2.5.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), A \to}\,[\wedge\mathrm{L}]}{A \to \neg(\neg A \wedge \neg B)}\,[\neg\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{B \to B}{\neg B, B \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), B \to}\,[\wedge\mathrm{L}]}{B \to \neg(\neg A \wedge \neg B)}\,[\neg\mathrm{R}]}{(A \vee B) \to \neg(\neg A \wedge \neg B)}\,[\vee\mathrm{L}]}{\to ((A \vee B) \supset \neg(\neg A \wedge \neg B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to (A \vee B)}\,[\vee\mathrm{R}]}{\to (A \vee B), \neg A}\,[\neg\mathrm{R}] \quad \dfrac{\dfrac{B \to B}{B \to (A \vee B)}\,[\vee\mathrm{R}]}{\to (A \vee B), \neg B}\,[\neg\mathrm{R}]}{\to (A \vee B), (\neg A \wedge \neg B)}\,[\wedge\mathrm{R}]}{\neg(\neg A \wedge \neg B) \to (A \vee B)}\,[\neg\mathrm{L}]}{\to (\neg(\neg A \wedge \neg B) \supset (A \vee B))}\,[\supset\mathrm{R}]}{\to ((A \vee B) \equiv \neg(\neg A \wedge \neg B))}\,[\wedge\mathrm{R}]$$

2.5.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\mathrm{R}]}{\to A, (\neg A \vee B)}\,[\vee\mathrm{R}]}{\to (\neg A \vee B), A}\,[\mathrm{XR}] \quad \dfrac{B \to B}{B \to (\neg A \vee B)}\,[\vee\mathrm{R}]}{(A \supset B) \to (\neg A \vee B), (\neg A \vee B)}\,[\supset\mathrm{L}]}{(A \supset B) \to (\neg A \vee B)}\,[\mathrm{CR}]}{\to ((A \supset B) \supset (\neg A \vee B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{\neg A, A \to B}\,[\mathrm{WR}]}{A, \neg A \to B}\,[\mathrm{XL}]}{\neg A \to (A \supset B)}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{B \to B}{A, B \to B}\,[\mathrm{WL}]}{B \to (A \supset B)}\,[\supset\mathrm{R}]}{(\neg A \vee B) \to (A \supset B)}\,[\vee\mathrm{L}]}{\to ((\neg A \vee B) \supset (A \supset B))}\,[\supset\mathrm{R}]}{\to ((A \supset B) \equiv (\neg A \vee B))}\,[\wedge\mathrm{R}]$$

2.5.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\neg F(a), F(a) \to}\,[\neg\mathrm{L}]}{\forall y \neg F(y), F(a) \to}\,[\forall\mathrm{L}]}{F(a) \to \neg\forall y \neg F(y)}\,[\neg\mathrm{R}]}{\exists x F(x) \to \neg\forall y \neg F(y)}\,[\exists\mathrm{L}]}{\to (\exists x F(x) \supset \neg\forall y \neg F(y))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\to F(a), \neg F(a)}\,[\neg\mathrm{R}]}{\to \neg F(a), F(a)}\,[\mathrm{XR}]}{\to \neg F(a), \exists x F(x)}\,[\exists\mathrm{R}]}{\to \exists x F(x), \neg F(a)}\,[\mathrm{XR}]}{\to \exists x F(x), \forall y \neg F(y)}\,[\forall\mathrm{R}]}{\neg\forall y \neg F(y) \to \exists x F(x)}\,[\neg\mathrm{L}]}{\to (\neg\forall y \neg F(y) \supset \exists x F(x))}\,[\supset\mathrm{R}]}{\to (\exists x F(x) \equiv \neg\forall y \neg F(y))}\,[\wedge\mathrm{R}]$$

