Chapter 2

NOTATION

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$: syntactical variables over terms.
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$: syntactical variables over formulas.
$\mathbf{e}$: syntactical variables over constant symbols.
$\mathbf{f}$, $\mathbf{g}$: syntactical variables over function symbols.
$\mathbf{i}$, $\mathbf{j}$: syntactical variables over names.
$\mathbf{p}$, $\mathbf{q}$: syntactical variables over predicate symbols.
$\mathbf{u}$, $\mathbf{v}$: syntactical variables over expressions.
$\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$: syntactical variables over (individual) variables.

DEFINITIONS

- A *first-order language* has as symbols:
  a) the *variables*: x, y, z, w, x', y', z', w', x'', y'', z'', w'', ...
  b) for each n, the *n-ary function symbols* and the *n-ary predicate symbols*.
  c) the symbols $\neg$, $\vee$ and $\exists$.
- A *term* is defined inductively as:
  i) $\mathbf{x}$ is a term;
  ii) if $\mathbf{f}$ is n-ary, then $\mathbf{f}\mathbf{a}_1 \dots \mathbf{a}_n$ is a term.
- A *formula* is defined inductively as:
  i) if $\mathbf{p}$ is n-ary, then an atomic formula $\mathbf{p}\mathbf{a}_1 \dots \mathbf{a}_n$ is a formula;
  ii) $\neg\mathbf{A}$ is a formula;
  iii) $\vee\mathbf{A}\mathbf{B}$ is a formula;
  iv) $\exists\mathbf{x}\mathbf{A}$ is a formula.
- A *designator* is an expression which is either a term or a formula.
- A *structure* $\mathcal{A}$ for a first-order language L consist of:
  i) A nonempty set $|\mathcal{A}|$, the *universe* and its *individuals*.
  ii) For each n-ary function symbol $\mathbf{f}$ of L, an n-ary function $\mathbf{f}_{\mathcal{A}} : |\mathcal{A}|^n \to |\mathcal{A}|$. (In particular, for each constant $\mathbf{e}$ of L, $\mathbf{e}_{\mathcal{A}}$ is an individual of $\mathcal{A}$.)
  iii) For each n-ary predicate symbol $\mathbf{p}$ of L other than $=$, an n-ary predicate $\mathbf{p}_{\mathcal{A}}$ in $|\mathcal{A}|$.

  Also, $\mathcal{A}(\mathbf{a})$ designates an individual and $\mathcal{A}(\mathbf{A})$ designates a truth value.
- A formula $\mathbf{A}$ is *valid* in a structure $\mathcal{A}$ if $\mathcal{A}(\mathbf{A}') = \top$ for every $\mathcal{A}$-instance $\mathbf{A}'$ of $\mathbf{A}$. In particular, a closed formula $\mathbf{A}$ is valid in $\mathcal{A}$ iff $\mathcal{A}(\mathbf{A}) = \top$.
- A formula $\mathbf{A}$ is *logically valid* if it's valid in every structure.
- A formula $\mathbf{A}$ is a *consequence* of a set Γ of formulas if the validity of $\mathbf{A}$ follows from the validity of the formulas in Γ.
- A formula $\mathbf{A}$ is a *logical consequence* of a set Γ of formulas if $\mathbf{A}$ is valid in every structure for L in which all of the formulas in Γ are valid.
- A *first-order theory* is a formal system T such that
  i) the language of T is a first-order language;
  ii) the axioms of T are the logical axioms of $L(T)$ and certain further axioms, the *nonlogical axioms*;
  iii) the rules of T are Expansion, Contraction, Associative, Cut and $\exists$-Introduction.
- A *model* of a theory T, is a structure for $L(T)$ in which all the nonlogical axioms of T are valid.
- A formula $\mathbf{A}$ is *valid* in a theory T if it is valid in every model of T.

LOGICAL AXIOMS

Propositional: $\neg\mathbf{A} \vee \mathbf{A}$
Substitution: $\mathbf{A_x}[\mathbf{a}] \to \exists\mathbf{x}\mathbf{A}$
Identity: $\mathbf{x} = \mathbf{x}$
Equality: $\mathbf{x}_1 = \mathbf{y}_1 \to \cdots \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n$
$\mathbf{x}_1 = \mathbf{y}_1 \to \cdots \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{p}\mathbf{x}_1 \ldots \mathbf{x}_n \to \mathbf{p}\mathbf{y}_1 \ldots \mathbf{y}_n$

RULES OF INFERENCE

Expansion. Infer $\mathbf{B} \vee \mathbf{A}$ from $\mathbf{A}$.
Contraction. Infer $\mathbf{A}$ from $\mathbf{A} \vee \mathbf{A}$.
Associative. Infer $(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$ from $\mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$.
Cut. Infer $\mathbf{B} \vee \mathbf{C}$ from $\mathbf{A} \vee \mathbf{B}$ and $\neg\mathbf{A} \vee \mathbf{C}$.
∃-Introduction. If $\mathbf{x}$ is not free in $\mathbf{B}$, infer $\exists\mathbf{x}\mathbf{A} \to \mathbf{B}$ from $\mathbf{A} \to \mathbf{B}$.

RESULTS

§2.4

Lemma 1. *If $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}'_1, \ldots, \mathbf{u}'_n$ are designators and $\mathbf{u}_1 \ldots \mathbf{u}_n$ and $\mathbf{u}'_1 \ldots \mathbf{u}'_n$ are compatible, then $\mathbf{u}_i$ is $\mathbf{u}'_i$ for $i = 1, \ldots, n$.*

Formation Theorem. *Every designator can be written in the form $\mathbf{u}\mathbf{v}_1 \ldots \mathbf{v}_n$, where $\mathbf{u}$ is a symbol of index n and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are designators, in one and only one way.*

Lemma 2. *Every occurrence of a symbol in a designator $\mathbf{u}$ begins an occurrence of a designator in $\mathbf{u}$.*

Occurrence Theorem. *Let $\mathbf{u}$ be a symbol of index n, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be designators. Then any occurrence of a designator $\mathbf{v}$ in $\mathbf{u}\mathbf{v}_1 \ldots \mathbf{v}_n$ is either all of $\mathbf{u}\mathbf{v}_1 \ldots \mathbf{v}_n$ or a part of one of the $\mathbf{v}_i$.*

§2.5

Lemma. *Let $\mathcal{A}$ be a structure for L; $\mathbf{a}$ a variable-free term in $L(\mathcal{A})$; $\mathbf{i}$ the name of $\mathcal{A}(\mathbf{a})$. If $\mathbf{b}$ is a term of $L(\mathcal{A})$ in which no variable except $\mathbf{x}$ occurs, then $\mathcal{A}(\mathbf{b_x}[\mathbf{a}]) = \mathcal{A}(\mathbf{b_x}[\mathbf{i}])$. If $\mathbf{A}$ is a formula of $L(\mathcal{A})$ in which no variable except $\mathbf{x}$ is free, then $\mathcal{A}(\mathbf{A_x}[\mathbf{a}]) = \mathcal{A}(\mathbf{A_x}[\mathbf{i}])$*

Validity Theorem. *If T is a theory, then every theorem of T is valid in T.*

EXERCISES

1.

