

2.5.1)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{(\neg A \wedge \neg B), A \rightarrow}{A \rightarrow \neg(\neg A \wedge \neg B)} [\neg R]}{\frac{(A \vee B) \rightarrow \neg(\neg A \wedge \neg B)}{\rightarrow ((A \vee B) \supset \neg(\neg A \wedge \neg B))} [\supset R]} \quad \frac{\frac{B \rightarrow B}{\neg B, B \rightarrow} [\neg L] \quad \frac{(\neg A \wedge \neg B), B \rightarrow}{B \rightarrow \neg(\neg A \wedge \neg B)} [\neg R]}{\frac{(A \vee B) \rightarrow \neg(\neg A \wedge \neg B)}{\rightarrow ((A \vee B) \supset \neg(\neg A \wedge \neg B))} [\supset R]} \\
\frac{\frac{A \rightarrow A}{A \rightarrow (A \vee B)} [\vee R] \quad \frac{B \rightarrow B}{B \rightarrow (A \vee B)} [\vee R]}{\frac{\rightarrow (A \vee B), \neg A}{\rightarrow (A \vee B), \neg B} [\wedge R]} \quad \frac{\frac{\rightarrow (A \vee B), \neg A}{\rightarrow (A \vee B), \neg B} [\wedge R] \quad \frac{\rightarrow (A \vee B), (\neg A \wedge \neg B)}{\neg(\neg A \wedge \neg B) \rightarrow (A \vee B)} [\neg L]}{\frac{\neg(\neg A \wedge \neg B) \rightarrow (A \vee B)}{\rightarrow (\neg(\neg A \wedge \neg B) \supset (A \vee B))} [\supset R]} \\
\frac{\rightarrow ((A \vee B) \supset \neg(\neg A \wedge \neg B)) \quad \rightarrow (\neg(\neg A \wedge \neg B) \supset (A \vee B))}{\rightarrow ((A \vee B) \equiv \neg(\neg A \wedge \neg B))} [\wedge R]
\end{array}$$

2.5.2)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R] \quad \frac{\rightarrow A, (\neg A \vee B)}{\rightarrow (\neg A \vee B), A} [\vee R]}{\frac{(A \supset B) \rightarrow (\neg A \vee B), (\neg A \vee B)}{(A \supset B) \rightarrow (\neg A \vee B)} [\wedge R]} \quad \frac{\frac{B \rightarrow B}{B \rightarrow (\neg A \vee B)} [\vee R] \quad \frac{(A \supset B) \rightarrow (\neg A \vee B), (\neg A \vee B)}{\rightarrow ((A \supset B) \supset (\neg A \vee B))} [\supset R]}{\frac{(A \supset B) \rightarrow (\neg A \vee B)}{\rightarrow ((A \supset B) \supset (\neg A \vee B))} [\supset R]} \\
\frac{\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L] \quad \frac{\neg A, A \rightarrow B}{A, \neg A \rightarrow B} [\wedge R] \quad \frac{B \rightarrow B}{A, B \rightarrow B} [\wedge R]}{\frac{\neg A \rightarrow (A \supset B)}{B \rightarrow (A \supset B)} [\supset R]} \quad \frac{\frac{B \rightarrow B}{A, B \rightarrow B} [\wedge R] \quad \frac{B \rightarrow B}{B \rightarrow (A \supset B)} [\supset R]}{\frac{(A \supset B) \rightarrow (\neg A \vee B)}{\rightarrow ((A \supset B) \supset (\neg A \vee B))} [\supset R]} \\
\frac{\rightarrow ((A \supset B) \supset (\neg A \vee B)) \quad \rightarrow ((\neg A \vee B) \supset (A \supset B))}{\rightarrow ((A \supset B) \equiv (\neg A \vee B))} [\wedge R]
\end{array}$$