2.5.4)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\to F(a), \neg F(a)}\,[\neg\mathrm{R}]}{\to F(a), \exists x \neg F(x)}\,[\exists\mathrm{R}]}{\to \exists x \neg F(x), F(a)}\,[\mathrm{XR}]}{\to \exists x \neg F(x), \forall y F(y)}\,[\forall\mathrm{R}]}{\neg\forall y F(y) \to \exists x \neg F(x)}\,[\neg\mathrm{L}]}{\to (\neg\forall y F(y) \supset \exists x \neg F(x))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\forall y F(y) \to F(a)}\,[\forall\mathrm{L}]}{\neg F(a), \forall y F(y) \to}\,[\neg\mathrm{L}]}{\forall y F(y), \neg F(a) \to}\,[\mathrm{XL}]}{\neg F(a) \to \neg\forall y F(y)}\,[\neg\mathrm{R}]}{\exists x \neg F(x) \to \neg\forall y F(y)}\,[\exists\mathrm{L}]}{\to (\exists x \neg F(x) \supset \neg\forall y F(y))}\,[\supset\mathrm{R}]}{\to (\neg\forall y F(y) \equiv \exists x \neg F(x))}\,[\wedge\mathrm{R}]$$

2.5.5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\mathrm{R}]}{\to A, (\neg A \vee \neg B)}\,[\vee\mathrm{R}]}{\to (\neg A \vee \neg B), A}\,[\mathrm{XR}] \quad \dfrac{\dfrac{\dfrac{B \to B}{\to B, \neg B}\,[\neg\mathrm{R}]}{\to B, (\neg A \vee \neg B)}\,[\vee\mathrm{R}]}{\to (\neg A \vee \neg B), B}\,[\mathrm{XR}]}{\to (\neg A \vee \neg B), (A \wedge B)}\,[\wedge\mathrm{R}]}{\to \neg(A \wedge B) \to (\neg A \vee \neg B)}\,[\neg\mathrm{L}]}{\to (\neg(A \wedge B) \supset (\neg A \vee \neg B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\mathrm{L}]}{\to A, \neg(A \wedge B)}\,[\neg\mathrm{R}]}{\to \neg(A \wedge B), A}\,[\mathrm{XR}]}{\neg A \to \neg(A \wedge B)}\,[\neg\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{B \to B}{(A \wedge B) \to B}\,[\wedge\mathrm{L}]}{\to B, \neg(A \wedge B)}\,[\neg\mathrm{R}]}{\to \neg(A \wedge B), B}\,[\mathrm{XR}]}{\neg B \to \neg(A \wedge B)}\,[\neg\mathrm{L}]}{(\neg A \vee \neg B) \to \neg(A \wedge B)}\,[\vee\mathrm{L}]}{\to ((\neg A \vee \neg B) \supset \neg(A \wedge B))}\,[\supset\mathrm{R}]}{\to (\neg(A \wedge B) \equiv (\neg A \vee \neg B))}\,[\wedge\mathrm{R}]$$

2.6.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A, B(a)}\,[\mathrm{WR}]}{A \to B(a), A}\,[\mathrm{XR}] \qquad B(a) \to B(a)}{(A \supset B(a)), A \to B(a), B(a)}\,[\supset\mathrm{L}]}{(A \supset B(a)), A \to B(a)}\,[\mathrm{CR}]}{(A \supset B(a)), A \to \exists x B(x)}\,[\exists\mathrm{R}]}{A, (A \supset B(a)) \to \exists x B(x)}\,[\mathrm{XL}]}{(A \supset B(a)) \to (A \supset \exists x B(x))}\,[\supset\mathrm{R}]}{\exists x(A \supset B(x)) \to (A \supset \exists x B(x))}\,[\exists\mathrm{L}]}{\to (\exists x(A \supset B(x)) \supset (A \supset \exists x B(x)))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{A \to A \qquad \exists x B(x) \to \exists x B(x)}{(A \supset \exists x B(x)) \to (\neg A \vee \exists x B(x))}\,[2.5.2.\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{\neg A, A \to B(a)}\,[\mathrm{WR}]}{A, \neg A \to B(a)}\,[\mathrm{XL}]}{\neg A \to (A \supset B(a))}\,[\supset\mathrm{R}]}{\neg A \to \exists x(A \supset B(x))}\,[\exists\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{B(a) \to B(a)}{A, B(a) \to B(a)}\,[\mathrm{WL}]}{B(a) \to (A \supset B(a))}\,[\supset\mathrm{R}]}{B(a) \to \exists x(A \supset B(x))}\,[\exists\mathrm{R}]}{\exists x B(x) \to \exists x(A \supset B(x))}\,[\exists\mathrm{L}]}{(\neg A \vee \exists x B(x)) \to \exists x(A \supset B(x))}\,[\vee\mathrm{L}]}{(A \supset \exists x B(x)) \to \exists x(A \supset B(x))}\,[\mathrm{Cut}]}{\to ((A \supset \exists x B(x)) \supset \exists x(A \supset B(x)))}\,[\supset\mathrm{R}]}{\to (\exists x(A \supset B(x)) \equiv (A \supset \exists x B(x)))}\,[\wedge\mathrm{R}]$$

