Mathematical model

A modified version of the lifting line theory is used to model the wing-propeller system. The effect of the propeller is taken into account adding to the Prandtl equation the velocity that it induces on the wing. This velocity consists of two components: the axial velocity and the circumferential velocity.
The axial velocity is the velocity induced in the x direction and it is computed with the disk actuator theory:

$$\Delta v = \frac{1}{2}\left[-v_0 + \sqrt{{v_0}^2 + \frac{8T}{\pi\rho D^2}}\right] \tag{1}$$

The circumferential velocity is the velocity induced in the z direction, it is the swirl induced by the propeller (Ferrari, 1957).

$$v_{swirl} = \frac{2v_\infty \Delta v}{\omega r} \tag{2}$$

The aerodynamic parameters of the wing are then computed inserting in the standard Prandtl equation the velocity induced by the propellers.
The wing is subdivided in discrete lifting line segments each one with its own profile, chord and swirl. The lift coefficient of each lifting line segment is computed as:

$$cl = C_{L\alpha}\left(\alpha_{eff} - \alpha_{L=0}\right) \tag{3}$$

Where α_{eff} is the effective angle of attack and takes into account the geometric angle of attack also called flight angle of attack, the twist and the propeller effect:

$$\alpha_{eff} = \alpha_g - \alpha_t - \frac{w_w(y) + w_p(y)}{V(y)} \tag{4}$$

$w_w(y)$ is the downwash induced by the wing still unknown and $w_p(y)$ is the downwash induced by the propeller, earlier called v_{swirl}. We are making the assumption of small angles $w_w(y) + w_p(y) \ll V(y)$.

$$tan\left(\frac{w_w(y) + w_p(y)}{V(y)}\right) \approx \frac{w_w(y) + w_p(y)}{V(y)} \tag{12}$$

The other effect of the propeller, the velocity induced in the x direction is inside $V(y)$. Writing the equality of lift computed as airfoil lift and as 3D vortex lift we find another expression of the lift coefficient:

$$\frac{1}{2}\rho_\infty V(y)^2 c(y) cl(y) = \rho_\infty V(y)\Gamma(y) \tag{13}$$

$$cl = \frac{2\Gamma(y)}{V(y)c(y)} \tag{14}$$

Then with some more steps we can get to the complete Prandtl equation that takes into account the propellers effect. We rearrange the first cl expression (10) an substitute into it the second one (13):

$$\alpha_{eff} = \frac{cl}{a_0} + \alpha_{L=0} \tag{15}$$

$$\alpha_{eff} = \frac{2\Gamma(y)}{V(y)c(y)C_{L\alpha}} + \alpha_{L=0} \tag{16}$$

Then we insert the induced wing downwash into the effective angle of attack formula (11):

$$w_w(y) = \frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{\frac{d\Gamma}{dy}}{y_0 - y} dy \tag{175}$$

And we finally get the complete Prandtl equation:

$$\frac{2\Gamma(y)}{V(y)c(y)C_{L\alpha}} + \alpha_{L=0} = \alpha_g - \alpha_t - \frac{1}{V(y)}\left[\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{\frac{d\Gamma}{dy}}{y_0 - y} dy + w_p(y)\right] \tag{18}$$

In order to solve it numerically we proceed with the following coordinate transformation:

$$y = -\frac{b}{2}\cos\theta \tag{19}$$

We assume a circuitation distribution gamma in the form of Fourier series expansion:

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin n\theta \tag{20}$$

And we insert it into the Prandtl equation. The derivative of the circuitation distribution is:

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2bV_\infty \sum_{n=1}^{N} nA_n \cos(n\theta)\frac{d\theta}{dy} \tag{21}$$