2.5.3)

$$\begin{array}{c}
\frac{\frac{F(a) \rightarrow F(a)}{\neg F(a), F(a) \rightarrow} [\neg L] \quad \frac{\forall y \neg F(y), F(a) \rightarrow}{F(a) \rightarrow \neg \forall y \neg F(y)} [\neg R]}{\frac{F(a) \rightarrow \neg \forall y \neg F(y)}{\exists x F(x) \rightarrow \neg \forall y \neg F(y)} [\supset R]} \quad \frac{\frac{F(a) \rightarrow F(a)}{\rightarrow F(a), \neg F(a)} [\neg R] \quad \frac{\rightarrow F(a), \neg F(a)}{\rightarrow \neg F(a), F(a)} [\wedge R]}{\frac{\rightarrow \neg F(a), F(a)}{\rightarrow \neg F(a), \exists x F(x)} [\wedge R]} \\
\frac{\frac{\rightarrow \neg F(a), \exists x F(x)}{\rightarrow \exists x F(x), \neg F(a)} [\wedge R] \quad \frac{\rightarrow \exists x F(x), \neg F(a)}{\rightarrow \exists x F(x), \forall y \neg F(y)} [\wedge R]}{\frac{\rightarrow \exists x F(x), \forall y \neg F(y)}{\neg \forall y \neg F(y) \rightarrow \exists x F(x)} [\neg L]} \quad \frac{\frac{\neg \forall y \neg F(y) \rightarrow \exists x F(x)}{\rightarrow (\neg \forall y \neg F(y) \supset \exists x F(x))} [\supset R]}{\frac{\rightarrow (\exists x F(x) \supset \neg \forall y \neg F(y)) \quad \rightarrow (\neg \forall y \neg F(y) \supset \exists x F(x))}{\rightarrow (\exists x F(x) \equiv \neg \forall y \neg F(y))} [\wedge R]}
\end{array}$$

2.5.4)

$$\begin{array}{c}
\frac{\frac{F(a) \rightarrow F(a)}{\rightarrow F(a), \neg F(a)} [\neg R] \quad \frac{\rightarrow F(a), \neg F(a)}{\rightarrow F(a), \exists x \neg F(x)} [\wedge R]}{\frac{\rightarrow F(a), \exists x \neg F(x)}{\rightarrow \exists x \neg F(x), F(a)} [\wedge R]} \quad \frac{\frac{F(a) \rightarrow F(a)}{\forall y F(y) \rightarrow F(a)} [\wedge L] \quad \frac{\forall y F(y) \rightarrow F(a)}{\neg F(a), \forall y F(y) \rightarrow} [\wedge R]}{\frac{\neg F(a), \forall y F(y) \rightarrow}{\neg F(a) \rightarrow \neg \forall y F(y)} [\wedge R]} \\
\frac{\frac{\rightarrow \exists x \neg F(x), F(a)}{\rightarrow \exists x \neg F(x), \forall y F(y)} [\wedge R] \quad \frac{\rightarrow \exists x \neg F(x), \forall y F(y)}{\neg \forall y F(y) \rightarrow \exists x \neg F(x)} [\wedge R]}{\frac{\neg \forall y F(y) \rightarrow \exists x \neg F(x)}{\rightarrow (\neg \forall y F(y) \supset \exists x \neg F(x))} [\supset R]} \quad \frac{\frac{\neg \forall y F(y) \rightarrow \exists x \neg F(x)}{\rightarrow (\neg \forall y F(y) \supset \exists x \neg F(x))} [\supset R]}{\frac{\rightarrow (\neg \forall y F(y) \supset \exists x \neg F(x)) \quad \rightarrow (\exists x \neg F(x) \supset \neg \forall y F(y))}{\rightarrow (\neg \forall y F(y) \equiv \exists x \neg F(x))} [\wedge R]}
\end{array}$$

