Convergence of Random Variables

Chapter 6

6.1

Let (X_n) be a sequence of independent random variables such that:

$$X_n = \begin{cases} 2^{-n} & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases}$$

Show that $X_n \xrightarrow{\mathcal{L}^1} 0$ and $X_n \xrightarrow{a.s.} 0$. Deduce that $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

Convergence in $\mathcal{L}^1$

We need to prove:

$$\mathbb{E}[|X_n - 0|] \to 0$$

We have:

$$\mathbb{E}[|X_n - 0|] = \mathbb{E}[X_n] = \frac{1}{2^{n+1}}$$

And:

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} = 0$$

Hence, we have $\mathcal{L}^1$ convergence.

Almost Sure Convergence

We need to prove:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\right\}\right] = 1$$

We have that $0 \leq X_n \leq 2^{-n}$, hence:

$$0 \leq \lim_{n\to\infty} X_n(\omega) \leq \lim_{n\to\infty} 2^{-n} = 0$$

By the sandwich rule:

$$\lim_{n\to\infty} X_n(\omega) = 0$$

and hence:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\right\}\right] = 1$$

Convergence in Probability and Distribution

Since almost sure convergence implies convergence in probability, and convergence in probability implies convergence in distribution (lemma 6.1.2), we also have $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

6.2

Let X_n, X be random variables.

(a) suppose that$X_n \xrightarrow{\mathcal{L}^1} X$ as $n \to \infty$. Show that $\mathbb{E}[X_n] \to \mathbb{E}[X]$

By assumption, we have that:

$$\mathbb{E}[|X_n - X|] \to 0$$

By the Absolute Value property of Expectation and linearity, we have:

$$0 \leq |\mathbb{E}[X_n - X]| = |\mathbb{E}[X_n] - \mathbb{E}[X]| \leq \mathbb{E}[|X_n - X|] \to 0$$

Hence, by the sandwich rule:

$$|\mathbb{E}[X_n] - \mathbb{E}[X]| \to 0$$

And, hence, $\mathbb{E}[X_n] \to \mathbb{E}[X]$

(b) Give an example where $\mathbb{E}[X_n] \to \mathbb{E}[X]$ but X_n does not converge to X in $\mathcal{L}^1$

A pretty trivial counterexample is any sequence of random variables X_n such that $\mathbb{E}[X_n] = 0$, but $\mathbb{E}[|X_n|] \neq 0$ and does not converge to zero. In that case, by setting $X = 0$, we have that $\mathbb{E}[X_n] = 0 \to 0 = \mathbb{E}[X]$, but $\mathbb{E}[|X_n - 0|]$ does not converge to zero. For example:

$$X_n = \begin{cases} n & \text{with probability } \frac{1}{2} \\ -n & \text{with probability } \frac{1}{2} \end{cases}$$

We have that $\mathbb{E}[X_n] = 0$, but:

$$\mathbb{E}[|X_n - 0|] = \mathbb{E}[|X_n|] = n$$

which does not converge to $\mathbb{E}[0] = 0$.

6.3

Let U be a random variable such that $\mathbb{P}[U = 0] = \mathbb{P}[U = 1] = \mathbb{P}[U = 2] = \frac{1}{3}$. Let:

$$X_n = \begin{cases} 1 + \frac{1}{n} & \text{if } U = 0 \\ 1 - \frac{1}{n} & \text{if } U = 1 \\ 0 & \text{if } U = 2 \end{cases}$$

and:

$$X = \begin{cases} 1 & \text{if } U \in \{0, 1\} \\ 0 & \text{if } U = 2 \end{cases}$$

Show that $X_n \xrightarrow{\mathcal{L}^1} 0$ and $X_n \xrightarrow{a.s.} 0$. Deduce that $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

Convergence in $\mathcal{L}^1$

We need to prove:

$$\mathbb{E}[|X_n - X|] \to 0$$

We have that:

$$0 \leq |X_n - X| \leq \frac{1}{n}$$

Hence, by monotonicity of expectation:

$$0 \leq \mathbb{E}[|X_n - X|] \leq \frac{1}{n}$$

And by the sandwich rule:

$$\mathbb{E}[|X_n - X|] \to 0$$

Almost Sure convergence

We need to prove:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right] = 1$$