(a) Let $F(a_1, \ldots, a_n)$ be any truth function. We can construct another function

$$F'(a_1, \ldots, a_n) = H_{d,m}(H_{c,n}(a_1^1, \ldots, a_n^1), \ldots, H_{c,n}(a_1^m, \ldots, a_n^m))$$

where the $a_1^i, \ldots, a_n^i$ are all the tuples of truth values such that $F(a_1^i, \ldots, a_n^i) = \top$. Thus, $a_j^i = a_j$ or $a_j^i = H_\neg(a_j)$, for some values of i and j. Now, we can see that F and F' are the same function, since any truth assignment $a_1', \ldots, a_n'$ that satisfies (falsifies) F, also satisfies (falsifies) F', respectively. This is called *Disjunctive Normal Form (DNF)*.

We can also construct a similar function

$$\begin{aligned} F''(a_1, \ldots, a_n) &= H_{c,m}(H_\neg(H_{c,n}(a_1^1, \ldots, a_n^1)), \ldots, H_\neg(H_{c,n}(a_1^m, \ldots, a_n^m))) \\ &= H_{c,m}(H_{d,n}(H_\neg(a_1^1), \ldots, H_\neg(a_n^1)), \ldots, H_{d,n}(H_\neg(a_1^m), \ldots, H_\neg(a_n^m))) \end{aligned}$$

where the $a_1^i, \ldots, a_n^i$ are all the tuples of truth values such that $F(a_1^i, \ldots, a_n^i) = \bot$. It can be seen by a reasoning similar to above, that F and F'' are the same function. This is called *Conjunctive Normal Form (CNF)*.

(b) It can be seen that

$$\begin{aligned} H_{c,n} &= H_\wedge(a_1, H_\wedge(a_2, \ldots)) \\ H_{d,n} &= H_\vee(a_1, H_\vee(a_2, \ldots)). \end{aligned}$$

This means we can define any truth function F in terms of $H_\neg$, $H_\vee$ and $H_\wedge$, due to (a). Additionally, we can convert each instance of $H_\wedge(a, b)$ into $H_\neg(H_\vee(H_\neg(a), H_\neg(b)))$. Thus, every truth function is definable in terms of $H_\neg$ and $H_\vee$.

(c) Since $H_\vee(a, b)$ can be defined as $H_\rightarrow(H_\neg(a), b)$, every truth function is definable in terms of $H_\neg$ and $H_\rightarrow$, due to (b).

(d) Since $H_\vee(a, b)$ can be defined as $H_\neg(H_\wedge(H_\neg(a), H_\neg(b)))$, every truth function is definable in terms of $H_\neg$ and $H_\wedge$, due to (b).

(e) Consider the following identities, which can be easily verified e.g. via their truth tables

$$H_\vee(a, a) = a, \quad H_\vee(a, \top) = \top$$

$$H_\wedge(a, a) = a, \quad H_\wedge(a, \top) = a$$

$$H_\rightarrow(a, a) = \top, \quad H_\rightarrow(a, \top) = \top, \quad H_\rightarrow(\top, a) = a$$

$$H_\leftrightarrow(a, a) = \top, \quad H_\leftrightarrow(a, \top) = a, \quad H_\leftrightarrow(\top, a) = a.$$

Thus, any formula consisting of only those connectives and the free variable a can be inductively reduced to either a or $\top$ and can never define $H_\neg$. Those connectives can only define monotone functions while negation is not monotone. Note that allowing constants in the expression would allow to define negation as e.g. $H_\neg(a) = H_\rightarrow(a, \bot)$.

2.

(a) Note that $H_d(a, b) = H_\wedge(H_\neg(a), H_\neg(b))$. We can then define

$$\begin{aligned} H_\neg(a) &= H_d(a, a) \\ H_\vee(a, b) &= H_d(H_d(a, b), H_d(a, b)) \end{aligned}$$

and thus every truth function is definable in terms of H_d (using result from 1.1(b)).

(b) Note that $H_s(a, b) = H_\neg(H_\wedge(a, b))$. We can then define

$$\begin{aligned} H_\neg(a) &= H_s(a, a) \\ H_\vee(a, b) &= H_s(H_s(a, a), H_s(b, b)) \end{aligned}$$

and thus every truth function is definable in terms of H_s (using result from 1.1(b)).

(c) Let H be singulary with $H(a_1, \ldots, a_n) = H'(a_i)$. The syntax of every truth function $F(a_1, \ldots, a_m)$ definable in terms of H can be inductively defined by

$$e ::= a_j | H(e_1, \ldots, e_n)$$

where $1 \leq j \leq m$ and $e_1, \ldots, e_n$ are valid expressions.

We can then reduce every expression to an equivalent expression that involves a single a_j: as long as the expression has the form $H(e_1, \ldots, e_n)$, we can replace it with $H'(e_i)$ and inductively reduce e_i. Thus, every truth function F definable in terms of H is singulary and furthermore

$$F(a_1, \ldots, a_m) = {H'}^k(a_j)$$

for some integers $k \geq 0$ and $1 \leq j \leq m$.

(d) Note that since any n-ary truth function is completely determined by its truth table, there are 2^{2^n} of them. So we know there are $2^{2^2} = 16$ binary truth functions. Let's analyze them:

- Consider the four binary truth functions H such that

  $$H(a, a) = a.$$

  It is easy to see that any function definable in terms of such H can be inductively reduced to a, in a similar fashion as before. Thus, none of these four functions can define every truth function (e.g. negation $H_\neg$ cannot be defined).
- Consider the four binary truth functions H such that

  $$H(a, a) = \perp.$$

  For each of these four functions, we have

  $$H(a, \perp) \in \{a, \perp\}, \quad H(\perp, a) \in \{a, \perp\}$$

  and thus none of these four functions can define every truth function (e.g. negation $H_\neg$ cannot be defined).
- Consider the four binary truth functions H such that

  $$H(a, a) = \top.$$

  This case is symmetric to the previous one. For each of these four functions, we have

  $$H(a, \top) \in \{a, \top\}, \quad H(\top, a) \in \{a, \top\}$$

  and thus none of these four functions can define every truth function (e.g. negation $H_\neg$ cannot be defined).
- For the four remaining binary truth functions, we have

  $$H(\top, \top) = \perp, \quad H(\perp, \perp) = \top.$$

  Two of those functions

  $$H_1(\top, \perp) = \top, \quad H_1(\perp, \top) = \perp$$
  $$H_2(\top, \perp) = \perp, \quad H_2(\perp, \top) = \top$$

  are singulary and thus cannot define functions such as $H_\vee$, due to the result from 2.2(c). The two remaining functions are H_d and H_s, presented in 2.2(a) and 2.2(b), respectively.

3. If $\mathbf{v}$ is empty, then trivially neither $\mathbf{u}$ or $\mathbf{v'}$ are empty, and they are both designators.

Let's assume that $\mathbf{v}$ is not empty and that the designator $\mathbf{uv}$ has the form $\mathbf{t}\mathbf{t}_1 \ldots \mathbf{t}_n$. Since $\mathbf{uv}$ and $\mathbf{vv}'$ are designators, they both begin with a symbol: thus $\mathbf{v}$ also begins with a symbol, since it is a non-empty prefix of $\mathbf{vv}'$. The occurrence of this symbol in $\mathbf{uv}$ begins the occurrence of a designator $\mathbf{u}'$ in $\mathbf{uv}$ (by Lemma 2), which is compatible with $\mathbf{v}$. Moreover, the occurrence of $\mathbf{u}'$ in $\mathbf{uv}$ is either all of $\mathbf{uv}$ or part of one of the $\mathbf{t}_i$ (by the Occurrence Theorem). In the former case, it means that $\mathbf{v}$ is a designator and $\mathbf{u}$ and $\mathbf{v}'$ are empty. On the other hand, if $\mathbf{u}'$ is part of one of the $\mathbf{t}_i$, it means that $\mathbf{vv}'$ begins with $\mathbf{u}'$, and thus $\mathbf{u}'$ and $\mathbf{v}$ are the same (by the Formation Theorem) and $\mathbf{v}'$ is empty.

