}_i at all its free occurrences: if $f(\mathbf{A}''_x[\mathbf{e}_i]) = \top$ for some i , then $f(\exists \mathbf{x} \mathbf{A}'') = \top$ by the definition of f . Otherwise, $f(\mathbf{A}''_x[\mathbf{e}_i]) = \perp$ for all i and thus $\neg f(\mathbf{A}''_x[\mathbf{e}_i]) = \top$;
- an identity axiom: $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$ for any substitution of \mathbf{x} by some \mathbf{e}_i ;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}')$ with $f(\mathbf{A}') = \top$ for every closed formula \mathbf{A}'' obtained from \mathbf{A}' by replacing each variable by some \mathbf{e}_i at all its free occurrences, by the induction hypothesis;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = \top$ for every closed formula \mathbf{A}'' obtained from \mathbf{A}' by replacing each variable by some \mathbf{e}_i at all its free occurrences, by the induction hypothesis;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = \top$ for every closed formulas \mathbf{A}' , \mathbf{B}' and \mathbf{C}' obtained from \mathbf{A}' , \mathbf{B}' and \mathbf{C}' , respectively, by replacing each variable by some \mathbf{e}_i at all its free occurrences, by the induction hypothesis;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}') = \top$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}') = \top$ for every closed formulas \mathbf{A}' , \mathbf{B}' and \mathbf{C}' obtained from \mathbf{A}' , \mathbf{B}' and \mathbf{C}' , respectively, by replacing each variable by some \mathbf{e}_i at all its free occurrences, by the induction hypothesis. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \perp$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x} \mathbf{A}' \rightarrow \mathbf{B}') = \neg f(\exists \mathbf{x} \mathbf{A}') \vee f(\mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \neg f(\mathbf{A}') \vee f(\mathbf{B}') = \top$ for every closed formula \mathbf{A}'' and \mathbf{B}'' obtained from \mathbf{A}' and \mathbf{B}' , respectively, by replacing each variable by some \mathbf{e}_i at all its free occurrences, by the induction hypothesis. If $f(\mathbf{B}') = \top$, then $f(\mathbf{A}) = \top$ follows trivially. Otherwise, we must have $f(\mathbf{A}') = \perp$ for all closed formulas \mathbf{A}'' obtained from \mathbf{A}' as described above. This implies that $f(\exists \mathbf{x} \mathbf{A}') = \perp$ and thus $f(\mathbf{A}) = \top$.

Thus, if \mathbf{A} is provable without equality axioms, we have $f(\mathbf{A}') = \top$ for every formula \mathbf{A}' obtained from \mathbf{A} by replacing each variable by some \mathbf{e}_i at all its free occurrences. But $f(x = y \rightarrow x = z \rightarrow x = x \rightarrow y = z) = \neg f(x = y) \vee \neg f(x = z) \vee \neg f(x = x) \vee f(y = z) = \perp$ since it does not hold for the substitution $[\mathbf{x}, \mathbf{y}, \mathbf{z}] \rightarrow [\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2]$ and so it is not provable without equality axioms.

(e) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic}; \\ f(\neg \mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \leftrightarrow \neg f(\mathbf{B}); \\ f(\exists \mathbf{x} \mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if \mathbf{A} is provable without the expansion rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_x[\mathbf{a}] \vee \exists \mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_x[\mathbf{a}]) \leftrightarrow \neg f(\mathbf{A}') = \top$ (see below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \leftrightarrow \neg f(\mathbf{x}_2 = \mathbf{y}_2) \leftrightarrow \neg f(\mathbf{x}_1 = \mathbf{x}_2) \leftrightarrow \neg f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. However, this is a contradiction since $f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{A}') = \perp$ for any \mathbf{A}' so it's not possible to have a proof where the contraction rule is applied (???);
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \leftrightarrow \neg(f(\mathbf{B}') \leftrightarrow \neg f(\mathbf{C}')) = f(\mathbf{A}') \leftrightarrow (f(\mathbf{B}') \leftrightarrow f(\mathbf{C}'))$ and $f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \leftrightarrow \neg f(\mathbf{C}') = (f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')) \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}') \leftrightarrow f(\mathbf{C}')$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{C}')$, and thus $f(\mathbf{B}') \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}) = \top$;
- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x}\mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}')$ and $f(\neg\exists \mathbf{x}\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}')$.