We have:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right] =$$
$$\mathbb{P}\left[\lim_{n\to\infty} X_n = X \middle| U = 0\right]\mathbb{P}[U=0] + \mathbb{P}\left[\lim_{n\to\infty} X_n = X \middle| U = 1\right]\mathbb{P}[U=1] + \mathbb{P}\left[\lim_{n\to\infty} X_n = X \middle| U = 2\right]\mathbb{P}[U=2]$$
$$\mathbb{P}\left[\lim_{n\to\infty} 1 + \frac{1}{n} = 1 \middle| U = 0\right]\frac{1}{3} + \mathbb{P}\left[\lim_{n\to\infty} 1 - \frac{1}{n} = 1 \middle| U = 1\right]\frac{1}{3} + \mathbb{P}\left[\lim_{n\to\infty} 0 = 0 \middle| U = 2\right]\frac{1}{3} = 1$$

Convergence in Probability and Distribution

Since almost sure convergence implies convergence in probability, and convergence in probability implies convergence in distribution (lemma 6.1.2), we also have $X_n \xrightarrow{\mathbb{P}} 0$ and $X_n \xrightarrow{d} 0$.

6.4

Let X_1 be a random variable with distribution given by $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = 0] = \frac{1}{2}$. Set $X_n = X_1$ for all $n \geq 2$. Set $Y = 1 - X_1$. Show that $X_n \to Y$ in distribution, but not in probability.

To show convergence in distribution, we need to prove:

$$\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[Y \leq x] \;\forall\; x \in \mathbb{R} : \mathbb{P}[Y = x] = 0$$

We have:

$$\mathbb{P}[Y \leq x] = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x \in [0, 1) \\ 1 & \text{if } x \geq 1 \end{cases}$$

Hence, we need to prove convergence in the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

- For $(-\infty, 0)$, we have $\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X_1 \leq x] = 0$.
- For $(0, 1)$, we have $\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X_1 \leq x] = \frac{1}{2}$.
- For $(1, \infty)$, we have $\lim_{n\to\infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X_1 \leq x] = 1$.

Hence, we have convergence in distribution. Let's now look at convergence in probability, which means:

$$\lim_{n\to\infty} \mathbb{P}[|X_n - Y| > a] = 0,\ \forall\ a > 0$$

We have:

$$|X_n - Y| = |X_1 - Y| = |X_1 - 1 + X_1| = |2X_1 - 1|$$

If $X_1 = 1$, than we have that $|X_n - Y| = 1$. But also if $X_1 = 0$, than we have that $|X_n - Y| = 1$. So, for any $a \in (0, 1]$:

$$\lim_{n\to\infty} \mathbb{P}[|X_n - Y| > a] = \mathbb{P}[|2X_1 - 1| > a] \neq 0$$

And we don't have convergence in probability.

6.4

Let (X_n) be the sequence of random variables from 6.1. Define:

$$Y_n = \sum_{i=1}^{n} X_i$$

- (a) show that $\forall\ \omega \in \Omega$, the sequence $Y_n(\omega)$ is increasing and bounded.

We have that $X_n(\omega) \geq 0, \forall\ \omega \in \Omega, \forall\ n \geq 0$. Hence we have that, for any n and ω:

$$Y_{n+1}(\omega) = Y_n(\omega) + X_{n+1}(\omega) \geq Y_n(\omega)$$

That is, the sequence $Y_n(\omega)$ is increasing.

Also, we have that:

$$Y_n(\omega) \leq \sum_{i=1}^{n} \frac{1}{2^i}$$

which, being a geometric series, is bounded. Hence, $Y_n(\omega)$ is bounded.

- (b) deduce that there exists a random variable Y such that $Y_n \xrightarrow{a.s.} Y$

Given that $Y_n(\omega)$ is increasing and bounded for any ω, for the Monotone Convergence Theorem for real valued sequence, we have:

$$\lim_{n\to\infty} Y_n(\omega) = Y_\infty(\omega)$$

for some $Y_\infty(\omega)$. By proposition 2.2.6, we know that $Y_\infty(\omega)$ is a random variable.