2.6.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to A(a), B}\,[\mathrm{WR}]}{A(a) \to B, A(a)}\,[\mathrm{XR}] \qquad B \to B}{(A(a) \supset B), A(a) \to B, B}\,[\supset\mathrm{L}]}{(A(a) \supset B), A(a) \to B}\,[\mathrm{CR}]}{A(a), (A(a) \supset B) \to B}\,[\mathrm{XL}]}{\forall x A(x), (A(a) \supset B) \to B}\,[\forall\mathrm{L}]}{(A(a) \supset B) \to (\forall x A(x) \supset B)}\,[\supset\mathrm{R}]}{\exists x(A(x) \supset B) \to (\forall x A(x) \supset B)}\,[\exists\mathrm{L}]}{\to (\exists x(A(x) \supset B) \supset (\forall x A(x) \supset B))}\,[\supset\mathrm{R}] \quad \dfrac{\dfrac{\dfrac{\forall x A(x) \to \forall x A(x) \qquad B \to B}{(\forall x A(x) \supset B) \to (\neg \forall x A(x) \vee B)}\,[2.5.2.\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to A(a), B}\,[\mathrm{WR}]}{\to A(a), (A(a) \supset B)}\,[\supset\mathrm{R}]}{\to A(a), \exists x(A(x) \supset B)}\,[\exists\mathrm{R}]}{\to \exists x(A(x) \supset B), A(a)}\,[\mathrm{XR}]}{\to \exists x(A(x) \supset B), \forall x A(x)}\,[\forall\mathrm{R}]}{\neg \forall x A(x) \to \exists x(A(x) \supset B)}\,[\neg\mathrm{L}] \quad \dfrac{\dfrac{\dfrac{B \to B}{A(a), B \to B}\,[\mathrm{WL}]}{B \to (A(a) \supset B)}\,[\supset\mathrm{R}]}{B \to \exists x(A(x) \supset B)}\,[\exists\mathrm{R}]}{(\neg \forall x A(x) \vee B) \to \exists x(A(x) \supset B)}\,[\vee\mathrm{L}]}{(\forall x A(x) \supset B) \to \exists x(A(x) \supset B)}\,[\mathrm{Cut}]}{\to ((\forall x A(x) \supset B) \supset \exists x(A(x) \supset B))}\,[\supset\mathrm{R}]}{\to (\exists x(A(x) \supset B) \equiv (\forall x A(x) \supset B))}\,[\wedge\mathrm{R}]$$

2.6.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\to A(a)\qquad B(a)\to B(a)}{A(a),(A(a)\supset B(a))\to B(a)}\,[\text{detachment}]}{\forall xA(x),(A(a)\supset B(a))\to B(a)}\,[\forall\text{L}]}{\forall xA(x),(A(a)\supset B(a))\to \exists xB(x)}\,[\exists\text{R}]}{(A(a)\supset B(a))\to(\forall xA(x)\supset\exists xB(x))}\,[\supset\text{R}]}{\exists x(A(x)\supset B(x))\to(\forall xA(x)\supset\exists xB(x))}\,[\exists\text{L}]}{\to(\exists x(A(x)\supset B(x))\supset(\forall xA(x)\supset\exists xB(x)))}\,[\supset\text{R}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\to A(a)}{A(a)\to A(a),B(a)}\,[\text{WR}]}{\to A(a),(A(a)\supset B(a))}\,[\supset\text{R}]}{\to A(a),\exists x(A(x)\supset B(x))}\,[\exists\text{R}]}{\to\exists x(A(x)\supset B(x)),A(a)}\,[\text{XR}]}{\to\exists x(A(x)\supset B(x)),\forall xA(x)}\,[\forall\text{R}]}{\neg\forall xA(x)\to\exists x(A(x)\supset B(x))}\,[\neg\text{L}]\qquad \dfrac{\dfrac{\dfrac{B(a)\to B(a)}{A(a),B(a)\to B(a)}\,[\text{WL}]}{B(a)\to(A(a)\supset B(a))}\,[\supset\text{R}]}{B(a)\to\exists x(A(x)\supset B(x))}\,[\exists\text{R}]}{\exists xB(x)\to\exists x(A(x)\supset B(x))}\,[\exists\text{L}]$$

