MAS6052 - HW2

Chapter 6

6.1

Let (X_n) be a sequence of independent random variables such that:

$$X_n = \begin{cases} 2^{-n} & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases}$$

Show that $X_n \xrightarrow{\mathcal{L}^1} 0$ and $X_n \xrightarrow{a.s.} 0$. Deduce that $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

Convergence in $\mathcal{L}^1$

We need to prove:

$$\mathbb{E}[|X_n - 0|] \to 0$$

We have:

$$\mathbb{E}[|X_n - 0|] = \mathbb{E}[X_n] = \frac{1}{2^{n+1}}$$

And:

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} = 0$$

Hence, we have $\mathcal{L}^1$ convergence.

Almost Sure Convergence

We need to prove:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\right\}\right] = 1$$

We have that $0 \le X_n \le 2^{-n}$, hence:

$$0 \le \lim_{n\to\infty} X_n(\omega) \le \lim_{n\to\infty} 2^{-n} = 0$$

By the sandwich rule:

$$\lim_{n\to\infty} X_n(\omega) = 0$$

and hence:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\right\}\right] = 1$$

Convergence in Probability and Distribution

Since almost sure convergence implies convergence in probability, and convergence in probability implies convergence in distribution (lemma 6.1.2), we also have $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

6.2

Let X_n, X be random variables.

(a) suppose that$X_n \xrightarrow{\mathcal{L}^1} X$ as $n \to \infty$. Show that $\mathbb{E}[X_n] \to \mathbb{E}[X]$

By assumption, we have that:

$$\mathbb{E}[|X_n - X|] \to 0$$

By the Absolute Value property of Expectation and linearity, we have:

$$0 \leq |\mathbb{E}[X_n - X]| = |\mathbb{E}[X_n] - \mathbb{E}[X]| \leq \mathbb{E}[|X_n - X|] \to 0$$

Hence, by the sandwich rule:

$$|\mathbb{E}[X_n] - \mathbb{E}[X]| \to 0$$

And, hence, $\mathbb{E}[X_n] \to \mathbb{E}[X]$

(b) Give an example where $\mathbb{E}[X_n] \to \mathbb{E}[X]$ but X_n does not converge to X in $\mathcal{L}^1$

A pretty trivial counterexample is any sequence of random variables X_n such that $\mathbb{E}[X_n] = 0$, but $\mathbb{E}[|X_n|] \neq 0$ and does not converge to zero. In that case, by setting $X = 0$, we have that $\mathbb{E}[X_n] = 0 \to 0 = \mathbb{E}[X]$, but $\mathbb{E}[|X_n - 0|]$ does not converge to zero. For example:

$$X_n = \begin{cases} n & \text{with probability } \frac{1}{2} \\ -n & \text{with probability } \frac{1}{2} \end{cases}$$

We have that $\mathbb{E}[X_n] = 0$, but:

$$\mathbb{E}[|X_n - 0|] = \mathbb{E}[|X_n|] = n$$

which does not converge to $\mathbb{E}[0] = 0$.

6.3

Let U be a random variable such that $\mathbb{P}[U = 0] = \mathbb{P}[U = 1] = \mathbb{P}[U = 2] = \frac{1}{3}$. Let:

$$X_n = \begin{cases} 1 + \frac{1}{n} & \text{if } U = 0 \\ 1 - \frac{1}{n} & \text{if } U = 1 \\ 0 & \text{if } U = 2 \end{cases}$$

and:

$$X = \begin{cases} 1 & \text{if } U \in \{0, 1\} \\ 0 & \text{if } U = 2 \end{cases}$$

Show that $X_n \xrightarrow{\mathcal{L}^1} 0$ and $X_n \xrightarrow{a.s.} 0$. Deduce that $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

Convergence in $\mathcal{L}^1$

We need to prove:

$$\mathbb{E}[|X_n - X|] \to 0$$

We have that:

$$0 \leq |X_n - X| \leq \frac{1}{n}$$

Hence, by monotonicity of expectation:

$$0 \leq \mathbb{E}[|X_n - X|] \leq \frac{1}{n}$$

And by the sandwich rule:

$$\mathbb{E}[|X_n - X|] \to 0$$

Almost Sure convergence

We need to prove:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right] = 1$$

We have:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right] =$$
$$\mathbb{P}\left[\lim_{n\to\infty} X_n = X \middle| U = 0\right] \mathbb{P}[U = 0] + \mathbb{P}\left[\lim_{n\to\infty} X_n = X \middle| U = 1\right] \mathbb{P}[U = 1] + \mathbb{P}\left[\lim_{n\to\infty} X_n = X \middle| U = 2\right] \mathbb{P}[U = 2]$$
$$\mathbb{P}\left[\lim_{n\to\infty} 1 + \frac{1}{n} = 1 \middle| U = 0\right] \frac{1}{3} + \mathbb{P}\left[\lim_{n\to\infty} 1 - \frac{1}{n} = 1 \middle| U = 1\right] \frac{1}{3} + \mathbb{P}\left[\lim_{n\to\infty} 0 = 0 \middle| U = 2\right] \frac{1}{3} = 1$$

Convergence in Probability and Distribution

Since almost sure convergence implies convergence in probability, and convergence in probability implies convergence in distribution (lemma 6.1.2), we also have $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

6.4

Let X_1 be a random variable with distribution given by $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = 0] = \frac{1}{2}$. Set $X_n = X_1$ for all $n \geq 2$. Set $Y = 1 - X_1$. Show that $X_n \to Y$ in distribution, but not in probability.

To show convergence in distribution, we need to prove:

$$\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[Y \leq x] \;\forall\; x \in \mathbb{R} : \mathbb{P}[Y = x] = 0$$

We have:

$$\mathbb{P}[Y \leq x] = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x \in [0, 1) \\ 1 & \text{if } x \geq 1 \end{cases}$$

Hence, we need to prove convergence in the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

- For $(-\infty, 0)$, we have $\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X_1 \leq x] = 0$.
- For $(0, 1)$, we have $\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X_1 \leq x] = \frac{1}{2}$.
- For $(1, \infty)$, we have $\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X_1 \leq x] = 1$.

Hence, we have convergence in distribution. Let's now look at convergence in probability, which means:

$$\lim_{n\to\infty} \mathbb{P}[|X_n - Y| > a] = 0, \ \forall \ a > 0$$

We have:

$$|X_n - Y| = |X_1 - Y| = |X_1 - 1 + X_1| = |2X_1 - 1|$$

If $X_1 = 1$, than we have that $|X_n - Y| = 1$. But also if $X_1 = 0$, than we have that $|X_n - Y| = 1$. So, for any $a \in (0, 1]$:

$$\lim_{n\to\infty} \mathbb{P}[|X_n - Y| > a] = \mathbb{P}[|2X_1 - 1| > a] \neq 0$$

And we don't have convergence in probability.