To treat substitution axioms, let's show that $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of \mathbf{A} :

- for \mathbf{A} atomic with form $\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1x}[\mathbf{a}] \dots \mathbf{b}_{nx}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n) = \top$.
- for \mathbf{A} with form $\neg\mathbf{A}'$: we have $f(\mathbf{A}_x[\mathbf{a}]) = \neg f(\mathbf{A}'_x[\mathbf{a}])$ and $f(\mathbf{A}) = \neg f(\mathbf{A}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for \mathbf{A} with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}]) \leftrightarrow \neg f(\mathbf{B}'_x[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_x[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for \mathbf{A} with form $\exists \mathbf{y}\mathbf{A}'$: we have $f(\exists \mathbf{y}\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}])$ and $f(\exists \mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if \mathbf{A} is provable without the expansion rule, we have $f(\mathbf{A}) = \top$. But $f(x = x \vee (\neg(x = x) \vee (x = x))) = f(x = x) \leftrightarrow \neg(\neg f(x = x) \leftrightarrow \neg f(x = x)) = \perp$ and so it is not provable without the expansion rule.

(f) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \perp; \\ f(\mathbf{A} \vee \mathbf{B}) &= \top; \\ f(\exists \mathbf{x}\mathbf{A}) &= \perp. \end{aligned}$$

Let's prove that if \mathbf{A} is provable without the contraction rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_x[\mathbf{a}] \vee \exists \mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = \top$;

- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x} \mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg \exists \mathbf{x} \mathbf{A}' \vee \mathbf{B}') = \top$.

Thus, if \mathbf{A} is provable without the contraction rule, we have $f(\mathbf{A}) = \top$. But $f(\neg \neg(x = x)) = \perp$ and so it is not provable without the contraction rule.

(g) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= 0, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg \mathbf{A}) &= 1 - f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \cdot f(\mathbf{B}) \cdot (1 - f(\mathbf{A}) - f(\mathbf{B})); \\ f(\exists \mathbf{x} \mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if \mathbf{A} is provable without the associative rule then $f(\mathbf{A}) = 0$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}' \vee \mathbf{A}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{A}') \cdot (1 - (1 - f(\mathbf{A}')) - f(\mathbf{A}')) = (1 - f(\mathbf{A}')) \cdot f(\mathbf{A}') \cdot (f(\mathbf{A}') - f(\mathbf{A}')) = 0$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists \mathbf{x} \mathbf{A}') = (1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])) \cdot f(\mathbf{A}') \cdot (1 - (1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])) - f(\mathbf{A}')) = 0$ (see below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = 0$;
- an equality axiom: we have

$$\begin{aligned} f(\mathbf{A}) &= f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) \\ &= f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) \\ &= (1 - f(\mathbf{x}_1 = \mathbf{y}_1)) \cdot f(\mathbf{A}') \cdot (f(\mathbf{x}_1 = \mathbf{y}_1) - f(\mathbf{A}')) \end{aligned}$$

$$\begin{aligned} f(\mathbf{A}') &= f(\mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = (1 - f(\mathbf{x}_2 = \mathbf{y}_2)) \cdot f(\mathbf{A}'') \cdot (f(\mathbf{x}_2 = \mathbf{y}_2) - f(\mathbf{A}'')) \\ f(\mathbf{A}'') &= f(\mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = (1 - f(\mathbf{x}_1 = \mathbf{x}_2)) \cdot f(\mathbf{y}_1 = \mathbf{y}_2) \cdot (f(\mathbf{x}_1 = \mathbf{x}_2) - f(\mathbf{y}_1 = \mathbf{y}_2)) = 0; \end{aligned}$$

- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}') \cdot f(\mathbf{A}') \cdot (1 - f(\mathbf{B}') - f(\mathbf{A}')) = 0$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \cdot f(\mathbf{A}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{A}')) = 0$ and the only integer solution is $f(\mathbf{A}') = 0$ and thus $f(\mathbf{A}) = 0$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = 0$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = 0$ by the induction hypothesis. Consider the equations

$$f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{B}')) = 0 \quad (1)$$

$$f(\neg \mathbf{A}' \vee \mathbf{C}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{C}') \cdot (f(\mathbf{A}') - f(\mathbf{C}')) = 0 \quad (2)$$

$$f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \cdot f(\mathbf{C}') \cdot (1 - f(\mathbf{B}') - f(\mathbf{C}')) = 0 \quad (3)$$

and the possible cases that satisfy equation (2). First, $(1 - f(\mathbf{A}')) = 0$ implies that $f(\mathbf{A}') = 1$ and substituting in equation (1) we obtain $f(\mathbf{B}') \cdot (-f(\mathbf{B}')) = 0$ which means that $f(\mathbf{B}') = 0$ which satisfies equation (3). Second, $f(\mathbf{C}') = 0$, which trivially satisfies equation (3). Third, $f(\mathbf{A}') - f(\mathbf{C}') = 0$ which implies $f(\mathbf{A}') = f(\mathbf{C}')$ and substituting in equation (1) we obtain $f(\mathbf{C}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{C}') - f(\mathbf{B}')) = 0$. Thus, equation (3) is satisfied in all cases;

- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x} \mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg \exists \mathbf{x} \mathbf{A}' \vee \mathbf{B}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{B}') \cdot (f(\mathbf{A}') - f(\mathbf{B}')) = f(\mathbf{A}' \rightarrow \mathbf{B}') = 0$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of \mathbf{A} :

- for \mathbf{A} atomic with form $\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \dots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = 0$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n) = 0$.
- for \mathbf{A} with form $\neg \mathbf{A}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = 1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = 1 - f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

- for \mathbf{A} with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}]) \cdot f(\mathbf{B}'_x[\mathbf{a}]) \cdot (1 - f(\mathbf{A}'_x[\mathbf{a}]) - f(\mathbf{B}'_x[\mathbf{a}]))$ and $f(\mathbf{A}) = f(\mathbf{A}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{B}'))$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_x[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for \mathbf{A} with form $\exists \mathbf{y} \mathbf{A}'$: we have $f(\exists \mathbf{y} \mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}])$ and $f(\exists \mathbf{y} \mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if \mathbf{A} is provable without the associative rule, we have $f(\mathbf{A}) = 0$. But $f(\neg(\neg(x = x) \vee \neg(x = x))) = 1 - f(\neg(x = x) \vee \neg(x = x)) = 1 - ((1 - f(x = x))^2 \cdot (1 - 2 \cdot (1 - f(x = x)))) = 1 - (1 - 2) = 2$ and so it is not provable without the associative rule.

(h) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg \mathbf{A}) &= \begin{cases} \top, & \text{if } f(\mathbf{A}) = \perp \text{ or } \mathbf{A} \text{ is atomic;} \\ \perp, & \text{otherwise.} \end{cases} \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists \mathbf{x} \mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if \mathbf{A} is provable without the cut rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}' \vee \mathbf{A}') = f(\neg \mathbf{A}') \vee f(\mathbf{A}') = \top$ (since if $f(\mathbf{A}') = \perp$, then $f(\neg \mathbf{A}') = \top$ from the definition of f);
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}'_x[\mathbf{a}] \vee \exists \mathbf{x} \mathbf{A}') = f(\neg \mathbf{A}'_x[\mathbf{a}]) \vee f(\mathbf{A}') = \top$ (since if $f(\mathbf{A}') = \perp$, then $f(\neg \mathbf{A}') = \top$ from the definition of f and $f(\neg \mathbf{A}'_x[\mathbf{a}]) = f(\neg \mathbf{A}')$. See below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1)) \vee f(\neg(\mathbf{x}_2 = \mathbf{y}_2)) \vee f(\neg(\mathbf{x}_1 = \mathbf{x}_2)) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = \top$ and thus $f(\mathbf{A}) = f(\mathbf{A}') = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the \exists -introduction rule: we have $f(\mathbf{A}) = f(\exists \mathbf{x} \mathbf{A}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{A}) = f(\neg \exists \mathbf{x} \mathbf{A}' \vee \mathbf{B}') = f(\neg \exists \mathbf{x} \mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg \mathbf{A}') \vee f(\mathbf{B}') = \top$. So either $f(\neg \mathbf{A}') = \top$ or $f(\mathbf{B}') = \top$. In the latter case, it follows trivially that $f(\mathbf{A}) = \top$. In the former case, note that since $f(\exists \mathbf{x} \mathbf{A}) = f(\mathbf{A})$ and $\exists \mathbf{x} \mathbf{A}$ is not atomic, then $f(\neg \exists \mathbf{x} \mathbf{A}') = f(\neg \mathbf{A}')$.