This implies that:

$$\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y_\infty(\omega)\right\} = \Omega$$

And hence:

$$\mathbb{P}\left[\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y_\infty(\omega)\right\}\right] = 1$$

That is, Y_n converges a.s. to $Y = Y_\infty$.

- (c) Write down the distribution of Y_1, Y_2, Y_3
- (d) Suggest why we might guess that Y has a uniform distribution on $[0,1]$
- (e) Prove that Y_n has a uniform distribution on $\{k2^{-n}; k = 0, 1, ..., 2^{n-1}\}$
- (r) Prove that Y has a uniform distribution on $[0,1]$

Chapter 10

10.4

(a). Show that $\mathbb{E}[B_t^n] = t(n-1)\mathbb{E}[B_t^{n-2}] \;\; \forall n \geq 2$

To simplify notation, let's denote $X = B_t$. By the definition of Brownian motion, we know that $X \sim \mathcal{N}(0,t)$. Using the law of the unconscious statistician:

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

Note that what we want is to isolate a term corresponding to $\mathbb{E}[X^{n-2}]$. In order to do so, we need to maintain the term corresponding to the normal density. Notice that such term has derivative:

$$\frac{d}{dx} e^{-\frac{x^2}{2t}} = -\frac{x}{t} e^{-\frac{x^2}{2t}}.$$

Hence, we can proceed integrating by parts:

$$\int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} -tx^{n-1}(-\frac{x}{t} e^{-\frac{x^2}{2t}})dx$$

$$= \frac{1}{\sqrt{2\pi t}} \left[\left[-tx^{n-1} e^{-\frac{x^2}{2t}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -t(n-1)x^{n-2} e^{-\frac{x^2}{2t}} \right] =$$

$$= t(n-1) \int_{-\infty}^{\infty} x^{n-2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = t(n-1)\mathbb{E}[X^{n-2}] = t(n-1)\mathbb{E}[B_t^{n-2}]$$

(b). Deduce that $\mathbb{E}[B_t^2] = t$ and $\mathbb{V}ar[B_t^2] = 2t^2$

We have:

$$\mathbb{E}[B_t^2] = t\mathbb{E}[B_t^0] = t$$
$$\mathbb{V}ar[B_t^2] = \mathbb{E}[(B_t^2)^2] - \mathbb{E}[B_t^2]^2 = \mathbb{E}[B_t^4] - t^2 = 3t\mathbb{E}[B_t^2] - t^2 = 2t^2$$

(c). Write down $\mathbb{E}[B_t^n]$ for any $n \in \mathbb{N}$

We have that

- $\mathbb{E}[B_t^n] = 0$ if n is odd
- $\mathbb{E}[B_t^n] = (n-1)(n-3)...(1)t^{n/2}$ if n is even

We can prove it by induction. For n odd, let's start by observing that $\mathbb{E}[B_t] = 0$. We have our base case, $n = 1$. For our inductive case, let's assume that for an odd $n-2$, $\mathbb{E}[B_t^{n-2}] = 0$. Since $n-2$ is odd, so is n. We have:

$$\mathbb{E}[B_t^n] = t(n-1)\mathbb{E}[B_t^{n-2}] = 0$$

And we're done. As for n even, similarly, we have our base case for $n = 2$:

$$\mathbb{E}[B_t^2] = t\mathbb{E}[B_t^0] = t = (1)t^{2/2}$$

As for the inductive case, let's assume that, for n - 2 even:

$$\mathbb{E}[B_t^{n-2}] = (n-3)(n-5)...(1)t^{(n-2)/2}$$

Then n is also even, and:

$$\begin{aligned}\mathbb{E}[B_t^n] = t(n-1)\mathbb{E}[B_t^{n-2}] &= t(n-1)(n-3)(n-5)...(1)t^{(n-2)/2}\\ &= (n-1)(n-3)(n-5)...(1)t^{n/2}\end{aligned}$$

(d). Show that $B_t^n \in L^1$ for all $n \in \mathbb{N}$

We know that $\mathbb{V}ar[B_t^n] = \mathbb{E}[B_t^{2n}] - \mathbb{E}[B_t^n]^2 < \infty$, as both terms on the right hand side are, as we've seen in part (c). We also know that L^2 is the set of random variables with finite variance, and hence $B_t^n \in L^2$. Finally, we know that if a random variable is in L^2, then it is also in L^1. Hence, $B_t^n \in L^1$.

