CHAPTER 2
FIRST-ORDER THEORIES

NOTATION

a, **b**, **c**, **d**: syntactical variables over terms.
A, **B**, **C**, **D**: syntactical variables over formulas.
e: syntactical variables over constant symbols.
f, **g**: syntactical variables over function symbols.
i, **j**: syntactical variables over names.
p, **q**: syntactical variables over predicate symbols.
r, **s**, **t**: syntactical variables over special constants.
u, **v**: syntactical variables over expressions.
x, **y**, **z**, **w**: syntactical variables over (individual) variables.

DEFINITIONS

- A *first-order language* has as symbols:
  a) the *variables*: x, y, z, w, x', y', z', w', x'', y'', z'', w'', ...
  b) for each n, the *n-ary function symbols* and the *n-ary predicate symbols*.
  c) the symbols $\neg$, $\vee$ and $\exists$.
- A *term* is defined inductively as:
  i) $\mathbf{x}$ is a term;
  ii) if $\mathbf{f}$ is n-ary, then $\mathbf{fa}_1 \ldots \mathbf{a}_n$ is a term.
- A *formula* is defined inductively as:
  i) if $\mathbf{p}$ is n-ary, then an atomic formula $\mathbf{pa}_1 \ldots \mathbf{a}_n$ is a formula;
  ii) $\neg\mathbf{A}$ is a formula;
  iii) $\vee\mathbf{AB}$ is a formula;
  iv) $\exists\mathbf{xA}$ is a formula.
- A *designator* is an expression which is either a term or a formula.
- A *structure* $\mathcal{A}$ for a first-order language L consist of:
  i) A nonempty set $|\mathcal{A}|$, the *universe* and its *individuals*.
  ii) For each n-ary function symbol $\mathbf{f}$ of L, an n-ary function $\mathbf{f}_{\mathcal{A}} : |\mathcal{A}|^n \to |\mathcal{A}|$. (In particular, for each constant $\mathbf{e}$ of L, $\mathbf{e}_{\mathcal{A}}$ is an individual of $\mathcal{A}$.)
  iii) For each n-ary predicate symbol $\mathbf{p}$ of L other than $=$, an n-ary predicate $\mathbf{p}_{\mathcal{A}}$ in $|\mathcal{A}|$.

  Also, $\mathcal{A}(\mathbf{a})$ designates an individual and $\mathcal{A}(\mathbf{A})$ designates a truth value.
- A formula $\mathbf{A}$ is *valid* in a structure $\mathcal{A}$ if $\mathcal{A}(\mathbf{A}') = \top$ for every $\mathcal{A}$-instance $\mathbf{A}'$ of $\mathbf{A}$. In particular, a closed formula $\mathbf{A}$ is valid in $\mathcal{A}$ iff $\mathcal{A}(\mathbf{A}) = \top$.
- A formula $\mathbf{A}$ is *logically valid* if it's valid in every structure.
- A formula $\mathbf{A}$ is a *consequence* of a set Γ of formulas if the validity of $\mathbf{A}$ follows from the validity of the formulas in Γ.
- A formula $\mathbf{A}$ is a *logical consequence* of a set Γ of formulas if $\mathbf{A}$ is valid in every structure for L in which all of the formulas in Γ are valid.
- A *first-order theory* is a formal system T such that
  i) the language of T is a first-order language;
  ii) the axioms of T are the logical axioms of $L(T)$ and certain further axioms, the *nonlogical axioms*;
  iii) the rules of T are Expansion, Contraction, Associative, Cut and $\exists$-Introduction.
- A *model* of a theory T, is a structure for $L(T)$ in which all the nonlogical axioms of T are valid.
- A formula $\mathbf{A}$ is *valid* in a theory T if it is valid in every model of T.

LOGICAL AXIOMS

Propositional: $\neg\mathbf{A} \vee \mathbf{A}$
Substitution: $\mathbf{A_x}[\mathbf{a}] \rightarrow \exists\mathbf{x}\mathbf{A}$
Identity: $\mathbf{x} = \mathbf{x}$
Equality: $\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \cdots \rightarrow \mathbf{x}_n = \mathbf{y}_n \rightarrow \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n$
$\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \cdots \rightarrow \mathbf{x}_n = \mathbf{y}_n \rightarrow \mathbf{p}\mathbf{x}_1 \ldots \mathbf{x}_n \rightarrow \mathbf{p}\mathbf{y}_1 \ldots \mathbf{y}_n$

RULES OF INFERENCE

Expansion. Infer $\mathbf{B} \vee \mathbf{A}$ from $\mathbf{A}$.
Contraction. Infer $\mathbf{A}$ from $\mathbf{A} \vee \mathbf{A}$.
Associative. Infer $(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$ from $\mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$.
Cut. Infer $\mathbf{B} \vee \mathbf{C}$ from $\mathbf{A} \vee \mathbf{B}$ and $\neg\mathbf{A} \vee \mathbf{C}$.
$\exists$-Introduction. If $\mathbf{x}$ is not free in $\mathbf{B}$, infer $\exists\mathbf{x}\mathbf{A} \rightarrow \mathbf{B}$ from $\mathbf{A} \rightarrow \mathbf{B}$.

RESULTS

§2.4

Lemma 1. *If $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}'_1, \ldots, \mathbf{u}'_n$ are designators and $\mathbf{u}_1 \ldots \mathbf{u}_n$ and $\mathbf{u}'_1 \ldots \mathbf{u}'_n$ are compatible, then $\mathbf{u}_i$ is $\mathbf{u}'_i$ for $i = 1, \ldots, n$.*

Formation Theorem. *Every designator can be written in the form $\mathbf{u}\mathbf{v}_1 \ldots \mathbf{v}_n$, where $\mathbf{u}$ is a symbol of index n and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are designators, in one and only one way.*

Lemma 2. *Every occurrence of a symbol in a designator $\mathbf{u}$ begins an occurrence of a designator in $\mathbf{u}$.*

Occurrence Theorem. *Let $\mathbf{u}$ be a symbol of index n, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be designators. Then any occurrence of a designator $\mathbf{v}$ in $\mathbf{u}\mathbf{v}_1 \ldots \mathbf{v}_n$ is either all of $\mathbf{u}\mathbf{v}_1 \ldots \mathbf{v}_n$ or a part of one of the $\mathbf{v}_i$.*

§2.5

Lemma. *Let $\mathcal{A}$ be a structure for L; $\mathbf{a}$ a variable-free term in $L(\mathcal{A})$; $\mathbf{i}$ the name of $\mathcal{A}(\mathbf{a})$. If $\mathbf{b}$ is a term of $L(\mathcal{A})$ in which no variable except $\mathbf{x}$ occurs, then $\mathcal{A}(\mathbf{b_x}[\mathbf{a}]) = \mathcal{A}(\mathbf{b_x}[\mathbf{i}])$. If $\mathbf{A}$ is a formula of $L(\mathcal{A})$ in which no variable except $\mathbf{x}$ is free, then $\mathcal{A}(\mathbf{A_x}[\mathbf{a}]) = \mathcal{A}(\mathbf{A_x}[\mathbf{i}])$.*

Validity Theorem. *If T is a theory, then every theorem of T is valid in T.*

PROBLEMS

1.

(a) Let $F(a_1, \ldots, a_n)$ be any truth function. We can construct another function

$$F'(a_1, \ldots, a_n) = H_{d,m}(H_{c,n}(a_1^1, \ldots, a_n^1), \ldots, H_{c,n}(a_1^m, \ldots, a_n^m))$$

where the $a_1^i, \ldots, a_n^i$ are all the tuples of truth values such that $F(a_1^i, \ldots, a_n^i) = \top$. Thus, $a_j^i = a_j$ or $a_j^i = H_\neg(a_j)$, for some values of i and j. Now, we can see that F and F' are the same function, since any truth assignment $a_1', \ldots, a_n'$ that satisfies (falsifies) F, also satisfies (falsifies) F', respectively. This is called *Disjunctive Normal Form (DNF).*

We can also construct a similar function

$$\begin{aligned} F''(a_1, \ldots, a_n) &= H_{c,m}(H_\neg(H_{c,n}(a_1^1, \ldots, a_n^1)), \ldots, H_\neg(H_{c,n}(a_1^m, \ldots, a_n^m))) \\ &= H_{c,m}(H_{d,n}(H_\neg(a_1^1), \ldots, H_\neg(a_n^1)), \ldots, H_{d,n}(H_\neg(a_1^m), \ldots, H_\neg(a_n^m))) \end{aligned}$$

where the $a_1^i, \ldots, a_n^i$ are all the tuples of truth values such that $F(a_1^i, \ldots, a_n^i) = \bot$. It can be seen by a reasoning similar to above, that F and F'' are the same function. This is called *Conjunctive Normal Form (CNF).*

(b) It can be seen that

$$\begin{aligned} H_{c,n} &= H_\wedge(a_1, H_\wedge(a_2, \ldots)) \\ H_{d,n} &= H_\vee(a_1, H_\vee(a_2, \ldots)). \end{aligned}$$

This means we can define any truth function F in terms of $H_\neg$, $H_\vee$ and $H_\wedge$, due to (a). Additionally, we can convert each instance of $H_\wedge(a, b)$ into $H_\neg(H_\vee(H_\neg(a), H_\neg(b)))$. Thus, every truth function is definable in terms of $H_\neg$ and $H_\vee$.

(c) Since $H_\vee(a, b)$ can be defined as $H_\to(H_\neg(a), b)$, every truth function is definable in terms of $H_\neg$ and $H_\to$, due to (b).

(d) Since $H_\vee(a, b)$ can be defined as $H_\neg(H_\wedge(H_\neg(a), H_\neg(b)))$, every truth function is definable in terms of $H_\neg$ and $H_\wedge$, due to (b).

(e) Consider the following identities, which can be easily verified e.g. via their truth tables

$$H_\vee(a, a) = a, \quad H_\vee(a, \top) = \top$$

$$H_\wedge(a, a) = a, \quad H_\wedge(a, \top) = a$$

$$H_\to(a, a) = \top, \quad H_\to(a, \top) = \top, \quad H_\to(\top, a) = a$$

$$H_\leftrightarrow(a, a) = \top, \quad H_\leftrightarrow(a, \top) = a, \quad H_\leftrightarrow(\top, a) = a.$$

Thus, any formula consisting of only those connectives and the free variable a can be inductively reduced to either a or $\top$ and can never define $H_\neg$. Those connectives can only define monotone functions while negation is not monotone. Note that allowing constants in the expression would allow to define negation as e.g. $H_\neg(a) = H_\to(a, \bot)$.

2.

(a) Note that $H_d(a, b) = H_\wedge(H_\neg(a), H_\neg(b))$. We can then define

$$\begin{aligned} H_\neg(a) &= H_d(a, a) \\ H_\vee(a, b) &= H_d(H_d(a, b), H_d(a, b)) \end{aligned}$$

and thus every truth function is definable in terms of H_d (using result from 1.1(b)).

(b) Note that $H_s(a, b) = H_\neg(H_\wedge(a, b))$. We can then define

$$\begin{aligned} H_\neg(a) &= H_s(a, a) \\ H_\vee(a, b) &= H_s(H_s(a, a), H_s(b, b)) \end{aligned}$$

and thus every truth function is definable in terms of H_s (using result from 1.1(b)).

(c) Let H be singulary with $H(a_1, \ldots, a_n) = H'(a_i)$. The syntax of every truth function $F(a_1, \ldots, a_m)$ definable in terms of H can be inductively defined by

$$e ::= a_j | H(e_1, \ldots, e_n)$$

where $1 \leq j \leq m$ and $e_1, \ldots, e_n$ are valid expressions.

We can then reduce every expression to an equivalent expression that involves a single a_j: as long as the expression has the form $H(e_1, \ldots, e_n)$, we can replace it with $H'(e_i)$ and inductively reduce e_i. Thus, every truth function F definable in terms of H is singulary and furthermore

$$F(a_1, \ldots, a_m) = {H'}^k(a_j)$$

for some integers $k \geq 0$ and $1 \leq j \leq m$.

(d) Note that since any n-ary truth function is completely determined by its truth table, there are 2^{2^n} of them. So we know there are $2^{2^2} = 16$ binary truth functions. Let's analyze them:

- Consider the four binary truth functions H such that

  $$H(a, a) = a.$$

  It is easy to see that any function definable in terms of such H can be inductively reduced to a, in a similar fashion as before. Thus, none of these four functions can define every truth function (e.g. negation $H_\neg$ cannot be defined).
- Consider the four binary truth functions H such that

  $$H(a, a) = \bot.$$

  For each of these four functions, we have

  $$H(a, \bot) \in \{a, \bot\}, \quad H(\bot, a) \in \{a, \bot\}$$

  and thus none of these four functions can define every truth function (e.g. negation $H_\neg$ cannot be defined).
- Consider the four binary truth functions H such that

  $$H(a, a) = \top.$$

  This case is symmetric to the previous one. For each of these four functions, we have

  $$H(a, \top) \in \{a, \top\}, \quad H(\top, a) \in \{a, \top\}$$

  and thus none of these four functions can define every truth function (e.g. negation $H_\neg$ cannot be defined).
- For the four remaining binary truth functions, we have

  $$H(\top, \top) = \bot, \quad H(\bot, \bot) = \top.$$

  Two of those functions

  $$H_1(\top, \bot) = \top, \quad H_1(\bot, \top) = \bot$$
  $$H_2(\top, \bot) = \bot, \quad H_2(\bot, \top) = \top$$

  are singulary and thus cannot define functions such as $H_\vee$, due to the result from 2.2(c). The two remaining functions are H_d and H_s, presented in 2.2(a) and 2.2(b), respectively.

3. If $\mathbf{v}$ is empty, then trivially neither $\mathbf{u}$ or $\mathbf{v}'$ are empty, and they are both designators.