### 2.5.5)

$$\begin{array}{c}
\frac{\frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg R] \quad \frac{B \rightarrow B}{\rightarrow B, \neg B} [\neg R]}{\rightarrow A, (\neg A \vee \neg B)} [\vee R] \quad \frac{\frac{A \rightarrow A}{(A \wedge B) \rightarrow A} [\wedge L] \quad \frac{B \rightarrow B}{(A \wedge B) \rightarrow B} [\wedge L]}{\rightarrow A, \neg(A \wedge B)} [\neg R] \\
\frac{\rightarrow A, (\neg A \vee \neg B)}{\rightarrow (\neg A \vee \neg B), A} [XR] \quad \frac{\rightarrow B, (\neg A \vee \neg B)}{\rightarrow (\neg A \vee \neg B), B} [XR] \\
\frac{\rightarrow (\neg A \vee \neg B), A}{\rightarrow (\neg A \vee \neg B), (A \wedge B)} [\wedge R] \quad \frac{\rightarrow (\neg A \vee \neg B), B}{\rightarrow (\neg A \vee \neg B), \neg(A \wedge B)} [\neg L] \\
\frac{\rightarrow (\neg A \vee \neg B), (A \wedge B)}{\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)} [\neg L] \quad \frac{\rightarrow (\neg A \vee \neg B), \neg(A \wedge B)}{\neg A \rightarrow \neg(A \wedge B)} [\neg L] \\
\frac{\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)}{\rightarrow (\neg(A \wedge B) \supset (\neg A \vee \neg B))} [\supset R] \quad \frac{\neg A \rightarrow \neg(A \wedge B)}{(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)} [\supset R] \\
\frac{\rightarrow (\neg(A \wedge B) \supset (\neg A \vee \neg B))}{\rightarrow (\neg(A \wedge B) \equiv (\neg A \vee \neg B))} [\wedge R]
\end{array}$$

### 2.6.1)

$\frac{A \rightarrow A}{A \rightarrow A, B(a)}$ [WR]		$\frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg L]$		$\frac{B(a) \rightarrow B(a)}{A, B(a) \rightarrow B(a)}$ [WL]
$\frac{A \rightarrow A, B(a)}{A \rightarrow B(a), A}$ [XR]		$\frac{\neg A, A \rightarrow B(a)}{A, \neg A \rightarrow B(a)}$ [XL]		$\frac{A, B(a) \rightarrow B(a)}{B(a) \rightarrow (A \supset B(a))}$ [ $\supset R$ ]
$\frac{B(a) \rightarrow B(a)}{(A \supset B(a)), A \rightarrow B(a), B(a)}$ [ $\supset L$ ]		$\frac{A, \neg A \rightarrow B(a)}{\neg A \rightarrow (A \supset B(a))}$ [ $\supset R$ ]		$\frac{B(a) \rightarrow (A \supset B(a))}{B(a) \rightarrow \exists x(A \supset B(x))}$ [ $\exists R$ ]
$\frac{(A \supset B(a)), A \rightarrow B(a), B(a)}{(A \supset B(a)), A \rightarrow B(a)}$ [CR]		$\frac{\neg A \rightarrow (A \supset B(a))}{\neg A \rightarrow \exists x(A \supset B(x))}$ [ $\exists R$ ]		$\frac{B(a) \rightarrow \exists x(A \supset B(x))}{\exists x B(x) \rightarrow \exists x(A \supset B(x))}$ [ $\forall R$ ]
$\frac{(A \supset B(a)), A \rightarrow B(a)}{(A \supset B(a)), A \rightarrow \exists x B(x)}$ [ $\exists R$ ]		$\frac{\neg A \rightarrow \exists x(A \supset B(x))}{(\neg A \vee \exists x B(x)) \rightarrow \exists x(A \supset B(x))}$ [ $\vee L$ ]		
$\frac{(A \supset B(a)), A \rightarrow \exists x B(x)}{A, (A \supset B(a)) \rightarrow \exists x B(x)}$ [XL]				
$\frac{A, (A \supset B(a)) \rightarrow \exists x B(x)}{(A \supset B(a)) \rightarrow (A \supset \exists x B(x))}$ [ $\supset R$ ]				
$\frac{(A \supset B(a)) \rightarrow (A \supset \exists x B(x))}{\exists x(A \supset B(x)) \rightarrow (A \supset \exists x B(x))}$ [ $\forall R$ ]				
$\frac{\exists x(A \supset B(x)) \rightarrow (A \supset \exists x B(x))}{\rightarrow (\exists x(A \supset B(x)) \supset (A \supset \exists x B(x)))}$ [ $\supset R$ ]				
$\frac{\rightarrow (\exists x(A \supset B(x)) \supset (A \supset \exists x B(x)))}{\rightarrow (\exists x(A \supset B(x)) \equiv (A \supset \exists x B(x)))}$ [ $\wedge R$ ]				

