S&P 500 Index Modeling

ISE:6760

Prepared by
Group 7: Alec Countryman (33.3%), Elliott Newman (33.3%), Kelsey Petrich (33.3%)
Contribution listed in parenthesis

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1 Introduction

At a high level, a Markov Chain Model is a mathematical model based on the Markov property. The Markov Property suggests a situation in which a current time/action/presence (state) is the only feature that has an impact on future states. More concisely, the probability of an action occurring is not impacted by past sequences of events. These models have wide ranged applications, though appear frequently in statistics and finance.

Volatility models forecast risk associated with an asset through asset volatility. For this project, volatility is the statistical dispersion of returns on a market index. A high volatility indicates a high risk for an investment. Volatility is estimated using the price of stocks or derivatives of stock prices. Volatility changes over time, continuously.

An autoregressive conditional heteroscedasticity (ARCH) model is used to model a change in variance in a time series that is time dependent. The generalized ARCH or GARCH model is used when the error variance follows an autoregressive moving average (ARMA) model.

Markov-Switching GARCH (MSGARCH) models are used to predict regime changes in dynamic variances of time series data. MSGARCH models were developed to fit the residuals of the daily log returns of the S&P 500 Index. The model results were used to test conditional volatility and value-at-risk (VaR) of the S&P 500 Index.

2 Analysis

The daily log returns of the S&P 500 Index from January 1, 1990 to April 15, 2020 were used to develop MSGARCH models. Figure 2-1 shows the autocorrelations for the daily log returns of the S&P 500. Based on visual inspection of the autocorrelations, the daily log returns appear to be stationary.

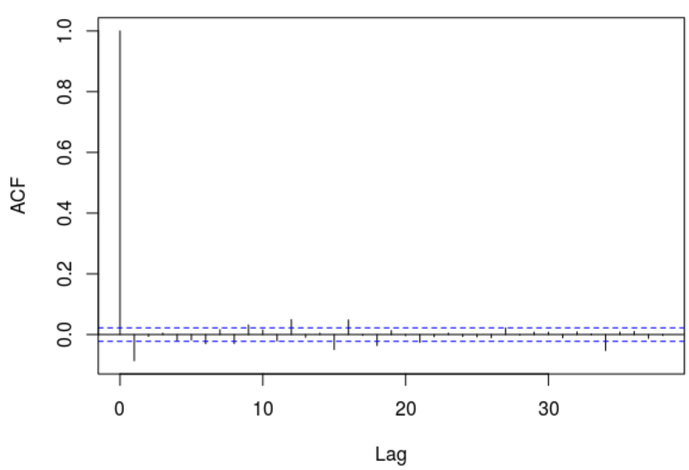

Figure 2-1 Autocorrelation of Daily Log Returns of S&P 500

2.1 Model Fitting

The daily log returns of the adjusted closing prices of the S&P 500 Index are best described by an ARMA(0,1) model when optimizing for the lowest BIC.

The autocorrelations of the residuals of the ARMA(0,1) model is shown in Figure 2-2. Based on visual inspection of the autocorrelations, the residuals appear to be stationary. From the Ljung-Box test of the residuals, the p-value was < 0.05 at a t-statistic of 90.78 meaning that there are ARCH effects present in the residuals.