To treat substitution axioms, let's show that $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of \mathbf{A} :

- for \mathbf{A} atomic with form $\mathbf{p} \mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{p} \mathbf{b}_{1x}[\mathbf{a}] \dots \mathbf{b}_{nx}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\mathbf{p} \mathbf{b}_1 \dots \mathbf{b}_n) = \top$.
- for \mathbf{A} with form $\neg \mathbf{A}'$ with \mathbf{A}' atomic: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\neg \mathbf{A}'_x[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\neg \mathbf{A}') = \top$.
- for \mathbf{A} with form $\neg \mathbf{A}'$ with \mathbf{A}' not atomic: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\neg \mathbf{A}'_x[\mathbf{a}])$ and $f(\mathbf{A}) = f(\neg \mathbf{A}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for \mathbf{A} with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}]) \vee f(\mathbf{B}'_x[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_x[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for \mathbf{A} with form $\exists \mathbf{y} \mathbf{A}'$: we have $f(\exists \mathbf{y} \mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}'_x[\mathbf{a}])$ and $f(\exists \mathbf{y} \mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_x[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if \mathbf{A} is provable without the cut rule, we have $f(\mathbf{A}) = \top$. But $f(\neg \neg(x = x)) = \perp$ since $f(\neg(x = x)) = \top$ and so it is not provable without the cut rule.

(i) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg \mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists \mathbf{x} \mathbf{A}) &= \top. \end{aligned}$$

Let's prove that if \mathbf{A} is provable without the \exists -introduction rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of \mathbf{A} , if \mathbf{A} was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg \mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists \mathbf{x} \mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\exists \mathbf{x} \mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg \mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \perp$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;

Thus, if \mathbf{A} is provable without the \exists -introduction rule, we have $f(\mathbf{A}) = \top$. But $f(\exists y \neg(x = x) \rightarrow \neg(x = x)) = \neg f(\exists y \neg(x = x)) \vee \neg f(x = x) = \perp$ and so it is not provable without the \exists -introduction rule.

Chapter 3

DEFINITIONS

- A formula is *elementary* if it is either atomic or an instantiation.
- A *truth valuation* for T is a mapping from the set of elementary formulas in T to the set of truth values.
- \mathbf{B} is a *tautological consequence* of $\mathbf{A}_1, \dots, \mathbf{A}_n$ if $V(\mathbf{B}) = \top$ for every truth valuation V such that $V(\mathbf{A}_1) = \dots = V(\mathbf{A}_n) = \top$.
- A formula \mathbf{A} is a *tautology* if it is a tautological consequence of the empty sequence of formulas, i.e. if $V(\mathbf{A}) = \top$ for every truth valuation V .
- \mathbf{A}' is an *instance* of \mathbf{A} if \mathbf{A}' is of the form $\mathbf{A}_{\mathbf{x}_1, \dots, \mathbf{x}_n}[\mathbf{a}_1, \dots, \mathbf{a}_n]$.
- Let \mathbf{A} be a formula and $\mathbf{x}_1, \dots, \mathbf{x}_n$ its free variables in alphabetical order. The *closure* of \mathbf{A} is the formula $\forall \mathbf{x}_1 \dots \forall \mathbf{x}_n \mathbf{A}$.
- \mathbf{A}' is a *variant* of \mathbf{A} if \mathbf{A}' can be obtained from \mathbf{A} by a sequence of replacements of the following type: replace a part $\exists \mathbf{x} \mathbf{B}$ by $\exists \mathbf{y} \mathbf{B}_{\mathbf{x}}[\mathbf{y}]$, where \mathbf{y} is a variable not free in \mathbf{B} .

RESULTS

§3.1

Tautology Theorem. If \mathbf{B} is a tautological consequence of $\mathbf{A}_1, \dots, \mathbf{A}_n$, and $\vdash \mathbf{A}_1, \dots, \vdash \mathbf{A}_n$, then $\vdash \mathbf{B}$.

Corollary. Every tautology is a theorem.

Lemma 1. If $\vdash \mathbf{A} \vee \mathbf{B}$, then $\vdash \mathbf{B} \vee \mathbf{A}$.

Detachment Rule. If $\vdash \mathbf{A}$ and $\vdash \mathbf{A} \rightarrow \mathbf{B}$, then $\vdash \mathbf{B}$.

Corollary. If $\vdash \mathbf{A}_1, \dots, \vdash \mathbf{A}_n$, and $\vdash \mathbf{A}_1 \rightarrow \dots \rightarrow \mathbf{A}_n \rightarrow \mathbf{B}$, then $\vdash \mathbf{B}$.

Lemma 2. If $n \geq 2$, and $\mathbf{A}_1 \vee \dots \vee \mathbf{A}_n$ is a tautology, then $\vdash \mathbf{A}_1 \vee \dots \vee \mathbf{A}_n$.

§3.2

\forall -Introduction Rule. If $\vdash \mathbf{A} \rightarrow \mathbf{B}$ and \mathbf{x} is not free in \mathbf{A} , then $\vdash \mathbf{A} \rightarrow \forall \mathbf{x} \mathbf{B}$.