4. If a term is:

i) a variable $\mathbf{x}'$, then the substitution result is $\mathbf{x}$ itself, which is also a term.

ii) a function application $\mathbf{f}\mathbf{a}_1 \ldots \mathbf{a}_n$, then $\mathbf{a}$ is one of the $\mathbf{a}_i$ and the substitution result is also a term, or $\mathbf{a}$ is substituted in one of the terms $\mathbf{a}_i$, and it remains a term, by the induction hypothesis.

If a formula is:

i) an atomic formula $\mathbf{p}\mathbf{a}_1 \ldots \mathbf{a}_n$, then substituting $\mathbf{a}$ in any of the $\mathbf{a}_i$ results in a term, as previously shown. Thus it remains a formula.

ii) $\neg\mathbf{A}$, then substituting $\mathbf{a}$ in $\mathbf{A}$ remains a formula by the induction hypothesis.

iii) $\vee\mathbf{AB}$, then substituting $\mathbf{a}$ in $\mathbf{A}$ or $\mathbf{B}$ remains a formula by the induction hypothesis.

iv) $\exists\mathbf{y}\mathbf{A}$, then substituting $\mathbf{a}$ in $\mathbf{A}$ remains a formula by the induction hypothesis.

5.

(a) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \bot; \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= \top. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without propositional axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = f(\exists\mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\neg\mathbf{A}' \vee \mathbf{C}') = f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \to \mathbf{B}') = f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{B}') = f(\mathbf{A}) = \top$.

Thus, if $\mathbf{A}$ is provable without propositional axioms, we have $f(\mathbf{A}) = \top$. But $f(\neg\neg(x = x) \vee \neg(x = x)) = f(\neg(x = x)) = \bot$ and so it is not provable without propositional axioms.

(b) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= \bot. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without substitution axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = \neg f(\exists\mathbf{x}\mathbf{A}') \vee f(\mathbf{B}') = \top$.

Thus, if $\mathbf{A}$ is provable without substitution axioms, we have $f(\mathbf{A}) = \top$. But $f(x = x \to \exists x(x = x)) = \neg f(x = x) \vee f(\exists x(x = x)) = \bot$ and so it is not provable without substitution axioms.

(c) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \bot, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without identity axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\mathbf{A}') = \top$ (see below for this case);
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = \neg f(\mathbf{A}') \vee f(\mathbf{B}') = f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}' \to \mathbf{B}') = \top$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \ldots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \bot$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n) = \bot$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = \neg f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \vee f(\mathbf{B}'_\mathbf{x}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without identity axioms, we have $f(\mathbf{A}) = \top$. But $f(x = x) = \bot$ and so it is not provable without identity axioms.

(d) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{e}_i = \mathbf{e}_j) &= \top \quad \text{iff } i \le j; \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= \top \quad \text{iff } f(\mathbf{A_x}[\mathbf{e}_i]) = \top \text{ for some } i. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without equality axioms then $f(\mathbf{A}') = \top$ for every formula obtained from $\mathbf{A}$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$ for every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \vee f(\exists\mathbf{x}\mathbf{A}')$. For every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable (except $\mathbf{x}$) by some $\mathbf{e}_i$ at all its free occurrences: if $f(\mathbf{A}''_\mathbf{x}[\mathbf{e}_i]) = \top$ for some i, then $f(\exists\mathbf{x}\mathbf{A}'') = \top$ by the definition of f. Otherwise, $f(\mathbf{A}''_\mathbf{x}[\mathbf{e}_i]) = \bot$ for all i and thus $\neg f(\mathbf{A}''_\mathbf{x}[\mathbf{e}_i]) = \top$;
- an identity axiom: $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$ for any substitution of $\mathbf{x}$ by some $\mathbf{e}_i$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}')$ with $f(\mathbf{A}') = \top$ for every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = \top$ for every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = \top$ for every closed formulas $\mathbf{A}''$, $\mathbf{B}''$ and $\mathbf{C}''$ obtained from $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$, respectively, by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}') = \top$ for every closed formulas $\mathbf{A}''$, $\mathbf{B}''$ and $\mathbf{C}''$ obtained from $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$, respectively, by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}') = \neg f(\exists\mathbf{x}\mathbf{A}') \vee f(\mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \neg f(\mathbf{A}') \vee f(\mathbf{B}') = \top$ for every closed formula $\mathbf{A}''$ and $\mathbf{B}''$ obtained from $\mathbf{A}'$ and $\mathbf{B}'$, respectively, by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis. If $f(\mathbf{B}') = \top$, then $f(\mathbf{A}) = \top$ follows trivially. Otherwise, we must have $f(\mathbf{A}') = \bot$ for all closed formulas $\mathbf{A}''$ obtained from $\mathbf{A}'$ as described above. This implies that $f(\exists\mathbf{x}\mathbf{A}') = \bot$ and thus $f(\mathbf{A}) = \top$.

Thus, if $\mathbf{A}$ is provable without equality axioms, we have $f(\mathbf{A}') = \top$ for every formula $\mathbf{A}'$ obtained from $\mathbf{A}$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurences. But $f(x = y \to x = z \to x = x \to y = z) = \neg f(x = y) \vee \neg f(x = z) \vee \neg f(x = x) \vee f(y = z) = \bot$ since it does not hold for the substitution $[\mathbf{x}, \mathbf{y}, \mathbf{z}] \to [\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2]$ and so it is not provable without equality axioms.

(e) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic}; \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \leftrightarrow \neg f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the expansion rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \leftrightarrow \neg f(\mathbf{A}') = \top$ (see below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \leftrightarrow \neg f(\mathbf{x}_2 = \mathbf{y}_2) \leftrightarrow \neg f(\mathbf{x}_1 = \mathbf{x}_2) \leftrightarrow \neg f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. However, this is a contradiction since $f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{A}') = \bot$ for any $\mathbf{A}'$ so it's not possible to have a proof where the contraction rule is applied (???);
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \leftrightarrow \neg(f(\mathbf{B}') \leftrightarrow \neg f(\mathbf{C}')) = f(\mathbf{A}') \leftrightarrow (f(\mathbf{B}') \leftrightarrow f(\mathbf{C}'))$ and $f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \leftrightarrow \neg f(\mathbf{C}') = (f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')) \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}') \leftrightarrow f(\mathbf{C}')$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')$ and $f(\neg\mathbf{A} \vee \mathbf{C}') = \neg f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{C}')$, and thus $f(\mathbf{B}') \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}')$ and $f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}')$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \dots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n) = \top$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = \neg f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \leftrightarrow \neg f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without the expansion rule, we have $f(\mathbf{A}) = \top$. But $f(x = x \vee (\neg(x = x) \vee (x = x))) = f(x = x) \leftrightarrow \neg(\neg f(x = x) \leftrightarrow \neg f(x = x)) = \bot$ and so it is not provable without the expansion rule.