6.4

Let (X_n) be the sequence of random variables from 6.1. Define:

$$Y_n = \sum_{i=1}^{n} X_i$$

- (a) show that $\forall \ \omega \in \Omega$, the sequence $Y_n(\omega)$ is increasing and bounded.

We have that $X_n(\omega) \geq 0, \forall \ \omega \in \Omega, \forall \ n \geq 0$. Hence we have that, for any n and ω:

$$Y_{n+1}(\omega) = Y_n(\omega) + X_{n+1}(\omega) \geq Y_n(\omega)$$

That is, the sequence $Y_n(\omega)$ is increasing.

Also, we have that:

$$Y_n(\omega) \leq \sum_{i=1}^{n} \frac{1}{2^i}$$

which, being a geometric series, is bounded. Hence, $Y_n(\omega)$ is bounded.

- (b) deduce that there exists a random variable Y such that $Y_n \xrightarrow{a.s.} Y$

Given that $Y_n(\omega)$ is increasing and bounded for any ω, for the Monotone Convergence Theorem for real valued sequence, we have:

$$\lim_{n\to\infty} Y_n(\omega) = Y_\infty(\omega)$$

for some $Y_\infty(\omega)$. By proposition 2.2.6, we know that $Y_\infty(\omega)$ is a random variable.

This implies that:

$$\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y_\infty(\omega)\right\} = \Omega$$

And hence:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y_\infty(\omega)\right\}\right] = 1$$

That is, Y_n converges a.s. to $Y = Y_\infty$.

- (c) Write down the distribution of Y_1, Y_2, Y_3
- (d) Suggest why we might guess that Y has a uniform distribution on $[0, 1]$
- (e) Prove that Y_n has a uniform distribution on $\{k2^{-n}; k = 0, 1, ..., 2^{n-1}\}$
- (r) Prove that Y has a uniform distribution on $[0, 1]$

Chapter 10

In all the following questions, B_t denotes Brownian motion.

10.1

Consider the process:

$$C_t = \mu t + \sigma B_t, \quad t \geq 0, \;\; \mu \in \mathbb{R}, \;\; \sigma > 0$$

(a). Find the mean and variance of C_t

We have:

$$\mathbb{E}[C_t] = \mathbb{E}[C_t] = \mathbb{E}[\mu t + \sigma B_t] = \mu t + \sigma \mathbb{E}[B_t] = \mu t$$

and:

$$\mathbb{E}[C_t^2] = \mathbb{E}[(\mu t)^2 + 2\mu t \sigma B_t + (\sigma B_t)^2] = (\mu t)^2 + 2\mu t \sigma \mathbb{E}[B_t] + \sigma^2 \mathbb{E}[B_t^2] = (\mu t)^2 + \sigma^2 t$$

and, finally:

$$\mathbb{V}ar[C_t] = \mathbb{E}[C_t^2] - \mathbb{E}[C_t]^2 = (\mu t)^2 + \sigma^2 t - (\mu t)^2 = \sigma^2 t$$

This one needs checking: can't I also state that $C_t \sim \mathbb{N}(\mu t, \sigma^2 t)$?

(b). Let $0 \leq u \leq t$. What is the distribution of $C_t - C_u$

We have that:

$$C_t - C_u = \mu t + \sigma B_t - \mu u + \sigma B_u = \mu(t-u) + \sigma(B_t - B_u)$$

We know that:

$$B_t - B_u \sim \mathcal{N}(0, t-u)$$

And, by the properties of the normal distribution:

$$C_t - C_u \sim \mathcal{N}(\mu(t-u), \sigma^2(t-u))$$

(c). Is C_t a random continuous function?

Yes, since it is a the result of a continuous function applied to a random continuous function, i.e. Brownian motion.

This one needs checking

(d). Is C_t a Brownian motion?

No, as $C_t - C_u$ has not mean 0.

10.2

Let $0 \leq u \leq t$. Use the properties of Brownian motion to show that $\text{cov}(B_t, B_u) = u$

We have:

$$\mathbb{C}\text{ov}[B_t, B_u] = \mathbb{E}[B_t B_u] - \mathbb{E}[B_t]\mathbb{E}[B_u] = \mathbb{E}[B_t B_u] = \mathbb{E}[B_t B_u - B_u^2 + B_u^2] =$$
$$= \mathbb{E}[B_u(B_t - B_u)] + \mathbb{E}[B_u^2] =$$

Since, for any $0 \leq u \leq t$, the random variable $B_t - B_u$ is independent of $\sigma(B_v; v \leq u)$:

$$= \mathbb{E}[B_u]\mathbb{E}[B_t - B_u] + \mathbb{E}[B_u^2] = u$$

10.3

Let $u \geq 0$ and $t \geq 0$. Show that $\mathbb{E}[B_u|\mathcal{F}_t] = B_{\min(u,t)}$.

- if $u \geq t$, then, by the martingale property of Brownian motion, $\mathbb{E}[B_u|\mathcal{F}_t] = B_t$
- if $u < t$, then $B_u \in m\mathcal{F}_t$. Hence, by the measurability property of conditional expectation, $\mathbb{E}[B_u|\mathcal{F}_t] = B_u$

Combining the two cases together, we get the desired result.

10.4

(a). Show that $\mathbb{E}[B_t^n] = t(n-1)\mathbb{E}[B_t^{n-2}] \quad \forall n \geq 2$

To simplify notation, let's denote $X = B_t$. By the definition of Brownian motion, we know that $X \sim \mathcal{N}(0, t)$. Using the law of the unconscious statistician:

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

Note that what we want is to isolate a term corresponding to $\mathbb{E}[X^{n-2}]$. In order to do so, we need to maintain the term corresponding to the normal density. Notice that such term has derivative:

$$\frac{d}{dx} e^{-\frac{x^2}{2t}} = -\frac{x}{t} e^{-\frac{x^2}{2t}}.$$

Hence, we can proceed integrating by parts:

$$\int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} -tx^{n-1}(-\frac{x}{t} e^{-\frac{x^2}{2t}})dx$$

$$= \frac{1}{\sqrt{2\pi t}} \left[\left[-tx^{n-1} e^{-\frac{x^2}{2t}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -t(n-1)x^{n-2} e^{-\frac{x^2}{2t}} \right] =$$

$$= t(n-1) \int_{-\infty}^{\infty} x^{n-2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = t(n-1)\mathbb{E}[X^{n-2}] = t(n-1)\mathbb{E}[B_t^{n-2}]$$