Generalization Rule. If $\vdash \mathbf{A}$, then $\vdash \forall \mathbf{x} \mathbf{A}$.

Substitution Rule. If $\vdash \mathbf{A}$ and \mathbf{A}' is an instance of \mathbf{A} , then $\vdash \mathbf{A}'$.

Substitution Theorem.

$$\begin{aligned} & \vdash \mathbf{A}_{\mathbf{x}_1, \dots, \mathbf{x}_n}[\mathbf{a}_1, \dots, \mathbf{a}_n] \rightarrow \exists \mathbf{x}_1 \dots \exists \mathbf{x}_n \mathbf{A} \\ & \vdash \forall \mathbf{x}_1 \dots \forall \mathbf{x}_n \mathbf{A} \rightarrow \mathbf{A}_{\mathbf{x}_1, \dots, \mathbf{x}_n}[\mathbf{a}_1, \dots, \mathbf{a}_n] \end{aligned}$$

Distribution Rule. If $\vdash \mathbf{A} \rightarrow \mathbf{B}$, then $\vdash \exists \mathbf{x} \mathbf{A} \rightarrow \exists \mathbf{x} \mathbf{B}$ and $\vdash \forall \mathbf{x} \mathbf{A} \rightarrow \forall \mathbf{x} \mathbf{B}$.

Closure Theorem. If \mathbf{A}' is the closure of \mathbf{A} , then $\vdash \mathbf{A}'$ iff $\vdash \mathbf{A}$.

Corollary. If \mathbf{A}' is the closure of \mathbf{A} , then \mathbf{A}' is valid in a structure \mathcal{A} iff \mathbf{A} is valid in \mathcal{A} .

§3.3

Deduction Theorem. Let \mathbf{A} be a closed formula in T . For every formula \mathbf{B} of T , $\vdash_T \mathbf{A} \rightarrow \mathbf{B}$ iff \mathbf{B} is a theorem of $T[\mathbf{A}]$.

Corollary. Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be closed formulas in T . For every formula \mathbf{B} in T , $\vdash_T \mathbf{A}_1 \rightarrow \dots \rightarrow \mathbf{A}_n \rightarrow \mathbf{B}$ iff \mathbf{B} is a theorem of $T[\mathbf{A}_1, \dots, \mathbf{A}_n]$.

Theorem on Constants. Let T' be obtained from T by adding new constants (but no new nonlogical axioms). For every formula \mathbf{A} of T and every sequence $\mathbf{e}_1, \dots, \mathbf{e}_n$ of distinct new constants, $\vdash_{T'} \mathbf{A}[\mathbf{e}_1, \dots, \mathbf{e}_n]$.

§3.4

Equivalence Theorem. Let \mathbf{A}' be obtained from \mathbf{A} by replacing some occurrences of $\mathbf{B}_1, \dots, \mathbf{B}_n$ by $\mathbf{B}'_1, \dots, \mathbf{B}'_n$, respectively. If

$$\vdash \mathbf{B}_1 \leftrightarrow \mathbf{B}'_1, \dots, \vdash \mathbf{B}_n \leftrightarrow \mathbf{B}'_n$$

then

$$\vdash \mathbf{A} \leftrightarrow \mathbf{A}'.$$

Variant Theorem. If \mathbf{A}' is a variant of \mathbf{A} , then $\vdash \mathbf{A} \leftrightarrow \mathbf{A}'$.

Symmetry Theorem. $\vdash \mathbf{a} = \mathbf{b} \leftrightarrow \mathbf{b} = \mathbf{a}$.

Equality Theorem. Let \mathbf{b}' be obtained from \mathbf{b} by replacing some occurrences of $\mathbf{a}_1, \dots, \mathbf{a}_n$ not immediately following \exists or \forall by $\mathbf{a}'_1, \dots, \mathbf{a}'_n$ respectively, and let \mathbf{A}' be obtained from \mathbf{A} by the same type of replacements. If $\vdash \mathbf{a}_1 = \mathbf{a}'_1, \dots, \vdash \mathbf{a}_n = \mathbf{a}'_n$ then $\vdash \mathbf{b} = \mathbf{b}'$ and $\vdash \mathbf{A} \leftrightarrow \mathbf{A}'$.

Corollary 1. $\vdash \mathbf{a}_1 = \mathbf{a}'_1 \rightarrow \dots \rightarrow \mathbf{a}_n = \mathbf{a}'_n \rightarrow \mathbf{b}[\mathbf{a}_1, \dots, \mathbf{a}_n] = \mathbf{b}[\mathbf{a}'_1, \dots, \mathbf{a}'_n]$.