(f) The hinted function is defined as:

$$
\begin{aligned}
f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\
f(\neg\mathbf{A}) &= \bot; \\
f(\mathbf{A} \vee \mathbf{B}) &= \top; \\
f(\exists\mathbf{x}\mathbf{A}) &= \bot.
\end{aligned}
$$

Let's prove that if $\mathbf{A}$ is provable without the contraction rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = \top$;

- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = \top$.

Thus, if $\mathbf{A}$ is provable without the contraction rule, we have $f(\mathbf{A}) = \top$. But $f(\neg\neg(x = x)) = \bot$ and so it is not provable without the contraction rule.

(g) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= 0, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= 1 - f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \cdot f(\mathbf{B}) \cdot (1 - f(\mathbf{A}) - f(\mathbf{B})); \\ f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the associative rule then $f(\mathbf{A}) = 0$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{A}') \cdot (1 - (1 - f(\mathbf{A}')) - f(\mathbf{A}')) = (1 - f(\mathbf{A}')) \cdot f(\mathbf{A}') \cdot (f(\mathbf{A}') - f(\mathbf{A}')) = 0$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = (1 - f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])) \cdot f(\mathbf{A}') \cdot (1 - (1 - f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])) - f(\mathbf{A}')) = 0$ (see below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = 0$;
- an equality axiom: we have

$$\begin{aligned} f(\mathbf{A}) &= f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) \\ &= f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) \\ &= (1 - f(\mathbf{x}_1 = \mathbf{y}_1)) \cdot f(\mathbf{A}') \cdot (f(\mathbf{x}_1 = \mathbf{y}_1) - f(\mathbf{A}')) \end{aligned}$$

$$f(\mathbf{A}') = f(\mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = (1 - f(\mathbf{x}_2 = \mathbf{y}_2)) \cdot f(\mathbf{A}'') \cdot (f(\mathbf{x}_2 = \mathbf{y}_2) - f(\mathbf{A}''))$$

$$f(\mathbf{A}'') = f(\mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = (1 - f(\mathbf{x}_1 = \mathbf{x}_2)) \cdot f(\mathbf{y}_1 = \mathbf{y}_2) \cdot (f(\mathbf{x}_1 = \mathbf{x}_2) - f(\mathbf{y}_1 = \mathbf{y}_2)) = 0;$$

- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}') \cdot f(\mathbf{A}') \cdot (1 - f(\mathbf{B}') - f(\mathbf{A}')) = 0$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \cdot f(\mathbf{A}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{A}')) = 0$ and the only integer solution is $f(\mathbf{A}') = 0$ and thus $f(\mathbf{A}) = 0$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = 0$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = 0$ by the induction hypothesis. Consider the equations

$$f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{B}')) = 0 \tag{1}$$
$$f(\neg\mathbf{A}' \vee \mathbf{C}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{C}') \cdot (f(\mathbf{A}') - f(\mathbf{C}')) = 0 \tag{2}$$
$$f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \cdot f(\mathbf{C}') \cdot (1 - f(\mathbf{B}') - f(\mathbf{C}')) = 0 \tag{3}$$

  and the possible cases that satisfy equation (2). First, $(1 - f(\mathbf{A}')) = 0$ implies that $f(\mathbf{A}') = 1$ and substituting in equation (1) we obtain $f(\mathbf{B}') \cdot (-f(\mathbf{B}')) = 0$ which means that $f(\mathbf{B}') = 0$ which satisfies equation (3). Second, $f(\mathbf{C}') = 0$, which trivially satifies equation (3). Third, $f(\mathbf{A}') - f(\mathbf{C}') = 0$ which implies $f(\mathbf{A}') = f(\mathbf{C}')$ and substituting in equation (1) we obtain $f(\mathbf{C}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{C}') - f(\mathbf{B}')) = 0$. Thus, equation (3) is satisfied in all cases;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{B}') \cdot (f(\mathbf{A}') - f(\mathbf{B}')) = f(\mathbf{A}' \to \mathbf{B}') = 0$.

To treat substitution axioms, let's show that $f(\mathbf{A}_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n$: we have $f(\mathbf{A}_\mathbf{x}[\mathbf{a}]) = f(\mathbf{p}{\mathbf{b}_1}_\mathbf{x}[\mathbf{a}] \ldots {\mathbf{b}_n}_\mathbf{x}[\mathbf{a}]) = 0$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n) = 0$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$: we have $f(\mathbf{A}_\mathbf{x}[\mathbf{a}]) = 1 - f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\mathbf{A}) = 1 - f(\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \cdot f(\mathbf{B}'_\mathbf{x}[\mathbf{a}]) \cdot (1 - f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) - f(\mathbf{B}'_\mathbf{x}[\mathbf{a}]))$ and $f(\mathbf{A}) = f(\mathbf{A}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{B}'))$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without the associative rule, we have $f(\mathbf{A}) = 0$. But $f(\neg(\neg(x = x) \vee \neg(x = x))) = 1 - f(\neg(x = x) \vee \neg(x = x)) = 1 - ((1 - f(x = x))^2 \cdot (1 - 2 \cdot (1 - f(x = x)))) = 1 - (1 - 2) = 2$ and so it is not provable without the associative rule.

(h) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \begin{cases} \top, & \text{if } f(\mathbf{A}) = \bot \text{ or } \mathbf{A} \text{ is atomic;} \\ \bot, & \text{otherwise.} \end{cases} \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the cut rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = f(\neg\mathbf{A}') \vee f(\mathbf{A}') = \top$ (since if $f(\mathbf{A}') = \bot$, then $f(\neg\mathbf{A}') = \top$ from the definition of f);
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \vee f(\mathbf{A}') = \top$ (since if $f(\mathbf{A}') = \bot$, then $f(\neg\mathbf{A}') = \top$ from the definition of f and $f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\neg\mathbf{A}')$. See below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1)) \vee f(\neg(\mathbf{x}_2 = \mathbf{y}_2)) \vee f(\neg(\mathbf{x}_1 = \mathbf{x}_2)) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = \top$ and thus $f(\mathbf{A}) = f(\mathbf{A}') = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = f(\neg\exists\mathbf{x}\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}') \vee f(\mathbf{B}') = \top$. So either $f(\neg\mathbf{A}') = \top$ or $f(\mathbf{B}') = \top$. In the latter case, it follows trivially that $f(\mathbf{A}) = \top$. In the former case, note that since $f(\exists\mathbf{x}\mathbf{A}) = f(\mathbf{A})$ and $\exists\mathbf{x}\mathbf{A}$ is not atomic, then $f(\neg\exists\mathbf{x}\mathbf{A}') = f(\neg\mathbf{A}')$.

To treat substitution axioms, let's show that $f(\mathbf{A_x}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \dots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n) = \top$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$ with $\mathbf{A}'$ atomic: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\neg\mathbf{A}') = \top$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$ with $\mathbf{A}'$ not atomic: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\neg\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \vee f(\mathbf{B}'_\mathbf{x}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without the cut rule, we have $f(\mathbf{A}) = \top$. But $f(\neg\neg(x = x)) = \bot$ since $f(\neg(x = x)) = \top$ and so it is not provable without the cut rule.