(b). Deduce that $\mathbb{E}[B_t^2] = t$ and $\mathbb{V}ar[B_t^2] = 2t^2$

We have:

$$\mathbb{E}[B_t^2] = t\mathbb{E}[B_t^0] = t$$

$$\mathbb{V}ar[B_t^2] = \mathbb{E}[(B_t^2)^2] - \mathbb{E}[B_t^2]^2 = \mathbb{E}[B_t^4] - t^2 = 3t\mathbb{E}[B_t^2] - t^2 = 2t^2$$

(c). Write down $\mathbb{E}[B_t^n]$ for any $n \in \mathbb{N}$

We have that

- $\mathbb{E}[B_t^n] = 0$ if n is odd
- $\mathbb{E}[B_t^n] = (n-1)(n-3)...(1)t^{n/2}$ if n is even

We can prove it by induction. For n odd, let's start by observing that $\mathbb{E}[B_t] = 0$. We have our base case, $n = 1$. For our inductive case, let's assume that for an odd $n-2$, $\mathbb{E}[B_t^{n-2}] = 0$. Since $n-2$ is odd, so is n. We have:

$$\mathbb{E}[B_t^n] = t(n-1)\mathbb{E}[B_t^{n-2}] = 0$$

And we're done. As for n even, similarly, we have our base case for $n = 2$:

$$\mathbb{E}[B_t^2] = t\mathbb{E}[B_t^0] = t = (1)t^{2/2}$$

As for the inductive case, let's assume that, for n - 2 even:

$$\mathbb{E}[B_t^{n-2}] = (n-3)(n-5)...(1)t^{(n-2)/2}$$

Then n is also even, and:

$$\begin{aligned}\mathbb{E}[B_t^n] &= t(n-1)\mathbb{E}[B_t^{n-2}] = t(n-1)(n-3)(n-5)...(1)t^{(n-2)/2} \\ &= (n-1)(n-3)(n-5)...(1)t^{n/2}\end{aligned}$$

(d). Show that $B_t^n \in L^1$ for all $n \in \mathbb{N}$

We know that $\mathbb{V}ar[B_t^n] = \mathbb{E}[B_t^{2n}] - \mathbb{E}[B_t^n]^2 < \infty$, as both terms on the right hand side are, as we've seen in part (c). We also know that L^2 is the set of random variables with finite variance, and hence $B_t^n \in L^2$. Finally, we know that if a random variable is in L^2, then it is also in L^1. Hence, $B_t^n \in L^1$.

10.5

Let $Z \sim \mathcal{N}(\mu, \sigma^2)$. Show that $\mathbb{E}[\mathrm{e}^X] = \mathrm{e}^{\mu + \frac{1}{2}\sigma^2}$. *(Hint: complete the square!)*

Chapter 11

11.1 Using (11.8) find $\int_v^t 1dB_u$, where $0 \leq v \leq t$

By the consistency property of Ito integrals:

$$\int_0^t 1dB_u = \int_0^v 1dB_u + \int_v^t 1dB_u \Rightarrow \int_v^t 1dB_u = \int_0^t 1dB_u - \int_0^v 1dB_u$$

By (10.8):

$$\int_0^v 1dB_u = B_v; \quad \int_0^t 1dB_u = B_t.$$

Hence:

$$\int_v^t 1dB_u = B_t - B_u$$

11.2 Show that the process e^{B_t} is in $\mathcal{H}^2$. (Hint: use (10.2))

We work onver a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t = \sigma(B_t)$. We need to show that:

- e^{B_t} is continuous
- e^{B_t} is adapted to $\mathcal{F}_t$
- e^{B_u} is locally square integrable, i.e. $\int_0^t \mathbb{E}[(e^{B_u})^2]du < \infty$

We have that since B_t and e^x are continuous, their composition is continuous. Furthermore, given that B_t is adapted by construction, by 2.2.6, so is e^{B_t}.

As for local square integrability, since B_t is a standard Brownian motion, we have that $B_t \sim \mathcal{N}(0, t)$, and, hence, $2B_t \sim \mathcal{N}(0, 4t)$.

Thus:

$$\mathbb{E}[(e^{B_u})^2] = \mathbb{E}[e^{2B_u}] = e^{2u}$$

Where we used (10.2) to calculate the expected value. Hence:

$$\int_0^t \mathbb{E}[(e^{B_u})^2]du = \int_0^t e^{2u}du = \left[\frac{1}{2}e^{2u}\right]_0^t = \frac{1}{2}(e^{2t} - 1) < \infty$$

Which proves that $e^{B_t} \in \mathcal{H}^2$.

11.3

(a). Let $Z \sim \mathcal{N}(0, 1)$. Show that the expectation of $e^{\frac{z^2}{2}}$ is infinite.

We have:

$$\mathbb{E}\left[e^{\frac{z^2}{2}}\right] = \int_{-\infty}^{\infty} e^{\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} dz = \infty$$

(b). Give an example of a continuous, adapted stochastic process that is not in $\mathcal{H}^2$

We know that, $X \sim \mathcal{N}(0, t) \Rightarrow \frac{1}{\sqrt{t}}X \sim \mathcal{N}(0, 1)$. We also know that $B_t \sim \mathcal{N}(0, t)$. Hence, $F_t = \frac{1}{\sqrt{t}}B_t \sim \mathcal{N}(0, 1)$.

Now, if we take the process $X_t = e^{\frac{1}{4}F_t^2}$ we have:

- $X_t \in m\mathcal{F}_t$, since $B_t \in m\mathcal{F}_t \Rightarrow F_t \in m\mathcal{F}_t$, by proposition 2.2.6, and $F_t \in m\mathcal{F}_t \Rightarrow X_t \in m\mathcal{F}_t$, also by proposition 2.2.6. Hence, X_t is adapted.
- Since B_t is continuous, and $x^2, e^x, \frac{x}{\sqrt{t}}$ are continuos, so is X_t
- However, we have:

$$\mathbb{E}[X_t^2] = \mathbb{E}[e^{\frac{1}{2}F_t^2}] = \infty$$

And, hence,

$$\int_0^t \mathbb{E}[X_u^2]du = \infty$$

And X_t is not in $\mathcal{H}^2$. *Note that we can't simply use the stochastic process $X_t = Z$, because we can't guarantee that such process is adapted.*

Chapter 12

12.13 - Challenge Question

Assignment 1

1. Recall the one-period market, and its parameters r, u, d, p_u, p_d and s. We assume that $d < 1 + r < u$.

   a. At time $t = 0$ our portfolio contains 2 unit of cash and 3 units of stock. What is the value of our portfolio at time $t = 0$? If we hold this portfolio until time $t = 1$, what is its new value?