Corollary 2. $\vdash \mathbf{a}_1 = \mathbf{a}'_1 \rightarrow \dots \rightarrow \mathbf{a}_n = \mathbf{a}'_n \rightarrow (\mathbf{A}[\mathbf{a}_1, \dots, \mathbf{a}_n] \leftrightarrow \mathbf{A}[\mathbf{a}'_1, \dots, \mathbf{a}'_n])$.

Corollary 3. If \mathbf{x} does not occur in \mathbf{a} , then

$$\vdash \mathbf{A}_{\mathbf{x}}[\mathbf{a}] \leftrightarrow \exists \mathbf{x}(\mathbf{x} = \mathbf{a} \wedge \mathbf{A})$$

Theories

N (Natural Numbers)

Nonlogical symbols:

- constant 0
- unary function symbol S , the successor function
- binary function symbols $+$ and \cdot
- binary predicate symbol $<$

Nonlogical axioms:

- N1.** $Sx \neq 0$
 - N2.** $Sx = Sy \rightarrow x = y$
 - N3.** $x + 0 = x$
 - N4.** $x + Sy = S(x + y)$
 - N5.** $x \cdot 0 = 0$
 - N6.** $x \cdot Sy = (x \cdot y) + x$
 - N7.** $\neg(x < 0)$
 - N8.** $x < Sy \leftrightarrow x < y \vee x = y$
 - N9.** $x < y \leftrightarrow \vee x = y \vee y < x$
-

G (Elementary Theory of Groups)

Nonlogical symbols:

- binary function symbol \cdot

Nonlogical axioms:

- G1.** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
 - G2.** $\exists x(\forall y(x \cdot y = y) \wedge \forall y \exists z(z \cdot y = x))$
-

Proofs

Chapter 2 - Exercise 5(a)

(1) $\neg\neg(x = x) \vee \neg(x = x)$ [axiom: propositional]

Chapter 2 - Exercise 5(b)

(1) $\neg(x = x) \vee \exists x(x = x)$ [axiom: substitution]

Chapter 2 - Exercise 5(c)

(1) $(x = x)$ [axiom: identity]

Chapter 2 - Exercise 5(d)

(1) $\neg(x = y) \vee (\neg(x = z) \vee (\neg(x = x) \vee (y = z)))$ [axiom: equality]

Chapter 2 - Exercise 5(e)

(1) $(x = x)$ [axiom: identity]
(2) $\neg(x = x) \vee (x = x)$ [rule: expansion: (1)]
(3) $(x = x) \vee (\neg(x = x) \vee (x = x))$ [rule: expansion: (2)]

Chapter 2 - Exercises 5(f) and 5(h)

(1) $(x = x)$ [axiom: identity]
(2) $\neg\neg(x = x) \vee (x = x)$ [rule: expansion: (1)]
(3) $\neg\neg\neg(x = x) \vee \neg\neg(x = x)$ [axiom: propositional]
(4) $(x = x) \vee \neg\neg(x = x)$ [rule: cut: (2) (3)]
(5) $\neg\neg(x = x) \vee \neg(x = x)$ [axiom: propositional]
(6) $\neg(x = x) \vee \neg\neg(x = x)$ [rule: cut: (5) (3)]
(7) $\neg\neg(x = x) \vee \neg\neg(x = x)$ [rule: cut: (4) (6)]
(8) $\neg\neg(x = x)$ [rule: contraction: (7)]

Chapter 2 - Exercise 5(g)

(1) $(x = x)$ [axiom: identity]
(2) $\neg(\neg(x = x) \vee \neg(x = x)) \vee (\neg(x = x) \vee \neg(x = x))$ [axiom: propositional]
(3) $(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee \neg(x = x)$ [rule: associative: (2)]
(4) $(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee (x = x)$ [rule: expansion: (1)]
(5) $\neg(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [axiom: propositional]
(6) $\neg(x = x) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [rule: cut: (3) (5)]
(7) $(x = x) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [rule: cut: (4) (5)]
(8) $(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [rule: cut: (7) (6)]
(9) $\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)$ [rule: contraction: (8)]
(10) $\neg(\neg(x = x) \vee \neg(x = x)) \vee (x = x)$ [rule: expansion: (1)]
(11) $\neg\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [axiom: propositional]
(12) $\neg(x = x) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [rule: cut: (9) (11)]
(13) $(x = x) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [rule: cut: (10) (11)]
(14) $\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [rule: cut: (13) (12)]
(15) $\neg(\neg(x = x) \vee \neg(x = x))$ [rule: contraction: (14)]

