The Binomial Model

Contents

1 Introductory definitions, assumptions and notation

A **market** is any setting in where we can exchange one or more **asset** through time. In order to define a model of the market, hence, we need to specify how we model time, what assets are available to exchange, and how they change values through time.

When modelling markets in mathematical finance, we usually make a set of basic assumptions:

- **Not moving the market**: our actions (short or sell assets) do not affect the price of the asset. This is obviously false in absolute sense: demand affects the market price. However, if we are dealing with relatively small quantities, the effect is usually negligible
- **Liquidity**: at any point in time we can exchange any amount of any asset at its current market price. A market that satisfies this assumptions is called a **liquid** market.
- **Shorting**: we can sell assets that we do not hold, i.e., we can hold negative quantities of assets. In practice, this is usually allowed.
- **Fractional Quantities**: we can sell or bay fractional quantities of assets. This is usually not allowed, but when dealing with large quantities of money, the error is usually negligible.
- **No transaction costs (a.k.a., no frictions assumption)**: we can buy or sell assets without costs. This is usually not the case, as we have to kind of costs. First, usually brokers ask some money to execute transactions for us. But also, usually sell prices are lower than buy prices (this is called the bid-offer spread). The size of the spread is correlated with the liquidity of the market. Indeed, assuming a **liquid market** implies assuming no bid-offer spread.

Furthermore, in the Binomial Model, we will use discrete time: $t = 0, 1, 2, ..., T$.

1.1 Assets

1.1.1 Cash

Cash is a riskless asset, i.e., its value changes through time in a deterministic way. In particular, let r > 0 be a deterministic constant, the **interest rate**. If we put x cash in the bank at time zero, its value at time t will be $x(1+r)^t$. The same formula applies when x is a negative value: in that case we would be borrowing, rather than depositing (that is, lending) cash.

1.1.2 Stock

With stock we mean a collection of shares, i.e., proportions of rights of ownership of a company. As for cash, we allow ourselves to own negative amount of stock. However we assume that we don't pay any interest for borrowing it.

Stock, as opposed to cash, is not riskless: it changes its value through time randomly. Hence, we model it as the **stochastic process**:

$$(S_t)_{t=0}^{\infty}$$

But how shall we define such process? Stock prices cannot take negative values, so a multiplicative process seems a good model.

Let $u, d, s > 0$ with $u > d$. At time zero, it costs $S_0 = s$ cash to buy one unit of stock. After that, in each time step, the value of our stock is multiplied by a random variable Z:

$$S_t = \begin{cases} s & \text{if } t = 0 \\ Z_t S_{t-1} & \text{otherwise} \end{cases}$$

Where the random variable Z_t has distribution:

$$\mathbb{P}[Z_t = u] = p_u$$
$$\mathbb{P}[Z_t = d] = p_d$$

For each time step, we'll assue the Z_t's will be independent.

1.1.3 Contingent claims

Finally, A **contingent claim** is any random variable of the form:

$$X = \Phi(S_t)$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a deterministic function. We can use contingent claim to model an additional kind of assets available on the market: **derivatives**, i.e. assets whose value depends on the value of the underlying stock.

1.2 Portfolios and Value Processes

As we will model the decisions of market participants on what to trade at each time point, we have to also model what information is available to the them through time. We'll model it using a **filtration**:

$$\mathcal{F}_t = \sigma(Z_1, Z_2, ..., Z_t)$$

i.e. $\mathcal{F}_t$ is the **generated filtration** of (Z_T). That is, $\mathcal{F}_t$ contains information around changes in stock prices up to and including time t. So, we have:

$$Z_t \in m\mathcal{F}_t \Rightarrow S_t \in m\mathcal{F}_t$$

But:

$$Z_{t+1} \notin m\mathcal{F}_t \Rightarrow S_{t+1} \notin m\mathcal{F}_t$$

A **portfolio strategy** is a stochastic process:

$$h_t = (x_t, y_t) \in \mathcal{F}_{t-1}$$

where x_t is the amount of cash held during the transition $t-1 \to t$, and y_t is the units of stock held during the transition $t-1 \to t$.