$$\dfrac{\dfrac{\dfrac{\forall xA(x)\to\forall xA(x)\qquad\exists xB(x)\to\exists xB(x)}{(\forall xA(x)\supset\exists xB(x))\to(\neg\forall xA(x)\vee\exists xB(x))}\,[2.5.2.\text{L}]\qquad(\neg\forall xA(x)\vee\exists xB(x))\to\exists x(A(x)\supset B(x))}{(\forall xA(x)\supset\exists xB(x))\to\exists x(A(x)\supset B(x))}\,[\text{Cut}]}{\to((\forall xA(x)\supset\exists xB(x))\supset\exists x(A(x)\supset B(x)))}\,[\supset\text{R}]$$

$$\to(\exists x(A(x)\supset B(x))\equiv(\forall xA(x)\supset\exists xB(x)))\,[\wedge\text{R}]$$

2.6.4)

$$\dfrac{\dfrac{\dfrac{\neg A\to\neg A\qquad B\to B}{(\neg A\supset B)\to(\neg\neg A\vee B)}\,[2.5.2.\text{L}]\qquad\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg B,A\to A}\,[\text{WL}]}{A\to(\neg B\supset A)}\,[\supset\text{R}]}{\to(\neg B\supset A),\neg A}\,[\neg\text{R}]}{\neg\neg A\to(\neg B\supset A)}\,[\neg\text{L}]\qquad\dfrac{\dfrac{\dfrac{B\to B}{B\to B,A}\,[\text{WR}]}{B\to A,B}\,[\text{XR}]}{\neg B,B\to A}\,[\neg\text{L}]}{B\to(\neg B\supset A)}\,[\supset\text{R}]}{(\neg\neg A\vee B)\to(\neg B\supset A)}\,[\vee\text{L}]}{(\neg A\supset B)\to(\neg B\supset A)}\,[\text{Cut}]$$

2.6.5)

$$\dfrac{\dfrac{\dfrac{\neg A\to\neg A\qquad\neg B\to\neg B}{(\neg A\supset\neg B)\to(\neg\neg A\vee\neg B)}\,[2.5.2.\text{L}]\qquad\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{B,A\to A}\,[\text{WL}]}{A\to(B\supset A)}\,[\supset\text{R}]}{\to(B\supset A),\neg A}\,[\neg\text{R}]}{\neg\neg A\to(B\supset A)}\,[\neg\text{L}]\qquad\dfrac{\dfrac{\dfrac{\dfrac{B\to B}{B\to B,A}\,[\text{WR}]}{B\to A,B}\,[\text{XR}]}{\neg B,B\to A}\,[\neg\text{L}]}{B,\neg B\to A}\,[\text{XL}]}{\neg B\to(B\supset A)}\,[\supset\text{R}]}{(\neg\neg A\vee\neg B)\to(B\supset A)}\,[\vee\text{L}]}{(\neg A\supset\neg B)\to(B\supset A)}\,[\text{Cut}]$$

2.7)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\to A(a)}{A(a)\to A(a),B}\,[\text{WR}]}{\to A(a),(A(a)\supset B)}\,[\supset\text{R}]}{\to A(a),\exists x(A(x)\supset B)}\,[\exists\text{R}]}{\to\exists x(A(x)\supset B),A(a)}\,[\text{XR}]}{\to\exists x(A(x)\supset B),\forall xA(x)}\,[\forall\text{R}]\qquad\dfrac{\dfrac{\dfrac{B\to B}{A(a),B\to B}\,[\text{WL}]}{B\to(A(a)\supset B)}\,[\supset\text{R}]}{B\to\exists x(A(x)\supset B)}\,[\exists\text{R}]}{(\forall xA(x)\supset B)\to\exists x(A(x)\supset B),\exists x(A(x)\supset B)}\,[\supset\text{L}]$$

$$\dfrac{(\forall xA(x)\supset B)\to\exists x(A(x)\supset B),\exists x(A(x)\supset B)}{(\forall xA(x)\supset B)\to\exists x(A(x)\supset B)}\,[\text{CR}]$$