Let's assume that $\mathbf{v}$ is not empty and that the designator $\mathbf{uv}$ has the form $\mathbf{tt}_1 \ldots \mathbf{t}_n$. Since $\mathbf{uv}$ and $\mathbf{vv}'$ are designators, they both begin with a symbol: thus $\mathbf{v}$ also begins with a symbol, since it is a non-empty prefix of $\mathbf{vv}'$. The occurrence of this symbol in $\mathbf{uv}$ begins the occurrence of a designator $\mathbf{u}'$ in $\mathbf{uv}$ (by Lemma 2), which is compatible with $\mathbf{v}$. Moreover, the occurrence of $\mathbf{u}'$ in $\mathbf{uv}$ is either all of $\mathbf{uv}$ or part of one of the $\mathbf{t}_i$ (by the Occurrence Theorem). In the former case, it means that $\mathbf{v}$ is a designator and $\mathbf{u}$ and $\mathbf{v}'$ are empty. On the other hand, if $\mathbf{u}'$ is part of one of the $\mathbf{t}_i$, it means that $\mathbf{vv}'$ begins with $\mathbf{u}'$, and thus $\mathbf{u}'$ and $\mathbf{v}$ are the same (by the Formation Theorem) and $\mathbf{v}'$ is empty.

4. If a term is:

i) a variable $\mathbf{x}'$, then the substitution result is $\mathbf{x}$ itself, which is also a term.

ii) a function application $\mathbf{fa}_1 \ldots \mathbf{a}_n$, then $\mathbf{a}$ is one of the $\mathbf{a}_i$ and the substitution result is also a term, or $\mathbf{a}$ is substituted in one of the terms $\mathbf{a}_i$, and it remains a term, by the induction hypothesis.

If a formula is:

i) an atomic formula $\mathbf{pa}_1 \ldots \mathbf{a}_n$, then substituting $\mathbf{a}$ in any of the $\mathbf{a}_i$ results in a term, as previously shown. Thus it remains a formula.

ii) $\neg\mathbf{A}$, then substituting $\mathbf{a}$ in $\mathbf{A}$ remains a formula by the induction hypothesis.

iii) $\vee\mathbf{AB}$, then substituting $\mathbf{a}$ in $\mathbf{A}$ or $\mathbf{B}$ remains a formula by the induction hypothesis.

iv) $\exists\mathbf{yA}$, then substituting $\mathbf{a}$ in $\mathbf{A}$ remains a formula by the induction hypothesis.

5. The sections of this problem show that each axiom and rule allows proving theorems that wouldn't be provable without it. The common strategy is finding a suitable mapping f from formulas to truth values and a specific theorem $\mathbf{B}$, such that if a theorem $\mathbf{A}$ is provable without the axiom or rule in question, then $f(\mathbf{A}) = \top$, but $f(\mathbf{B}) = \bot$ and it is thus no provable without it. This means that none of the axioms or rules are redundant.

(a) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \bot; \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{B}); \\ f(\exists\mathbf{xA}) &= \top. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without propositional axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}] \vee \exists\mathbf{xA}') = f(\exists\mathbf{xA}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \rightarrow \mathbf{x}_2 = \mathbf{y}_2 \rightarrow \mathbf{x}_1 = \mathbf{x}_2 \rightarrow \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\neg\mathbf{A}' \vee \mathbf{C}') = f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{xA}' \rightarrow \mathbf{B}')$ with $f(\mathbf{A}' \rightarrow \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \rightarrow \mathbf{B}') = f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{B}') = f(\mathbf{A}) = \top$.

Thus, if $\mathbf{A}$ is provable without propositional axioms, we have $f(\mathbf{A}) = \top$. But $f(\neg\neg(x = x) \vee \neg(x = x)) = f(\neg(x = x)) = \bot$ and so it is not provable without propositional axioms.

(b) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists\mathbf{xA}) &= \bot. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without substitution axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = \neg f(\exists\mathbf{x}\mathbf{A}') \vee f(\mathbf{B}') = \top$.

Thus, if $\mathbf{A}$ is provable without substitution axioms, we have $f(\mathbf{A}) = \top$. But $f(x = x \to \exists x(x = x)) = \neg f(x = x) \vee f(\exists x(x = x)) = \bot$ and so it is not provable without substitution axioms.

(c) The hinted function is defined as:

$$
\begin{aligned}
f(\mathbf{A}) &= \bot, \quad \text{for } \mathbf{A} \text{ atomic;}\\
f(\neg\mathbf{A}) &= \neg f(\mathbf{A});\\
f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B});\\
f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}).
\end{aligned}
$$

Let's prove that if $\mathbf{A}$ is provable without identity axioms then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\mathbf{A}') = \top$ (see below for this case);
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = \neg f(\mathbf{A}') \vee f(\mathbf{B}') = f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}' \to \mathbf{B}') = \top$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \ldots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \bot$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n) = \bot$.

- for $\mathbf{A}$ with form $\neg\mathbf{A}'$: we have $f(\mathbf{A_x}[\mathbf{a}]) = \neg f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\mathbf{A}) = \neg f(\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A_x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \vee f(\mathbf{B}'_\mathbf{x}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}'_\mathbf{x}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without identity axioms, we have $f(\mathbf{A}) = \top$. But $f(x = x) = \bot$ and so it is not provable without identity axioms.

(d) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{e}_i = \mathbf{e}_j) &= \top \quad \text{iff } i \leq j; \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= \top \quad \text{iff } f(\mathbf{A_x}[\mathbf{e}_i]) = \top \text{ for some } i. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without equality axioms then $f(\mathbf{A}') = \top$ for every formula obtained from $\mathbf{A}$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$ for every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_\mathbf{x}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_\mathbf{x}[\mathbf{a}]) \vee f(\exists\mathbf{x}\mathbf{A}')$. For every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable (except $\mathbf{x}$) by some $\mathbf{e}_i$ at all its free occurrences: if $f(\mathbf{A}''_\mathbf{x}[\mathbf{e}_i]) = \top$ for some i, then $f(\exists\mathbf{x}\mathbf{A}'') = \top$ by the definition of f. Otherwise, $f(\mathbf{A}''_\mathbf{x}[\mathbf{e}_i]) = \bot$ for all i and thus $\neg f(\mathbf{A}''_\mathbf{x}[\mathbf{e}_i]) = \top$;
- an identity axiom: $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$ for any substitution of $\mathbf{x}$ by some $\mathbf{e}_i$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}')$ with $f(\mathbf{A}') = \top$ for every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = \top$ for every closed formula $\mathbf{A}''$ obtained from $\mathbf{A}'$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = \top$ for every closed formulas $\mathbf{A}''$, $\mathbf{B}''$ and $\mathbf{C}''$ obtained from $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$, respectively, by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}') = \top$ for every closed formulas $\mathbf{A}''$, $\mathbf{B}''$ and $\mathbf{C}''$ obtained from $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$, respectively, by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}') = \neg f(\exists\mathbf{x}\mathbf{A}') \vee f(\mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \neg f(\mathbf{A}') \vee f(\mathbf{B}') = \top$ for every closed formula $\mathbf{A}''$ and $\mathbf{B}''$ obtained from $\mathbf{A}'$ and $\mathbf{B}'$, respectively, by replacing each variable by some $\mathbf{e}_i$ at all its free occurrences, by the induction hypothesis. If $f(\mathbf{B}') = \top$, then $f(\mathbf{A}) = \top$ follows trivially. Otherwise, we must have $f(\mathbf{A}') = \bot$ for all closed formulas $\mathbf{A}''$ obtained from $\mathbf{A}'$ as described above. This implies that $f(\exists\mathbf{x}\mathbf{A}') = \bot$ and thus $f(\mathbf{A}) = \top$.

Thus, if $\mathbf{A}$ is provable without equality axioms, we have $f(\mathbf{A}') = \top$ for every formula $\mathbf{A}'$ obtained from $\mathbf{A}$ by replacing each variable by some $\mathbf{e}_i$ at all its free occurences. But $f(x = y \to x = z \to x = x \to y = z) = \neg f(x = y) \vee \neg f(x = z) \vee \neg f(x = x) \vee f(y = z) = \bot$ since it does not hold for the substitution $[\mathbf{x}, \mathbf{y}, \mathbf{z}] \to [\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2]$ and so it is not provable without equality axioms.

(e) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \neg f(\mathbf{A}); \\ f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \leftrightarrow \neg f(\mathbf{B}); \\ f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}). \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the expansion rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \leftrightarrow \neg f(\mathbf{A}') = \top$ (see below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \leftrightarrow \neg f(\mathbf{x}_2 = \mathbf{y}_2) \leftrightarrow \neg f(\mathbf{x}_1 = \mathbf{x}_2) \leftrightarrow \neg f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. However, this is a contradiction since $f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{A}') = \bot$ for any $\mathbf{A}'$ so it's not possible to have a proof where the contraction rule is applied (???);
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \leftrightarrow \neg(f(\mathbf{B}') \leftrightarrow \neg f(\mathbf{C}')) = f(\mathbf{A}') \leftrightarrow (f(\mathbf{B}') \leftrightarrow f(\mathbf{C}'))$ and $f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \leftrightarrow \neg f(\mathbf{C}') = (f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')) \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}') \leftrightarrow f(\mathbf{C}')$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')$ and $f(\neg\mathbf{A} \vee \mathbf{C}') = \neg f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{C}')$, and thus $f(\mathbf{B}') \leftrightarrow \neg f(\mathbf{C}') = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}) = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\neg\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}')$ and $f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \leftrightarrow f(\mathbf{B}')$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \ldots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \ldots \mathbf{b}_n) = \top$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = \neg f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \leftrightarrow \neg f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \leftrightarrow \neg f(\mathbf{B}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without the expansion rule, we have $f(\mathbf{A}) = \top$. But $f(x = x \vee (\neg(x = x) \vee (x = x))) = f(x = x) \leftrightarrow \neg(\neg f(x = x) \leftrightarrow \neg f(x = x)) = \bot$ and so it is not provable without the expansion rule.

(f) The hinted function is defined as:

$$\begin{aligned} f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;} \\ f(\neg\mathbf{A}) &= \bot; \\ f(\mathbf{A} \vee \mathbf{B}) &= \top; \\ f(\exists\mathbf{x}\mathbf{A}) &= \bot. \end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the contraction rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \top$;

- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = \top$.

Thus, if $\mathbf{A}$ is provable without the contraction rule, we have $f(\mathbf{A}) = \top$. But $f(\neg\neg(x = x)) = \bot$ and so it is not provable without the contraction rule.

(g) The hinted function is defined as:

$$\begin{aligned}
f(\mathbf{A}) &= 0, \quad \text{for } \mathbf{A} \text{ atomic};\\
f(\neg\mathbf{A}) &= 1 - f(\mathbf{A});\\
f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \cdot f(\mathbf{B}) \cdot (1 - f(\mathbf{A}) - f(\mathbf{B}));\\
f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}).
\end{aligned}$$

Let's prove that if $\mathbf{A}$ is provable without the associative rule then $f(\mathbf{A}) = 0$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{A}') \cdot (1 - (1 - f(\mathbf{A}')) - f(\mathbf{A}')) = (1 - f(\mathbf{A}')) \cdot f(\mathbf{A}') \cdot (f(\mathbf{A}') - f(\mathbf{A}')) = 0$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = (1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])) \cdot f(\mathbf{A}') \cdot (1 - (1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])) - f(\mathbf{A}')) = 0$ (see below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = 0$;
- an equality axiom: we have

$$\begin{aligned}
f(\mathbf{A}) &= f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2)\\
&= f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2))\\
&= (1 - f(\mathbf{x}_1 = \mathbf{y}_1)) \cdot f(\mathbf{A}') \cdot (f(\mathbf{x}_1 = \mathbf{y}_1) - f(\mathbf{A}'))
\end{aligned}$$

$$f(\mathbf{A}') = f(\mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = (1 - f(\mathbf{x}_2 = \mathbf{y}_2)) \cdot f(\mathbf{A}'') \cdot (f(\mathbf{x}_2 = \mathbf{y}_2) - f(\mathbf{A}''))$$

$$f(\mathbf{A}'') = f(\mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = (1 - f(\mathbf{x}_1 = \mathbf{x}_2)) \cdot f(\mathbf{y}_1 = \mathbf{y}_2) \cdot (f(\mathbf{x}_1 = \mathbf{x}_2) - f(\mathbf{y}_1 = \mathbf{y}_2)) = 0;$$

- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}') \cdot f(\mathbf{A}') \cdot (1 - f(\mathbf{B}') - f(\mathbf{A}')) = 0$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \cdot f(\mathbf{A}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{A}')) = 0$ and the only integer solution is $f(\mathbf{A}') = 0$ and thus $f(\mathbf{A}) = 0$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = 0$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = 0$ by the induction hypothesis. Consider the equations

$$f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{B}')) = 0 \tag{1}$$
$$f(\neg\mathbf{A}' \vee \mathbf{C}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{C}') \cdot (f(\mathbf{A}') - f(\mathbf{C}')) = 0 \tag{2}$$
$$f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \cdot f(\mathbf{C}') \cdot (1 - f(\mathbf{B}') - f(\mathbf{C}')) = 0 \tag{3}$$

  and the possible cases that satisfy equation (2). First, $(1 - f(\mathbf{A}')) = 0$ implies that $f(\mathbf{A}') = 1$ and substituting in equation (1) we obtain $f(\mathbf{B}') \cdot (-f(\mathbf{B}')) = 0$ which means that $f(\mathbf{B}') = 0$ which satisfies equation (3). Second, $f(\mathbf{C}') = 0$, which trivially satifies equation (3). Third, $f(\mathbf{A}') - f(\mathbf{C}') = 0$ which implies $f(\mathbf{A}') = f(\mathbf{C}')$ and substituting in equation (1) we obtain $f(\mathbf{C}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{C}') - f(\mathbf{B}')) = 0$. Thus, equation (3) is satisfied in all cases;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = 0$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = (1 - f(\mathbf{A}')) \cdot f(\mathbf{B}') \cdot (f(\mathbf{A}') - f(\mathbf{B}')) = f(\mathbf{A}' \to \mathbf{B}') = 0$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \dots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = 0$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n) = 0$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = 1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = 1 - f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \cdot f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}]) \cdot (1 - f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) - f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}]))$ and $f(\mathbf{A}) = f(\mathbf{A}') \cdot f(\mathbf{B}') \cdot (1 - f(\mathbf{A}') - f(\mathbf{B}'))$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without the associative rule, we have $f(\mathbf{A}) = 0$. But $f(\neg(\neg(x = x) \vee \neg(x = x))) = 1 - f(\neg(x = x) \vee \neg(x = x)) = 1 - ((1 - f(x = x))^2 \cdot (1 - 2 \cdot (1 - f(x = x)))) = 1 - (1 - 2) = 2$ and so it is not provable without the associative rule.