### 2.6.2)

$$\begin{array}{c}
\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B} [\text{WR}] \\
\frac{A(a) \rightarrow B, A(a)}{A(a) \rightarrow B, A(a)} [\text{XR}] \\
\frac{B \rightarrow B}{A(a) \rightarrow B, A(a)} [\supset\text{L}] \\
\frac{(A(a) \supset B), A(a) \rightarrow B, B}{(A(a) \supset B), A(a) \rightarrow B} [\text{CR}] \\
\frac{(A(a) \supset B), A(a) \rightarrow B}{A(a), (A(a) \supset B) \rightarrow B} [\text{XL}] \\
\frac{A(a), (A(a) \supset B) \rightarrow B}{\forall x A(x), (A(a) \supset B) \rightarrow B} [\forall\text{L}] \\
\frac{\forall x A(x), (A(a) \supset B) \rightarrow B}{(A(a) \supset B) \rightarrow (\forall x A(x) \supset B)} [\supset\text{R}] \\
\frac{(A(a) \supset B) \rightarrow (\forall x A(x) \supset B)}{\exists x (A(x) \supset B) \rightarrow (\forall x A(x) \supset B)} [\forall\text{R}] \\
\frac{\exists x (A(x) \supset B) \rightarrow (\forall x A(x) \supset B)}{\rightarrow (\exists x (A(x) \supset B) \supset (\forall x A(x) \supset B))} [\supset\text{R}] \\
\hline
\rightarrow (\exists x (A(x) \supset B) \equiv (\forall x A(x) \supset B))
\end{array}
\qquad
\begin{array}{c}
\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B} [\text{WR}] \\
\frac{A(a) \rightarrow A(a), B}{\rightarrow A(a), (A(a) \supset B)} [\supset\text{R}] \\
\frac{\rightarrow A(a), (A(a) \supset B)}{\rightarrow A(a), \exists x (A(x) \supset B)} [\exists\text{R}] \\
\frac{\rightarrow A(a), \exists x (A(x) \supset B)}{\rightarrow \exists x (A(x) \supset B), A(a)} [\text{XR}] \\
\frac{\rightarrow \exists x (A(x) \supset B), A(a)}{\rightarrow \exists x (A(x) \supset B), \forall x A(x)} [\forall\text{R}] \\
\frac{\rightarrow \exists x (A(x) \supset B), \forall x A(x)}{\neg \forall x A(x) \rightarrow \exists x (A(x) \supset B)} [\neg\text{L}] \\
\frac{\neg \forall x A(x) \rightarrow \exists x (A(x) \supset B)}{(\neg \forall x A(x) \vee B) \rightarrow \exists x (A(x) \supset B)} [\text{Cut}] \\
\frac{(\neg \forall x A(x) \vee B) \rightarrow \exists x (A(x) \supset B)}{\rightarrow ((\forall x A(x) \supset B) \supset \exists x (A(x) \supset B))} [\supset\text{R}] \\
\hline
\rightarrow ((\forall x A(x) \supset B) \supset \exists x (A(x) \supset B)) \\
\hline
\rightarrow (\exists x (A(x) \supset B) \equiv (\forall x A(x) \supset B))
\end{array}$$

