

The following is a derivation of the equation for a two dimensional density distribution of a Bose-Einstein Condensate (BEC) and thermal cloud.

We begin by recognizing that our density distribution is simply the sum of the density distribution of the thermal cloud, n_{ex} , and the density distribution of the BEC, n_0 . Thus:

$$n(x, y) = n_0(x, y) + n_{ex}(x, y) \quad (1)$$

Let's derive an equation for the BEC's density distribution first. Pethick and Smith Eq. 2.32 gives us that:

$$n_0(x, y) = N_0 |\psi_0(x, y)|^2 \quad (2)$$

where N_0 is the number of particles in the BEC.

Furthermore, based on Pethick and Smith Eq. 2.33's three dimensional formula, we can describe the ground-state wave function of a two dimensional anisotropic harmonic oscillator with the following equation:

$$\psi_0(x, y) = \frac{1}{\sqrt{\pi a_x a_y}} e^{\frac{-x^2}{2a_x^2}} e^{\frac{-y^2}{2a_y^2}} \quad (3)$$

where a_i represents the width of the wave function in the direction i given by the square root of Pethick and Smith Eq. 2.34:

$$a_i = \sqrt{\frac{\hbar}{m\omega_i}} \quad (4)$$

Squaring this wavefunction gives us:

$$|\psi_0(x, y)|^2 = \frac{1}{\pi a_x a_y} e^{\frac{-x^2}{a_x^2}} e^{\frac{-y^2}{a_y^2}} \quad (5)$$

Pethick and Smith Eq. 2.29 tells us that the number of particles in a BEC for a two dimensional harmonic oscillator potential is given by:

$$N_0 = N \left[1 - \left(\frac{T}{T_c} \right)^2 \right] \quad (6)$$

where N is the total number of particles in the system and T_c is the transition temperature, which Pethick and Smith define as “the highest temperature at which the macroscopic occupation of the lowest-energy state appears.” When T is greater than T_c , we don’t expect to see any particles in the condensate.

Now, substituting Eq. 5 and Eq. 6 into Eq. 2, we find that:

$$n_0(x, y) = N \left[1 - \left(\frac{T}{T_c} \right)^2 \right] \frac{1}{\pi a_x a_y} e^{-\frac{x^2}{a_x^2}} e^{-\frac{y^2}{a_y^2}} \quad (7)$$

Before continuing further, we want to make all of our quantities dimensionless for easier analysis. Let’s establish these dimensionless quantities:

$$k\tilde{T} \equiv \frac{kT}{\hbar\omega} \quad (8)$$

$$k\tilde{T}_c \equiv \frac{kT_c}{\hbar\omega} \quad (9)$$

$$\tilde{x} \equiv \frac{x}{a_x} \quad (10)$$

$$\tilde{y} \equiv \frac{y}{a_y} \quad (11)$$

As a small aside, using Pethick and Smith Eq. 2.19, we can define the unitless transition temperature energy, $k\tilde{T}_c$, in a two dimensional system as:

$$k\tilde{T}_c = \left(\frac{3N}{\pi^2} \right)^{\frac{1}{2}} \quad (12)$$

Let’s substitute these dimensionless quantities into Eq. 7:

$$n_0(x, y) = N \left[1 - \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \right] \frac{1}{\pi a_x a_y} e^{-\tilde{x}^2} e^{-\tilde{y}^2} \quad (13)$$

Multiplying Eq. 13 by $a_x a_y$, we find a dimensionless quantity, \tilde{n}_0 , for a two dimensional BEC density distribution:

$$\tilde{n}_0(x, y) = N \left[1 - \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \right] \frac{1}{\pi} e^{-\tilde{x}^2} e^{-\tilde{y}^2} \quad (14)$$

where:

$$\tilde{n}_0(x, y) = a_x a_y n_0(x, y) \quad (15)$$

Now, let's derive an equation for the density distribution of the thermal cloud. Pethick and Smith 2.39 gives us:

$$n_{ex}(x, y) = \frac{N_{ex}}{\pi R_x R_y} e^{\frac{-x^2}{R_x^2}} e^{\frac{-y^2}{R_y^2}} \quad (16)$$

where R_i is given by Pethick and Smith 2.40:

$$R_i^2 = \frac{2kT}{m\omega_i^2} \quad (17)$$

and i represents direction.

After multiplying by $\frac{\hbar\omega}{\hbar\omega}$ and rearranging, Eq. 17 can be rewritten as:

$$R_i^2 = \frac{2kT}{\hbar\omega_i} \frac{\hbar}{m\omega_i} \quad (18)$$

This can be further simplified using Eq. 4 and Eq. 8:

$$R_i^2 = 2k\tilde{T}a_i^2 \quad (19)$$

Finally, substituting Eq. 10 and Eq. 11 into this equation and rearranging, we get:

$$\frac{x^2}{R_x^2} = \frac{\tilde{x}^2}{2k\tilde{T}} \quad (20)$$

$$\frac{y^2}{R_y^2} = \frac{\tilde{y}^2}{2k\tilde{T}} \quad (21)$$

Now, let's substitute Eq. 20 and Eq. 21 into Eq. 16:

$$n_{ex}(x, y) = \frac{N_{ex}\tilde{x}\tilde{y}}{2\pi k\tilde{T}xy} e^{\frac{-\tilde{x}^2}{2k\tilde{T}}} e^{\frac{-\tilde{y}^2}{2k\tilde{T}}} \quad (22)$$

Pethick and Smith Eq. 2.27 gives us an equation for the number of particles in the thermal cloud:

$$N_{ex} = N \left(\frac{T}{T_c} \right)^2 \quad (23)$$

where we can easily replace the temperatures with unitless quantities, giving us:

$$N_{ex} = N \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \quad (24)$$

Substituting Eq. 24 into Eq. 22 gives:

$$n_{ex}(x, y) = N \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \frac{\tilde{x}\tilde{y}}{2\pi k\tilde{T}xy} e^{\frac{-\tilde{x}^2}{2k\tilde{T}}} e^{\frac{-\tilde{y}^2}{2k\tilde{T}}} \quad (25)$$

Multiplying Eq. 25 by $a_x a_y$, we find a dimensionless quantity, \tilde{n}_{ex} , for a two dimensional thermal cloud density distribution:

$$\tilde{n}_{ex}(x, y) = N \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \frac{1}{2\pi k\tilde{T}} e^{\frac{-\tilde{x}^2}{2k\tilde{T}}} e^{\frac{-\tilde{y}^2}{2k\tilde{T}}} \quad (26)$$

where:

$$\tilde{n}_{ex}(x, y) = a_x a_y n_{ex}(x, y) \quad (27)$$

Finally, we can substitute Eq. 14 and Eq. 26 into Eq. 1 to find a dimensionless quantity, \tilde{n} , for a two dimensional density distribution including a BEC and thermal cloud:

$$\tilde{n}(x, y) = N \left[1 - \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \right] \frac{1}{\pi} e^{\tilde{x}^2} e^{\tilde{y}^2} + N \left(\frac{k\tilde{T}}{k\tilde{T}_c} \right)^2 \frac{1}{2\pi k\tilde{T}} e^{\frac{-\tilde{x}^2}{2k\tilde{T}}} e^{\frac{-\tilde{y}^2}{2k\tilde{T}}} \quad (28)$$

where:

$$\tilde{n}(x, y) = a_x a_y n(x, y) \quad (29)$$