(h) The hinted function is defined as:

$$
\begin{aligned}
f(\mathbf{A}) &= \top \quad \text{for } \mathbf{A} \text{ atomic;}\\
f(\neg\mathbf{A}) &= \begin{cases} \top, & \text{if } f(\mathbf{A}) = \bot \text{ or } \mathbf{A} \text{ is atomic;}\\ \bot, & \text{otherwise.} \end{cases}\\
f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B});\\
f(\exists\mathbf{x}\mathbf{A}) &= f(\mathbf{A}).
\end{aligned}
$$

Let's prove that if $\mathbf{A}$ is provable without the cut rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = f(\neg\mathbf{A}') \vee f(\mathbf{A}') = \top$ (since if $f(\mathbf{A}') = \bot$, then $f(\neg\mathbf{A}') = \top$ from the definition of f);
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\mathbf{A}') = \top$ (since if $f(\mathbf{A}') = \bot$, then $f(\neg\mathbf{A}') = \top$ from the definition of f and $f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\neg\mathbf{A}')$. See below for this case);
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1)) \vee f(\neg(\mathbf{x}_2 = \mathbf{y}_2)) \vee f(\neg(\mathbf{x}_1 = \mathbf{x}_2)) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = \top$ and thus $f(\mathbf{A}) = f(\mathbf{A}') = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the $\exists$-introduction rule: we have $f(\mathbf{A}) = f(\exists\mathbf{x}\mathbf{A}' \to \mathbf{B}')$ with $f(\mathbf{A}' \to \mathbf{B}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{A}) = f(\neg\exists\mathbf{x}\mathbf{A}' \vee \mathbf{B}') = f(\neg\exists\mathbf{x}\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}') \vee f(\mathbf{B}') = \top$. So either $f(\neg\mathbf{A}') = \top$ or $f(\mathbf{B}') = \top$. In the latter case, it follows trivially that $f(\mathbf{A}) = \top$. In the former case, note that since $f(\exists\mathbf{x}\mathbf{A}) = f(\mathbf{A})$ and $\exists\mathbf{x}\mathbf{A}$ is not atomic, then $f(\neg\exists\mathbf{x}\mathbf{A}') = f(\neg\mathbf{A}')$.

To treat substitution axioms, let's show that $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A})$ by induction on the length of $\mathbf{A}$:

- for $\mathbf{A}$ atomic with form $\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{p}\mathbf{b}_{1\mathbf{x}}[\mathbf{a}] \dots \mathbf{b}_{n\mathbf{x}}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\mathbf{p}\mathbf{b}_1 \dots \mathbf{b}_n) = \top$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$ with $\mathbf{A}'$ atomic: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = \top$ and $f(\mathbf{A}) = f(\neg\mathbf{A}') = \top$.
- for $\mathbf{A}$ with form $\neg\mathbf{A}'$ with $\mathbf{A}'$ not atomic: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\neg\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\mathbf{A}' \vee \mathbf{B}'$: we have $f(\mathbf{A}_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\mathbf{A}) = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ and $f(\mathbf{B}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{B}')$ by the induction hypothesis.
- for $\mathbf{A}$ with form $\exists\mathbf{y}\mathbf{A}'$: we have $f(\exists\mathbf{y}\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}])$ and $f(\exists\mathbf{y}\mathbf{A}') = f(\mathbf{A}')$ and $f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) = f(\mathbf{A}')$ by the induction hypothesis.

Thus, if $\mathbf{A}$ is provable without the cut rule, we have $f(\mathbf{A}) = \top$. But $f(\neg\neg(x = x)) = \bot$ since $f(\neg(x = x)) = \top$ and so it is not provable without the cut rule.

(i) The hinted function is defined as:

$$
\begin{aligned}
f(\mathbf{A}) &= \top, \quad \text{for } \mathbf{A} \text{ atomic;}\\
f(\neg\mathbf{A}) &= \neg f(\mathbf{A});\\
f(\mathbf{A} \vee \mathbf{B}) &= f(\mathbf{A}) \vee f(\mathbf{B});\\
f(\exists \mathbf{x} \mathbf{A}) &= \top.
\end{aligned}
$$

Let's prove that if $\mathbf{A}$ is provable without the $\exists$-introduction rule then $f(\mathbf{A}) = \top$, by induction on theorems. In a proof of $\mathbf{A}$, if $\mathbf{A}$ was obtained from:

- a propositional axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}' \vee \mathbf{A}') = \neg f(\mathbf{A}') \vee f(\mathbf{A}') = \top$;
- a substitution axiom: we have $f(\mathbf{A}) = f(\neg\mathbf{A}'_{\mathbf{x}}[\mathbf{a}] \vee \exists\mathbf{x}\mathbf{A}') = \neg f(\mathbf{A}'_{\mathbf{x}}[\mathbf{a}]) \vee f(\exists\mathbf{x}\mathbf{A}') = \top$;
- an identity axiom: since it's an atomic formula, $f(\mathbf{A}) = f(\mathbf{x} = \mathbf{x}) = \top$;
- an equality axiom: we have $f(\mathbf{A}) = f(\mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{x}_2 \to \mathbf{y}_1 = \mathbf{y}_2) = f(\neg(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg(\mathbf{x}_1 = \mathbf{x}_2) \vee (\mathbf{y}_1 = \mathbf{y}_2)) = \neg f(\mathbf{x}_1 = \mathbf{y}_1) \vee \neg f(\mathbf{x}_2 = \mathbf{y}_2) \vee \neg f(\mathbf{x}_1 = \mathbf{x}_2) \vee f(\mathbf{y}_1 = \mathbf{y}_2) = \top$;
- the expansion rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}')$ with $f(\mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{A}') = f(\mathbf{B}') \vee f(\mathbf{A}') = \top$;
- the contraction rule: we have $f(\mathbf{A}) = f(\mathbf{A}')$ with $f(\mathbf{A}' \vee \mathbf{A}') = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee \mathbf{A}') = f(\mathbf{A}') \vee f(\mathbf{A}') = f(\mathbf{A}') = f(\mathbf{A}) = \top$;
- the associative rule: we have $f(\mathbf{A}) = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = \top$ by the induction hypothesis. In this case $f(\mathbf{A}' \vee (\mathbf{B}' \vee \mathbf{C}')) = f(\mathbf{A}') \vee f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{A}') \vee f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}' \vee \mathbf{B}') \vee f(\mathbf{C}') = f((\mathbf{A}' \vee \mathbf{B}') \vee \mathbf{C}') = \top$;
- the cut rule: we have $f(\mathbf{A}) = f(\mathbf{B}' \vee \mathbf{C}')$ with $f(\mathbf{A}' \vee \mathbf{B}') = \top$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \top$ by the induction hypothesis. In this case we have $f(\mathbf{B}' \vee \mathbf{C}') = f(\mathbf{B}') \vee f(\mathbf{C}')$, $f(\mathbf{A}' \vee \mathbf{B}') = f(\mathbf{A}') \vee f(\mathbf{B}')$ and $f(\neg\mathbf{A}' \vee \mathbf{C}') = \neg f(\mathbf{A}') \vee f(\mathbf{C}')$. If $f(\mathbf{A}') = \top$, then $f(\mathbf{C}') = \top$. If $f(\mathbf{A}') = \bot$, then $f(\mathbf{B}') = \top$. Thus $f(\mathbf{B}') \vee f(\mathbf{C}') = f(\mathbf{A}) = \top$;

Thus, if $\mathbf{A}$ is provable without the $\exists$-introduction rule, we have $f(\mathbf{A}) = \top$. But $f(\exists y \neg(x = x) \to \neg(x = x)) = \neg f(\exists y \neg(x = x)) \vee \neg f(x = x) = \bot$ and so it is not provable without the $\exists$-introduction rule.

CHAPTER 3
THEOREMS IN FIRST-ORDER THEORIES

DEFINITIONS

- $\mathbf{A}$ is *elementary* if it is either atomic or an instantiation.
- A *truth valuation* for T is a mapping from the set of elementary formulas in T to the set of truth values.
- $\mathbf{B}$ is a *tautological consequence* of $\mathbf{A}_1, \ldots, \mathbf{A}_n$ if $V(\mathbf{B}) = \top$ for every truth valuation V such that $V(\mathbf{A}_1) = \cdots = V(\mathbf{A}_n) = \top$.
- $\mathbf{A}$ is a *tautology* if it is a tautological consequence of the empty sequence of formulas, i.e. if $V(\mathbf{A}) = \top$ for every truth valuation V.
- $\mathbf{A}'$ is an *instance* of $\mathbf{A}$ if $\mathbf{A}'$ is of the form $\mathbf{A}_{\mathbf{x}_1,\ldots,\mathbf{x}_n}[\mathbf{a}_1, \ldots, \mathbf{a}_2]$.
- Let $\mathbf{A}$ be a formula and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ its free variables in alphabetical order. The *closure* of $\mathbf{A}$ is the formula $\forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n \mathbf{A}$.
- $\mathbf{A}'$ is a *variant* of $\mathbf{A}$ if $\mathbf{A}'$ can be obtained from $\mathbf{A}$ by a sequence of replacements of the following type: replace a part $\exists \mathbf{x} \mathbf{B}$ by $\exists \mathbf{y} \mathbf{B}_{\mathbf{x}}[\mathbf{y}]$, where $\mathbf{y}$ is a variable not free in $\mathbf{B}$.
- $\mathbf{A}$ is *open* if it contains no quantifiers.
- $\mathbf{A}$ is in *prenex form* if it has the form $Q\mathbf{x}_1 \ldots Q\mathbf{x}_n \mathbf{B}$ where each $Q\mathbf{x}_i$ is either $\exists \mathbf{x}_i$ or $\forall \mathbf{x}_i$; $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are distinct; and $\mathbf{B}$ is open.

RESULTS

§3.1

Tautology Theorem. *If* $\mathbf{B}$ *is a tautological consequence of* $\mathbf{A}_1, \ldots, \mathbf{A}_n$, *and* $\vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_n$, *then* $\vdash \mathbf{B}$.

Corollary. *Every tautology is a theorem.*

Lemma 1. *If* $\vdash \mathbf{A} \vee \mathbf{B}$, *then* $\vdash \mathbf{B} \vee \mathbf{A}$.

Detachment Rule. *If* $\vdash \mathbf{A}$ *and* $\vdash \mathbf{A} \to \mathbf{B}$, *then* $\vdash \mathbf{B}$.

Corollary. *If* $\vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_n$, *and* $\vdash \mathbf{A}_1 \to \ldots \to \mathbf{A}_n \to \mathbf{B}$, *then* $\vdash \mathbf{B}$.

Lemma 2. *If* $n \geq 2$, *and* $\mathbf{A}_1 \vee \cdots \vee \mathbf{A}_n$ *is a tautology, then* $\vdash \mathbf{A}_1 \vee \cdots \vee \mathbf{A}_n$.

§3.2

$\forall$-Introduction Rule. *If* $\vdash \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{x}$ *is not free in* $\mathbf{A}$, *then* $\vdash \mathbf{A} \to \forall \mathbf{x} \mathbf{B}$.

Generalization Rule. *If* $\vdash \mathbf{A}$, *then* $\vdash \forall \mathbf{x} \mathbf{A}$.

Substitution Rule. *If* $\vdash \mathbf{A}$ *and* $\mathbf{A}'$ *is an instance of* $\mathbf{A}$, *then* $\vdash \mathbf{A}'$.

Substitution Theorem.

$$\vdash \mathbf{A}_{\mathbf{x}_1,\ldots,\mathbf{x}_n}[\mathbf{a}_1, \ldots, \mathbf{a}_n] \to \exists \mathbf{x}_1 \ldots \exists \mathbf{x}_n \mathbf{A}$$
$$\vdash \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n \mathbf{A} \to \mathbf{A}_{\mathbf{x}_1,\ldots,\mathbf{x}_n}[\mathbf{a}_1, \ldots, \mathbf{a}_n]$$

Distribution Rule. *If* $\vdash \mathbf{A} \to \mathbf{B}$, *then* $\vdash \exists \mathbf{x} \mathbf{A} \to \exists \mathbf{x} \mathbf{B}$ *and* $\vdash \forall \mathbf{x} \mathbf{A} \to \forall \mathbf{x} \mathbf{B}$.

Closure Theorem. *If* $\mathbf{A}'$ *is the closure of* $\mathbf{A}$, *then* $\vdash \mathbf{A}'$ *iff* $\vdash \mathbf{A}$.

Corollary. *If* $\mathbf{A}'$ *is the closure of* $\mathbf{A}$, *then* $\mathbf{A}'$ *is valid in a structure* $\mathcal{A}$ *iff* $\mathbf{A}$ *is valid in* $\mathcal{A}$.

§3.3

Deduction Theorem. *Let* $\mathbf{A}$ *be a closed formula in* T. *For every formula* $\mathbf{B}$ *of* T, $\vdash_T \mathbf{A} \to \mathbf{B}$ *iff* $\mathbf{B}$ *is a theorem of* $T[\mathbf{A}]$.

Corollary. *Let* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *be closed formulas in* T. *For every formula* $\mathbf{B}$ *in* T, $\vdash_T \mathbf{A}_1 \to \cdots \to \mathbf{A}_n \to \mathbf{B}$ *iff* $\mathbf{B}$ *is a theorem of* $T[\mathbf{A}_1, \ldots, \mathbf{A}_n]$.

Theorem on Constants. *Let* T' *be obtained from* T *by adding new constants (but no new nonlogical axioms). For every formula* $\mathbf{A}$ *of* T *and every sequence* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *of distinct new constants,* $\vdash_T \mathbf{A}$ *iff* $\vdash_{T'} \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$.