   In our model, the cost of a unit of stock at time 0 is $S_0 = s$. Hence, at time $t = 0$, the value of our portfolio is 2 + 3s. At time t = 1, the value of the cash will be $2(1 + r)$, while the value of stock is a random variable. The value of the portfolio will be:

$$V_1 = \begin{cases} 2(1+r) + 3su, & \text{with probability } p_u \\ 2(1+r) + 3sd, & \text{with probability } p_d \end{cases} \tag{1}$$

   b. A rival investor holds a portfolio containing 3 units of cash and 2 unit of stock. Under what condition (on the parameters) can we be certain that our own portfolio will have a strictly greater value at time $t = 1$?

   The portfolio of the rival investor, at time $t = 1$, will be worth:

$$V_1^* = \begin{cases} 3(1+r) + 2su, & \text{with probability } p_u \\ 3(1+r) + 2sd, & \text{with probability } p_d \end{cases} \tag{2}$$

   In order to have the certainty that our portfolio will have a strictly greater value at time 1, it will have to have strictly greater value in every possible state of the world. That is, we want:

$$\mathbb{P}[V > V^*] = 1$$

   Hence, we need:

$$\begin{cases} 2(1+r) + 3su > 3(1+r) + 2su \\ 2(1+r) + 3sd > 3(1+r) + 2sd \end{cases} \tag{3}$$

   Which means that we need $su > 1 + r$ and $sd > 1 + r$. Given our assumption $d < 1 + r < u$, we know that su > sd, and the first inequality is redundant. Thus we have that our portfolio will certainly be worth more, at time $t = 1$, if and only if:

$$sd > 1 + r \tag{4}$$

   In other words, if and only if a unit of stock is worth more than cash in every state of the world at time 1. Note that, since we are assuming $r > 0$, this implies that s > 1, i.e. the stock at time 0 costs more than one unit of cash. Also, note that this is different from our no-arbitrage assumption (i.e. that $d < 1 + r < u$), as that only requires a non-null probability that the stock will **grow** lower than cash, it doesn't impose a condition on the final value.

2. Let $\Omega = \{HH, HT, TH, TT\}$, representing two coin tosses each of which may show either H (head) or T (tail). Let $X : \Omega \to \mathbb{R}$ be the toss in which the first head occurred, or zero if no heads occurred.

$$X = \begin{cases} 0, & \text{if } \omega = TT \\ 1, & \text{if } \omega = HT \text{ or } \omega = HH \\ 2, & \text{if } \omega = TH \end{cases} \tag{5}$$

Let Y be the total number of heads that occurred in both tosses.

a. Write down the sets $X^{-1}(0)$, $X^{-1}(1)$ and $X^{-1}(2)$

- $X^{-1}(0) = \{TT\}$
- $X^{-1}(1) = \{HT, HH\}$
- $X^{-1}(2) = \{TH\}$

b. State the definition of $\sigma(X)$, and ist its elements

$\sigma(X)$ is the σ-field generated by X, i.e., it is the smallest σ-field containing all the preimages of any subinterval of $\mathbb{R}$ under X. As X contains a finite set of values {0, 1, 2}:

$$\sigma(X) = \{\emptyset, \Omega, \{TT\}, \{HT, HH\}, \{TH\}, ...$$
$$..., \{HT, HH, TH\}, \{TT, TH\}, \{TT, HT, HH\}\}$$

c. Is Y measurable with respect to $\sigma(X)$? Why, or why not?

As Y takes a finite set of values {0, 1, 2}, we can use lemma 2.2.2, and state that it is measurable w.r.t. $\sigma(X)$ iff:

$$Y^{-1}(0), Y^{-1}(1), Y^{-1}(2) \in \sigma(X)$$

We can see that while $Y^{-1}(0) = \{TT\} \in \sigma(X), Y^{-1}(1) = \{HT, TH\} \notin \sigma(X)$, and $Y^{-1}(2) = \{HH\} \notin \sigma(X)$. Hence, $Y \notin m\sigma(X)$. Note that this makes intuitive sense. $\sigma(X)$ allows us to measure when the first head comes up. It doesn't need to distinguish, for example, HH and HT - this information is not relevant for X, as in both these outcomes a head comes up in the first throw. On the other hand, HH and HT need to be distinguished by Y, as they imply different numbers of heads.

3. Let $\Omega = \{1, 2, 3, 4, 5\}$, representing one roll of a five sided dice. Let $\mathcal{F}$ be the set of all subsets of Ω. Describe, in words, the information contained within the following sub-σ-fields of $\mathcal{F}$:

a. $\mathcal{G}_1 = \{\emptyset, \Omega, \{1\}, \{2, 3, 4, 5\}\}$

$\mathcal{G}_1$ contains information around whether 1 came up or not.

b. $\mathcal{G}_2 = \{\emptyset, \Omega, \{1, 3, 5\}, \{2, 4\}\}$

$\mathcal{G}_2$ contains information around whether the number that came up in the roll is odd or even.

c. $\mathcal{G}_3 = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}, \{5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 5\}\}$

$\mathcal{G}_3$ contains information around whether the outcome was 1, 5 or a value in the middle of the range.

4. Let X be a random variable.

a. Show that $Y = \cos X$ is also a random variable

We know that:

$$\cos(X) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-1)^n}{2n!} X^{2n}$$

From proposition 2.2.6, we know that

- $Z_n = X^{2n}$, as a product of random variables, is a random variable
- $\frac{(-1)^n}{2n!} Z_n = \alpha Z_n$, as a product of a real number and a random variable, is a random variable
- $Z_N = \sum_{n=0}^{N} \alpha Z_n$, as a sum of random variables, is a random variable
- $Y = \lim_{N \to \infty} Z_N$ exists for all $Z_N \in \mathbb{R}$, hence is a random variable.

b. For which $p \in [1, \infty)$ do we have $Y \in L^p$?

We have that $|Y| \leq 1$, hence Y is bounded. Thus, using monotonicity, we have that $\mathbb{E}[|Y|^p| \leq 1 \leq \infty$ for any p. Hence, by the definition of L^p spaces, $Y \in L^p$ for any $p \in [1, \infty)$.