(i) The hinted function is defined as:

$$\begin{aligned}
f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\
f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\
f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\
f(\exists\mathbf{x}\mathbf{A}) &= \top.
\end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the $\exists$-introduction rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\exists\mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;

Thus, if $\mathbf{A}$ is provable without the $\exists$-introduction rule, we have $f(\mathbf{A}) = \top$. But $f(\exists y \neg(x = x) \to \neg(x = x)) = \neg f(\exists y \neg(x = x)) \vee \neg f(x = x) = \bot$ and so it is not provable without the $\exists$-introduction rule.

Chapter 3

DEFINITIONS

- $\mathbf{A}$ is *elementary* if it is either atomic or an instantiation.
- A *truth valuation* for T is a mapping from the set of elementary formulas in T to the set of truth values.
- $\mathbf{B}$ is a *tautological consequence* of $\mathbf{A}_1, \dots, \mathbf{A}_n$ if $V(\mathbf{B}) = \top$ for every truth valuation V such that $V(\mathbf{A}_1) = \cdots = V(\mathbf{A}_n) = \top$.
- $\mathbf{A}$ is a *tautology* if it is a tautological consequence of the empty sequence of formulas, i.e. if $V(\mathbf{A}) = \top$ for every truth valuation V.
- $\mathbf{A}'$ is an *instance* of $\mathbf{A}$ if $\mathbf{A}'$ is of the form $\mathbf{A}_{\mathbf{x}_1,\dots,\mathbf{x}_n}[\mathbf{a}_1, \dots, \mathbf{a}_2]$.
- Let $\mathbf{A}$ be a formula and $\mathbf{x}_1, \dots, \mathbf{x}_n$ its free variables in alphabetical order. The *closure* of $\mathbf{A}$ is the formula $\forall \mathbf{x}_1 \dots \forall \mathbf{x}_n \mathbf{A}$.
- $\mathbf{A}'$ is a *variant* of $\mathbf{A}$ if $\mathbf{A}'$ can be obtained from $\mathbf{A}$ by a sequence of replacements of the following type: replace a part $\exists \mathbf{x} \mathbf{B}$ by $\exists \mathbf{y} \mathbf{B}_{\mathbf{x}}[\mathbf{y}]$, where $\mathbf{y}$ is a variable not free in $\mathbf{B}$.
- $\mathbf{A}$ is *open* if it contains no quantifiers.
- $\mathbf{A}$ is in *prenex form* if it has the form $Q\mathbf{x}_1 \dots Q\mathbf{x}_n \mathbf{B}$ where each $Q\mathbf{x}_i$ is either $\exists \mathbf{x}_i$ or $\forall \mathbf{x}_i$; $\mathbf{x}_1, \dots, \mathbf{x}_n$ are distinct; and $\mathbf{B}$ is open.

RESULTS

§**3.1**

Tautology Theorem. *If $\mathbf{B}$ is a tautological consequence of $\mathbf{A}_1, \dots, \mathbf{A}_n$, and $\vdash \mathbf{A}_1, \dots, \vdash \mathbf{A}_n$, then $\vdash \mathbf{B}$.*

Corollary. *Every tautology is a theorem.*

Lemma 1. *If $\vdash \mathbf{A} \vee \mathbf{B}$, then $\vdash \mathbf{B} \vee \mathbf{A}$.*

Detachment Rule. *If $\vdash \mathbf{A}$ and $\vdash \mathbf{A} \to \mathbf{B}$, then $\vdash \mathbf{B}$.*

Corollary. *If $\vdash \mathbf{A}_1, \dots, \vdash \mathbf{A}_n$, and $\vdash \mathbf{A}_1 \to \dots \to \mathbf{A}_n \to \mathbf{B}$, then $\vdash \mathbf{B}$.*

Lemma 2. *If $n \geq 2$, and $\mathbf{A}_1 \vee \cdots \vee \mathbf{A}_n$ is a tautology, then $\vdash \mathbf{A}_1 \vee \cdots \vee \mathbf{A}_n$.*

§**3.2**

$\forall$-Introduction Rule. *If $\vdash \mathbf{A} \to \mathbf{B}$ and $\mathbf{x}$ is not free in $\mathbf{A}$, then $\vdash \mathbf{A} \to \forall \mathbf{x} \mathbf{B}$.*

Generalization Rule. *If $\vdash \mathbf{A}$, then $\vdash \forall \mathbf{x} \mathbf{A}$.*

Substitution Rule. *If $\vdash \mathbf{A}$ and $\mathbf{A}'$ is an instance of $\mathbf{A}$, then $\vdash \mathbf{A}'$.*

Substitution Theorem.

$$\vdash \mathbf{A}_{\mathbf{x}_1,\dots,\mathbf{x}_n}[\mathbf{a}_1, \dots, \mathbf{a}_n] \to \exists \mathbf{x}_1 \dots \exists \mathbf{x}_n \mathbf{A}$$
$$\vdash \forall \mathbf{x}_1 \dots \forall \mathbf{x}_n \mathbf{A} \to \mathbf{A}_{\mathbf{x}_1,\dots,\mathbf{x}_n}[\mathbf{a}_1, \dots, \mathbf{a}_n]$$

Distribution Rule. *If $\vdash \mathbf{A} \to \mathbf{B}$, then $\vdash \exists \mathbf{x} \mathbf{A} \to \exists \mathbf{x} \mathbf{B}$ and $\vdash \forall \mathbf{x} \mathbf{A} \to \forall \mathbf{x} \mathbf{B}$.*

Closure Theorem. *If $\mathbf{A}'$ is the closure of $\mathbf{A}$, then $\vdash \mathbf{A}'$ iff $\vdash \mathbf{A}$.*

Corollary. *If $\mathbf{A}'$ is the closure of $\mathbf{A}$, then $\mathbf{A}'$ is valid in a structure $\mathcal{A}$ iff $\mathbf{A}$ is valid in $\mathcal{A}$.*

§**3.3**

Deduction Theorem. *Let $\mathbf{A}$ be a closed formula in T. For every formula $\mathbf{B}$ of T, $\vdash_T \mathbf{A} \to \mathbf{B}$ iff $\mathbf{B}$ is a theorem of $T[\mathbf{A}]$.*

Corollary. *Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be closed formulas in T. For every formula $\mathbf{B}$ in T, $\vdash_T \mathbf{A}_1 \to \cdots \to \mathbf{A}_n \to \mathbf{B}$ iff $\mathbf{B}$ is a theorem of $T[\mathbf{A}_1, \dots, \mathbf{A}_n]$.*

Theorem on Constants. *Let T' be obtained from T by adding new constants (but no new nonlogical axioms). For every formula $\mathbf{A}$ of T and every sequence $\mathbf{e}_1, \dots, \mathbf{e}_n$ of distinct new constants, $\vdash_T \mathbf{A}$ iff $\vdash_{T'} \mathbf{A}[\mathbf{e}_1, \dots, \mathbf{e}_n]$.*

§**3.4**

Equivalence Theorem. *Let* $\mathbf{A}'$ *be obtained from* $\mathbf{A}$ *by replacing some occurrences of* $\mathbf{B}_1, \ldots, \mathbf{B}_n$ *by* $\mathbf{B}'_1, \ldots, \mathbf{B}'_n$, *respectively. If*

$$\vdash \mathbf{B}_1 \leftrightarrow \mathbf{B}'_1, \ldots, \vdash \mathbf{B}_n \leftrightarrow \mathbf{B}'_n$$

then

$$\vdash \mathbf{A} \leftrightarrow \mathbf{A}'.$$

Variant Theorem. *If* $\mathbf{A}'$ *is a variant of* $\mathbf{A}$, *then* $\vdash \mathbf{A} \leftrightarrow \mathbf{A}'$.