§3.4

Equivalence Theorem. *Let* $\mathbf{A}'$ *be obtained from* $\mathbf{A}$ *by replacing some occurrences of* $\mathbf{B}_1, \ldots, \mathbf{B}_n$ *by* $\mathbf{B}'_1, \ldots, \mathbf{B}'_n$*, respectively. If*

$$\vdash \mathbf{B}_1 \leftrightarrow \mathbf{B}'_1, \ldots, \vdash \mathbf{B}_n \leftrightarrow \mathbf{B}'_n$$

then

$$\vdash \mathbf{A} \leftrightarrow \mathbf{A}'.$$

Variant Theorem. *If* $\mathbf{A}'$ *is a variant of* $\mathbf{A}$*, then* $\vdash \mathbf{A} \leftrightarrow \mathbf{A}'$*.*

Symmetry Theorem. $\vdash \mathbf{a} = \mathbf{b} \leftrightarrow \mathbf{b} = \mathbf{a}$.

Equality Theorem. *Let* $\mathbf{b}'$ *be obtained from* $\mathbf{b}$ *by replacing some occurrences of* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ *not immediately following* $\exists$ *or* $\forall$ *by* $\mathbf{a}'_1, \ldots, \mathbf{a}'_n$ *respectively, and let* $\mathbf{A}'$ *be obtained from* $\mathbf{A}$ *by the same type of replacements. If* $\vdash \mathbf{a}_1 = \mathbf{a}'_1, \ldots, \vdash \mathbf{a}_n = \mathbf{a}'_n$ *then* $\vdash \mathbf{b} = \mathbf{b}'$ *and* $\vdash \mathbf{A} \leftrightarrow \mathbf{A}'$*.*

Corollary 1. $\vdash \mathbf{a}_1 = \mathbf{a}'_1 \to \cdots \to \mathbf{a}_n = \mathbf{a}'_n \to \mathbf{b}[\mathbf{a}_1, \ldots, \mathbf{a}_n] = \mathbf{b}[\mathbf{a}'_1, \ldots, \mathbf{a}'_n]$.

Corollary 2. $\vdash \mathbf{a}_1 = \mathbf{a}'_1 \to \cdots \to \mathbf{a}_n = \mathbf{a}'_n \to (\mathbf{A}[\mathbf{a}_1, \ldots, \mathbf{a}_n] \leftrightarrow \mathbf{A}[\mathbf{a}'_1, \ldots, \mathbf{a}'_n])$.

Corollary 3. *If* $\mathbf{x}$ *does not occur in* $\mathbf{a}$*, then*

$$\vdash \mathbf{A_x}[\mathbf{a}] \leftrightarrow \exists \mathbf{x}(\mathbf{x} = \mathbf{a} \wedge \mathbf{A})$$

PROBLEMS

1. Let's prove it by induction on theorems (as in §3.1). If $\mathbf{A}$ is a theorem provable without use of substitution axioms, identity axioms, equality axioms, nonlogical axioms or the $\exists$-introduction rule, then it is a tautological consequence of some theorems $\mathbf{B}_1, \ldots, \mathbf{B}_n$. If $n = 0$, then $\mathbf{A}$ is a tautology, since it's a tautological consequence of the empty sequence of formulas. Otherwise, by the induction hypothesis, if $\mathbf{B}_1, \ldots, \mathbf{B}_n$ can be proven without the use of substitution axioms, identity axioms, equality axioms, nonlogical axioms or the $\exists$-introduction rule, they are also tautologies. This means that $V(\mathbf{B}_i) = \top$ for all i and truth valuations V, which implies that $V(\mathbf{A}) = \top$ for all truth valuations and thus $\mathbf{A}$ is also a tautology.

2. First note, that if a formula $\mathbf{A}$ is a tautology, then it's a tautological consequence of any set of formulas Γ since $V(\mathbf{A}) = \top$ for every truth valuation V. We use induction on theorems (as in §3.1):

- If $\mathbf{A}$ is an identity axiom $\mathbf{x} = \mathbf{x}$, then $\mathbf{A}^*$ is $\mathbf{e} = \mathbf{e}$ which is a tautological consequence of $\mathbf{e} = \mathbf{e}$.
- If $\mathbf{A}$ is an equality axiom

  $$\mathbf{x}_1 = \mathbf{y}_1 \to \cdots \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n,$$

  then $\mathbf{A}^*$ is

  $$\mathbf{e} = \mathbf{e} \to \cdots \to \mathbf{e} = \mathbf{e} \to \mathbf{e} = \mathbf{e},$$

  which is a tautology.
- If $\mathbf{A}$ is an equality axiom

  $$\mathbf{x}_1 = \mathbf{y}_1 \to \cdots \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{p}\mathbf{x}_1 \ldots \mathbf{x}_n \to \mathbf{p}\mathbf{y}_1 \ldots \mathbf{y}_n,$$

  then $\mathbf{A}^*$ is

  $$\mathbf{e} = \mathbf{e} \to \cdots \to \mathbf{e} = \mathbf{e} \to \mathbf{p}\mathbf{e} \ldots \mathbf{e} \to \mathbf{p}\mathbf{e} \ldots \mathbf{e},$$

  which is a tautology.
- If $\mathbf{A}$ is a substitution axiom $\mathbf{B}_{\mathbf{x}}[\mathbf{a}] \to \exists\mathbf{x}\mathbf{B}$, then $\mathbf{A}^*$ is $(\mathbf{B}_{\mathbf{x}}[\mathbf{a}])^* \to \mathbf{B}^*$. First, let's show that $(\mathbf{B}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{B}^*$ are the same, by induction on the length of $\mathbf{B}$:
  - If $\mathbf{B}$ is an atomic formula $\mathbf{p}\mathbf{a}_1 \ldots \mathbf{a}_n$, then any occurrence of $\mathbf{x}$ in $\mathbf{B}$ is completely inside one of the $\mathbf{a}_i$. Hence, both $(\mathbf{B}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{B}^*$ are $\mathbf{p}\mathbf{e} \ldots \mathbf{e}$.
  - If $\mathbf{B}$ is $\neg\mathbf{C}$, then $(\mathbf{B}_{\mathbf{x}}[\mathbf{a}])^*$ is $\neg(\mathbf{C}_{\mathbf{x}}[\mathbf{a}])^*$, $\mathbf{B}^*$ is $\neg\mathbf{C}^*$, and $(\mathbf{C}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{C}^*$ are the same by the induction hypothesis.
  - If $\mathbf{B}$ is $\mathbf{C} \vee \mathbf{D}$, then $(\mathbf{B}_{\mathbf{x}}[\mathbf{a}])^*$ is $(\mathbf{C}_{\mathbf{x}}[\mathbf{a}])^* \vee (\mathbf{D}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{B}^*$ is $\mathbf{C}^* \vee \mathbf{D}^*$. Finally, $(\mathbf{C}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{C}^*$ are the same, and $(\mathbf{D}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{D}^*$ are the same, by the induction hypothesis.
  - Suppose $\mathbf{B}$ is $\exists\mathbf{y}\mathbf{C}$: if $\mathbf{x}$ and $\mathbf{y}$ are the same, then obviously $\mathbf{B}_{\mathbf{x}}[\mathbf{a}]$ and $\mathbf{B}$ are the same since $\mathbf{B}$ has no free occurrences of $\mathbf{x}$. Otherwise, $(\mathbf{B}_{\mathbf{x}}[\mathbf{a}])^*$ is $(\mathbf{C}_{\mathbf{x}}[\mathbf{a}])^*$, $\mathbf{B}^*$ is $\mathbf{C}^*$, and $(\mathbf{C}_{\mathbf{x}}[\mathbf{a}])^*$ and $\mathbf{C}^*$ are the same by the induction hypothesis.

  Hence, $\mathbf{A}^*$ is $\mathbf{B}^* \to \mathbf{B}^*$ which is a tautology.
- If $\mathbf{A}$ was obtained from $\mathbf{B} \to \mathbf{C}$ by the $\exists$-introduction rule, then $\mathbf{A}$ is $\exists\mathbf{x}\mathbf{B} \to \mathbf{C}$, $\mathbf{A}^*$ is $\mathbf{B}^* \to \mathbf{C}^*$, and $\mathbf{B}^* \to \mathbf{C}^*$ is a tautological consequence of $\mathbf{e} = \mathbf{e}$ by the induction hypothesis.
- If $\mathbf{A}$ is a tautological consequence of formulas $\mathbf{B}_1, \ldots, \mathbf{B}_n$, then the $\mathbf{B}_1^*, \ldots, \mathbf{B}_n^*$ are tautological consequences of $\mathbf{e} = \mathbf{e}$ by the induction hypothesis. Hence, $\mathbf{A}^*$ is a tautological consequence of $\mathbf{B}_1^*, \ldots, \mathbf{B}_n^*$ (see the *Remark* in §3.1, page 30), so $\mathbf{A}^*$ is also a tautological consequence of $\mathbf{e} = \mathbf{e}$.

Now, let's assume that there is a formula $\mathbf{A}$ such that $\vdash_T \mathbf{A}$ and $\vdash_T \neg\mathbf{A}$. Using the result just proved, the first condition implies that $\mathbf{A}^*$ is a tautological consequence of $\mathbf{e} = \mathbf{e}$ and the second condition implies that $\neg\mathbf{A}^*$ is also a tautological consequence of $\mathbf{e} = \mathbf{e}$. Hence, for any truth valuation V such that $V(\mathbf{e} = \mathbf{e}) = \top$, we must have $V(\mathbf{A}^*) = \top$ and $V(\neg\mathbf{A}^*) = \neg V(\mathbf{A}^*) = \top$. This is a contradiction. Thus, there is no such formula.

This result is interesting, since it basically says that we can not prove two theorems that contradict each other in T.

3.

(a) Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ the free variables of $\forall\mathbf{x}(\mathbf{A} \to \mathbf{B})$ and $\exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}\mathbf{B}$; let T' be a theory obtained from T by adding n new constants $\mathbf{e}_1, \ldots, \mathbf{e}_n$; let $\mathbf{C}$ be $\forall\mathbf{x}(\mathbf{A} \to \mathbf{B})$ and let $\mathbf{D}$ be $\exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}\mathbf{B}$. Note that

$$\vdash_T \mathbf{C} \to \mathbf{D} \quad \text{iff} \quad \vdash_{T'} \mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

by the Theorem on Constants, and

$$\vdash_{T'} \mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \quad \text{iff} \quad \vdash_{T'[\mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n]]} \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

by the Deduction Theorem. Hence, in $T'[\mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n]]$, we have

$\vdash \mathbf{C}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [the added nonlogical axiom]
$\vdash \forall\mathbf{x}(\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n])$ [by the definition of $\mathbf{C}$]
$\vdash \forall\mathbf{x}(\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n]) \to (\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n])$ [Substitution Theorem]
$\vdash \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [Detachment Rule]
$\vdash \exists\mathbf{x}\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_n] \to \exists\mathbf{x}\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [Distribution Rule]
$\vdash \mathbf{D}[\mathbf{e}_1, \ldots, \mathbf{e}_n]$ [by the definition of $\mathbf{D}$]

(b) As in (a), but using the universal-quantifier form of the Distribution Rule.

4.

(a) For the forward implication we have
(1) $\vdash \mathbf{A} \to \exists\mathbf{x}\mathbf{A}$ [Substitution Theorem]
(2) $\vdash \mathbf{B} \to \exists\mathbf{x}\mathbf{B}$ [Substitution Theorem]
(3) $\vdash \mathbf{A} \vee \mathbf{B} \to \exists\mathbf{x}\mathbf{A} \vee \exists\mathbf{x}\mathbf{B}$ [from (1) and (2) by the Tautology Theorem]
(4) $\vdash \exists\mathbf{x}(\mathbf{A} \vee \mathbf{B}) \to \exists\mathbf{x}\mathbf{A} \vee \exists\mathbf{x}\mathbf{B}$ [from (3) by the $\exists$-Introduction Rule]
and for the reverse implication we have
(1) $\vdash \mathbf{A} \to \mathbf{A} \vee \mathbf{B}$ [Tautology Theorem]
(2) $\vdash \mathbf{B} \to \mathbf{A} \vee \mathbf{B}$ [Tautology Theorem]
(3) $\vdash \exists\mathbf{x}\mathbf{A} \to \exists\mathbf{x}(\mathbf{A} \vee \mathbf{B})$ [from (1) by the Distribution Rule]
(4) $\vdash \exists\mathbf{x}\mathbf{B} \to \exists\mathbf{x}(\mathbf{A} \vee \mathbf{B})$ [from (2) by the Distribution Rule]
(5) $\vdash \exists\mathbf{x}\mathbf{A} \vee \exists\mathbf{x}\mathbf{B} \to \exists\mathbf{x}(\mathbf{A} \vee \mathbf{B})$ [from (3) and (4) by the Tautology Theorem]

(b) For the forward implication we have
(1) $\vdash \mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$ [Tautology Theorem]
(2) $\vdash \mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$ [Tautology Theorem]
(3) $\vdash \forall\mathbf{x}(\mathbf{A} \wedge \mathbf{B}) \to \forall\mathbf{x}\mathbf{A}$ [from (1) by the Distribution Rule]
(4) $\vdash \forall\mathbf{x}(\mathbf{A} \wedge \mathbf{B}) \to \forall\mathbf{x}\mathbf{B}$ [from (2) by the Distribution Rule]
(5) $\vdash \forall\mathbf{x}(\mathbf{A} \wedge \mathbf{B}) \to \forall\mathbf{x}\mathbf{A} \wedge \forall\mathbf{x}\mathbf{B}$ [from (3) and (4) by the Tautology Theorem]
and for the reverse implication we have
(1) $\vdash \forall\mathbf{x}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]
(2) $\vdash \forall\mathbf{x}\mathbf{B} \to \mathbf{B}$ [Substitution Theorem]
(3) $\vdash \forall\mathbf{x}\mathbf{A} \wedge \forall\mathbf{x}\mathbf{B} \to \mathbf{A} \wedge \mathbf{B}$ [from (1) and (2) by the Tautology Theorem]
(4) $\vdash \forall\mathbf{x}\mathbf{A} \wedge \forall\mathbf{x}\mathbf{B} \to \forall\mathbf{x}(\mathbf{A} \wedge \mathbf{B})$ [from (3) by the $\forall$-Introduction Rule]

(c)
(1) $\vdash \mathbf{A} \to \exists\mathbf{x}\mathbf{A}$ [Substitution Theorem]
(2) $\vdash \mathbf{B} \to \exists\mathbf{x}\mathbf{B}$ [Substitution Theorem]
(3) $\vdash \mathbf{A} \wedge \mathbf{B} \to \exists\mathbf{x}\mathbf{A} \wedge \exists\mathbf{x}\mathbf{B}$ [from (1) and (2) by the Tautology Theorem]
(4) $\vdash \exists\mathbf{x}(\mathbf{A} \wedge \mathbf{B}) \to \exists\mathbf{x}\mathbf{A} \wedge \exists\mathbf{x}\mathbf{B}$ [from (3) by the $\exists$-Introduction Rule]

(d)
(1) $\vdash \forall\mathbf{x}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]
(2) $\vdash \forall\mathbf{x}\mathbf{B} \to \mathbf{B}$ [Substitution Theorem]
(3) $\vdash \forall\mathbf{x}\mathbf{A} \vee \forall\mathbf{x}\mathbf{B} \to \mathbf{A} \vee \mathbf{B}$ [from (1) and (2) by the Tautology Theorem]
(4) $\vdash \forall\mathbf{x}\mathbf{A} \vee \forall\mathbf{x}\mathbf{B} \to \forall\mathbf{x}(\mathbf{A} \vee \mathbf{B})$ [from (3) by the $\forall$-Introduction Rule]

(e) Consider the formula

$$\forall x(x = 0 \vee 0 < x) \to \forall x(x = 0) \vee \forall x(0 < x).$$

The left side says *"every number is either zero or greater than zero"* while the right side says *"every number is zero or every number is greater than zero"*.