Chapter 11

11.1 Using (11.8) find $\int_v^t 1dB_u$, where $0 \leq v \leq t$

By the consistency property of Ito integrals:

$$\int_0^t 1dB_u = \int_0^v 1dB_u + \int_v^t 1dB_u \Rightarrow \int_v^t 1dB_u = \int_0^t 1dB_u - \int_0^v 1dB_u$$

By (10.8):

$$\int_0^v 1dB_u = B_v; \quad \int_0^t 1dB_u = B_t.$$

Hence:

$$\int_v^t 1dB_u = B_t - B_u$$

11.2 Show that the process e^{B_t} is in $\mathcal{H}^2$. (Hint: use (10.2))

We work onver a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t = \sigma(B_t)$. We need to show that:

- e^{B_t} is continuous
- e^{B_t} is adapted to $\mathcal{F}_t$
- e^{B_u} is locally square integrable, i.e. $\int_0^t \mathbb{E}[(e^{B_u})^2]du < \infty$

We have that since B_t and e^x are continuous, their composition is continuous. Furthermore, given that B_t is adapted by construction, by 2.2.6, so is e^{B_t}.

As for local square integrability, since B_t is a standard Brownian motion, we have that $B_t \sim \mathcal{N}(0, t)$, and, hence, $2B_t \sim \mathcal{N}(0, 4t)$.

Thus:

$$\mathbb{E}[(e^{B_u})^2] = \mathbb{E}[e^{2B_u}] = e^{2u}$$

Where we used (10.2) to calculate the expected value. Hence:

$$\int_0^t \mathbb{E}[(e^{B_u})^2]du = \int_0^t e^{2u}du = \left[\frac{1}{2}e^{2u}\right]_0^t = \frac{1}{2}(e^{2t} - 1) < \infty$$

Which proves that $e^{B_t} \in \mathcal{H}^2$.

11.3

(a). Let $Z \sim \mathcal{N}(0, 1)$. Show that the expectation of $e^{\frac{z^2}{2}}$ is infinite.

We have:

$$\mathbb{E}\left[e^{\frac{z^2}{2}}\right] = \int_{-\infty}^{\infty} e^{\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} dz = \infty$$

(b). Give an example of a continuous, adapted stochastic process that is not in $\mathcal{H}^2$

We know that, $X \sim \mathcal{N}(0, t) \Rightarrow \frac{1}{\sqrt{t}}X \sim \mathcal{N}(0, 1)$. We also know that $B_t \sim \mathcal{N}(0, t)$. Hence, $F_t = \frac{1}{\sqrt{t}}B_t \sim \mathcal{N}(0, 1)$.

Now, if we take the process $X_t = e^{\frac{1}{4}F_t^2}$ we have:

- $X_t \in m\mathcal{F}_t$, since $B_t \in m\mathcal{F}_t \Rightarrow F_t \in m\mathcal{F}_t$, by proposition 2.2.6, and $F_t \in m\mathcal{F}_t \Rightarrow X_t \in m\mathcal{F}_t$, also by proposition 2.2.6. Hence, X_t is adapted.
- Since B_t is continuous, and $x^2, e^x, \frac{x}{\sqrt{t}}$ are continuos, so is X_t
- However, we have:

$$\mathbb{E}[X_t^2] = \mathbb{E}[e^{\frac{1}{2}F_t^2}] = \infty$$

And, hence,

$$\int_0^t \mathbb{E}[X_u^2]du = \infty$$

And X_t is not in $\mathcal{H}^2$. *Note that we can't simply use the stochastic process $X_t = Z$, because we can't guarantee that such process is adapted.*