Symmetry Theorem. $\vdash \mathbf{a} = \mathbf{b} \leftrightarrow \mathbf{b} = \mathbf{a}$.

Equality Theorem. *Let* $\mathbf{b}'$ *be obtained from* $\mathbf{b}$ *by replacing some occurrences of* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ *not immediately following* $\exists$ *or* $\forall$ *by* $\mathbf{a}'_1, \ldots, \mathbf{a}'_n$ *respectively, and let* $\mathbf{A}'$ *be obtained from* $\mathbf{A}$ *by the same type of replacements. If* $\vdash \mathbf{a}_1 = \mathbf{a}'_1, \ldots \vdash \mathbf{a}_n = \mathbf{a}'_n$ *then* $\vdash \mathbf{b} = \mathbf{b}'$ *and* $\vdash \mathbf{A} \leftrightarrow \mathbf{A}'$.

Corollary 1. $\vdash \mathbf{a}_1 = \mathbf{a}'_1 \to \cdots \to \mathbf{a}_n = \mathbf{a}'_n \to \mathbf{b}[\mathbf{a}_1, \ldots, \mathbf{a}_n] = \mathbf{b}[\mathbf{a}'_1, \ldots, \mathbf{a}'_n]$.

Corollary 2. $\vdash \mathbf{a}_1 = \mathbf{a}'_1 \to \cdots \to \mathbf{a}_n = \mathbf{a}'_n \to (\mathbf{A}[\mathbf{a}_1, \ldots, \mathbf{a}_n] \leftrightarrow \mathbf{A}[\mathbf{a}'_1, \ldots, \mathbf{a}'_n])$.

Corollary 3. *If* $\mathbf{x}$ *does not occur in* $\mathbf{a}$, *then*

$$\vdash \mathbf{A}_{\mathbf{x}}[\mathbf{a}] \leftrightarrow \exists \mathbf{x}(\mathbf{x} = \mathbf{a} \wedge \mathbf{A})$$

EXERCISES

1. Let's prove it by induction on theorems (as in §3.1). If $\mathbf{A}$ is a theorem provable without use of substitution axioms, identity axioms, equality axioms, nonlogical axioms or the $\exists$-introduction rule, then it is a tautological consequence of some theorems $\mathbf{B}_1, \ldots, \mathbf{B}_n$. If $n = 0$, then $\mathbf{A}$ is a tautology, since it's a tautological consequence of the empty sequence of formulas. Otherwise, by the induction hypothesis, if $\mathbf{B}_1, \ldots, \mathbf{B}_n$ can be proven without the use of substitution axioms, identity axioms, equality axioms, nonlogical axioms or the $\exists$-introduction rule, they are also tautologies. This means that $V(\mathbf{B}_i) = \top$ for all i and truth valuations V, which implies that $V(\mathbf{A}) = \top$ for all truth valuations and thus $\mathbf{A}$ is also a tautology.

3.

(a) Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ the free variables of $\forall\mathbf{x}(\mathbf{A} \to \mathbf{B})$ and $\exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}\mathbf{B}$; let T' be a theory obtained from T by adding n new constants $\mathbf{e}_1, \ldots, \mathbf{e}_n$; let $\mathbf{C}$ be $\forall\mathbf{x}(\mathbf{A} \to \mathbf{B})$ and let $\mathbf{D}$ be $\exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}\mathbf{B}$. Note that

$$\vdash_T \mathbf{C} \to \mathbf{D} \quad \text{iff} \quad \vdash_{T'} \mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

by the Theorem on Constants, and

$$\vdash_{T'} \mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \quad \text{iff} \quad \vdash_{T'[\mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n]]} \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

by the Deduction Theorem. Hence, in $T'[\mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n]]$, we have

$\vdash \mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [the added nonlogical axiom]
$\vdash \forall\mathbf{x}(\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n])$ [by the definition of $\mathbf{C}$]
$\vdash \forall\mathbf{x}(\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n]) \to (\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n])$ [Substitution Theorem]
$\vdash \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [Detachment Rule]
$\vdash \exists\mathbf{x}\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \exists\mathbf{x}\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [Distribution Rule]
$\vdash \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [by the definition of $\mathbf{D}$]

(b) As in (a), but using the universal-quantifier form of the Distribution Rule.

5. The existential form
(1) $\vdash \mathbf{A} \to \exists\mathbf{x}\mathbf{A}$ [Substitution Theorem or Substitution Axiom]
(2) $\vdash \mathbf{A} \to \mathbf{A}$ [Propositional Axiom and definition of $\to$]
(3) $\vdash \exists\mathbf{x}\mathbf{A} \to \mathbf{A}$ [$\exists$-Introduction Rule]
(4) $\vdash \exists\mathbf{x}\mathbf{A} \leftrightarrow \mathbf{A}$ [from (1) and (3) and the definition of $\leftrightarrow$]
and the universal form
(1) $\vdash \forall\mathbf{x}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]
(2) $\vdash \mathbf{A} \to \mathbf{A}$ [Propositional Axiom and definition of $\to$]
(3) $\vdash \mathbf{A} \to \forall\mathbf{x}\mathbf{A}$ [$\forall$-Introduction Rule]
(4) $\vdash \forall\mathbf{x}\mathbf{A} \leftrightarrow \mathbf{A}$ [from (1) and (3) and the definition of $\leftrightarrow$]

6.

(a)
(1) $\vdash \mathbf{A} \to \exists\mathbf{x}\exists\mathbf{y}\mathbf{A}$ [Substitution Theorem]
(2) $\vdash \exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}\exists\mathbf{y}\mathbf{A}$ [$\exists$-Introduction Rule]
(3) $\vdash \exists\mathbf{y}\exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}\exists\mathbf{y}\mathbf{A}$ [$\exists$-Introduction Rule]

The reverse implication is obtained in a similar fashion. Note that it is also possible to obtain $\exists\mathbf{y}\exists\mathbf{x}\mathbf{A}$ as a variant of $\exists\mathbf{x}\exists\mathbf{y}\mathbf{A}$: first obtain $\exists\mathbf{x}\exists\mathbf{x}'\mathbf{A}'$ where $\mathbf{A}' = \mathbf{A}_y[\mathbf{x}']$ and $\mathbf{x}'$ is a new variable not appearing in $\mathbf{A}$; then obtain $\exists\mathbf{y}\exists\mathbf{x}'\mathbf{A}'_{\mathbf{x}}[\mathbf{y}]$ and finally $\exists\mathbf{y}\exists\mathbf{x}\mathbf{A}$ by substituting $\mathbf{x}'$ to $\mathbf{x}$. This would imply the result by the Variant Theorem.

(b)
(1) $\vdash \forall\mathbf{x}\forall\mathbf{y}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]
(2) $\vdash \forall\mathbf{x}\forall\mathbf{y}\mathbf{A} \to \forall\mathbf{x}\mathbf{A}$ [$\forall$-Introduction Rule]
(3) $\vdash \forall\mathbf{x}\forall\mathbf{y}\mathbf{A} \to \forall\mathbf{y}\forall\mathbf{x}\mathbf{A}$ [$\forall$-Introduction Rule]

The reverse implication is obtained in a similar fashion.