**2.6.3)**

$\frac{\frac{\frac{A(a) \rightarrow A(a) \quad B(a) \rightarrow B(a)}{A(a), (A(a) \supset B(a)) \rightarrow B(a)} [\text{detachment}]}{\forall x A(x), (A(a) \supset B(a)) \rightarrow B(a)} [\forall L]}{\frac{\forall x A(x), (A(a) \supset B(a)) \rightarrow \exists x B(x)}{(A(a) \supset B(a)) \rightarrow (\forall x A(x) \supset \exists x B(x))} [\exists R]} \supset R$	$\frac{\frac{\forall x A(x) \rightarrow \forall x A(x) \quad \exists x B(x) \rightarrow \exists x B(x)}{(\forall x A(x) \supset \exists x B(x)) \rightarrow (\neg \forall x A(x) \vee \exists x B(x))} [\text{2.5.2.L}]}{\frac{(\forall x A(x) \supset \exists x B(x)) \rightarrow \exists x (A(x) \supset B(x))}{\rightarrow ((\forall x A(x) \supset \exists x B(x)) \supset \exists x (A(x) \supset B(x)))} [\supset R]} \wedge R$	$\frac{\frac{\frac{\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B(a)} [\text{WR}]}{\rightarrow A(a), (A(a) \supset B(a))} [\supset R]}{\rightarrow A(a), \exists x (A(x) \supset B(x))} [\exists R]} \supset R$ $\frac{\frac{\frac{\frac{B(a) \rightarrow B(a)}{A(a), B(a) \rightarrow B(a)} [\text{WL}]}{B(a) \rightarrow (A(a) \supset B(a))} [\supset R]}{B(a) \rightarrow \exists x (A(x) \supset B(x))} [\exists R]}{\frac{\exists x B(x) \rightarrow \exists x (A(x) \supset B(x))}{(\neg \forall x A(x) \vee \exists x B(x)) \rightarrow \exists x (A(x) \supset B(x))} [\forall R]} \vee R$ $\frac{\frac{\frac{\frac{\frac{\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B(a)} [\text{WR}]}{\rightarrow A(a), (A(a) \supset B(a))} [\supset R]}{\rightarrow A(a), \exists x (A(x) \supset B(x))} [\exists R]}{\rightarrow \exists x (A(x) \supset B(x)), A(a)} [\text{XR}]}{\rightarrow \exists x (A(x) \supset B(x)), \forall x A(x)} [\forall R]} \neg L$ $\frac{\frac{\frac{\frac{\frac{B(a) \rightarrow B(a)}{A(a), B(a) \rightarrow B(a)} [\text{WL}]}{B(a) \rightarrow (A(a) \supset B(a))} [\supset R]}{B(a) \rightarrow \exists x (A(x) \supset B(x))} [\exists R]}{\exists x B(x) \rightarrow \exists x (A(x) \supset B(x))} [\forall R]} \neg L$ $\frac{\frac{\frac{\frac{\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B(a)} [\text{WR}]}{\rightarrow A(a), (A(a) \supset B(a))} [\supset R]}{\rightarrow A(a), \exists x (A(x) \supset B(x))} [\exists R]}{\rightarrow \exists x (A(x) \supset B(x)), \forall x A(x)} [\forall R]}{\rightarrow \exists x (A(x) \supset B(x)), \exists x (A(x) \supset B(x))} [\text{Cut}]$
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### 2.6.4)

$$\begin{array}{c}
\frac{\frac{\neg A \rightarrow \neg A \quad B \rightarrow B}{(\neg A \supset B) \rightarrow (\neg \neg A \vee B)} [2.5.2.L] \quad \frac{\frac{\frac{\frac{A \rightarrow A}{\neg B, A \rightarrow A} [WL] \quad \frac{B \rightarrow B}{B \rightarrow B, A} [WR]}{A \rightarrow (\neg B \supset A)} [\supset R] \quad \frac{B \rightarrow A, B}{\neg B, B \rightarrow A} [XR]}{\rightarrow (\neg B \supset A), \neg A} [\neg R] \quad \frac{B \rightarrow (\neg B \supset A)}{B \rightarrow (\neg B \supset A)} [\supset R]} \\
\frac{\frac{\neg \neg A \rightarrow (\neg B \supset A)}{\neg \neg A \rightarrow (\neg B \supset A)} [\neg L] \quad \frac{B \rightarrow (\neg B \supset A)}{B \rightarrow (\neg B \supset A)} [\supset R]}{(\neg \neg A \vee B) \rightarrow (\neg B \supset A)} [Cut] \\
\frac{}{(\neg A \supset B) \rightarrow (\neg B \supset A)} [Cut]
\end{array}$$