3.9.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{\neg A, A \to B}\,[\mathrm{WR}]}{A, \neg A \to B}\,[\mathrm{XL}]}{\neg A \to (A \supset B)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{B \to B}{A, B \to B}\,[\mathrm{WL}]}{B \to (A \supset B)}\,[\supset\mathrm{R}]}{(\neg A \vee B) \to (A \supset B)}\,[\vee\mathrm{L}]$$

3.9.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\neg F(a), F(a) \to}\,[\neg\mathrm{L}]}{\forall y \neg F(y), F(a) \to}\,[\forall\mathrm{L}]}{F(a) \to \neg\forall y \neg F(y)}\,[\neg\mathrm{R}]}{\exists x F(x) \to \neg\forall y \neg F(y)}\,[\exists\mathrm{L}]$$

3.9.3)

$$\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\mathrm{L}]$$

3.9.4)

$$\dfrac{A \to A}{A \to (A \vee B)}\,[\vee\mathrm{R}]$$

3.9.5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\mathrm{L}]}{\neg A, (A \wedge B) \to}\,[\neg\mathrm{L}]}{(A \wedge B), \neg A \to}\,[\mathrm{XL}]}{\neg A \to \neg(A \wedge B)}\,[\neg\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{(A \wedge B) \to B}\,[\wedge\mathrm{L}]}{\neg B, (A \wedge B) \to}\,[\neg\mathrm{L}]}{(A \wedge B), \neg B \to}\,[\mathrm{XL}]}{\neg B \to \neg(A \wedge B)}\,[\neg\mathrm{R}]}{(\neg A \vee \neg B) \to \neg(A \wedge B)}\,[\vee\mathrm{L}]$$

3.9.6)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to (A \vee B)}\,[\vee\mathrm{R}]}{\neg(A \vee B), A \to}\,[\neg\mathrm{L}]}{A, \neg(A \vee B) \to}\,[\mathrm{XL}]}{\neg(A \vee B) \to \neg A}\,[\neg\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B \to (A \vee B)}\,[\vee\mathrm{R}]}{\neg(A \vee B), B \to}\,[\neg\mathrm{L}]}{B, \neg(A \vee B) \to}\,[\mathrm{XL}]}{\neg(A \vee B) \to \neg B}\,[\neg\mathrm{R}]}{\neg(A \vee B) \to (\neg A \wedge \neg B)}\,[\wedge\mathrm{R}]}{\to (\neg(A \vee B) \supset (\neg A \wedge \neg B))}\,[\supset\mathrm{R}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), A \to}\,[\wedge\mathrm{L}]}{A, (\neg A \wedge \neg B) \to}\,[\mathrm{XL}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\neg B, B \to}\,[\neg\mathrm{L}]}{(\neg A \wedge \neg B), B \to}\,[\wedge\mathrm{L}]}{B, (\neg A \wedge \neg B) \to}\,[\mathrm{XL}]}{(A \vee B), (\neg A \wedge \neg B) \to}\,[\vee\mathrm{L}]}{(\neg A \wedge \neg B) \to \neg(A \vee B)}\,[\neg\mathrm{R}]}{\to ((\neg A \wedge \neg B) \supset \neg(A \vee B))}\,[\supset\mathrm{R}]$$

$$\dfrac{\to (\neg(A \vee B) \supset (\neg A \wedge \neg B)) \qquad \to ((\neg A \wedge \neg B) \supset \neg(A \vee B))}{\to (\neg(A \vee B) \equiv (\neg A \wedge \neg B))}\,[\wedge\mathrm{R}]$$

3.9.7)