Then, consider the formula

$$\exists x(x = 0) \wedge \exists x(0 < x) \to \exists x(x = 0 \wedge 0 < x).$$

The left side says *"there is a number which is equal to zero and there is a number that is greater than zero"* while the right side says *"there is a number which is both equal to zero and greater than zero"*.

5. The existential form

(1) $\vdash \mathbf{A} \to \exists \mathbf{x}\mathbf{A}$ [Substitution Theorem or Substitution Axiom]

(2) $\vdash \mathbf{A} \to \mathbf{A}$ [Propositional Axiom and definition of $\to$]

(3) $\vdash \exists \mathbf{x}\mathbf{A} \to \mathbf{A}$ [from (2) by the $\exists$-Introduction Rule]

(4) $\vdash \exists \mathbf{x}\mathbf{A} \leftrightarrow \mathbf{A}$ [from (1) and (3) and the definition of $\leftrightarrow$]

and the universal form

(1) $\vdash \forall \mathbf{x}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]

(2) $\vdash \mathbf{A} \to \mathbf{A}$ [Propositional Axiom and definition of $\to$]

(3) $\vdash \mathbf{A} \to \forall \mathbf{x}\mathbf{A}$ [from (2) by the $\forall$-Introduction Rule]

(4) $\vdash \forall \mathbf{x}\mathbf{A} \leftrightarrow \mathbf{A}$ [from (1) and (3) and the definition of $\leftrightarrow$]

6.

(a)

$\vdash \mathbf{A} \to \exists \mathbf{x}\exists \mathbf{y}\mathbf{A}$ [Substitution Theorem]

$\vdash \exists \mathbf{x}\mathbf{A} \to \exists \mathbf{x}\exists \mathbf{y}\mathbf{A}$ [$\exists$-Introduction Rule]

$\vdash \exists \mathbf{y}\exists \mathbf{x}\mathbf{A} \to \exists \mathbf{x}\exists \mathbf{y}\mathbf{A}$ [$\exists$-Introduction Rule]

The reverse implication is obtained in a similar fashion. Note that it is also possible to obtain $\exists \mathbf{y}\exists \mathbf{x}\mathbf{A}$ as a variant of $\exists \mathbf{x}\exists \mathbf{y}\mathbf{A}$: first obtain $\exists \mathbf{x}\exists \mathbf{x}'\mathbf{A}'$ where $\mathbf{A}' = \mathbf{A}_y[\mathbf{x}']$ and $\mathbf{x}'$ is a new variable not appearing in $\mathbf{A}$; then obtain $\exists \mathbf{y}\exists \mathbf{x}'\mathbf{A}'_{\mathbf{x}}[\mathbf{y}]$ and finally $\exists \mathbf{y}\exists \mathbf{x}\mathbf{A}$ by substituting $\mathbf{x}'$ to $\mathbf{x}$. This would imply the result by the Variant Theorem.

(b)

$\vdash \forall \mathbf{x}\forall \mathbf{y}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]

$\vdash \forall \mathbf{x}\forall \mathbf{y}\mathbf{A} \to \forall \mathbf{x}\mathbf{A}$ [$\forall$-Introduction Rule]

$\vdash \forall \mathbf{x}\forall \mathbf{y}\mathbf{A} \to \forall \mathbf{y}\forall \mathbf{x}\mathbf{A}$ [$\forall$-Introduction Rule]

The reverse implication is obtained in a similar fashion.

(c)

$\vdash \mathbf{A} \to \exists \mathbf{x}\mathbf{A}$ [Substitution Theorem]

$\vdash \forall \mathbf{y}\mathbf{A} \to \forall \mathbf{y}\exists \mathbf{x}\mathbf{A}$ [Distribution Rule]

$\vdash \exists \mathbf{x}\forall \mathbf{y}\mathbf{A} \to \forall \mathbf{y}\exists \mathbf{x}\mathbf{A}$ [$\exists$-Introduction Rule]

Note that it might seem that using the dual results in the above proof, the opposite implication could be obtained (i.e. $\vdash \forall \mathbf{y}\exists \mathbf{x}\mathbf{A} \to \exists \mathbf{x}\forall \mathbf{y}\mathbf{A}$). However this is not the case, as they result in an alternative proof of the same result as above:

$\vdash \forall \mathbf{y}\mathbf{A} \to \mathbf{A}$ [Substitution Theorem]

$\vdash \exists \mathbf{x}\forall \mathbf{y}\mathbf{A} \to \exists \mathbf{x}\mathbf{A}$ [Distribution Rule]

$\vdash \exists \mathbf{x}\forall \mathbf{y}\mathbf{A} \to \forall \mathbf{y}\exists \mathbf{x}\mathbf{A}$ [$\forall$-Introduction Rule]

(d) Consider the formula

$$\forall x\exists y(Sx = y) \to \exists y\forall x(Sx = y).$$

The left side can be interpreted as *"every number has a successor"*, while the right side can be interpreted as *"there is a number that is the successor of every number"*.

7.

(a) Let $\mathbf{y}_1, \ldots, \mathbf{y}_m$ include the variables free in $\forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{B} \leftrightarrow \mathbf{B}')$ and let T' be obtained from T by adding new constants $\mathbf{e}_1, \ldots, \mathbf{e}_m$. Then

$$\vdash_T \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{B} \leftrightarrow \mathbf{B}') \to (\mathbf{A} \leftrightarrow \mathbf{A}')$$

if and only if

$$\vdash_{T'} \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{B}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]) \to (\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])$$

by the Theorem on Constants. And furthermore

$$\vdash_{T'} \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{B}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]) \to (\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])$$

if and only if

$$\vdash_{T'[\forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{B}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])]} \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]$$

by the Deduction Theorem. We can obtain the latter result as follows

$\vdash \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{B}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])$ [the added axiom]
$\vdash \mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{B}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]$ [Closure Theorem]
$\vdash \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]$ [Equivalence Theorem]

Note that the universal quantifiers on $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are really necessary. For if a variable appears free in $\mathbf{B}$ or $\mathbf{B}'$ and bound in $\mathbf{A}$ or $\mathbf{A}'$, the original occurrences of $\mathbf{B}$ in $\mathbf{A}$ are not necessarily the occurrences of $\mathbf{B}[\mathbf{e}_1, \ldots, \mathbf{e}_m]$ in $\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m]$ and $\mathbf{A}'$ would not be obtained from $\mathbf{A}$ as described. This would mean the Equivalence Theorem is not applicable.

(b) Consider the formula

$$(y < Sx \leftrightarrow x = 2 \cdot y) \to (\exists y(y < Sx) \leftrightarrow \exists y(x = 2 \cdot y))$$

and an instance with $x = 3, y = 4$. Then

$$\begin{aligned} \mathcal{A}(4 < 4) &= \bot \\ \mathcal{A}(3 = 2 \cdot 4) &= \bot \\ \mathcal{A}(\exists y(y < 4)) &= \top \\ \mathcal{A}(\exists y(3 = 2 \cdot y)) &= \bot \end{aligned}$$

and hence the formula is not valid in $\mathfrak{N}$.

8.

(a) The proof parallels 7(a). Let $\mathbf{y}_1, \ldots, \mathbf{y}_m$ include the variables free in $\forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{a} = \mathbf{a}')$ and let T' be obtained from T by adding new constants $\mathbf{e}_1, \ldots, \mathbf{e}_m$. Then

$$\vdash_T \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{a} = \mathbf{a}') \to (\mathbf{A} \leftrightarrow \mathbf{A}')$$

if and only if

$$\vdash_{T'} \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{a}[\mathbf{e}_1, \ldots, \mathbf{e}_m] = \mathbf{a}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]) \to (\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])$$

by the Theorem on Constants. And furthermore

$$\vdash_{T'} \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{a}[\mathbf{e}_1, \ldots, \mathbf{e}_m] = \mathbf{a}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]) \to (\mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])$$

if and only if

$$\vdash_{T'[\forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{a}[\mathbf{e}_1, \ldots, \mathbf{e}_m] = \mathbf{a}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])]} \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]$$

by the Deduction Theorem. We can obtain the latter result as follows

$\vdash \forall \mathbf{x}_1 \ldots \forall \mathbf{x}_n(\mathbf{a}[\mathbf{e}_1, \ldots, \mathbf{e}_m] = \mathbf{a}'[\mathbf{e}_1, \ldots, \mathbf{e}_m])$ [the added axiom]
$\vdash \mathbf{a}[\mathbf{e}_1, \ldots, \mathbf{e}_m] = \mathbf{a}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]$ [Closure Theorem]
$\vdash \mathbf{A}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \leftrightarrow \mathbf{A}'[\mathbf{e}_1, \ldots, \mathbf{e}_m]$ [Equality Theorem]

Note that the universal quantifiers on $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are really necessary. For if a variable appears in $\mathbf{a}$ or $\mathbf{a}'$ and is bound in $\mathbf{A}$ or $\mathbf{A}'$, the original occurrences of $\mathbf{a}$ in $\mathbf{A}$ are not necessarily the occurrences of

$\mathbf{a}[\mathbf{e}_1,\ldots,\mathbf{e}_m]$ in $\mathbf{A}[\mathbf{e}_1,\ldots,\mathbf{e}_m]$ and $\mathbf{A}'$ would not be obtained from $\mathbf{A}$ as described. This would mean the Equality Theorem is not applicable.

(b) Consider the formula

$$(Sx = 2\cdot x) \to (\exists x(Sx = y)) \leftrightarrow \exists x(2\cdot x = y)$$

and an instance with $x = 1, y = 3$. Then

$$\begin{aligned}\mathcal{A}(2 = 2\cdot 1) &= \top\\ \mathcal{A}(\exists x(Sx = 3)) &= \top\\ \mathcal{A}(\exists x(2\cdot x = 3)) &= \bot\end{aligned}$$

and hence the formula is not valid in $\mathfrak{N}$.

9. First, notice that any formula $\mathbf{x} = \mathbf{y} \to \mathbf{A} \to \mathbf{A}_{\mathbf{x}}[\mathbf{y}]$ for $\mathbf{A}$ atomic, is a theorem of T: this follows from 8(a) and the Tautology Theorem by noticing that $\mathbf{A}$ has no bound variables and thus no universal quantifiers are needed. So T has the same or more theorems than T'.

To prove that this relation is tight (i.e. that T and T' have the same theorems), let's show that any formula that is an equality axiom in T, is a theorem of T'. Note that all the results from this chapter (except the Symmetry Theorem, the Equality Theorem and its corollaries) can be proven without using the equality axioms and hence are also applicable to T'.

First, let's handle the equality axioms of the form

$$\mathbf{x}_1 = \mathbf{y}_1 \to \cdots \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{p}\mathbf{x}_1\ldots\mathbf{x}_n \to \mathbf{p}\mathbf{y}_1\ldots\mathbf{y}_n.$$

Using the new axiom, we can obtain

$$\begin{aligned}&\vdash \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{p}\mathbf{x}_1\mathbf{y}_2\ldots\mathbf{y}_n \to \mathbf{p}\mathbf{y}_1\ldots\mathbf{y}_n\\ &\vdash \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{p}\mathbf{x}_1\mathbf{x}_2\mathbf{y}_3\ldots\mathbf{y}_n \to \mathbf{p}\mathbf{x}_1\mathbf{y}_2\ldots\mathbf{y}_n\\ &\qquad\cdots\\ &\vdash \mathbf{x}_n = \mathbf{y}_n \to \mathbf{p}\mathbf{x}_1\ldots\mathbf{x}_n \to \mathbf{p}\mathbf{x}_1\ldots\mathbf{x}_{n-1}\mathbf{y}_n\end{aligned}$$

and then, by the Tautology Theorem

$$\begin{aligned}&\vdash \mathbf{p}\mathbf{x}_1\mathbf{y}_2\ldots\mathbf{y}_n \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{p}\mathbf{y}_1\ldots\mathbf{y}_n\\ &\vdash \mathbf{p}\mathbf{x}_1\mathbf{x}_2\mathbf{y}_3\ldots\mathbf{y}_n \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{p}\mathbf{x}_1\mathbf{y}_2\ldots\mathbf{y}_n\\ &\qquad\cdots\\ &\vdash \mathbf{p}\mathbf{x}_1\ldots\mathbf{x}_n \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{p}\mathbf{x}_1\ldots\mathbf{x}_{n-1}\mathbf{y}_n.\end{aligned}$$

Also by the Tautology Theorem, if $\vdash \mathbf{A} \to \mathbf{B} \to \mathbf{C}$ and $\vdash \mathbf{C} \to \mathbf{D}$, then $\vdash \mathbf{A} \to \mathbf{B} \to \mathbf{D}$. Using this and starting from the first formula, we find successively

$$\begin{aligned}&\vdash \mathbf{p}\mathbf{x}_1\mathbf{x}_2\mathbf{y}_3\ldots\mathbf{y}_n \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{p}\mathbf{y}_1\ldots\mathbf{y}_n\\ &\vdash \mathbf{p}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{y}_4\ldots\mathbf{y}_n \to \mathbf{x}_3 = \mathbf{y}_3 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{p}\mathbf{y}_1\ldots\mathbf{y}_n\\ &\qquad\cdots\\ &\vdash \mathbf{p}\mathbf{x}_1\ldots\mathbf{x}_n \to \mathbf{x}_n = \mathbf{y}_n \to \cdots \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{p}\mathbf{y}_1\ldots\mathbf{y}_n\end{aligned}$$

and the equality axiom for predicates follows from the last formula and the Tautology Theorem. Note that the Symmetry Theorem can now be proven in T'.