Note that we defined portfolio stategies in terms of their composition of cash and stock, without allowing ourselves to hold contingent claims. This is a restrictive assumption, but useful to simplify the notation giving the argument that we will use to derive the price of the contingent claims (see the replication argument for pricing).

Note also that how we defined the measurability of the portfolio strategy makes sense: we make decisions on what portfolio to hold during the transition $t-1 \to t$ using the information around how prices changed up to time $t-1$.

The **value process** or **price process** of the portfolio strategy $(h_t)_{t=1}^T$ is the process (V^t) given by:

$$V_0^h = x_1 + y_1 S_0$$
$$V_t^h = x_t(1+r) + y_t S_t$$

This makes sense:

- as x_t is the amount of cash held during the transition $t-1 \to t$, it's "deposited" at time $t-1$, and, at time t, it grows thanks to the interest rate
- as y_t is the amount of stock held during the transition $t-1 \to t$, its value at time t is just the amount of stock multiplied by the stock price at time t

In general, we'll be interested in portfolio strategies that require an initial investment at time 0, but after that any changes in cash/stock held will pay for themselves. Such a portfolio is called **self-financing**. More formally, a portfolio strategy (h_t) is **self-financing** if:

$$V_t^h = x_{t+1} + y_{t+1} S_t$$

Let's examine this definition. What we are asking is that, at time t, we can compose the portfolio held during the transition $t \to t+1$ (i.e., x_{t+1} cash and y_{t+1} stock) just by using the value of the portfolio held during the previous transition evaluated at time t.

2 Arbitrage

When modelling a market, we want to encode in the structure of the model the fact that "there ain't no such thing as a free lunch". Basically what that means is that there aren't ways, for investors, of making money without risking any capital. We can formalise the idea: we say that a portfolio strategy (h_t) is an **arbitrage possibility** if it is **self-financing** and satisfies:

$$V_0^h = 0$$
$$\mathbb{P}[V_T^h \geq 0] = 1$$
$$\mathbb{P}[V_T^h > 0] > 0$$

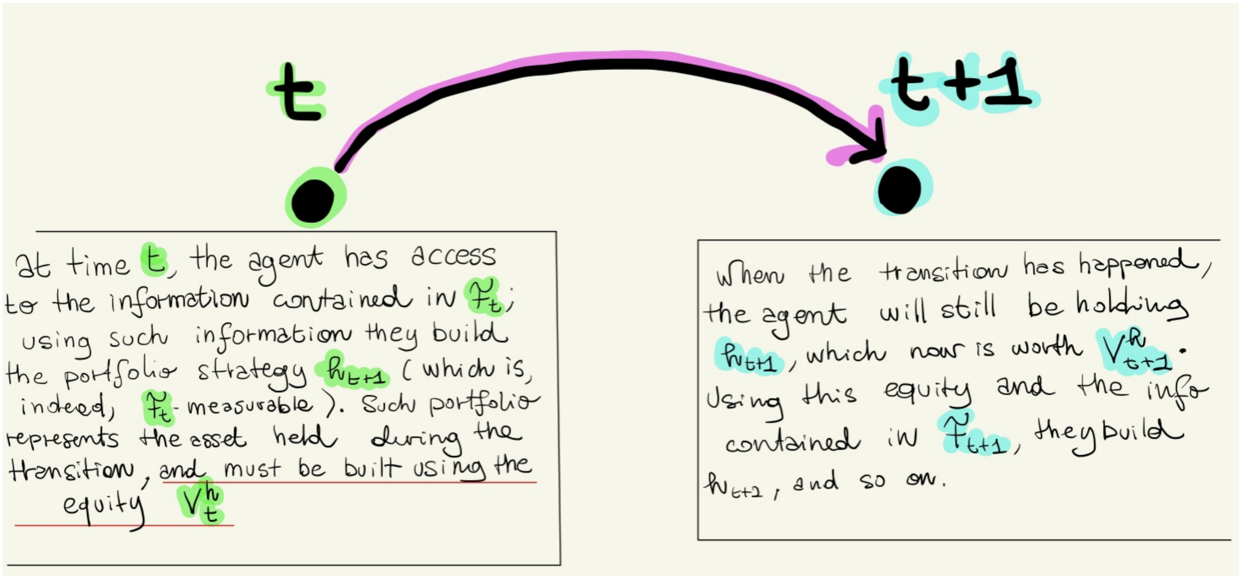

Figure 1: alt text here

2.1 Monotonicity Theorem [Joshi]

If we have two portfolio strategies h_t and k_t:

$$\mathbb{P}[V_T^h \geq V_T^k] = 1 \Rightarrow \mathbb{P}[V_t^h \geq V_t^k] = 1 \ \ \forall \ \ t < T$$

Furthermore, if:

$$\mathbb{P}[V_T^h > V_T^k] > 0$$

Then:

$$\mathbb{P}[V_t^h > V_t^k] = 1 \ \ \forall \ \ t < T$$

The proof is in [Joshi], page 27.

3 Three pricing arguments

Now that we defined our model of the market and set some notation, we can go start trying to solve our problem, that is, the pricing of derivatives. Note that, in the model we set up, this means finding the value of $\Phi(S_0)$, i.e. V_0^ϕ.

3.1 Replication

Note that a corollary of the monotonicity theorem is that:

$$\mathbb{P}[V_T^h = V_T^k] = 1 \Rightarrow \mathbb{P}[V_t^h = V_t^k] = 1 \ \ \forall \ \ t < T$$

Hence, if we can find a portfolio h_t whose value at time T is the same as the value of a portfolio composed only by the contingent claim, then their value must also be equal at time zero, i.e.:

$$\mathbb{P}[V_T^h = V_T^\phi] = 1 \Rightarrow \mathbb{P}[V_0^h = V_0^\phi] = 1 \ \ \forall \ \ t < T$$

So, if we can find such a portfolio, then we are done: the value of that portfolio at time zero is the only price for the derivative that guarantees no arbitrage. This is the argument that we will use in detail to derive the price for contingent claims. We now understand why we chose to model porfolio strategies as allocations of only stock and cash: we will use such portfolios to replicate the value of the contingent claim.

3.2 Hedging

Alternatively, let's imagine that we have sold a **derivative**, i.e., we are short on the contingent claim. What we can try to do is build a portfolio strategy h_t such that $h_T - \phi(S_T)$ (#todo rewrite with value processes) has the same value, in T, in every possible state of the world. That is, we can try to hedge our risk, replicating the deterministic behaviour of cash using stock and the derivative.

3.3 Risk-neutrality

We'll see that the two previous arguments lead to the same price for the contingent claim. Furthermore, we'll be able to express this price as the expected value of the contingent claim under a particular probability measure, $\mathbb{Q}$. Under this measure:

$$(1+r)S_0 = \mathbb{E}^{\mathbb{Q}}[S_1]$$

That is, the average rate of growth of the stock is the same as the rate of growth of cash.

This different from the real world, where investors require a higher growth for stock in order to assume the risk associated with it. Indeed, one of the key principles of finance is that there is a strong relationship between risk and return: investors require that a risky asset as a higher expected return than a riskless one.

However, the principle only holds for unavoidable risk, that is, risk that cannot be hedged or diversified. When Black, Scholes and Merton presented their argument for derivative evaluation, they gave the investors a tool to hedge their risk: Delta Hedging. Hence, the risk associated with stock was no longer unavoidable: one could build a portfolio, using stock and options, whose value in the future was (under particular assumptions) deterministic.