(c)
(1) $\vdash \mathbf{A} \to \exists\mathbf{x}\mathbf{A}$ [Substitution Theorem]

(2) $\vdash \forall \mathbf{y} \mathbf{A} \to \forall \mathbf{y} \exists \mathbf{x} \mathbf{A}$ [Distribution Rule]
(3) $\vdash \exists \mathbf{x} \forall \mathbf{y} \mathbf{A} \to \forall \mathbf{y} \exists \mathbf{x} \mathbf{A}$ [$\exists$-Introduction Rule]

Note that it might seem that using the dual results in the above proof, the opposite implication could be obtained (i.e. $\vdash \forall \mathbf{y} \exists \mathbf{x} \mathbf{A} \to \exists \mathbf{x} \forall \mathbf{y} \mathbf{A}$). However this is not the case, as they result in an alternative proof of the same result as above:
(1) $\vdash \forall \mathbf{y} \mathbf{A} \to \mathbf{A}$ [Substitution Theorem]
(2) $\vdash \exists \mathbf{x} \forall \mathbf{y} \mathbf{A} \to \exists \mathbf{x} \mathbf{A}$ [Distribution Rule]
(3) $\vdash \exists \mathbf{x} \forall \mathbf{y} \mathbf{A} \to \forall \mathbf{y} \exists \mathbf{x} \mathbf{A}$ [$\forall$-Introduction Rule]

(d) Consider the formula

$$\forall x \exists y (Sx = y) \to \exists y \forall x (Sx = y).$$

The left side can be interpreted as *"every number has a successor"*, while the right side can be interpreted as *"there is a number that is the successor of every number"*.

Theories

N (Natural Numbers)

Nonlogical symbols:
- constant 0
- unary function symbol S, the successor function
- binary function symbols $+$ and $\cdot$
- binary predicate symbol $<$

Nonlogical axioms:

N1. $Sx \neq 0$
N2. $Sx = Sy \rightarrow x = y$
N3. $x + 0 = x$
N4. $x + Sy = S(x + y)$
N5. $x \cdot 0 = 0$
N6. $x \cdot Sy = (x \cdot y) + x$
N7. $\neg(x < 0)$
N8. $x < Sy \leftrightarrow x < y \vee x = y$
N9. $x < y \leftrightarrow \vee x = y \vee y < x$

G (Elementary Theory of Groups)

Nonlogical symbols:
- binary function symbol $\cdot$

Nonlogical axioms:

G1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
G2. $\exists x(\forall y(x \cdot y = y) \wedge \forall y \exists z(z \cdot y = x))$

Proofs

Chapter 2 - Exercise 5(a)

(1) $\neg\neg(x = x) \vee \neg(x = x)$ [axiom: propositional]

Chapter 2 - Exercise 5(b)

(1) $\neg(x = x) \vee \exists x(x = x)$ [axiom: substitution]

Chapter 2 - Exercise 5(c)

(1) $(x = x)$ [axiom: identity]

Chapter 2 - Exercise 5(d)

(1) $\neg(x = y) \vee (\neg(x = z) \vee (\neg(x = x) \vee (y = z)))$ [axiom: equality]

Chapter 2 - Exercise 5(e)

(1) $(x = x)$ [axiom: identity]
(2) $\neg(x = x) \vee (x = x)$ [rule: expansion: (1)]
(3) $(x = x) \vee (\neg(x = x) \vee (x = x))$ [rule: expansion: (2)]

Chapter 2 - Exercises 5(f) and 5(h)

(1) $(x = x)$ [axiom: identity]
(2) $\neg\neg(x = x) \vee (x = x)$ [rule: expansion: (1)]
(3) $\neg\neg\neg(x = x) \vee \neg\neg(x = x)$ [axiom: propositional]
(4) $(x = x) \vee \neg\neg(x = x)$ [rule: cut: (2) (3)]
(5) $\neg\neg(x = x) \vee \neg(x = x)$ [axiom: propositional]
(6) $\neg(x = x) \vee \neg\neg(x = x)$ [rule: cut: (5) (3)]
(7) $\neg\neg(x = x) \vee \neg\neg(x = x)$ [rule: cut: (4) (6)]
(8) $\neg\neg(x = x)$ [rule: contraction: (7)]

Chapter 2 - Exercise 5(g)

(1) $(x = x)$ [axiom: identity]
(2) $\neg(\neg(x = x) \vee \neg(x = x)) \vee (\neg(x = x) \vee \neg(x = x))$ [axiom: propositional]
(3) $(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee \neg(x = x)$ [rule: associative: (2)]
(4) $(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee (x = x)$ [rule: expansion: (1)]
(5) $\neg(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [axiom: propositional]
(6) $\neg(x = x) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [rule: cut: (3) (5)]
(7) $(x = x) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [rule: cut: (4) (5)]
(8) $(\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)) \vee (\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x))$ [rule: cut: (7) (6)]
(9) $\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(x = x)$ [rule: contraction: (8)]
(10) $\neg(\neg(x = x) \vee \neg(x = x)) \vee (x = x)$ [rule: expansion: (1)]
(11) $\neg\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [axiom: propositional]
(12) $\neg(x = x) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [rule: cut: (9) (11)]
(13) $(x = x) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [rule: cut: (10) (11)]
(14) $\neg(\neg(x = x) \vee \neg(x = x)) \vee \neg(\neg(x = x) \vee \neg(x = x))$ [rule: cut: (13) (12)]
(15) $\neg(\neg(x = x) \vee \neg(x = x))$ [rule: contraction: (14)]

Chapter 2 - Exercise 5(i)

(1) $\neg\neg(x = x) \vee \neg(x = x)$ [axiom: propositional]
(2) $\neg\exists y\neg(x = x) \vee \neg(x = x)$ [rule: e-introduction: (1)]

Chapter 3 - §3.1 - Lemma 1

(1) $\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\mathbf{A} \vee \mathbf{A}$ [axiom: propositional]
(3) $\mathbf{B} \vee \mathbf{A}$ [rule: cut: (1) (2)]

Chapter 3 - §3.1 - Detachment Rule

(1)	$\mathbf{A}$	[premise]
(2)	$\neg\mathbf{A} \vee \mathbf{B}$	[premise]
(3)	$\mathbf{B} \vee \mathbf{A}$	[rule: expansion: (1)]
(4)	$\neg\mathbf{B} \vee \mathbf{B}$	[axiom: propositional]
(5)	$\mathbf{A} \vee \mathbf{B}$	[rule: cut: (3) (4)]
(6)	$\mathbf{B} \vee \mathbf{B}$	[rule: cut: (5) (2)]
(7)	$\mathbf{B}$	[rule: contraction: (6)]

Chapter 3 - §3.1 - Tautology Theorem - result (B)

(1)	$\mathbf{A} \vee \mathbf{B}$	[premise]
(2)	$\neg\neg\mathbf{A} \vee \neg\mathbf{A}$	[axiom: propositional]
(3)	$\neg\neg\neg\mathbf{A} \vee \neg\neg\mathbf{A}$	[axiom: propositional]
(4)	$\neg\mathbf{A} \vee \neg\neg\mathbf{A}$	[rule: cut: (2) (3)]
(5)	$\mathbf{B} \vee \neg\neg\mathbf{A}$	[rule: cut: (1) (4)]
(6)	$\neg\mathbf{B} \vee \mathbf{B}$	[axiom: propositional]
(7)	$\neg\neg\mathbf{A} \vee \mathbf{B}$	[rule: cut: (5) (6)]