Now let's turn our attention to the equality axioms of the form

$$\mathbf{x}_1 = \mathbf{y}_1 \to \cdots \to \mathbf{x}_n = \mathbf{y}_n \to \mathbf{f}\mathbf{x}_1\ldots\mathbf{x}_n = \mathbf{f}\mathbf{y}_1\ldots\mathbf{y}_n.$$

We need the following results

(i) If $\vdash \mathbf{A} \to \mathbf{A}'$ and $\vdash \mathbf{B} \to \mathbf{B}'$, then $\vdash \mathbf{A} \to \mathbf{B} \to (\mathbf{A}' \wedge \mathbf{B}')$

(ii) $\vdash (\mathbf{a} = \mathbf{b} \wedge \mathbf{b} = \mathbf{c}) \to \mathbf{a} = \mathbf{c}$

It is easy to verify that (i) follows from the Tautology Theorem. Let's prove (ii):

(1) $\vdash \mathbf{y} = \mathbf{x} \to \mathbf{y} = \mathbf{z} \to \mathbf{x} = \mathbf{z}$ [by the new axiom]

(2) $\vdash \mathbf{b} = \mathbf{a} \to \mathbf{b} = \mathbf{c} \to \mathbf{a} = \mathbf{c}$ [from (1) by the Substitution Rule]

(3) $\vdash \mathbf{a} = \mathbf{b} \to \mathbf{b} = \mathbf{c} \to \mathbf{a} = \mathbf{c}$ [from (2) by the Symmetry Theorem and the Equivalence Theorem]

(4) $\vdash \neg(\mathbf{a} = \mathbf{b}) \vee \neg(\mathbf{b} = \mathbf{c}) \vee (\mathbf{a} = \mathbf{c})$ [from (3) and the definition of $\to$]

(5) $\vdash (\neg(\mathbf{a} = \mathbf{b}) \vee \neg(\mathbf{b} = \mathbf{c})) \vee (\mathbf{a} = \mathbf{c})$ [from (4) by the Associative Rule]

(6) $\vdash \neg\neg(\neg(\mathbf{a} = \mathbf{b}) \vee \neg(\mathbf{b} = \mathbf{c})) \vee (\mathbf{a} = \mathbf{c})$ [from (5) by the Tautology Theorem]

(7) $\vdash (\mathbf{a} = \mathbf{b} \wedge \mathbf{b} = \mathbf{c}) \to (\mathbf{a} = \mathbf{c})$ [from (6) and the definition of $\wedge$ and $\to$]

It essentially says that equality is transitive.

Now let's move on to the main part of the proof. First, we obtain the following result for any i in $1 \ldots n$

(1) $\vdash \mathbf{x}_i = \mathbf{y}_i \to \mathbf{f}\mathbf{z}_1 \ldots \mathbf{z}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n \to \mathbf{f}\mathbf{z}_1 \ldots \mathbf{z}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_{i-1}\mathbf{y}_i\mathbf{x}_{i+1} \ldots \mathbf{x}_n$ [by the new axiom]

(2) $\vdash \mathbf{x}_i = \mathbf{y}_i \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_{i-1}\mathbf{y}_i\mathbf{x}_{i+1} \ldots \mathbf{x}_n$ [from (1) by the Substitution Rule]

(3) $\vdash \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n \to \mathbf{x}_i = \mathbf{y}_i \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_{i-1}\mathbf{y}_i\mathbf{x}_{i+1} \ldots \mathbf{x}_n$ [from (2) by the Tautology Theorem]

(4) $\vdash \mathbf{x} = \mathbf{x}$ [Identity Axiom]

(5) $\vdash \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n$ [from (4) by the Substitution Rule]

(6) $\vdash \mathbf{x}_i = \mathbf{y}_i \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_{i-1}\mathbf{y}_i\mathbf{x}_{i+1} \ldots \mathbf{x}_n$. [from (5) and (3) by the Detachment Rule]

and expanding for all i we can obtain

$$\begin{gathered}
\vdash \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{f}\mathbf{x}_1\mathbf{y}_2 \ldots \mathbf{y}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n \\
\vdash \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{f}\mathbf{x}_1\mathbf{x}_2\mathbf{y}_3 \ldots \mathbf{y}_n = \mathbf{f}\mathbf{x}_1\mathbf{y}_2 \ldots \mathbf{y}_n \\
\ldots \\
\vdash \mathbf{x}_n = \mathbf{y}_n \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_{n-1}\mathbf{y}_n.
\end{gathered}$$

Finally, and similar to previous proof (for predicate equality axioms), we use (i), (ii) and starting with the first formula we find successively

$$\begin{gathered}
\vdash \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{f}\mathbf{x}_1\mathbf{x}_2\mathbf{y}_3 \ldots \mathbf{y}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n \\
\vdash \mathbf{x}_3 = \mathbf{y}_3 \to \mathbf{x}_2 = \mathbf{y}_2 \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{f}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{y}_4 \ldots \mathbf{y}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n \\
\ldots \\
\vdash \mathbf{x}_n = \mathbf{y}_n \to \cdots \to \mathbf{x}_1 = \mathbf{y}_1 \to \mathbf{f}\mathbf{x}_1 \ldots \mathbf{x}_n = \mathbf{f}\mathbf{y}_1 \ldots \mathbf{y}_n
\end{gathered}$$

and the equality axiom for functions follows from the last formula and the Tautology Theorem. Hence, T and T' have the same theorems since the axioms in which they differ are provable in the other system, respectively.

10.

(a)

(1) $\vdash \mathbf{x} = \mathbf{a} \to (\mathbf{A} \leftrightarrow \mathbf{A}_\mathbf{x}[\mathbf{a}])$ [by the Equality Theorem]

(2) $\vdash \mathbf{x} = \mathbf{a} \to \mathbf{A} \to \mathbf{A}_\mathbf{x}[\mathbf{a}]$ [from (1) by the Tautology Theorem]

(3) $\vdash \mathbf{x} = \mathbf{a} \to \mathbf{A}_\mathbf{x}[\mathbf{a}] \to \mathbf{A}$ [from (1) by the Tautology Theorem]

(4) $\vdash \mathbf{A}_\mathbf{x}[\mathbf{a}] \to \mathbf{x} = \mathbf{a} \to \mathbf{A}$ [from (3) by the Tautology Theorem]

(5) $\vdash \mathbf{A}_\mathbf{x}[\mathbf{a}] \to \forall\mathbf{x}(\mathbf{x} = \mathbf{a} \to \mathbf{A})$ [from (4) by the $\forall$-Introduction Rule]

(6) $\vdash \forall\mathbf{x}(\mathbf{x} = \mathbf{a} \to \mathbf{A}) \to (\mathbf{a} = \mathbf{a} \to \mathbf{A}_\mathbf{x}[\mathbf{a}])$ [by the Substitution Theorem]

(7) $\vdash \mathbf{x} = \mathbf{x}$ [Identity Axiom]

(8) $\vdash \mathbf{a} = \mathbf{a}$ [from (7) by the Substitution Rule]

(9) $\vdash \forall\mathbf{x}(\mathbf{x} = \mathbf{a} \to \mathbf{A}) \to \mathbf{A}_\mathbf{x}[\mathbf{a}]$ [from (6) and (8) by the Tautology Theorem]

(10) $\vdash \mathbf{A}_\mathbf{x}[\mathbf{a}] \leftrightarrow \forall\mathbf{x}(\mathbf{x} = \mathbf{a} \to \mathbf{A})$ [from (5) and (9) by the Tautology Theorem]

(b) Consider the first formula when $\mathbf{A}$ is $x = 1$ and $\mathbf{a}$ is $x + 1$

$$x + 1 = 1 \leftrightarrow \exists x(x = x + 1 \wedge x = 1).$$

The right-hand side of the implication is always false, but the left-hand side is true if we pick an instance when x is 0. Hence this formula is not valid in $\mathfrak{N}$.

Next consider the second formula for the same $\mathbf{A}$ and $\mathbf{a}$

$$x + 1 = 1 \leftrightarrow \forall x(x = x + 1 \rightarrow x = 1).$$

In this case, the right-hand side is always true, but the left-hand can be falsified if we pick an instance when x is 1 (or any value not equal to 0). Hence this formula is also not valid in $\mathfrak{N}$.

11.

(a) Let $\mathbf{A}$ be a formula and let $\mathbf{B}_1, \ldots, \mathbf{B}_n$ be the elementary formulas having an occurrence in $\mathbf{A}$. Let $V_1, \ldots, V_m$ all the truth valuations such that $V_j(\mathbf{A}) = \top$. Let also

$$\mathbf{C}_i^j = \begin{cases} \mathbf{B}_i, & \text{if } V_j(\mathbf{B}_i) = \top \\ \neg\mathbf{B}_i, & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$. We can now define $\mathbf{A}'$ as

$$(\mathbf{C}_1^1 \wedge \ldots \wedge \mathbf{C}_n^1) \vee ... \vee (\mathbf{C}_1^m \wedge \ldots \wedge \mathbf{C}_n^m).$$

To verify that $\mathbf{A} \leftrightarrow \mathbf{A}'$ is a tautology, consider a truth valuation V. If $V(\mathbf{A}) = \top$, then $V = V_j$ for some j, by the definition of A'. Moreover, $V(\mathbf{C}_1^j \wedge \ldots \wedge \mathbf{C}_n^j) = V(\mathbf{C}_1^j) \wedge \ldots \wedge V(\mathbf{C}_n^j) = \top$ and hence $V(\mathbf{A}') = \top$ follows. If, on the other hand, $V(\mathbf{A}) = \perp$, then $V \neq V_j$ and $V(\mathbf{C}_1^j \wedge \ldots \wedge \mathbf{C}_n^j) = \perp$ for all j.

(b) Let's start from a formula $\mathbf{B}$ in disjunctive form such that $\neg\mathbf{A} \leftrightarrow \mathbf{B}$ is a tautology. We proceed as in (a): let $\mathbf{B}_1, \ldots, \mathbf{B}_n$ be the elementary formulas having an occurrence in $\mathbf{A}$, let $V_1, \ldots, V_m$ all the truth valuations such that $V_j(\neg\mathbf{A}) = \top$ and let

$$\mathbf{C}_i^j = \begin{cases} \mathbf{B}_i, & \text{if } V_j(\mathbf{B}_i) = \top \\ \neg\mathbf{B}_i, & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$. We can now define $\mathbf{B}$ as

$$(\mathbf{C}_1^1 \wedge \ldots \wedge \mathbf{C}_n^1) \vee ... \vee (\mathbf{C}_1^m \wedge \ldots \wedge \mathbf{C}_n^m)$$

and we have that $\neg\mathbf{A} \leftrightarrow \mathbf{B}$ is a tautology. Now, by the Tautology Theorem, we have that if $\neg\mathbf{A} \leftrightarrow \mathbf{B}$ is a tautology, then so are $\neg\neg\mathbf{A} \leftrightarrow \neg\mathbf{B}$ and $\mathbf{A} \leftrightarrow \neg\mathbf{B}$. By expanding $\neg\mathbf{B}$ and applying the fact that $\vdash \neg(\mathbf{A} \vee \mathbf{B}) \leftrightarrow (\neg\mathbf{A} \wedge \neg\mathbf{B})$ and $\vdash \neg(\mathbf{A} \wedge \mathbf{B}) \leftrightarrow (\neg\mathbf{A} \vee \neg\mathbf{B})$ (the De Morgan laws):

$$\begin{gathered} \neg((\mathbf{C}_1^1 \wedge \ldots \wedge \mathbf{C}_n^1) \vee ... \vee (\mathbf{C}_1^m \wedge \ldots \wedge \mathbf{C}_n^m)) \\ \neg(\mathbf{C}_1^1 \wedge \ldots \wedge \mathbf{C}_n^1) \wedge ... \wedge \neg(\mathbf{C}_1^m \wedge \ldots \wedge \mathbf{C}_n^m) \\ (\neg\mathbf{C}_1^1 \vee \ldots \vee \neg\mathbf{C}_n^1) \wedge ... \wedge (\neg\mathbf{C}_1^m \vee \ldots \vee \neg\mathbf{C}_n^m). \end{gathered}$$

Finally, we can replace each $\neg\mathbf{C}_i^j$ with a new $\mathbf{D}_i^j$ defined as

$$\mathbf{D}_i^j = \begin{cases} \mathbf{B}_i, & \text{if } \mathbf{C}_i^j = \neg\mathbf{B}_i \\ \neg\mathbf{B}_i, & \text{otherwise} \end{cases}$$

by the Tautology Theorem.

