Study Guide

Exam FM: Financial Mathematics
Society of Actuaries (SOA)

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1 Time Value of Money (10%-15%)

Information

Objective

The Candidate will understand and be able to perform calculations relating to present value, current value, and accumulated value.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Interest rate (rate of interest) ;
   - Simple interest ;
   - Compound interest ;
   - Accumulation function ;
   - Future value ;
   - Current value ;
   - Present value ;
   - Net present value ;
   - Discount factor ;
   - Discount rate (rate of discount) ;
   - Convertible m-thly $(\dots?)$;
   - Nominal rate ;
   - Effective rate ;
   - Inflation ;
   - Real rate of interest ;
   - Force of interest ;
   - Equation of value.

b) Given any 3 of :
   - Interest rate ;
   - Period of time ;
   - Present value ;
   - Current value ;
   - Future value,

   calculate the remaining item using *simple* or *compound* interest ;

   Solve time value of money equations involving variable force of interest ;

c) Given any 1 of :
   - Effective interest rate ;
   - Nominal interest rate convertible m-thly ;
   - Force of interst,

   calculate any of the other items ;

d) Write the equation of value given a set of cash flows and interest rate.

Related lessons ASM

Section 1 : Interest rates and Discount Rates

- **??**
- **??**
- **??**
- **??**
- **??**
- **??**
- **??**
- **??**

Section 2 : Practical Applications

- **??**
- **??**

Chapter summaries

1a. Basic Concepts

Effective rate of interest

$a(t)$ **Accumulation function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of \$1.00 at time 0.

- $a(0) \equiv 1$.
- Generally **continuous** and **increasing**.

$a(t) - a(t-1)$ ***Amount* of growth** in the t^{th} year.

- a.k.a. the interest earned

$\frac{a(t)-a(t-1)}{a(t-1)}$ ***Rate* of growth** in the t^{th} year.

- a.k.a. effective rate of interest denoted i_t.

$A(t)$ **Amount function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of \$$k$ at time 0.

- $A(t) = ka(t)$.

i_t **Effective rate of interest** defined as the rate of growth based on the amount in the fund at the ***beginning*** of the year.

- $i_t = \frac{A(t)-A(t-1)}{A(t-1)}$.
- We deduce $A(t) = (1 + i_t)A(t-1)$.

Effective Rate of Discount

d_t **Effective rate of discount** defined as the rate of growth based on the amount in the fund at the ***end*** of the year.

- $d_t = \frac{A(t)-A(t-1)}{A(t)}$.
- Although we could get by without it, it's useful to determine the amount to pay today for a specified amount in the future.

Discounting Finding the price we'd be willing to pay for the promise to receive a future amount.

- a.k.a. finding the present value which is why $i = \frac{d}{1-d}$.
- $v = (1-d) = \frac{1}{1+i}$.
- $d = \frac{i}{1+i}$.

Nominal Rates of Interest

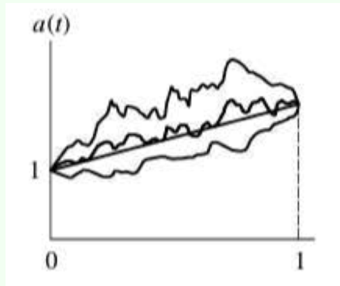
$i^{(m)}$ **Nominal** annual rate of of interest **compounded** m **times a year**.

$\frac{i^{(m)}}{m}$ **Effective** rate of of interest **for an** m^{th} **of a year**.

- Thus $(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m$.

1b. Why Do We Need a Force of Interest ?

- An effective rate of interest gives information about both the starting and ending values but none on the value in between.
- In contrast, the force of interest can give information **at any given time** about the rate of growth.
- The plot of four different funds' growth curves (with both the same starting and ending values) is a perfect visualization :



1c. Defining the Force of Interest

- To obtain a rate of growth *proportional to the amount invested*, the derivative is divided by the amount function.
- Two funds can have the same rate of change but different amounts originally invested.
- In this case, the fund growing with the smaller amount of money actually a *smaller* rate of change.

TRAP If given the derivative of the accumulation function, $a'(t)$, use the property that the fund at the beginning is 1, $a(0) = 1$, to define the $+C$ when integrating for $a(t)$.

1d. Finding the Fund in Terms of the Force of Interest

- If we want to find the accumulation, or amount, function from the force of interest we inverse the equation.
- To do so, recall that $\frac{\partial}{\partial x}\ln(f(x)) = \frac{f'(x)}{f(x)}$.
- Also $\int_0^t \frac{\partial}{\partial r}\ln(a(r))dr = \ln(a(r))\Big|_0^t = \ln(a(t))$.

Force of interest

Force of interest the rate of growth at a point in time.

- a.k.a. finding the present value which is why $i = \frac{d}{1-d}$.
- $v = (1-d) = \frac{1}{1+i}$.
- $d = \frac{i}{1+i}$.

δ_t The **Force of interest** at time t.