Assignment 2

1. Let (X_n) be a sequence of i.i.d. random variables, each with a uniform distribution on [-1, 1]. Define:

$$S_n = \sum_{i=1}^{n} X_i$$

Where $S_0 = 0$. Let $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$.

a. Show that S_n is a martingale with respect to the filtration $\mathcal{F}_n$

By definition 3.3.3, we need to show that:

1. (S_n) is adapted to $\mathcal{F}_n$, i.e., for each n, $S_n \in m\mathcal{F}_n$

By definition of σ-fields generated by random variables, we have that $X_1, X_2, ..., X_n \in m\mathcal{F}_n$. Also, by proposition 2.2.6, we know that:

$$X, Y \in m\mathcal{G} \Rightarrow (X + Y) \in m\mathcal{G}$$

Hence:

$$S_n = \sum_{i=1}^{n} X_i \in m\mathcal{F}_n$$

2. $S_n \in L^1$ $\forall n$ We have:

$$|S_n| = |\sum_{i=1}^{n} X_i| \leq \sum_{i=1}^{n} |X_i| \leq \sum_{i=1}^{n} 1 = n$$

That is, S_n is bounded. Hence, $S_n \in L^1$

3. $\mathbb{E}[S_{n+1}|\mathcal{F}_n] = S_n$ We have:

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1} + S_n|\mathcal{F}_n] =$$
$$= \mathbb{E}[X_{n+1}|\mathcal{F}_n] + \mathbb{E}[S_n|\mathcal{F}_n] = \mathbb{E}[X_{n+1}] + S_n = S_n$$

b. Find $\mathbb{E}[S_3^2|\mathcal{F}_2]$ in terms of X_2 and X_1, and hence show that:

$$\mathbb{E}[S_3^2|\mathcal{F}_2] = S_2^2 + \frac{1}{3}$$

We have that:

$$\mathbb{E}[S_3^2|\mathcal{F}_2] = \mathbb{E}[(X_1 + X_2 + X_3)^2|\mathcal{F}_2] =$$

By the linearity of conditional expectation:

$$= \mathbb{E}[X_1^2|\mathcal{F}_2] + \mathbb{E}[X_2^2|\mathcal{F}_2] + \mathbb{E}[X_3^2|\mathcal{F}_2] + 2\mathbb{E}[X_1X_2|\mathcal{F}_2] + 2\mathbb{E}[X_2X_3|\mathcal{F}_2] + 2\mathbb{E}[X_1X_3|\mathcal{F}_2]$$

By the measurability property of conditional expectation:

$$= X_1^2 + X_2^2 + \mathbb{E}[X_3^2|\mathcal{F}_2] + 2X_1X_2 + 2\mathbb{E}[X_2X_3|\mathcal{F}_2] + 2\mathbb{E}[X_1X_3|\mathcal{F}_2]$$

By taking out what it's known:

$$= X_1^2 + X_2^2 + \mathbb{E}[X_3^2|\mathcal{F}_2] + 2X_1X_2 + 2X_2\mathbb{E}[X_3|\mathcal{F}_2] + X_1 2\mathbb{E}[X_3|\mathcal{F}_2]$$

By independence:

$$= X_1^2 + X_2^2 + \mathbb{E}[X_3^2] + 2X_1X_2 + 2X_2\mathbb{E}[X_3] + X_1 2\mathbb{E}[X_3]$$

Since we know that $\mathbb{E}[X_3] = 0$:

$$\mathbb{E}[S_3^2|\mathcal{F}_2] = X_1^2 + X_2^2 + 2X_1X_2 = (X_1 + X_2)^2 = S_2^2 + \mathbb{E}[X_3^2]$$

And we have that:

$$\mathbb{E}[X_3^2] = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$$

Hence:

$$\mathbb{E}[S_3^2|\mathcal{F}_2] = X_1^2 + X_2^2 + 2X_1X_2 = (X_1 + X_2)^2 = S_2^2 + \frac{1}{3}$$

c. Write down a deterministic function $f : \mathbb{N} \to \mathbb{R}$ such that:

$$M_n = S_n^2 - f(n)$$

is a martingale.

In general, we'll have:

$$\mathbb{E}[S_{n+1}^2|\mathcal{F}_n] = \mathbb{E}\left[\left(\sum_{i=1}^{n+1} X_i\right)^2 \middle| \mathcal{F}_n\right] = \mathbb{E}\left[\left(X_{n+1} + S_n\right)^2 | \mathcal{F}_n\right] =$$

$$= \mathbb{E}\left[X_{n+1}^2 + 2X_{n+1}S_n + S_n^2 | \mathcal{F}_n\right] =$$

By linearity, taking out what is known and measurability:

$$= \mathbb{E}[X_{n+1}^2] + 2S_n\mathbb{E}[X_{n+1}] + S_n^2 = \mathbb{E}[X_{n+1}^2] + S_n^2 = \frac{1}{3} + S_n^2$$

Hence, if we define $f(n) = \frac{n}{3}$:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_{n+1} - \frac{n+1}{3}|\mathcal{F}_n]$$

By linearity:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_{n+1}^2|\mathcal{F}_n] - \frac{n+1}{3} = S_n^2 - \frac{n}{3} = S_n^2 - f(n) = M_n$$

It is easy to prove (but not requested) that M_n is adapted to $\mathcal{F}_n$ and $M_n \in L^1$, and hence is a Martingale.

2. Consider the one-period market with $r = \frac{1}{10}, s = 2, d = \frac{1}{2}, u = 3$, in our usual notation. A contract specifies that:

The holder of the contract will sell 2 units of stock, and be paid K units of cash, at time 1

a. Explain briefly why the contingent claim of this contract is:

$$\Phi(S_1) = K - 2S_1$$

The contingent claim of a contract is a random variable that describes the value of the contract as a function of the price of stock. In our case, the contract enforces the holder to sell two units of stock (hence the term $-2S_1$) in exchange for K cash (hence the term $+K$)

b. Find a replicating portfolio h for this contingent claim

A replicating portfolio h is a portfolio such that $V_1^h = \Phi(S_1)$, In other words, we want to find a portfolio that has the same value as the contingent claim in every possible state of the world, that is:

$$V_1^h = \begin{cases} \Phi(su) & \text{if } S_1 = su \\ \Phi(sd) & \text{if } S_1 = sd \end{cases}$$

That is, we want to solve the system:

$$\begin{cases} (1+r)x + suy = K - 2su \\ (1+r)x + sdy = K - 2sd \end{cases}$$

We can subtract the equations to find the amount of stock to hold y:

$$sy(u-d) = 2s(d-u) \Rightarrow y(u-d) = -2(u-d) \Rightarrow y = -2$$

And hence we can substitute this to find the amount of cash to hold x:

$$(1+r)x - 2su = K - 2su \Rightarrow x = \frac{K}{(1+r)} = \frac{10}{11}K$$

c. Write down the value V_0^h of h at time 0

$$V_0^h = x + sy = \frac{10}{11}K - 2s = \frac{10}{11}K - 4$$

d. Find the numerical values of risk-neutral probabilities:

$$q_u = \frac{(1+r)-d}{u-d} \quad \text{and} \quad q_d = \frac{u-(1+r)}{u-d}$$

Hence, check that $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)]$ and V_0^h have the same values.