12. Let's use induction on the length of $\mathbf{A}$ and also consider the defined symbols $\wedge$ and $\forall$ as part of the generalized inductive definition of formula:

- if $\mathbf{A}$ is of the form $\mathbf{B} \vee \mathbf{C}$:
  (1) $\vdash \mathbf{B}^* \leftrightarrow \neg\mathbf{B}$ [induction hypothesis]
  (2) $\vdash \mathbf{C}^* \leftrightarrow \neg\mathbf{C}$ [induction hypothesis]
  (3) $\vdash \mathbf{B}^* \wedge \mathbf{C}^* \leftrightarrow \neg\mathbf{B} \wedge \neg\mathbf{C}$ [from (1) and (2) by the Tautology Theorem]
  (4) $\vdash \neg\mathbf{B} \wedge \neg\mathbf{C} \leftrightarrow \neg(\mathbf{B} \vee \mathbf{C})$ [from (3) by the Tautology Theorem (De Morgan laws)]
  (5) $\vdash \mathbf{B}^* \wedge \mathbf{C}^* \leftrightarrow \neg(\mathbf{B} \vee \mathbf{C})$ [from (3) and (4) by the Tautology Theorem]
- if $\mathbf{A}$ is of the form $\mathbf{B} \wedge \mathbf{C}$ (symmetrical to the previous case):
  (1) $\vdash \mathbf{B}^* \leftrightarrow \neg\mathbf{B}$ [induction hypothesis]
  (2) $\vdash \mathbf{C}^* \leftrightarrow \neg\mathbf{C}$ [induction hypothesis]
  (3) $\vdash \mathbf{B}^* \vee \mathbf{C}^* \leftrightarrow \neg\mathbf{B} \vee \neg\mathbf{C}$ [from (1) and (2) by the Tautology Theorem]
  (4) $\vdash \neg\mathbf{B} \vee \neg\mathbf{C} \leftrightarrow \neg(\mathbf{B} \wedge \mathbf{C})$ [from (3) by the Tautology Theorem (De Morgan laws)]
  (5) $\vdash \mathbf{B}^* \vee \mathbf{C}^* \leftrightarrow \neg(\mathbf{B} \wedge \mathbf{C})$ [from (3) and (4) by the Tautology Theorem]
- if $\mathbf{A}$ is of the form $\exists\mathbf{x}\mathbf{B}$:
  (1) $\vdash \mathbf{B}^* \leftrightarrow \neg\mathbf{B}$ [induction hypothesis]
  (2) $\vdash \forall\mathbf{x}\mathbf{B}^* \leftrightarrow \forall\mathbf{x}\neg\mathbf{B}$ [from (1) by the Distribution Rule]
  (3) $\vdash \forall\mathbf{x}\mathbf{B}^* \leftrightarrow \neg\exists\mathbf{x}\neg\neg\mathbf{B}$ [from (2) expanding the definition of $\forall$]
  (4) $\vdash \forall\mathbf{x}\mathbf{B}^* \leftrightarrow \neg\exists\mathbf{x}\mathbf{B}$ [from (3) by the Equivalence Theorem and Tautology Theorem]
- if $\mathbf{A}$ is of the form $\forall\mathbf{x}\mathbf{B}$:
  (1) $\vdash \mathbf{B}^* \leftrightarrow \neg\mathbf{B}$ [induction hypothesis]
  (2) $\vdash \exists\mathbf{x}\mathbf{B}^* \leftrightarrow \exists\mathbf{x}\neg\mathbf{B}$ [from (1) by the Distribution Rule]
  (3) $\vdash \exists\mathbf{x}\mathbf{B}^* \leftrightarrow \neg\neg\exists\mathbf{x}\neg\mathbf{B}$ [from (2) by the Tautology Theorem]
  (4) $\vdash \exists\mathbf{x}\mathbf{B}^* \leftrightarrow \neg\forall\mathbf{x}\mathbf{B}$ [from (3) by the definition of $\forall$]
- if $\mathbf{A}$ is of the form $\neg\mathbf{B}$ and $\mathbf{B}$ is not atomic:
  (1) $\vdash \mathbf{B}^* \leftrightarrow \neg\mathbf{B}$ [induction hypothesis]
  (2) $\vdash \neg\mathbf{B}^* \leftrightarrow \neg\neg\mathbf{B}$ [from (1) by the Tautology Theorem]
- if $\mathbf{A}$ is of the form $\neg\mathbf{B}$ and $\mathbf{B}$ is atomic, then $\mathbf{A}^*$ is $\mathbf{A}^\circ$ and $\mathbf{A}^\circ$ is $\mathbf{B}$ by definition:
  (1) $\vdash \mathbf{A}^* \leftrightarrow \mathbf{B}$ [by definition]
  (2) $\vdash \mathbf{A}^* \leftrightarrow \neg\neg\mathbf{B}$ [from (1) by the Tautology Theorem]
  (3) $\vdash \mathbf{A}^* \leftrightarrow \neg\mathbf{A}$ [from (2) by the form of $\mathbf{A}$]
- finally, if $\mathbf{A}$ is atomic, then $\mathbf{A}^*$ is $\mathbf{A}^\circ$ and $\mathbf{A}^\circ$ is $\neg\mathbf{A}$ by definition:
  (1) $\vdash \mathbf{A}^* \leftrightarrow \neg\mathbf{A}$ [by definition]
  and this concludes the proof.

CHAPTER 4
THE CHARACTERIZATION PROBLEM

DEFINITIONS

- A first-order language L' is an *extension* of the first-order language L if every nonlogical symbol of L is a nonlogical symbol of L'.
- A theory T' is an *extension* of a theory T if $L(T')$ is an extension of $L(T)$ and every theorem of T is a theorem of T'.
- A *conservative* extension of T is an extension T' of T such that every formula of T which is a theorem of T' is also a theorem of T.
- The theories T and T' are *equivalent* if each is an extension of the other, i.e., if they have the same language and the same theorems.
- A theory T is *inconsistent* if every formula of T is a theorem of T; otherwise, T is *consistent*.
- If L' is an extension of L and $\mathcal{A}'$ is a structure for L', by omitting certain of the functions and predicates of $\mathcal{A}'$ a structure for L is obtained. Then $\mathcal{A}$ is the *restriction* of $\mathcal{A}'$ to L, denoted by $\mathcal{A}'|L$. Also $\mathcal{A}'$ is an *expansion* of $\mathcal{A}$ to L'.
- If T is a theory containing a constant, the canonical structure $\mathcal{A}$ for T is defined as:
  - $|\mathcal{A}|$ is the set of all equivalence classes of $\sim$
  - $\mathbf{f}_{\mathcal{A}}(\mathbf{a}_1{}^\circ, \ldots, \mathbf{a}_n{}^\circ) = (\mathbf{f}\mathbf{a}_1 \ldots \mathbf{a}_n)^\circ$
  - $\mathbf{p}_{\mathcal{A}}(\mathbf{a}_1{}^\circ, \ldots, \mathbf{a}_n{}^\circ)$ iff $\vdash_T \mathbf{p}\mathbf{a}_1 \ldots \mathbf{a}_n$
- T is a *Henkin theory* if for every closed instantiation $\exists\mathbf{x}\mathbf{A}$ of T, there is a constant $\mathbf{e}$ such that $\vdash_T \exists\mathbf{x}\mathbf{A} \to \mathbf{A}_{\mathbf{x}}[\mathbf{e}]$.
- A formula $\mathbf{A}$ of T is *undecidable* in T if neither $\mathbf{A}$ nor $\neg\mathbf{A}$ is a theorem of T. Otherwise, $\mathbf{A}$ is *decidable* in T.
- A theory T is *complete* if it is consistent and if every closed formula in T is decidable in T.
- Given a first-order language L, the *special constants* of *level* n are defined by induction on n. Assuming all special constants of all levels less than n have been defined and let $\exists\mathbf{x}\mathbf{A}$ be closed instantiation formed with these constants and the symbols of L. If $n > 0$, suppose also that $\exists\mathbf{x}\mathbf{A}$ contains at least one special constant of level $n-1$. Then the symbol $c_{\exists\mathbf{x}\mathbf{A}}$ is the special constant of level n for $\exists\mathbf{x}\mathbf{A}$. The language obtained from L by adding all special constants of all levels is L_c.
- If $\mathbf{r}$ is the special constant for $\exists\mathbf{x}\mathbf{A}$, then the formula $\exists\mathbf{x}\mathbf{A} \to \mathbf{A}_{\mathbf{x}}[\mathbf{r}]$ is the *special axiom* for $\mathbf{r}$. Let T be a theory with language L. Then T_c is the theory whose language is L_c is whose nonlogical axioms are those of T and the special axioms for the special constants of L_c.
- An extension T' of T is *simple* if T and T' have the same language.
- Let E be a set and J a class of subsets of E. We say that J has *finite character* if for every subset A of E, A is in J iff every finite subset of A is in J. A set A in J is a *maximal element* of J if A is not a subset of any other member of J.
- A theory is *open* if all of its nonlogical axioms are open.
- Let $\mathbf{r}$ be the special constant for $\exists\mathbf{x}\mathbf{A}$. A formula *belongs* to $\mathbf{r}$ if it is:
  - the special axiom for $\mathbf{r}$;
  - a closed substitution axiom in $L(T_c)$ of the form $\mathbf{A}_{\mathbf{x}}[\mathbf{a}] \to \exists\mathbf{x}\mathbf{A}$.
- $\Delta(T)$ is the set of formulas in T_c which either belong to some special constant or are closed instances of identity axioms, equality axioms, or nonlogical axioms of T.
- A formula is a *quasi-tautology* if it is a tautological consequence of instances of identity axioms and equality axioms.
- The *rank* of the special constant for $\exists\mathbf{x}\mathbf{A}$ is the number of occurrences of $\exists$ in $\exists\mathbf{x}\mathbf{A}$. It is at least 1.
- $\mathbf{A}_1, \ldots, \mathbf{A}_k$ is a *special sequence* if $\neg\mathbf{A}_1 \vee \cdots \vee \neg\mathbf{A}_k$ is a tautology.

RESULTS

§**4.1**

Reduction Theorem. *Let Γ be a set of formulas in the theory T, and let $\mathbf{A}$ be a formula of T. Then $\mathbf{A}$ is a theorem of $T[\Gamma]$ iff there is a theorem of T of the form $\mathbf{B}_1 \to \cdots \to \mathbf{B}_n \to \mathbf{A}$, where each $\mathbf{B}_i$ is the closure of a formula in Γ.*

Reduction Theorem for Consistency. *Let Γ be a nonempty set of formulas in the theory T. Then $T[\Gamma]$ is inconsistent iff there is a theorem of T which is a disjuction of negations of closures of distinct formulas in Γ.*

Corollary. *Let $\mathbf{A}'$ be the closure of $\mathbf{A}$. Then $\mathbf{A}$ is a theorem of T iff $T[\neg\mathbf{A}']$ is inconsistent.*

§**4.2**

Completeness Theorem, First Form. *A formula $\mathbf{A}$ of a theory T is a theorem of T iff it is valid in T.*

Completeness Theorem, Second Form. *A theory T is consistent iff it has a model.*

Lemma 1. *If T' is an extension of T, and $\mathcal{A}'$ is a model of T', then the restriction of $\mathcal{A}'$ to $L(T)$ is a model of T.*

Lemma 2. *Let T be a complete Henkin theory; $\mathcal{A}$ the canonical structure for T; $\mathbf{A}$ a closed formula of T. Then $\mathcal{A}(\mathbf{A}) = \top$ iff $\vdash_T \mathbf{A}$.*

Corollary. *If T is a complete Henkin theory, then the canonical structure for T is a model of T.*

Lemma 3. *T_c is a conservative extension of T.*

Teichmüller-Tukey Lemma. *If J is a nonempty class of subsets of E which is of finite character, then J contains a maximal element.*

Lindenbaum's Theorem. *If T is a consistent theory, then T has a complete simple extension.*

Lemma 4. *Let T be a theory, and let U be a consistent simple extension of T_c. Then U has a model $\mathcal{A}$ such that each individual of $\mathcal{A}$ is $\mathcal{A}(\mathbf{r})$ for infinitely many special constants $\mathbf{r}$.*

Corollary. *Let T and T' be theories with the same language. Then T' is an extension of T iff every model of T' is a model of T. Hence T and T' are equivalent iff they have the same models.*

§**4.3**

Lemma 1. *If $\vdash_T \mathbf{A}$ and $\mathbf{A}'$ is a closed instance of $\mathbf{A}$ in $L(T_c)$, then $\mathbf{A}'$ is a tautological consequence of formulas in $\Delta(T)$.*

Consistency Theorem. *An open theory T is inconsistent iff there is a quasi-tautology which is the disjunction of negations of instances of nonlogical axioms of T.*

PROBLEMS

APPENDIX A
COMPLETENESS OF PROPOSITIONAL LOGIC

The proof of the Tautology Theorem provides a direct way of proving the completeness of propositional logic, i.e. the fragment of the first-order system which does not contain quantifiers and employs only propositional variables.

Let's define this fragment more precisely. Consider a *propositional language* having as symbols the following:

a) the *propositional variables*: $x, y, z, w, x', y', z', w', x'', \ldots$;
b) the symbols $\neg$ and $\vee$.

In this case, the variables range over the set of truth values $\{\top, \bot\}$ and there are no terms. Instead, the formulas are defined by the generalized inductive definition:

i) a variable is a formula.
ii) if $\mathbf{u}$ is a formula, then $\neg\mathbf{u}$ is a formula.
iii) if $\mathbf{u}$ and $\mathbf{v}$ are formulas, then $\vee\mathbf{uv}$ is a formula.

There are no nonlogical symbols in this language. Moreover, all the remarks about defined symbols for first-order languages apply when appropriate. Any formula in this language can be assigned a unique truth value in the expected way and it is easy to see that every truth function is definable in this language (see problem 1.1). Finally, the propositional axiom is the only axiom of the system and the rules are the Expansion Rule, Contraction Rule, the Associative Rule and the Cut Rule (i.e. all the rules except the $\exists$-Introduction Rule). Let's call this formal system P.

We can see now that a formula $\mathbf{A}$ of P is valid iff every truth valuation of the propositional variables makes $\mathbf{A}$ true. This is analogous to a first-order tautology, since in P the propositional variables stand for the elementary formulas. Thus, given a propositional tautology $\mathbf{A}$ in P, the proof of the Tautology Theorem and its corollary provide a constructive procedure to produce a proof $\vdash_P \mathbf{A}$. Together with the Validity Theorem, this implies that this system is complete: that is, a formula is valid iff it is a theorem.