Chapter 2 - Exercise 5(i)

(1) $\neg\neg(x = x) \vee \neg(x = x)$ [axiom: propositional]
(2) $\neg\exists y\neg(x = x) \vee \neg(x = x)$ [rule: e-introduction: (1)]

Chapter 3 - §3.1 - Lemma 1

(1) $\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\mathbf{A} \vee \mathbf{A}$ [axiom: propositional]
(3) $\mathbf{B} \vee \mathbf{A}$ [rule: cut: (1) (2)]

Chapter 3 - §3.1 - Detachment Rule

(1) A	[premise]
(2) $\neg A \vee B$	[premise]
(3) $B \vee A$	[rule: expansion: (1)]
(4) $\neg B \vee B$	[axiom: propositional]
(5) $A \vee B$	[rule: cut: (3) (4)]
(6) $B \vee B$	[rule: cut: (5) (2)]
(7) B	[rule: contraction: (6)]

Chapter 3 - §3.1 - Tautology Theorem - result (B)

(1) $A \vee B$	[premise]
(2) $\neg\neg A \vee \neg A$	[axiom: propositional]
(3) $\neg\neg\neg A \vee \neg\neg A$	[axiom: propositional]
(4) $\neg A \vee \neg\neg A$	[rule: cut: (2) (3)]
(5) $B \vee \neg\neg A$	[rule: cut: (1) (4)]
(6) $\neg B \vee B$	[axiom: propositional]
(7) $\neg\neg A \vee B$	[rule: cut: (5) (6)]

Chapter 3 - §3.1 - Tautology Theorem - result (C)

(1) $\neg A \vee C$	[premise]
(2) $\neg B \vee C$	[premise]
(3) $\neg(A \vee B) \vee (A \vee B)$	[axiom: propositional]
(4) $(\neg(A \vee B) \vee A) \vee B$	[rule: associative: (3)]
(5) $\neg(\neg(A \vee B) \vee A) \vee (\neg(A \vee B) \vee A)$	[axiom: propositional]
(6) $B \vee (\neg(A \vee B) \vee A)$	[rule: cut: (4) (5)]
(7) $(\neg(A \vee B) \vee A) \vee C$	[rule: cut: (6) (2)]
(8) $C \vee (\neg(A \vee B) \vee A)$	[rule: cut: (7) (5)]
(9) $(C \vee \neg(A \vee B)) \vee A$	[rule: associative: (8)]
(10) $\neg(C \vee \neg(A \vee B)) \vee (C \vee \neg(A \vee B))$	[axiom: propositional]
(11) $A \vee (C \vee \neg(A \vee B))$	[rule: cut: (9) (10)]
(12) $(C \vee \neg(A \vee B)) \vee C$	[rule: cut: (11) (1)]
(13) $C \vee (C \vee \neg(A \vee B))$	[rule: cut: (12) (10)]
(14) $(C \vee C) \vee \neg(A \vee B)$	[rule: associative: (13)]
(15) $\neg(C \vee C) \vee (C \vee C)$	[axiom: propositional]
(16) $\neg(A \vee B) \vee (C \vee C)$	[rule: cut: (14) (15)]
(17) $(\neg(A \vee B) \vee C) \vee C$	[rule: associative: (16)]
(18) $\neg(\neg(A \vee B) \vee C) \vee (\neg(A \vee B) \vee C)$	[axiom: propositional]
(19) $C \vee (\neg(A \vee B) \vee C)$	[rule: cut: (17) (18)]
(20) $\neg(A \vee B) \vee (C \vee (\neg(A \vee B) \vee C))$	[rule: expansion: (19)]
(21) $(\neg(A \vee B) \vee C) \vee (\neg(A \vee B) \vee C)$	[rule: associative: (20)]
(22) $\neg(A \vee B) \vee C$	[rule: contraction: (21)]

Chapter 3 - §3.1 - Tautology Theorem - frequently used cases (ii)

(1) $\neg A \vee B$	[premise]
(2) $\neg B \vee C$	[premise]
(3) $\neg\neg A \vee \neg A$	[axiom: propositional]
(4) $B \vee \neg A$	[rule: cut: (1) (3)]
(5) $\neg A \vee C$	[rule: cut: (4) (2)]

Chapter 3 - §3.1 - Tautology Theorem - frequently used cases (vi)

(1) $\neg A \vee B$	[premise]
(2) $\neg\neg A \vee \neg A$	[axiom: propositional]
(3) $B \vee \neg A$	[rule: cut: (1) (2)]
(4) $\neg\neg B \vee \neg B$	[axiom: propositional]