Chapter 3 - §3.1 - Tautology Theorem - result (C)

(1)	$\neg\mathbf{A} \vee \mathbf{C}$	[premise]
(2)	$\neg\mathbf{B} \vee \mathbf{C}$	[premise]
(3)	$\neg(\mathbf{A} \vee \mathbf{B}) \vee (\mathbf{A} \vee \mathbf{B})$	[axiom: propositional]
(4)	$(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A}) \vee \mathbf{B}$	[rule: associative: (3)]
(5)	$\neg(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A}) \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A})$	[axiom: propositional]
(6)	$\mathbf{B} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A})$	[rule: cut: (4) (5)]
(7)	$(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A}) \vee \mathbf{C}$	[rule: cut: (6) (2)]
(8)	$\mathbf{C} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A})$	[rule: cut: (7) (5)]
(9)	$(\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B})) \vee \mathbf{A}$	[rule: associative: (8)]
(10)	$\neg(\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B})) \vee (\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B}))$	[axiom: propositional]
(11)	$\mathbf{A} \vee (\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B}))$	[rule: cut: (9) (10)]
(12)	$(\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B})) \vee \mathbf{C}$	[rule: cut: (11) (1)]
(13)	$\mathbf{C} \vee (\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B}))$	[rule: cut: (12) (10)]
(14)	$(\mathbf{C} \vee \mathbf{C}) \vee \neg(\mathbf{A} \vee \mathbf{B})$	[rule: associative: (13)]
(15)	$\neg(\mathbf{C} \vee \mathbf{C}) \vee (\mathbf{C} \vee \mathbf{C})$	[axiom: propositional]
(16)	$\neg(\mathbf{A} \vee \mathbf{B}) \vee (\mathbf{C} \vee \mathbf{C})$	[rule: cut: (14) (15)]
(17)	$(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}) \vee \mathbf{C}$	[rule: associative: (16)]
(18)	$\neg(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}) \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$	[axiom: propositional]
(19)	$\mathbf{C} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$	[rule: cut: (17) (18)]
(20)	$\neg(\mathbf{A} \vee \mathbf{B}) \vee (\mathbf{C} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}))$	[rule: expansion: (19)]
(21)	$(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}) \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$	[rule: associative: (20)]
(22)	$\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$	[rule: contraction: (21)]

Chapter 3 - §3.1 - Tautology Theorem - frequently used cases (ii)

(1)	$\neg\mathbf{A} \vee \mathbf{B}$	[premise]
(2)	$\neg\mathbf{B} \vee \mathbf{C}$	[premise]
(3)	$\neg\neg\mathbf{A} \vee \neg\mathbf{A}$	[axiom: propositional]
(4)	$\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (1) (3)]
(5)	$\neg\mathbf{A} \vee \mathbf{C}$	[rule: cut: (4) (2)]

Chapter 3 - §3.1 - Tautology Theorem - frequently used cases (vi)

(1)	$\neg\mathbf{A} \vee \mathbf{B}$	[premise]
(2)	$\neg\neg\mathbf{A} \vee \neg\mathbf{A}$	[axiom: propositional]
(3)	$\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (1) (2)]
(4)	$\neg\neg\mathbf{B} \vee \neg\mathbf{B}$	[axiom: propositional]

(5) $\neg\neg\neg\mathbf{B} \vee \neg\neg\mathbf{B}$ [axiom: propositional]
(6) $\neg\mathbf{B} \vee \neg\neg\mathbf{B}$ [rule: cut: (4) (5)]
(7) $\neg\mathbf{A} \vee \neg\neg\mathbf{B}$ [rule: cut: (3) (6)]
(8) $\neg\neg\mathbf{B} \vee \neg\mathbf{A}$ [rule: cut: (7) (2)]

Chapter 3 - §3.2 - ∀-Introduction Rule

(1) $\neg\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\neg\mathbf{A} \vee \neg\mathbf{A}$ [axiom: propositional]
(3) $\mathbf{B} \vee \neg\mathbf{A}$ [rule: cut: (1) (2)]
(4) $\neg\neg\mathbf{B} \vee \neg\mathbf{B}$ [axiom: propositional]
(5) $\neg\neg\neg\mathbf{B} \vee \neg\neg\mathbf{B}$ [axiom: propositional]
(6) $\neg\mathbf{B} \vee \neg\neg\mathbf{B}$ [rule: cut: (4) (5)]
(7) $\neg\mathbf{A} \vee \neg\neg\mathbf{B}$ [rule: cut: (3) (6)]
(8) $\neg\neg\mathbf{B} \vee \neg\mathbf{A}$ [rule: cut: (7) (2)]
(9) $\neg\exists\mathbf{x}\neg\mathbf{B} \vee \neg\mathbf{A}$ [rule: e-introduction: (8)]
(10) $\neg\neg\exists\mathbf{x}\neg\mathbf{B} \vee \neg\exists\mathbf{x}\neg\mathbf{B}$ [axiom: propositional]
(11) $\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{B}$ [rule: cut: (9) (10)]

Chapter 3 - §3.2 - Generalization Rule

(1) $\mathbf{A}$ [premise]
(2) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \mathbf{A}$ [rule: expansion: (1)]
(3) $\neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [axiom: propositional]
(4) $\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (2) (3)]
(5) $\neg\neg\mathbf{A} \vee \neg\mathbf{A}$ [axiom: propositional]
(6) $\neg\neg\neg\mathbf{A} \vee \neg\neg\mathbf{A}$ [axiom: propositional]
(7) $\neg\mathbf{A} \vee \neg\neg\mathbf{A}$ [rule: cut: (5) (6)]
(8) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\mathbf{A}$ [rule: cut: (4) (7)]
(9) $\neg\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (8) (3)]
(10) $\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: e-introduction: (9)]
(11) $\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$ [axiom: propositional]
(12) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (10) (11)]
(13) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [axiom: propositional]
(14) $\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (13) (3)]
(15) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (10) (14)]
(16) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: contraction: (15)]
(17) $\neg\neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [axiom: propositional]
(18) $\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (3) (17)]
(19) $\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (11) (18)]
(20) $\neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (19) (11)]
(21) $\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: cut: (12) (20)]
(22) $\neg\exists\mathbf{x}\neg\mathbf{A}$ [rule: contraction: (21)]

Chapter 3 - §3.2 - Distribution Rule

(1) $\neg\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\mathbf{B} \vee \exists\mathbf{x}\mathbf{B}$ [axiom: substitution]
(3) $\neg\neg\mathbf{A} \vee \neg\mathbf{A}$ [axiom: propositional]
(4) $\mathbf{B} \vee \neg\mathbf{A}$ [rule: cut: (1) (3)]
(5) $\neg\mathbf{A} \vee \exists\mathbf{x}\mathbf{B}$ [rule: cut: (4) (2)]
(6) $\neg\exists\mathbf{x}\mathbf{A} \vee \exists\mathbf{x}\mathbf{B}$ [rule: e-introduction: (5)]