We have that:

$$q_u = \frac{(1+r)-d}{u-d} = \frac{(1+\frac{1}{10}) - \frac{1}{2}}{3-\frac{1}{2}} = \frac{6}{25}$$

$$q_d = \frac{3-\frac{11}{10}}{3-\frac{1}{2}} = \frac{2}{5}\frac{19}{10} = \frac{19}{25}$$

Hence:

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)] = \frac{1}{1+r}\left[q_u\Phi(su) + q_d\Phi(sd)\right] = \frac{10}{11}\left[\frac{6}{25}(K-2su) + \frac{19}{25}(K-2sd)\right]$$

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)] = \frac{10}{11}\left[K - \frac{110}{25}\right] = \frac{10}{11}K - 4$$

e. For which K does the contract have zero value at time 0?

$$V_0^h = 0 \Rightarrow \frac{10}{11}K - 4 = 0 \Rightarrow K = 4.4$$

Assignment 3

1. Consider the binomial model with $r = \frac{1}{11}$, $u = 1.2$, $d = 0.9$, $s = 100$ and time steps $t = 0, 1, 2$

   a. Draw a recombining tree of the stock price process, for time $t = 0, 1, 2$

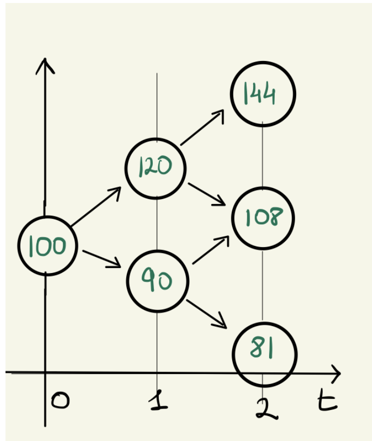


Figure 1: Recombining Tree of the stock process

   b. Find the value, at time t = 0, of a European call option that gives its holder the option to purchase one unit of stock at time $t = 2$ for a strike price $K = 90$. Write down the hedging strategy that replicatres the value of this contract, at all nodes of your tree.

We have that the risk neutral probabilities are:

- $q_u = \frac{(1+r)-d}{u-d} = 0.64$
- $q_d = \frac{u-(1+r)}{u-d} = 0.36$

A European Call has payoff $\Phi(S) = \max(S - K, 0)$. We can determine the value of the European Call at expiration:

- $S = 144 \Rightarrow \Phi(S) = 54$
- $S = 108 \Rightarrow \Phi(S) = 18$
- $S = 81 \Rightarrow \Phi(S) = 0$

Now we can proceed recursively and calculate the risk neutral expected value at each node. In general, if we are sitting in a node at time t, the value of the option at time t is:

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_{t+1}]$$

We find that the option value at time zero is 27.375.

By using the delta hedging formula, we can find our replicating portfolio at each node of the tree:

$$x = \frac{1}{1+r}\frac{u\Phi(sd) - d\Phi(su)}{u-d}$$
$$y = \frac{\Phi(su) - d\Phi(sd)}{su - sd}$$

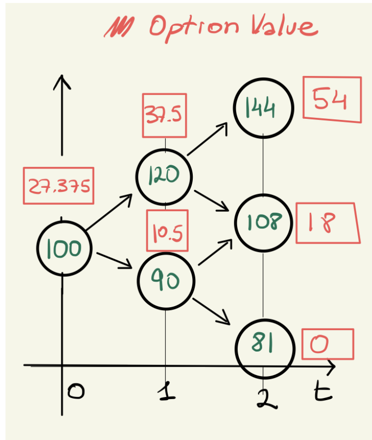
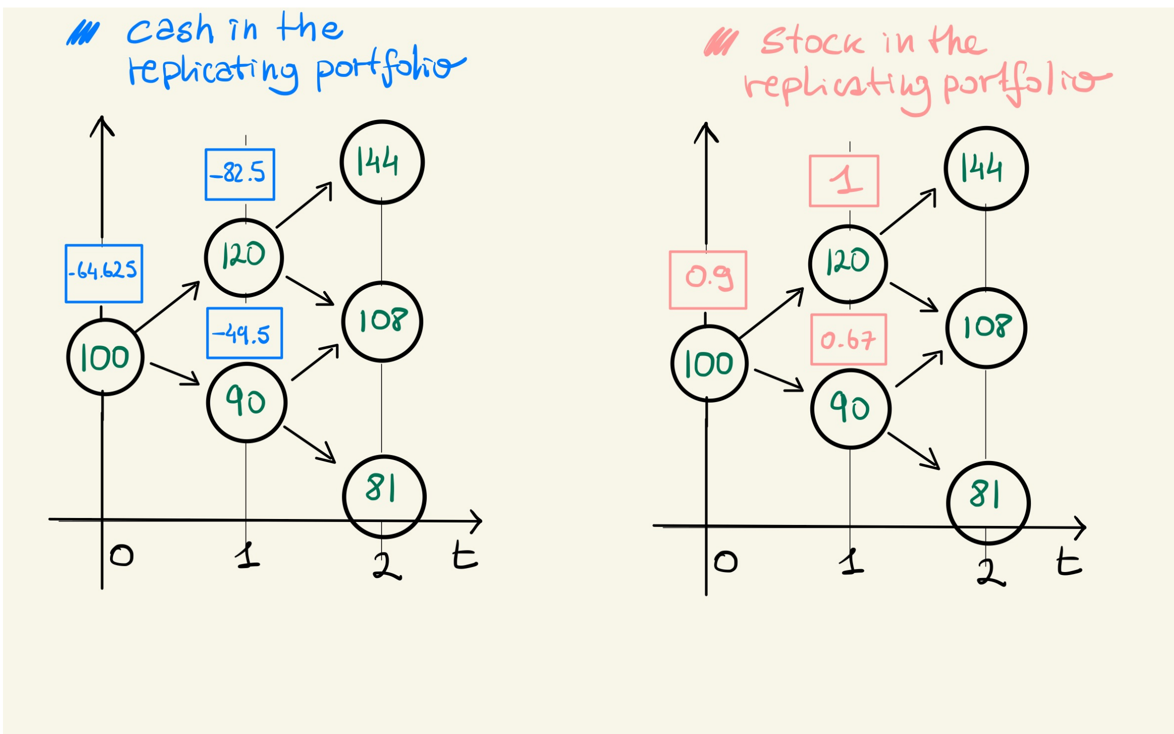


Figure 2: Value of the European Call over time



2. Let $S_n = \sum_{i=1}^{n} X_i$ be a random walk, in which $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common distribution $\mathbb{P}[X_i = \frac{1}{i^2}] = \mathbb{P}[X_i = -\frac{1}{i^2}] = \frac{1}{2}$

   a. Show that $\mathbb{E}[|S_n|] \leq \sum_{i=1}^{n} \frac{1}{i^2}$.