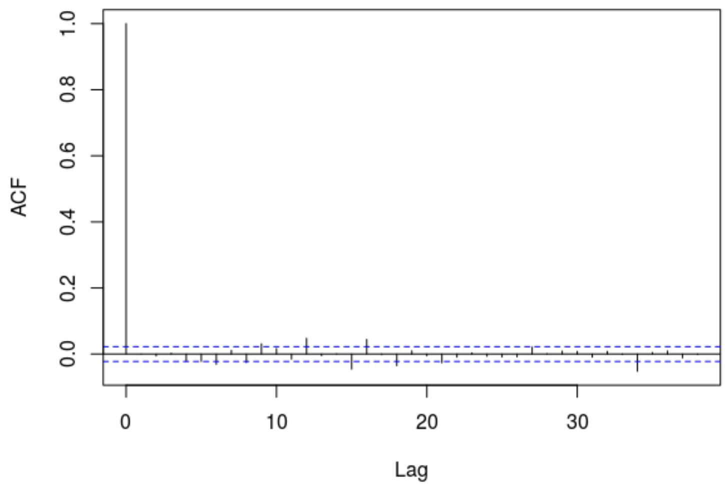

Figure 2-2 Autocorrelation of the ARMA Residuals

To adequately model the ARCH effects of the residuals, twelve models were developed based on the following possible volatility model settings for the residuals of the ARMA(0,1) model:

- Number of states/regimes (1 or 2)
- Conditional variance dynamics (GARCH model or GJR-GARCH model)
- Conditional distributions (Normal, Student-t, or skewed Student-t)

2.2 In-sample Comparison

Based on in-sample performance based on Bayesian Information Criterion (BIC), the GJR-GARCH model with two regimes and skewed Student-t distribution performed the best. Table 2-1 shows the ranking of the models from most preferred to least preferred based on lowest BIC to highest BIC.

The models using GJR-GARCH have better in-sample performance with the skewed Student-t distribution and Student-t distribution.

Table 2-1 MSGARCH Models Ranked by BIC

Rank	Model Parameters					
	GARCH (sGARCH)	GJR-GARCH	Normal Distribution	Student-t Distribution	Skewed Student-t Distribution	Number of Regimes/ States
1		x			x	2
2		x			x	1
3		x		x		2
4		x		x		1
5		x	x			2
6	x				x	1
7	x				x	2
8	x			x		1
9	x			x		2
10		x	x			1
11	x		x			2
12	x		x			1

2.3 Model Interpretation

The best model is a GJR-GARCH model with a skewed student t distribution and two regimes. The GJR-GARCH model can be described by the following equation:

$$h_{k,t} = \propto_{0,k} + \left(\propto_{1,k} + \propto_{2,k} \mathbb{I}\{y_{t-1} < 0\}\right) y_{t-1}^2 + \beta_k h_{k,t-1}$$

where $\mathbb{I}$ is an indicator function introduced to capture the leverage effect.

The best performing model is written out as:

$$h_{k,t} = (0.13\mathbb{I}\{y_{t-1} < 0\}) y_{t-1}^2 + 0.92 h_{2,t-1}$$

$\propto_{0,k}$ and $\propto_{1,,k}$ were both zero in the best performing model.

This model is heterogenous across the two regimes. The model parameters suggest low unconditional volatility in both regimes[1]. $\propto_{2,2}$ is greater than zero indicating the presence of the leverage effect and that previous negative returns have higher influence on the volatility. The volatility persistence is similar in both regimes suggesting low persistence of the volatility process.

[1] https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0198753

Figure 2-3 shows the smoothed state probability transitions over time for the best model.