4 The one-period market

Let's start our analysis using a very simple version of the model: on in which we only have two time points. Since there is only one time transition, we don't need a stochastic process to model portfolio strategies. Instead, we can model them as $h = (x, y) \in \mathbb{R}^2$, where x and y are the amounts of cash and stock that we hold during the transition, that is, that we compose at time zero.

Hence, the value process of the portfolio at time zero is:

$$V_0^h = x + yS_0 = x + ys$$

At time $t = 1$:

- the x units of cash that we held at time 0 become worth $x(1+r)$
- a single unit of stock changes value to:

$$S_1 = \begin{cases} su & \text{with probability } p_u \\ sd & \text{with probability } p_d \end{cases}$$

Hence, the value process of the portfolio at time one is:

$$V_1^h = x(1+r) + yS_1$$

In this simplified setting, the value process at time zero is a real number, and at time 1 is a random variable.

4.1 Arbitrage in the one-period market

Here we want to find out for what set of parameters our model is arbitrage free. We have that the one-period market is arbitrage free if and only if:

$$d < 1 + r < u$$

Proof in [Freeman]

4.2 Replication in the one-period market

Let's price a contingent claim, in the one-period market, using the replication argument. We want to find a portfolio $h(x, y)$ such that:

$$\mathbb{P}[V_1^h = \phi(S_1)] = 1$$

That is, we want our portfolio to have the same value as the contingent claim at time 1 in every possible state of the world:

$$V_1^h = \begin{cases} \phi(su), & \text{if } S_1 = su \\ \phi(sd), & \text{if } S_1 = sd \end{cases}$$

Hence, we want:

$$\begin{cases} x(1+r) + ysu = \phi(su) \\ x(1+r) + ysd = \phi(sd) \end{cases}$$

This is a pair of linear equations that we can easily solve:

$$\begin{pmatrix} 1+r & su \\ 1+r & sd \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \phi(su) \\ \phi(sd) \end{pmatrix}$$

The system of linear equations has a unique solution if and only if its determinant is not zero, that is $(1+r)sd - (1+r)su \neq 0 \Rightarrow u \neq d$. Given that a consequence of the no-arbitrage assumption is that $d < 0$, we have that our market is **complete**, i.e., we can price any contiget claim.

And we can solve the system:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{(1+r)sd - (1+r)su} \begin{pmatrix} \phi(su)sd - \phi(sd)su \\ (1+r)\phi(sd) - (1+r)\phi(su) \end{pmatrix}$$

that is:

$$x = \frac{1}{(1+r)} \frac{\phi(sd)u - \phi(su)d}{u - d}$$

$$y = \frac{\phi(su) - \phi(sd)}{su - sd}$$

So, we found our replicating portfolio. The latter formula is known as *delta hedging formula*

As mentioned before, as a consequence of the monotonicity theorem, we have that the value of this portfolio at time zero must be equal to the value of the contingent claim. We can easily find such value:

$$V_0^h = x + sy = \frac{1}{(1+r)} \frac{\phi(sd)u - \phi(su)d}{u - d} + \frac{\phi(su) - \phi(sd)}{u - d} =$$

$$= \frac{1}{(1+r)} \left(\frac{u-(1+r)}{u-d} \phi(sd) + \frac{(1+r)-d}{u-d} \phi(su) \right)$$

By defining:

$$q_d = \frac{u-(1+r)}{u-d}, \;\; q_u = \frac{(1+r)-d}{u-d}$$

We have:

$$V_0^{\phi} = V_0^h = \frac{1}{(1+r)} \left(q_d \phi(sd) + q_u \phi(su) \right)$$

we can interpret q_d and q_u as the risk neutral probabilities we introduced before, that is $\mathbb{Q}[S_1 = su] = q_u$ and $\mathbb{Q}[S_1 = sd] = q_d$.