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\dfrac{\dfrac{\dfrac{A \to A}{B, A \to A}\,[\text{WL}]}{A, B \to A}\,[\text{XL}] \quad \dfrac{B \to B}{A, B \to B}\,[\text{WL}]}{A, B \to (A \wedge B)}\,[\wedge\text{R}]
}{A, B \to ((A \wedge B) \vee C)}\,[\vee\text{R}]
\quad
\dfrac{\dfrac{\dfrac{C \to C}{B, C \to C}\,[\text{WL}]}{C, B \to C}\,[\text{XL}]}{C, B \to ((A \wedge B) \vee C)}\,[\vee\text{R}]
}{(A \vee C), B \to ((A \wedge B) \vee C)}\,[\vee\text{L}]
}{B, (A \vee C) \to ((A \wedge B) \vee C)}\,[\text{XL}]
\quad
\dfrac{\dfrac{\dfrac{C \to C}{C \to ((A \wedge B) \vee C)}\,[\vee\text{R}]}{(A \vee C), C \to ((A \wedge B) \vee C)}\,[\text{WL}]}{C, (A \vee C) \to ((A \wedge B) \vee C)}\,[\text{XL}]
}{(B \vee C), (A \vee C) \to ((A \wedge B) \vee C)}\,[\vee\text{L}]
}{(A \vee C), (B \vee C) \to ((A \wedge B) \vee C)}\,[\text{XL}]
}{((A \vee C) \wedge (B \vee C)), (B \vee C) \to ((A \wedge B) \vee C)}\,[\wedge\text{L}]
}{(B \vee C), ((A \vee C) \wedge (B \vee C)) \to ((A \wedge B) \vee C)}\,[\text{XL}]
}{((A \vee C) \wedge (B \vee C)), ((A \vee C) \wedge (B \vee C)) \to ((A \wedge B) \vee C)}\,[\wedge\text{L}]
}{((A \vee C) \wedge (B \vee C)) \to ((A \wedge B) \vee C)}\,[\text{CL}]
}{\to (((A \vee C) \wedge (B \vee C)) \supset ((A \wedge B) \vee C))}\,[\supset\text{R}]
\quad
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\dfrac{\dfrac{\dfrac{A \to A}{(A \wedge B) \to A}\,[\wedge\text{L}]}{(A \wedge B) \to (A \vee C)}\,[\vee\text{R}] \quad \dfrac{C \to C}{C \to (A \vee C)}\,[\vee\text{R}]}{((A \wedge B) \vee C) \to (A \vee C)}\,[\vee\text{L}]
\quad
\dfrac{\dfrac{B \to B}{(A \wedge B) \to B}\,[\wedge\text{L}]}{(A \wedge B) \to (B \vee C)}\,[\vee \ldots \quad \cdots \quad ((A \wedge B) \vee C) \ldots
}{((A \wedge B) \vee C) \to ((A \vee C) \wedge (B \vee C))}
}{\to (((A \wedge B) \vee C) \supset ((A \vee C) \wedge (B \vee C)))}\,[\supset\text{R}]
}{\to (((A \vee C) \wedge (B \vee C)) \equiv ((A \wedge B) \vee C))}\,[\wedge\text{R}]
$$

3.9.8)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{\neg F(a), F(a) \to}\,[\neg\text{L}]}{F(a), \neg F(a) \to}\,[\text{XL}]}{\forall x F(x), \neg F(a) \to}\,[\forall\text{L}]}{\neg F(a) \to \neg \forall x F(x)}\,[\neg\text{R}]}{\exists x \neg F(x) \to \neg \forall x F(x)}\,[\exists\text{L}]
$$

3.9.9)

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\dfrac{\dfrac{\dfrac{F(a) \to F(a)}{(F(a) \wedge G(a)) \to F(a)}\,[\wedge\text{L}]}{\forall x(F(x) \wedge G(x)) \to F(a)}\,[\forall\text{L}]}{\forall x(F(x) \wedge G(x)) \to \forall x F(x)}\,[\forall\text{R}]
\quad
\dfrac{\dfrac{\dfrac{G(a) \to G(a)}{(F(a) \wedge G(a)) \to G(a)}\,[\wedge\text{L}]}{\forall x(F(x) \wedge G(x)) \to G(a)}\,[\forall\text{L}]}{\forall x(F(x) \wedge G(x)) \to \forall x G(x)}\,[\forall\text{R}]
}{\forall x(F(x) \wedge G(x)) \to (\forall x F(x) \wedge \forall x G(x))}\,[\wedge\text{R}]
}{\to (\forall x(F(x) \wedge G(x)) \supset (\forall x F(x) \wedge \forall x G(x)))}\,[\supset\text{R}]
\quad
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\dfrac{\dfrac{F(a) \to F(a)}{\forall x F(x) \to F(a)}\,[\forall\text{L}]}{(\forall x F(x) \wedge \forall x G(x)) \to F(a)}\,[\wedge\text{L}]
\quad
\dfrac{\dfrac{G(a) \to G(a)}{\forall x G(x) \to G(a)}\,[\forall\text{L}]}{(\forall x F(x) \wedge \forall x G(x)) \to G(a)}\,[\wedge\text{L}]
}{(\forall x F(x) \wedge \forall x G(x)) \to (F(a) \wedge G(a))}\,[\wedge\text{R}]
}{(\forall x F(x) \wedge \forall x G(x)) \to \forall x(F(x) \wedge G(x))}\,[\forall\text{R}]
}{\to ((\forall x F(x) \wedge \forall x G(x)) \supset \forall x(F(x) \wedge G(x)))}\,[\supset\text{R}]
}{\to (\forall x(F(x) \wedge G(x)) \equiv (\forall x F(x) \wedge \forall x G(x)))}\,[\wedge\text{R}]
$$