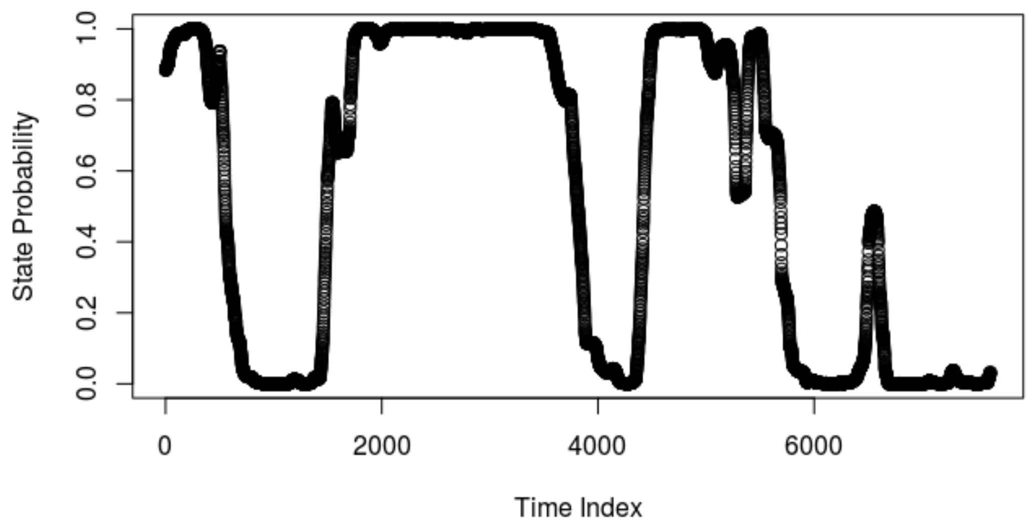

Figure 2-3 Smoothed state probability transitions over time for best model

2.4 Out-of-Sample Comparison

Out-of-sample testing was used to measure each model fit and prevent over-fitting. This enables better forecasting which is what is desired for time series financial data. The test period had rolling windows of 2000 observations. The models were re-estimated every 100 iterations. These tests were done at both the 1% and the 5% risk levels. Value at risk (VaR) was used for forecasting at the defined risk levels for 1-step ahead. Finally, the subsamples used for forecasting was on the data length (7629 observations) minus the test window resulting in samples of 5629 entries.

For each risk level backtesting was performed on all 12 models. A VaR curve was constructed based in its risk level and compared to the returns for each risk level. For the 1% risk level, models fit using a skewed Student-t distribution outperformed those with normal, Student-t, and their respective skewed distributions while the normal-distributed models did poorly in comparison. At the 5% risk level, some skewed Student-t distribution models perform well, however the best two models are those with normally distributed states. The model performance evaluation was based on the percent of actual residuals less than the VaR curve. Table 2-2 and Table 2-3 shows the percent of residuals less than the VaR curve at each risk level. The best model (Rank 1) is where the percent of residuals is closest to the risk level.

Table 2-2 Backtesting at 5% VaR results

	Model Parameters						
Rank	GARCH (sGARCH)	GJR-GARCH	Normal Dist'n	Student-t Dist'n	Skewed Student-t Dist'n	Number of Regimes/ States	5% VaR Risk (%)
1	x		x			1	5.03%
2		x	x			1	5.10%

Rank	Model Parameters						5% VaR Risk (%)
	GARCH (sGARCH)	GJR-GARCH	Normal Dist'n	Student-t Dist'n	Skewed Student-t Dist'n	Number of Regimes/ States	
3	x				x	1	4.80%
4		x			x	1	4.80%
5		x	x			2	5.24%
6	x			x		1	5.28%
7	x				x	2	4.67%
8		x			x	2	4.64%
9	x		x			2	5.40%
10		x		x		1	5.47%
11		x		x		2	5.47%
12	x			x		2	5.58%

Table 2-3 Backtesting at 1% VaR results

Rank	Model Parameters						1% VaR Risk (%)
	GARCH (sGARCH)	GJR-GARCH	Normal Dist'n	Student-t Dist'n	Skewed Student-t Dist'n	Number of Regimes/ States	
1		x			x	1	0.97%
2	x				x	1	0.96%
3		x			x	2	0.94%
4	x				x	2	1.07%
5		x		x		1	1.24%
6		x		x		2	1.26%
7	x			x		1	1.30%
8	x			x		2	1.37%
9	x		x			2	1.63%
10		x	x			2	1.65%
11		x	x			1	1.67%
12	x		x			1	1.81%

3 Conclusion

The S&P 500 Index can be best modeled using a GJR-GARCH model with a skewed Student-t distribution and two regimes based on in-sample comparison. However, for out-of-sample comparison, the S&P 500 Index is best modeled by a GARCH model with a normal distribution and 1 regime and a GJR-GARCH model with a skewed Student-t distribution and 1 regime for 5% and 1% VaR, respectively.