Inserting the circuitation distribution and its derivative into the complete Prandtl equation we have:

$$\frac{V_\infty}{V(y)}\frac{4b}{c(y)C_{L\alpha}}\sum_{n=1}^{N} A_n \sin n\theta + \alpha_{L=0}$$
$$= \alpha_g - \alpha_t - \frac{1}{V(y)}\left[\frac{1}{4\pi}\int_\pi^0 \frac{2bV_\infty \sum_{n=1}^{N} nA_n \cos(n\theta)}{-\frac{b}{2}\cos\theta_0 + \frac{b}{2}\cos\theta} d\theta + w_p(y)\right] \quad (226)$$

Rearranging we have:

$$\frac{V_\infty}{V(y)}\frac{4b}{c(y)C_{L\alpha}}\sum_{n=1}^{N} A_n \sin n\theta = \alpha - \frac{w_p(y)}{V(y)} - \frac{V_\infty}{V(y)}\frac{1}{\pi}\int_\pi^0 \frac{\sum_{n=1}^{N} nA_n \cos(n\theta)}{\cos\theta - \cos\theta_0} d\theta \tag{23}$$

Where α is:

$$\alpha = \alpha_g - \alpha_t - \alpha_{L=0} \tag{24}$$

Introducing the parameter $\mu = \frac{c(y)C_{L\alpha}}{4b}$ and rearranging the previous expressions a final expression can be derived as:

$$\sum_{n=1}^{N} A_n \sin n\theta(\sin\theta + n\mu) = \mu\frac{V_\infty}{V(\theta)}\left(\alpha - \frac{w_p(\theta)}{V(\theta)}\right)\sin\theta \tag{25}$$

The equation above is a system of equations that can be written in matrix form and solved directly with the MATLAB \ solver. Writing it in the standard form we have:

$$\begin{bmatrix} & M & \end{bmatrix}\begin{Bmatrix} A \end{Bmatrix} = \begin{Bmatrix} b \end{Bmatrix} \tag{26}$$

M is a *m x n* matrix, A is the n x 1 column vector of the Fourier series coefficients and b is a m x 1 column vector.

$$M(m,n) = \sin n\theta_m\left(\sin\theta_m + n\frac{c(\theta_m)C_{L\alpha}}{4b}\right) \tag{27}$$

$$b(m) = \frac{c(\theta_m)C_{L\alpha}}{4b}\frac{V_\infty}{V(\theta_m)}\left(\alpha - \frac{w_p(\theta_m)}{V(\theta_m)}\right)\sin\theta_m \tag{28}$$

Where:

$$m = 1, 2, \dots, M \quad (297)$$

$$n = 1, 2, \dots, N \quad (308)$$

M is the number of stations along the wingspan and N is the number of terms in the Fourier expansion. In order not to have the M matrix singular the number of terms in the Fourier expansion must be smaller than the number of stations along the wingspan, this due to the fact that M is the number of equations of our system and N is the number of unknowns.

The aerodynamic coefficients are then computed with the standard procedure as indicated in (Anderson, 2001):

$$\boldsymbol{C_L = A_1 \pi AR} \tag{31}$$

$$\boldsymbol{C_{Di} = \pi AR {A_1}^2 \left[1 + \sum_{n=2}^{N} n \left(\frac{A_n}{A_1}\right)^2\right]} \tag{32}$$

The circuitation is computed inserting the A_n into the expression:

$$\boldsymbol{\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin n\theta} \tag{33}$$

$$\boldsymbol{\Gamma(\theta) = \left[\quad G \quad\right]\left\{A\right\}} \tag{34}$$

And the *m x n* matrix G is:

$$\boldsymbol{M(m,n) = 2bV_\infty \sin n\theta_m} \tag{35}$$

The induced angle of attack is computed inserting the A_n into the expression:

$$\boldsymbol{\alpha_i(\theta) = \sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin \theta}} \tag{36}$$

$$\boldsymbol{\alpha_i(\theta) = \left[\quad Z \quad\right]\left\{A\right\}} \tag{37}$$

And the *m x n* matrix Z is:

$$\boldsymbol{Z(m,n) = n \frac{\sin n\theta}{\sin \theta}} \tag{38}$$