3.9.10)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \quad \dfrac{B \to B}{\neg B, B \to}\,[\neg\text{L}]}{(A \supset \neg B), A, B \to}\,[\supset\text{L}]}{A, (A \supset \neg B), B \to}\,[\text{XL}]}{A, B, (A \supset \neg B) \to}\,[\text{XL}]}{B, (A \supset \neg B) \to \neg A}\,[\neg\text{R}]}{(A \supset \neg B) \to (B \supset \neg A)}\,[\supset\text{R}]
$$

3.9.11)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B(a) \to B(a)}{}\,[\text{detachment}]}{A, (A \supset B(a)) \to B(a)}\,[\exists\text{R}]}{A, (A \supset B(a)) \to \exists x B(x)}\,[\supset\text{R}]}{(A \supset B(a)) \to (A \supset \exists x B(x))}\,[\exists\text{L}]}{\exists x(A \supset B(x)) \to (A \supset \exists x B(x))}$$

3.9.12)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a) \qquad B \to B}{}\,[\text{detachment}]}{A(a), (A(a) \supset B) \to B}\,[\forall\text{L}]}{\forall x A(x), (A(a) \supset B) \to B}\,[\supset\text{R}]}{(A(a) \supset B) \to (\forall x A(x) \supset B)}\,[\exists\text{L}]}{\exists x(A(x) \supset B) \to (\forall x A(x) \supset B)}$$

3.9.13)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a) \qquad B(a) \to B(a)}{}\,[\text{detachment}]}{A(a), (A(a) \supset B(a)) \to B(a)}\,[\exists\text{R}]}{A(a), (A(a) \supset B(a)) \to \exists x B(x)}\,[\forall\text{L}]}{\forall x A(x), (A(a) \supset B(a)) \to \exists x B(x)}\,[\supset\text{R}]}{(A(a) \supset B(a)) \to (\forall x A(x) \supset \exists x B(x))}\,[\exists\text{L}]}{\exists x(A(x) \supset B(x)) \to (\forall x A(x) \supset \exists x B(x))}$$

3.10.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{}\,[\text{detachment}]}{A, (A \supset B) \to B}\,[\text{XL}]}{(A \supset B), A \to B}\,[\neg\text{L}]}{\neg B, (A \supset B), A \to}\,[\text{XL}]}{(A \supset B), \neg B, A \to}\,[\neg\text{R}]}{\neg B, A \to \neg(A \supset B)}\,[\neg\text{L}]}{\neg\neg(A \supset B), \neg B, A \to}\,[\text{XL}]}{\neg B, \neg\neg(A \supset B), A \to}\,[\neg\text{R}]}{\neg\neg(A \supset B), A \to \neg\neg B}$$

3.10.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{}\,[3.10.1]}{\neg\neg(A \supset B), A \to \neg\neg B} \qquad B \to B}{}\,[\supset\text{L}]}{(\neg\neg B \supset B), \neg\neg(A \supset B), A \to B}\,[\text{XL}]}{(\neg\neg B \supset B), A, \neg\neg(A \supset B) \to B}\,[\text{XL}]}{A, (\neg\neg B \supset B), \neg\neg(A \supset B) \to B}\,[\supset\text{R}]}{(\neg\neg B \supset B), \neg\neg(A \supset B) \to (A \supset B)}$$