**2.6.5)**

$$\begin{array}{c}
\frac{\neg A \rightarrow \neg A \quad \neg B \rightarrow \neg B}{(\neg A \supset \neg B) \rightarrow (\neg A \vee \neg B)} [2.5.2.L] \\
\\
\frac{}{\neg A \rightarrow \neg A} [\text{WL}] \quad \frac{}{B \rightarrow B} [\text{WR}] \\
\frac{}{B, A \rightarrow A} [\text{XR}] \\
\frac{}{A \rightarrow (B \supset A)} [\supset R] \quad \frac{}{B \rightarrow A, B} [\text{XL}] \\
\frac{}{A \rightarrow (B \supset A), \neg A} [\neg R] \quad \frac{}{\neg B, B \rightarrow A} [\neg L] \\
\frac{}{\rightarrow (B \supset A), \neg A} [\neg L] \quad \frac{}{B, \neg B \rightarrow A} [\supset R] \\
\frac{}{\neg \neg A \rightarrow (B \supset A)} [\neg L] \quad \frac{}{\neg B \rightarrow (B \supset A)} [\supset R] \\
\frac{}{(\neg \neg A \vee \neg B) \rightarrow (B \supset A)} [\vee L] \\
\\
\frac{}{(\neg A \supset \neg B) \rightarrow (B \supset A)} [\text{Cut}]
\end{array}$$

2.7)

$$\begin{array}{c}
\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow A(a), B} [\text{WR}] \\
\frac{}{\rightarrow A(a), (A(a) \supset B)} [\supset\text{R}] \\
\frac{}{\rightarrow A(a), \exists x(A(x) \supset B)} [\exists\text{R}] \\
\frac{}{\rightarrow \exists x(A(x) \supset B), A(a)} [\text{XR}] \\
\frac{}{\rightarrow \exists x(A(x) \supset B), \forall x A(x)} [\forall\text{R}] \\
\hline
\frac{(\forall x A(x) \supset B) \rightarrow \exists x(A(x) \supset B), \exists x(A(x) \supset B)}{(\forall x A(x) \supset B) \rightarrow \exists x(A(x) \supset B)} [\text{CR}]
\end{array}
\qquad
\begin{array}{c}
\frac{B \rightarrow B}{A(a), B \rightarrow B} [\text{WL}] \\
\frac{}{B \rightarrow (A(a) \supset B)} [\supset\text{R}] \\
\frac{}{B \rightarrow \exists x(A(x) \supset B)} [\exists\text{R}] \\
\hline
\frac{}{B \rightarrow \exists x(A(x) \supset B)} [\supset\text{L}]
\end{array}$$

Detachment Rule)

$$\begin{array}{c}
 \frac{A \rightarrow A}{A \rightarrow A, B} [\text{WR}] \\
 \frac{A \rightarrow A, B}{A \rightarrow B, A} [\text{XR}] \\
 \frac{A \rightarrow B, A \quad B \rightarrow B}{(A \supset B), A \rightarrow B, B} [\supset\text{L}] \\
 \frac{(A \supset B), A \rightarrow B, B}{(A \supset B), A \rightarrow B} [\text{CR}] \\
 \frac{(A \supset B), A \rightarrow B}{A, (A \supset B) \rightarrow B} [\text{XL}]
 \end{array}$$

Double Negation)

$$\begin{array}{c}
 \frac{A \rightarrow A}{\neg A, A \rightarrow} [\neg\text{L}] \quad \frac{A \rightarrow A}{\rightarrow A, \neg A} [\neg\text{R}] \\
 \frac{\neg A, A \rightarrow}{A \rightarrow \neg\neg A} [\neg\text{R}] \quad \frac{\rightarrow A, \neg A}{\neg\neg A \rightarrow A} [\neg\text{L}] \\
 \frac{A \rightarrow \neg\neg A}{\rightarrow (A \supset \neg\neg A)} [\supset\text{R}] \quad \frac{\neg\neg A \rightarrow A}{\rightarrow (\neg\neg A \supset A)} [\supset\text{R}] \\
 \hline
 \rightarrow (A \equiv \neg\neg A) [\wedge\text{R}]
 \end{array}$$