Department of Computer Science, University College London

COMP0043 Numerical Methods for Finance

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In-class test 6 December 2023

Answer both questions for full marks. You have two hours.

Non-central chi-squared distribution (60 marks)

The Feller square-root process (1951) has a central role in finance for the modelling of short interest rates (Cox, Ingersoll, Ross 1983), volatilities (Heston 1993), etc. It has a scaled noncentral chi-squared distribution. Using $k = 5$ degrees of freedom and the noncentrality parameter $\lambda = 5$, write a script, e.g. starting from ncchisq.m, that

a) samples the noncentral chi-squared distribution with random numbers drawn from that distribution, e.g. using the library function `ncx2rnd`, builds a normed histogram, and overplots the sampled PDF with the theoretical PDF obtained from a library function, e.g. `pdf('ncx2',x,k,lambda)`;

b) overplots the PDF from the equation

$$f_{\chi'^2_{k,\lambda}}(x) = \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda}\right)^{\frac{k}{4}-\frac{1}{2}} I_{\frac{k}{2}-1}\big(\sqrt{\lambda x}\big),$$

where $I_\alpha(y)$ is the modified Bessel function of the first kind, provided e.g. by the library function `besseli`;

c) overplots the PDF from a numerical inverse Fourier transform of the characteristic function

$$\varphi_{\chi'^2_{k,\lambda}}(\xi) = \frac{e^{\frac{i\lambda\xi}{1-2i\xi}}}{(1-2i\xi)^{k/2}}.$$

European call and put options (40 marks)

The script bs.m prices European call and put options computing their discounted expected values,

$$\begin{aligned} v(x,t) &= e^{-r(T-t)} E[g(X(T) e^{-\alpha X(T)} | X(t) = x] \\ &= e^{-r(T-t)} \int_{-\infty}^{+\infty} g(x') e^{-\alpha x'} f_X(x', T|x,t) dx' \\ &= e^{-r(T-t)} \int_{-\infty}^{+\infty} \widehat{g}(\xi) \phi^*(\xi + i\alpha, T|x,t) d\xi, \end{aligned}$$

where $x = \log(S/S_0)$ is the log-price, $g(x)e^{-\alpha x} = \max(\theta(S_0 e^x - K), 0)$ is the undamped payoff, $\theta = 1$ for a call and -1 for a put, and K is the strike price. The first line of the above equation is computed with an average over the endpoints $X_n(T)$ of $n = 1, \dots, N_{\text{run}}$ Monte Carlo paths, the third line is computed with a numerical quadrature in Fourier space, and the second line is computed with Black and Scholes' (semi-)analytical solution. The step from the second to the third line is given by Plancherel's theorem.

The second line can also be computed with a numerical quadrature in log-price space of the product of the undamped payoff and of the PDF at maturity T of arithmetic Brownian motion conditional on its value at $t = 0$, when it is a Dirac delta function centered at $x_0 = 0$. Do this. To achieve the same accuracy, one needs a much larger grid size than for the correspoding integration in Fourier space, where 64 is sufficient. Write as a comment in your script a sentence or two why you believe this is necessary.