(5) $\neg\neg\neg B \vee \neg\neg B$	[axiom: propositional]
(6) $\neg B \vee \neg\neg B$	[rule: cut: (4) (5)]
(7) $\neg A \vee \neg\neg B$	[rule: cut: (3) (6)]
(8) $\neg\neg B \vee \neg A$	[rule: cut: (7) (2)]

Chapter 3 - §3.2 - \forall -Introduction Rule

(1) $\neg A \vee B$	[premise]
(2) $\neg\neg A \vee \neg A$	[axiom: propositional]
(3) $B \vee \neg A$	[rule: cut: (1) (2)]
(4) $\neg\neg B \vee \neg B$	[axiom: propositional]
(5) $\neg\neg\neg B \vee \neg\neg B$	[axiom: propositional]
(6) $\neg B \vee \neg\neg B$	[rule: cut: (4) (5)]
(7) $\neg A \vee \neg\neg B$	[rule: cut: (3) (6)]
(8) $\neg\neg B \vee \neg A$	[rule: cut: (7) (2)]
(9) $\neg\exists x\neg B \vee \neg A$	[rule: e-introduction: (8)]
(10) $\neg\neg\exists x\neg B \vee \neg\exists x\neg B$	[axiom: propositional]
(11) $\neg A \vee \neg\exists x\neg B$	[rule: cut: (9) (10)]

Chapter 3 - §3.2 - Generalization Rule

(1) A	[premise]
(2) $\neg\neg\neg\exists x\neg A \vee A$	[rule: expansion: (1)]
(3) $\neg\neg\neg\neg\exists x\neg A \vee \neg\neg\neg\exists x\neg A$	[axiom: propositional]
(4) $A \vee \neg\neg\neg\exists x\neg A$	[rule: cut: (2) (3)]
(5) $\neg\neg A \vee \neg A$	[axiom: propositional]
(6) $\neg\neg A \vee \neg\neg A$	[axiom: propositional]
(7) $\neg A \vee \neg\neg A$	[rule: cut: (5) (6)]
(8) $\neg\neg\neg\exists x\neg A \vee \neg\neg A$	[rule: cut: (4) (7)]
(9) $\neg\neg A \vee \neg\neg\neg\exists x\neg A$	[rule: cut: (8) (3)]
(10) $\neg\exists x\neg A \vee \neg\neg\neg\exists x\neg A$	[rule: e-introduction: (9)]
(11) $\neg\neg\exists x\neg A \vee \neg\exists x\neg A$	[axiom: propositional]
(12) $\neg\neg\neg\exists x\neg A \vee \neg\exists x\neg A$	[rule: cut: (10) (11)]
(13) $\neg\neg\neg\exists x\neg A \vee \neg\neg\exists x\neg A$	[axiom: propositional]
(14) $\neg\neg\exists x\neg A \vee \neg\neg\neg\exists x\neg A$	[rule: cut: (13) (3)]
(15) $\neg\neg\neg\exists x\neg A \vee \neg\neg\neg\exists x\neg A$	[rule: cut: (10) (14)]
(16) $\neg\neg\neg\exists x\neg A$	[rule: contraction: (15)]
(17) $\neg\neg\neg\neg\exists x\neg A \vee \neg\neg\neg\neg\exists x\neg A$	[axiom: propositional]
(18) $\neg\neg\neg\exists x\neg A \vee \neg\neg\neg\neg\exists x\neg A$	[rule: cut: (3) (17)]
(19) $\neg\exists x\neg A \vee \neg\neg\neg\neg\exists x\neg A$	[rule: cut: (11) (18)]
(20) $\neg\neg\neg\neg\exists x\neg A \vee \neg\exists x\neg A$	[rule: cut: (19) (11)]
(21) $\neg\exists x\neg A \vee \neg\exists x\neg A$	[rule: cut: (12) (20)]
(22) $\neg\exists x\neg A$	[rule: contraction: (21)]

Chapter 3 - §3.2 - Distribution Rule

(1) $\neg A \vee B$	[premise]
(2) $\neg B \vee \exists xB$	[axiom: substitution]
(3) $\neg\neg A \vee \neg A$	[axiom: propositional]
(4) $B \vee \neg A$	[rule: cut: (1) (3)]
(5) $\neg A \vee \exists xB$	[rule: cut: (4) (2)]
(6) $\neg\exists xA \vee \exists xB$	[rule: e-introduction: (5)]