By the triangle inequality, we have that:

$$\left|\sum_{i=1}^{n} X_i\right| \leq \sum_{i=1}^{n} |X_i|$$

Hence, by monotonicity of Expectation:

$$\mathbb{E}[|S_n|] = \mathbb{E}\left[\left|\sum_{i=1}^{n} X_i\right|\right] \leq \mathbb{E}\left[\sum_{i=1}^{n} |X_i|\right] = \sum_{i=1}^{n} \frac{1}{i^2}$$

b. Explain briefly why (a) means that S_n is bounded in L^1.

We have that $\sum_{i=1}^{n} \frac{1}{i^2}$ is bounded:

$$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \sum_{i=2}^{n} \frac{1}{i^2} < 1 + \sum_{i=2}^{n} \frac{1}{i(i-1)} = 1 + 1 - \frac{1}{n} < 2$$

Hence, (a) implies:

$$\mathbb{E}[|S_n|] < 2$$

Which means that S_n is bounded in L^1.

c. Show there exists a random variable S_∞ such that $S_n \xrightarrow{\text{a.s.}} S_\infty$ as $n \to \infty$.

We have shown that S_n is bounded in L^1, which implies that $S_n \in L^1$. Furthermore, if we define $\mathcal{F}_n = \sigma(X_!, ..., X_n)$, we have that (by proposition 2.2.6) $S_n \in m\mathcal{F}_n$, i.e., S_n is adapted.

Also, we have:

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[\sum_{i=1}^{n+1} X_i \middle| \mathcal{F}_n\right] = \mathbb{E}\left[X_{n+1} + \sum_{i=1}^{n} X_i \middle| \mathcal{F}_n\right]$$

By linearity of conditional expectation:

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] + \mathbb{E}\left[\sum_{i=1}^{n} X_i \middle| \mathcal{F}_n\right]$$

By measurability and independence property of conditional expectation:

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[X_{n+1}\right] + \sum_{i=1}^{n} X_i = S_n$$

Hence, S_n is a martingale, and, given that in (c) we also proved that is bounded in L_1, we can use the martingale convergence theorem to state that the limit $S_n \xrightarrow{\text{a.s.}} S_\infty$ exists.

d. Determine whether (S_n) is bounded in L^2, and briefly state what else (if anything) can be deduced about S_∞ as a consequence.

S_n is bounded in L^2 if there exists $M < \infty$ such that, for all n,

$$\mathbb{E}[|S_n|^2] \leq M$$

From the work done in (a), (b), we know that:

$$|S_n| = \left|\sum_{i=1}^{n} X_i\right| \leq \sum_{i=1}^{n} |X_i| = \sum_{i=1}^{n} \frac{1}{i^2} < 2$$

Hence, by monotonicity of expectation, we have:

$$\mathbb{E}[|S_n|^2] < 4$$

Hence, by taking $M = 4$, we have a suitable bound for S_n. By using Corollary 7.3.5, we can also deduce that:

- $\mathbb{E}[S_\infty] = \lim_{n\to\infty} \mathbb{E}[S_n] = \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^n X_i] = \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[X_i] = \lim_{n\to\infty} \sum_{i=1}^n 0$
- $\mathbb{V}[S_\infty] = \lim_{n\to\infty} \mathbb{V}[S_n] = \mathbb{E}[S_n^2] - \mathbb{E}[S_n]^2 = \mathbb{E}[S_n^2] = \sum_{i=1}^n \frac{1}{i^4}$

Assignment 4

1. Let B_t be a standard Brownian motion

   (a). Write down the distribution of B_t, and write down $\mathbb{E}[B_t]$ and $\mathbb{E}[B_t^2]$

   (b). Let $0 \leq u \leq t$. Show that $\mathbb{E}[(B_t - B_u)^2 | \mathcal{F}_u] = t - u$

(a). By the definition of standard Brownian motion, we have $B_t \sim \mathcal{N}(0, t)$. Hence, we know:

- $\mathbb{E}[B_t] = 0$
- $\mathbb{V}ar[B_t] = t \Rightarrow \mathbb{E}[B_t^2] - \mathbb{E}[B_t]^2 = t \Rightarrow \mathbb{E}[B_t^2] = t$

(b). By the definition of Brownian Motion, we have that $B_t - B_u$ is independent of $\mathcal{F}_u$. Hence, $(B_t - B_u)^2$ is independent of $\mathcal{F}_u$.

Hence, by the independence property of conditional expectation:

$$\mathbb{E}[(B_t - B_u)^2 | \mathcal{F}_u] = \mathbb{E}[(B_t - B_u)^2]$$

At this point we can show the result using two possible arguments. We can simply note that, by the definition of Brownian Motion, $B_t - B_u \sim \mathcal{N}(0, t-u)$. Hence, $\mathbb{V}ar[B_t - B_u] = t - u = \mathbb{E}[(B_t - B_u)^2]$.

Otherwise, we can proceed as follows. By linearity of conditional expectation:

$$\mathbb{E}[(B_t - B_u)^2] = \mathbb{E}[B_t^2 - 2B_tB_u + B_u^2] = \mathbb{E}[B_t^2] - 2\mathbb{E}[B_tB_u] + \mathbb{E}[B_u^2] =$$

Given the results of part (a):

$$= t - 2\mathbb{E}[B_tB_u] + u =$$

Using the "taking $\mathbb{E}$" property of conditional expectation:

$$= t - 2\mathbb{E}[\mathbb{E}[B_tB_u | \mathcal{F}_u]] + u =$$

Using the taking out what is known property of conditional expectation, and the fact the Brownian motion is a martingale:

$$= t - 2\mathbb{E}[B_u\mathbb{E}[B_t | \mathcal{F}_u]] + u = t - 2\mathbb{E}[B_u^2] + u = t - 2u + u = t - u$$

2. Write down the following stochastic differential equations in integral form, over the time interval $[0, t]$.

   (a). $dX_t = 2(X_t + 1)dt + 2B_t dB_t$.