APPENDIX B

HENKINS'S PROOF OUTLINE

1. Reduce the characterization of $T[\Gamma]$ to that of T (Reduction Theorem).
2. Reduce the validity form of the Completeness Theorem to the consistency form.
3. Show that the restriction of a model $\mathcal{A}'|L(T)$ is a model of T; so if we obtain a model for an extension, we also obtain a model for the base theory.
4. Given a consistent theory T we need to find a model via syntactical means.
5. Let the theorems of T say what is truth of the variable-free terms, which denote individuals.
6. Introduce the canonical structure $\mathcal{A}$ (also called the *term structure*) using the equivalence classes of variable-free terms as individuals.
7. Show that the atomic closed formulas are valid in the canonical structure iff they are theorems.
8. The two obstacles for showing that the same applies to general closed formulas are:
   (i) There might be not enough variable-free terms.
   (ii) There might be formulas which are undecidable.
9. To handle (i), Henkin theories are introduced which guarantee that a constant exists for every closed instantiation (also called the *witness property*).
10. Show that if T is a complete Henkin theory, then the canonical structure for T, is a model of T.
11. Show that we can construct a Henkin theory T_c from a consistent theory T using the special constants and axioms, and that T_c is a conservative extension of T.
12. Show that every consistent theory has a simple complete extension (Lindenbaum's Theorem).

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\text{extension}"] & T(\text{consistent}) \arrow[d, "\text{conservative extension}"] & \mathcal{A}|L \arrow[l, "\text{model}"'] \\
L_c \arrow[r] \arrow[dr] & T_c(\text{consistent, Henkin}) \arrow[d, "\text{simple extension}"] & \\
& U(\text{complete, Henkin}) & \mathcal{A} \arrow[l, "\text{model}"] \arrow[uu, "\text{restriction}"']
\end{tikzcd}$$

APPENDIX C
THEORIES

N (Natural Numbers)

Nonlogical symbols:
- constant 0
- unary function symbol S, the successor function
- binary function symbols $+$ and $\cdot$
- binary predicate symbol $<$

Nonlogical axioms:

N1. $Sx \neq 0$
N2. $Sx = Sy \rightarrow x = y$
N3. $x + 0 = x$
N4. $x + Sy = S(x + y)$
N5. $x \cdot 0 = 0$
N6. $x \cdot Sy = (x \cdot y) + x$
N7. $\neg(x < 0)$
N8. $x < Sy \leftrightarrow x < y \vee x = y$
N9. $x < y \vee x = y \vee y < x$

G (Elementary Theory of Groups)

Nonlogical symbols:
- binary function symbol $\cdot$

Nonlogical axioms:

G1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
G2. $\exists x(\forall y(x \cdot y = y) \wedge \forall y \exists z(z \cdot y = x))$

APPENDIX D
PROOFS

Chapter 2 - Problem 5(a)

(1) $\neg\neg(x = x) \lor \neg(x = x)$ [axiom: propositional]

Chapter 2 - Problem 5(b)

(1) $\neg(x = x) \lor \exists x(x = x)$ [axiom: substitution]

Chapter 2 - Problem 5(c)

(1) $(x = x)$ [axiom: identity]

Chapter 2 - Problem 5(d)

(1) $\neg(x = y) \lor (\neg(x = z) \lor (\neg(x = x) \lor (y = z)))$ [axiom: equality]

Chapter 2 - Problem 5(e)

(1) $(x = x)$ [axiom: identity]
(2) $\neg(x = x) \lor (x = x)$ [rule: expansion: (1)]
(3) $(x = x) \lor (\neg(x = x) \lor (x = x))$ [rule: expansion: (2)]

Chapter 2 - Problems 5(f) and 5(h)

(1) $(x = x)$ [axiom: identity]
(2) $\neg\neg(x = x) \lor (x = x)$ [rule: expansion: (1)]
(3) $\neg\neg\neg(x = x) \lor \neg\neg(x = x)$ [axiom: propositional]
(4) $(x = x) \lor \neg\neg(x = x)$ [rule: cut: (2) (3)]
(5) $\neg\neg(x = x) \lor \neg(x = x)$ [axiom: propositional]
(6) $\neg(x = x) \lor \neg\neg(x = x)$ [rule: cut: (5) (3)]
(7) $\neg\neg(x = x) \lor \neg\neg(x = x)$ [rule: cut: (4) (6)]
(8) $\neg\neg(x = x)$ [rule: contraction: (7)]

Chapter 2 - Problem 5(g)

(1) $(x = x)$ [axiom: identity]
(2) $\neg(\neg(x = x) \lor \neg(x = x)) \lor (\neg(x = x) \lor \neg(x = x))$ [axiom: propositional]
(3) $(\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x)) \lor \neg(x = x)$ [rule: associative: (2)]
(4) $(\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x)) \lor (x = x)$ [rule: expansion: (1)]
(5) $\neg(\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x)) \lor (\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x))$ [axiom: propositional]
(6) $\neg(x = x) \lor (\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x))$ [rule: cut: (3) (5)]
(7) $(x = x) \lor (\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x))$ [rule: cut: (4) (5)]
(8) $(\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x)) \lor (\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x))$ [rule: cut: (7) (6)]
(9) $\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(x = x)$ [rule: contraction: (8)]
(10) $\neg(\neg(x = x) \lor \neg(x = x)) \lor (x = x)$ [rule: expansion: (1)]
(11) $\neg\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(\neg(x = x) \lor \neg(x = x))$ [axiom: propositional]
(12) $\neg(x = x) \lor \neg(\neg(x = x) \lor \neg(x = x))$ [rule: cut: (9) (11)]
(13) $(x = x) \lor \neg(\neg(x = x) \lor \neg(x = x))$ [rule: cut: (10) (11)]
(14) $\neg(\neg(x = x) \lor \neg(x = x)) \lor \neg(\neg(x = x) \lor \neg(x = x))$ [rule: cut: (13) (12)]
(15) $\neg(\neg(x = x) \lor \neg(x = x))$ [rule: contraction: (14)]

Chapter 2 - Problem 5(i)

(1) $\neg\neg(x = x) \lor \neg(x = x)$ [axiom: propositional]
(2) $\neg\exists y \neg(x = x) \lor \neg(x = x)$ [rule: e-introduction: (1)]

Chapter 3 - §3.1 - Lemma 1

(1) $\mathbf{A} \lor \mathbf{B}$ [premise]
(2) $\neg\mathbf{A} \lor \mathbf{A}$ [axiom: propositional]

(3) $\mathbf{B} \vee \mathbf{A}$ [rule: cut: (1) (2)]

Chapter 3 - §3.1 - Detachment Rule

(1) $\mathbf{A}$ [premise]
(2) $\neg\mathbf{A} \vee \mathbf{B}$ [premise]
(3) $\mathbf{B} \vee \mathbf{A}$ [rule: expansion: (1)]
(4) $\neg\mathbf{B} \vee \mathbf{B}$ [axiom: propositional]
(5) $\mathbf{A} \vee \mathbf{B}$ [rule: cut: (3) (4)]
(6) $\mathbf{B} \vee \mathbf{B}$ [rule: cut: (5) (2)]
(7) $\mathbf{B}$ [rule: contraction: (6)]

Chapter 3 - §3.1 - Tautology Theorem - result (B)

(1) $\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\neg\mathbf{A} \vee \neg\mathbf{A}$ [axiom: propositional]
(3) $\neg\neg\neg\mathbf{A} \vee \neg\neg\mathbf{A}$ [axiom: propositional]
(4) $\neg\mathbf{A} \vee \neg\neg\mathbf{A}$ [rule: cut: (2) (3)]
(5) $\mathbf{B} \vee \neg\neg\mathbf{A}$ [rule: cut: (1) (4)]
(6) $\neg\mathbf{B} \vee \mathbf{B}$ [axiom: propositional]
(7) $\neg\neg\mathbf{A} \vee \mathbf{B}$ [rule: cut: (5) (6)]

Chapter 3 - §3.1 - Tautology Theorem - result (C)

(1) $\neg\mathbf{A} \vee \mathbf{C}$ [premise]
(2) $\neg\mathbf{B} \vee \mathbf{C}$ [premise]
(3) $\neg(\mathbf{A} \vee \mathbf{B}) \vee (\mathbf{A} \vee \mathbf{B})$ [axiom: propositional]
(4) $(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A}) \vee \mathbf{B}$ [rule: associative: (3)]
(5) $\neg(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A}) \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A})$ [axiom: propositional]
(6) $\mathbf{B} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A})$ [rule: cut: (4) (5)]
(7) $(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A}) \vee \mathbf{C}$ [rule: cut: (6) (2)]
(8) $\mathbf{C} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{A})$ [rule: cut: (7) (5)]
(9) $(\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B})) \vee \mathbf{A}$ [rule: associative: (8)]
(10) $\neg(\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B})) \vee (\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B}))$ [axiom: propositional]
(11) $\mathbf{A} \vee (\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B}))$ [rule: cut: (9) (10)]
(12) $(\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B})) \vee \mathbf{C}$ [rule: cut: (11) (1)]
(13) $\mathbf{C} \vee (\mathbf{C} \vee \neg(\mathbf{A} \vee \mathbf{B}))$ [rule: cut: (12) (10)]
(14) $(\mathbf{C} \vee \mathbf{C}) \vee \neg(\mathbf{A} \vee \mathbf{B})$ [rule: associative: (13)]
(15) $\neg(\mathbf{C} \vee \mathbf{C}) \vee (\mathbf{C} \vee \mathbf{C})$ [axiom: propositional]
(16) $\neg(\mathbf{A} \vee \mathbf{B}) \vee (\mathbf{C} \vee \mathbf{C})$ [rule: cut: (14) (15)]
(17) $(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}) \vee \mathbf{C}$ [rule: associative: (16)]
(18) $\neg(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}) \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$ [axiom: propositional]
(19) $\mathbf{C} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$ [rule: cut: (17) (18)]
(20) $\neg(\mathbf{A} \vee \mathbf{B}) \vee (\mathbf{C} \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}))$ [rule: expansion: (19)]
(21) $(\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}) \vee (\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$ [rule: associative: (20)]
(22) $\neg(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$ [rule: contraction: (21)]

Chapter 3 - §3.1 - Tautology Theorem - frequently used cases (ii)

(1) $\neg\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\mathbf{B} \vee \mathbf{C}$ [premise]
(3) $\neg\neg\mathbf{A} \vee \neg\mathbf{A}$ [axiom: propositional]
(4) $\mathbf{B} \vee \neg\mathbf{A}$ [rule: cut: (1) (3)]
(5) $\neg\mathbf{A} \vee \mathbf{C}$ [rule: cut: (4) (2)]

Chapter 3 - §3.1 - Tautology Theorem - frequently used cases (vi)

(1) $\neg\mathbf{A} \vee \mathbf{B}$ [premise]
(2) $\neg\neg\mathbf{A} \vee \neg\mathbf{A}$ [axiom: propositional]

(3)	$\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (1) (2)]
(4)	$\neg\neg\mathbf{B} \vee \neg\mathbf{B}$	[axiom: propositional]
(5)	$\neg\neg\neg\mathbf{B} \vee \neg\neg\mathbf{B}$	[axiom: propositional]
(6)	$\neg\mathbf{B} \vee \neg\neg\mathbf{B}$	[rule: cut: (4) (5)]
(7)	$\neg\mathbf{A} \vee \neg\neg\mathbf{B}$	[rule: cut: (3) (6)]
(8)	$\neg\neg\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (7) (2)]

Chapter 3 - §3.2 - ∀-Introduction Rule

(1)	$\neg\mathbf{A} \vee \mathbf{B}$	[premise]
(2)	$\neg\neg\mathbf{A} \vee \neg\mathbf{A}$	[axiom: propositional]
(3)	$\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (1) (2)]
(4)	$\neg\neg\mathbf{B} \vee \neg\mathbf{B}$	[axiom: propositional]
(5)	$\neg\neg\neg\mathbf{B} \vee \neg\neg\mathbf{B}$	[axiom: propositional]
(6)	$\neg\mathbf{B} \vee \neg\neg\mathbf{B}$	[rule: cut: (4) (5)]
(7)	$\neg\mathbf{A} \vee \neg\neg\mathbf{B}$	[rule: cut: (3) (6)]
(8)	$\neg\neg\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (7) (2)]
(9)	$\neg\exists\mathbf{x}\neg\mathbf{B} \vee \neg\mathbf{A}$	[rule: e-introduction: (8)]
(10)	$\neg\neg\exists\mathbf{x}\neg\mathbf{B} \vee \neg\exists\mathbf{x}\neg\mathbf{B}$	[axiom: propositional]
(11)	$\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{B}$	[rule: cut: (9) (10)]

Chapter 3 - §3.2 - Generalization Rule

(1)	$\mathbf{A}$	[premise]
(2)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \mathbf{A}$	[rule: expansion: (1)]
(3)	$\neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[axiom: propositional]
(4)	$\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (2) (3)]
(5)	$\neg\neg\mathbf{A} \vee \neg\mathbf{A}$	[axiom: propositional]
(6)	$\neg\neg\neg\mathbf{A} \vee \neg\neg\mathbf{A}$	[axiom: propositional]
(7)	$\neg\mathbf{A} \vee \neg\neg\mathbf{A}$	[rule: cut: (5) (6)]
(8)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\mathbf{A}$	[rule: cut: (4) (7)]
(9)	$\neg\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (8) (3)]
(10)	$\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: e-introduction: (9)]
(11)	$\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$	[axiom: propositional]
(12)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (10) (11)]
(13)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[axiom: propositional]
(14)	$\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (13) (3)]
(15)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (10) (14)]
(16)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: contraction: (15)]
(17)	$\neg\neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[axiom: propositional]
(18)	$\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (3) (17)]
(19)	$\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (11) (18)]
(20)	$\neg\neg\neg\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (19) (11)]
(21)	$\neg\exists\mathbf{x}\neg\mathbf{A} \vee \neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: cut: (12) (20)]
(22)	$\neg\exists\mathbf{x}\neg\mathbf{A}$	[rule: contraction: (21)]

Chapter 3 - §3.2 - Distribution Rule

(1)	$\neg\mathbf{A} \vee \mathbf{B}$	[premise]
(2)	$\neg\mathbf{B} \vee \exists\mathbf{x}\mathbf{B}$	[axiom: substitution]
(3)	$\neg\neg\mathbf{A} \vee \neg\mathbf{A}$	[axiom: propositional]
(4)	$\mathbf{B} \vee \neg\mathbf{A}$	[rule: cut: (1) (3)]
(5)	$\neg\mathbf{A} \vee \exists\mathbf{x}\mathbf{B}$	[rule: cut: (4) (2)]
(6)	$\neg\exists\mathbf{x}\mathbf{A} \vee \exists\mathbf{x}\mathbf{B}$	[rule: e-introduction: (5)]