- $\delta_t = \frac{A'(t)}{A(t)}$.
- $a(t) = \mathrm{e}^{\int_0^t \delta_r dr}$

1e. The Simplest Case : A Constant Force of Interest

Simple Force of interest

δ The constant force of interest.

- a.k.a. the nominal rate of interest compounded continuously.
- $\delta = \lim_{m\to\infty} i^{(m)} = i^{(\infty)} = \ln(1+i)$.
- $a(t) = \mathrm{e}^{\int_0^t \delta dr} = \mathrm{e}^{\delta t}$.

1f. Power Series

- Not really on past exams, section is « just in case ».

1g. The Variable Force of Interest Trap

- When we want the accumulated value of an amount not invested at the beginning, we integrate the force of interest over the respective integral.
- Alternatively, we can take the ratio of the accumulation function at both times.

Variable Force of interest

$$FV = e^{\int_{t_1}^{t_2} \delta_r dr}$$
$$\equiv \frac{a(t_2)}{a(t_1)}$$

1h. Equivalent Rates

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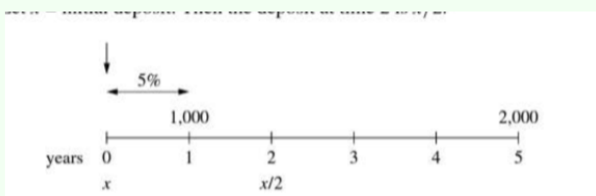
2a. Equations of Value, Time Value of Money, and Time Diagrams

Time value (equivalence principle) 1$ today is not equivalent to 1$ a year from now. However, 1$ today is equivalent to 1.05$ a year from now if the rate of interest is 5%.

Comparison date Date at which we solve the **equation of of value**.

- Important to use **time lines** to solve problems :



- We can treat a problem by **payment** or **interest** period.

2b. Unknown Time and Unknown Interest Rate

- Can approximate the time $\bar{t}$ by using a weighted average of the time of payment times the amount of payment divided by the total amount paid with the **method of equated time**. For example :

Time Due	Payment
1	5
3	1
10	15
Total Payments =	21

$$\bar{t} = \frac{1*5+3*1+10*15}{(5+1+15)}$$

2 Topic : Annuities / cash flows with non-contingent payments (15%-20%)

Information

Objective

The Candidate will be able to calculate present value, current value, and accumulated value for sequences of non-contingent payments.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Annuity-immediate ;
   - Annuity-due ;
   - Perpetuity ;
   - Payable m-thly or continously ;
   - Level payment annuity ;
   - Arithmetic increasing/decreasing annuity ;
   - Geometric increasing/decreasing annuity ;
   - Term of annuity ;

b) For each of the following types of annuity / cash flows, given sufficient information of :
   - Immediate or due ;
   - Present value ;
   - Future value ;
   - Current value ;
   - Interest rate ;
   - Payment amount ;

- Term of annuity,

calculate any remaining item.

The types are :

- Level annuity, finite term ;
- Level perpetuity ;
- Non-level annuities / cash flows ;
  - Arithmetic progression, finite term and perpetuity ;
  - Geometric progression, finite term and perpetuity ;
  - Other non-level annuities / cash flows.

Related lessons ASM

Section 3 : Annuities

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Section 4 : Complex Annuities

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Chapter summaries

3a. The Geometric Series Trap

Remember the formula for geometric series in words :

$$\begin{aligned} r^{10} + r^{20} + \cdots + r^{10n} &= r^{10}\frac{1-r^n}{1-r} \\ &= (\text{first term})\frac{1-(\text{ratio})^{\text{nb. of terms}}}{1-(\text{ratio})} \end{aligned}$$

3b. Annuity-Immediate and Annuity-Due

- The word annu(s) origins from latin meaning « yearly ».
- Standard annuity formulas for $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$.
- Interesting to note the relations between them however.

Formulas

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\
&= \frac{1 - v^n}{1 - v} \\
&= \frac{1 - v^n}{d} \\
a_{\overline{n}|} &= v + v^2 + \cdots + v^{n-1} + v^n \\
&= v\left(\frac{1 - v^n}{1 - v}\right) \\
&= \frac{1 - v^n}{i}
\end{aligned}$$

$$\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i) + \cdots + (1+i)^{n-1} + (1+i)^n \\
&= (1+i)\left(\frac{1 - (1+i)^n}{1 - (1+i)}\right) \\
&= \frac{(1+i)^n - 1}{d} \\
s_{\overline{n}|} &= 1 + (1+i) + \cdots + (1+i)^{n-1} \\
&= \frac{1 - (1+i)^n}{1 - (1+i)} \\
&= \frac{(1+i)^n - 1}{i}
\end{aligned}$$

Relations

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= (1+i)a_{\overline{n}|} \\
&= a_{\overline{n-1}|} - 1 \\
\ddot{s}_{\overline{n}|} &= (1+i)s_{\overline{n}|} \\
&= s_{\overline{n+1}|} - 1
\end{aligned}$$

3c. The Great Confusion : Annuity-Immediate and Annuity-Due

- Defining whether annuities are due or immediate based on *when payments are made* is deceptive, it is more precise to define it based on the *valuation date*.