It follows:

$$V_0^{\phi} = V_0^h = \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}}[S_1]$$

This is known as the **risk-neutral pricing formula**. We can note that even though we defined, in our model, actual probabilities of up and down moves, these do not affect the price. This is because our model provided us with a method to perfectly hedge our portfolio, and hence to make the outcome deterministic (at least in the Small World [McElreath]). Hence, the real-world probabilities do not need to affect us.

5 The multi-period market

Let's now extend our model and allow multiple time steps t $= 0, 1, 2, ..., T$.

5.1 Arbitrage in the one-period market

As for the one-period market, we want to find out for what set of parameters our model is arbitrage free. We have that the multi-period market is arbitrage free if and only if:

$$d < 1 + r < u$$

Proof??

Furthermore, as in the one-period market, we have that the multi-period market is complete if and only if is abritrage free.

Proof??

5.2 Replication in the multi-period market

We say that a portfolio strategy $(h_t)_{t=1}^T$ is a **replicating portfolio** or **hedging strategy** for the contingent claim $\Phi(S_T)$ if:

$$V_T^h = \Phi(S_T)$$

As in the one-period market, we can use the monotonicity theorem to state that if a contingent claim $\Phi(S_T)$ has a replicating portfolio $h = (h_t)_{t=1}^T$, then the price of such contingent claim at time zero must be equal to the value of h at time zero, that is, V_0^h.

6 The asset prices model implied by the Binomial Model

Now we can think about what are the statistical properties of the model that we defined for our model of asset prices (S_t).

6.1 Martingale Property

If $d < 1 + r < u$, then, under the probability measure $\mathbb{Q}$, the stochastic process:

$$M_t = \frac{1}{(1+r)^t} S_t$$

is a Martingale w.r.t. the filtration ($\mathcal{F}_t$).

Proof

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[M_{t+1}\mathbb{1}_{\{Z_{t+1}=u\}} + M_{t+1}\mathbb{1}_{\{Z_{t+1}=d\}}|\mathcal{F}_t]$$

Now, we know that:

$$M_{t+1} = \frac{S_{t+1}}{(1+r)^{t+1}} = \frac{S_t Z_{t+1}}{(1+r)^{t+1}} = \begin{cases} \frac{uS_t}{(1+r)^{t+1}} & \text{if} \quad Z_{t+1} = u \\ \frac{dS_t}{(1+r)^{t+1}} & \text{if} \quad Z_{t+1} = d \end{cases}$$

Hence:

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\frac{uS_t}{(1+r)^{t+1}}\mathbb{1}_{\{Z_{t+1}=u\}} + \frac{dS_t}{(1+r)^{t+1}}\mathbb{1}_{\{Z_{t+1}=d\}}\middle|\mathcal{F}_t\right]$$

For Linearity of Conditional Expectation:

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = u\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t}{(1+r)^{t+1}}\mathbb{1}_{\{Z_{t+1}=u\}}\middle|\mathcal{F}_t\right] + d\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t}{(1+r)^{t+1}}\mathbb{1}_{\{Z_{t+1}=d\}}\middle|\mathcal{F}_t\right]$$

We have that $S_t \in m\mathcal{F}_t$, so we can use the *Taking out what is known* property of Conditional Expectation:

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = \frac{S_t}{(1+r)^{t+1}}\left(u\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{Z_{t+1}=u\}}|\mathcal{F}_t\right] + d\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{Z_{t+1}=d\}}|\mathcal{F}_t\right]\right)$$

We also know that Z_t is independent of $\mathcal{F}_t$, hence:

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = \frac{S_t}{(1+r)^{t+1}}\left(u\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{Z_{t+1}=u\}}\right] + d\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{Z_{t+1}=d\}}\right]\right)$$

And, since for an indicator function $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$:

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = \frac{S_t}{(1+r)^{t+1}}(uq_u + dq_d)$$

Finally, noting that $uq_u + dq_d = (1+r)$:

$$\mathbb{E}^{\mathbb{Q}}[M_{t+1}|\mathcal{F}_t] = \frac{S_t}{(1+r)^t} = M_t$$

7 Implementation