   (b). $dY_t = 3Y_t dt$.

   Write down a differential equation satisfied by Y_t, and find its solution with the initial condition $Y_0 = 1$.

   Suppose that $X_0 = 1$. Show that $f(t) = \mathbb{E}[X_t]$ satisfies $f'(t) = 2f(t) + 2$ and hence find $f(t)$.

We have:

(a). $X_t = X_0 + \int_0^t 2(X_u + 1)du + \int_0^t 2B_u dB_u$.

(b). $Y_t = Y_0 + 3\int_0^t Y_u du$.

Using the fundamental theorem of calculus, we get:

$$\frac{dY_t}{dt} = 3Y_t$$

We can solve the equation by operating the substition $Z_t = \log(X_t)$:

$$\frac{dZ_t}{dt} = \frac{dZ_t}{dY_t}\frac{dY_t}{dt} = \frac{1}{Y_t}3Y_t = 3$$

And hence:

$$Z_t = 3t + C,\ C \in \mathbb{R}$$

By substituting back:

$$Y_t = e^{3t+C},\ C \in \mathbb{R}$$

This is the general solution, to find the particular solution we impose the initial condition:

$$Y_0 = e^C = 1 \Rightarrow C = 0$$

Hence, the particular solution is:

$$Y_t = e^{3t}$$

As for the second part of the assignment, we start by calculating $\mathbb{E}[X_t]$:

$$\mathbb{E}[X_t] = \mathbb{E}\left[X_0 + \int_0^t 2(X_u + 1)du + \int_0^t 2B_u dB_u\right]$$

Using linearity of expectation, lemma 11.4.2 to swap the expectation and the integration, and the fact the the Ito integral is a martingal with mean zero:

$$\mathbb{E}[X_t] = X_0 + \int_0^t \mathbb{E}\left[2(X_u + 1)\right] du = X_0 + \int_0^t (2\mathbb{E}\left[X_u\right] + 2)du$$

And hence:

$$f(t) = X_0 + \int_0^t 2f(u)du + 2t$$

Using the fundamental theorem of calculus:

$$f'(t) = 2f(t) + 2$$

This is a linear ordinary differential equation that we can solve by using an integrating factor:

$$\alpha(t) = e^{\int_0^t -2du} = e^{-2t}$$

Hence:

$$f(t) = \frac{1}{\alpha(x)}\left(f(0) + \int_0^t 2\alpha(u)du\right) = e^{2t}(1 + 2\int_0^t e^{-2t}du) = e^{2t}(1 - (e^{-2t} - 1)) = 2e^{2t-1}$$

3. Use Ito's formula to calculate the stochastic differential of dZ_t, where:

   (a). $Z_t = tB_t$

   (b). $Z_t = 1 + t^2 X_t$ where $dX_t = \mu dt + \sigma B_t dB_t$ and μ, σ are deterministic constants.

   (c). $Z_t = e^{-2t} S_t$ where $dS_t = 2S_t dt + 5S_t dB_t$.

   In which cases is Z_t a martingale?

(a). Using the lecture notes notation, we have:

- $X_t = B_t$, hence $F_t = 0, G_t = 1$
- $f(t,x) = tx$, hence $\frac{\partial f}{\partial t} = x$, $\frac{\partial f}{\partial x} = t$, $\frac{\partial^2 f}{\partial x^2} = 0$.

Using Ito-Doeblin formula:

$$dZ_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB_t = B_t dt + t dB_t$$

(b). Here we have:

- $F_t = \mu, G_t = \sigma B_t$
- $f(t,x) = 1 + t^2 x$, hence $\frac{\partial f}{\partial t} = 2tx$, $\frac{\partial f}{\partial x} = t^2$, $\frac{\partial^2 f}{\partial x^2} = 0$.

Using Ito-Doeblin formula:

$$dZ_t = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} +\right) dt + \sigma B_t \frac{\partial f}{\partial x} dB_t = \left(2tX_t + \mu t^2\right) dt + \sigma B_t t^2 dB_t$$

(c). Here we have:

- $F_t = 2S_t, G_t = 5S_t$
- $f(t,x) = xe^{-2t}$, hence $\frac{\partial f}{\partial t} = -2xe^{-2t}$, $\frac{\partial f}{\partial x} = e^{-2t}$, $\frac{\partial^2 f}{\partial x^2} = 0$.

Using Ito-Doeblin formula:

$$dZ_t = \left(\frac{\partial f}{\partial t} + 2S_t\frac{\partial f}{\partial x} +\right) dt + 5S_t \frac{\partial f}{\partial x} dB_t = \left(-2S_t e^{-2t} + 2S_t e^{-2t}\right) dt + 5S_t e^{-2t} dB_t = 5S_t e^{-2t} dB_t$$

Finally, we have that Z_t in (c), being an Ito integral, is a martingale.

4. Let S_t be a geometric Brownian motion, with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and (deterministic) initial condition S_0.

   (a). Find $\mathbb{E}[S_t]$ and deduce that S_t is not a Brownian motion when $\mu \neq 0$.

   (b). Is S_t a Brownian motion when $\mu = 0$?

(a). We have:

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dBu$$

Hence, using lemma 11.4.2 and the fact that Ito integrals are martingals with mean 0:

$$\mathbb{E}[S_t] = S_0 + \int_0^t \mu \mathbb{E}[S_u] du$$

Using the fundamental theorem of calculus:

$$\frac{d\mathbb{E}[S_t]}{dt} = \mu \mathbb{E}[S_t]$$

Operating the substitution $f(t) = \log \mathbb{E}[S_t]$:

$$\frac{df}{dt} = \frac{df}{d\mathbb{E}[S_t]} \frac{d\mathbb{E}[S_t]}{dt} = \mu$$

And thus:

$$f(t) = \mu t + C \Rightarrow \mathbb{E}[S_t] = C' e^{\mu t}$$

Hence, if $\mu \neq 0$, we have that S_t is not a martingale, hence it can't be Brownian Motion. Note that we could have directly worked with the expression for geometric Brownian Motion, rather than with the corresponding SDE.

(b). When $\mu = 0$, we have

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma B_t}$$

Hence, S_t is a log-normal random variable, and the process can't be a Brownian Motion.