3.10.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{A \to \neg\neg A}\,[\neg\mathrm{R}]}{\neg\neg\neg A, A \to}\,[\neg\mathrm{L}]}{A, \neg\neg\neg A \to}\,[\mathrm{XL}]}{\neg\neg\neg A \to \neg A}\,[\neg\mathrm{R}]}{\to (\neg\neg\neg A \supset \neg A)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{A, \neg A \to}\,[\mathrm{XL}]}{\neg A \to \neg A}\,[\neg\mathrm{R}]}{\neg\neg A, \neg A \to}\,[\neg\mathrm{L}]}{\neg A \to \neg\neg\neg A}\,[\neg\mathrm{R}]}{\to (\neg A \supset \neg\neg\neg A)}\,[\supset\mathrm{R}]}{\to (\neg\neg\neg A \equiv \neg A)}\,[\wedge\mathrm{R}]$$

3.11.∧)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{(B \wedge C) \to B}\,[\wedge\mathrm{L}]}{\neg B, (B \wedge C) \to}\,[\neg\mathrm{L}]}{(B \wedge C), \neg B \to}\,[\mathrm{XL}]}{\neg B \to \neg(B \wedge C)}\,[\neg\mathrm{R}]}{\neg\neg(B \wedge C), \neg B \to}\,[\neg\mathrm{L}]}{\neg B, \neg\neg(B \wedge C) \to}\,[\mathrm{XL}]}{\neg\neg(B \wedge C) \to \neg\neg B}\,[\neg\mathrm{R}] \qquad [\neg\neg B \to B]}{\neg\neg(B \wedge C) \to B}\,[\mathrm{Cut}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to C}{(B \wedge C) \to C}\,[\wedge\mathrm{L}]}{\neg C, (B \wedge C) \to}\,[\neg\mathrm{L}]}{(B \wedge C), \neg C \to}\,[\mathrm{XL}]}{\neg C \to \neg(B \wedge C)}\,[\neg\mathrm{R}]}{\neg\neg(B \wedge C), \neg C \to}\,[\neg\mathrm{L}]}{\neg C, \neg\neg(B \wedge C) \to}\,[\mathrm{XL}]}{\neg\neg(B \wedge C) \to \neg\neg C}\,[\neg\mathrm{R}] \qquad [\neg\neg C \to C]}{\neg\neg(B \wedge C) \to C}\,[\mathrm{Cut}]}{\neg\neg(B \wedge C) \to (B \wedge C)}\,[\wedge\mathrm{R}]$$

3.11.⊂)

$$\dfrac{\dfrac{[\neg\neg C \to C]}{\to (\neg\neg C \supset C)}\,[\supset\mathrm{R}] \qquad \dfrac{B \to B \qquad C \to C}{(\neg\neg C \supset C), \neg\neg(B \supset C) \to (B \supset C)}\,[3.10.2]}{\neg\neg(B \supset C) \to (B \supset C)}\,[\mathrm{Cut}]$$

3.11.∀)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B(a) \to B(a)}{\forall x B(x) \to B(a)}\,[\forall\mathrm{L}]}{\neg B(a), \forall x B(x) \to}\,[\neg\mathrm{L}]}{\forall x B(x), \neg B(a) \to}\,[\mathrm{XL}]}{\neg B(a) \to \neg\forall x B(x)}\,[\neg\mathrm{R}]}{\neg\neg\forall x B(x), \neg B(a) \to}\,[\neg\mathrm{L}]}{\neg B(a), \neg\neg\forall x B(x) \to}\,[\mathrm{XL}]}{\neg\neg\forall x B(x) \to \neg\neg B(a)}\,[\neg\mathrm{R}] \qquad [\neg\neg B(a) \to B(a)]}{\neg\neg\forall x B(x) \to B(a)}\,[\mathrm{Cut}]}{\neg\neg\forall x B(x) \to \forall x B(x)}\,[\forall\mathrm{R}]$$

Detachment Rule)

$$\dfrac{\dfrac{A \to A \qquad B \to B}{(A \supset B), A \to B}\,[\supset\mathrm{L}]}{A, (A \supset B) \to B}\,[\mathrm{XL}]$$

Double Negation)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\,[\neg\mathrm{L}]}{A \to \neg\neg A}\,[\neg\mathrm{R}]}{\to (A \supset \neg\neg A)}\,[\supset\mathrm{R}] \qquad \dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,[\neg\mathrm{R}]}{\neg\neg A \to A}\,[\neg\mathrm{L}]}{\to (\neg\neg A \supset A)}\,[\supset\mathrm{R}]}{\to (A \equiv \neg\neg A)}\,[\wedge\mathrm{R}]$$