Annuity

An annuity is called an **annuity-immediate** if, in determining its present value, the valuation date is ***one period before*** the first payment (symbol $a_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its present value, the valuation date is ***on*** the date of the first payment (symbol $\ddot{a}_{\overline{n}|}$).

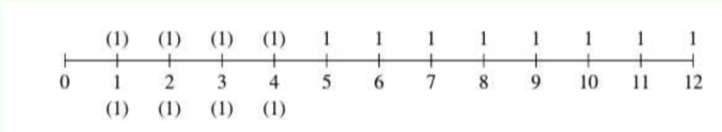
An annuity is called an **annuity-immediate** if, in determining its accumulated value, the valuation date is ***on*** the date of the last payment (symbol $s_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its accumulated value, the valuation date is ***one period after*** the date of the last payment (symbol $\ddot{s}_{\overline{n}|}$).

- Important to distinguish dates in time from the number of payments.

  For example, if we're the 1st of January in 2000 and annual payments are made on the 1st of January from 2006 to 2010 then the AV on the date of the last deposit is $s_{\overline{5}|}$ and not $s_{\overline{10}|} - s_{\overline{5}|}$ nor $s_{\overline{2010}|}$, etc.
- Better to set up equations of value with annuity-immediate than annuity-due.

3d. Deferred Annuities

- An n-year annuity deferred r years ${}_{r|}a_n = v^r a_n$.
- Can interpret as "go to to time r and start paying what the symbol to the right says".
- Can also interpret by playing "Now you see it ..." and redefining ${}_{r|}a_n = a_{n+r} - a_r$:



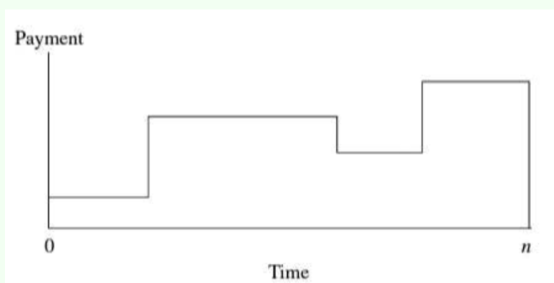
The value is thus ${}_{4|}a_8 = a_{12} - a_4$ in the example.

Deferred Annuities

$$
\begin{aligned}
{}_{r|}a_n &\equiv {}_{r+1|}\ddot{a}_n \\
v^r a_n &\equiv v^{r+1}\ddot{a}_n \\
a_{n+r} - a_r
\end{aligned}
$$

3e. A Short-Cut Method for Annuities with "Block" Payments

Block payments :



The long way is to calculate the payments by block and divide horizontally (first 8 payments, next 7 payments, etc.).

PV of first 8 payments:	$5a_{\overline{8}|}$
PV of next 7 payments:	$12(a_{\overline{15}|} - a_{\overline{8}|})$
PV of next 7 payments:	$10(a_{\overline{22}|} - a_{\overline{15}|})$
PV of next 6 payments:	$15(a_{\overline{28}|} - a_{\overline{22}|})$

Adding up all the present values, combining terms and writing in descending order of the periods:

$$PV = 15a_{\overline{28}|} - 5a_{\overline{22}|} + 2a_{\overline{15}|} - 7a_{\overline{8}|}$$

The short way starts from the end adding or decreasing annuities according to the change in payment amount.

We want the PV, so the comparison date is time 0. We start with the furthest payment ($15 at time 28) and immediately write $15a_{\overline{28}|}$. We move in closer from time 28 toward time 0 until there is a change. This occurs at time 22, when payments *decrease by $5* (from $15 to $10), so we write $-5a_{\overline{22}|}$. We move in closer, see another change at $t = 15$, when payments *increase by $2* (from $10 to $12) so we write $+2a_{\overline{15}|}$. Finally, the last change is at time 8, a *decrease of $7* (from $12 to $5), so we write $-7a_{\overline{8}|}$.

Putting all of this together, we have:

$$PV = 15a_{\overline{28}|} - 5a_{\overline{22}|} + 2a_{\overline{15}|} - 7a_{\overline{8}|}$$

The same idea is maintained for the AV :

Accumulated Value

The AV of annuities with block payments is obtained in much the same way as the PV. For example, consider the annuity just above. The comparison date is time 30 if we want the AV on the date of the last payment, so we start with the *furthest payment* from time 30, which is $5 at time 1. We immediately write $5s_{\overline{30}|}$. As we move toward the comparison date of time 30, we see that the first change is an *increase of $3* (from $5 to $8) at time 11, so our adjustment term is $+3s_{\overline{20}|}$. (20 is the number of payments that we must increase by $3, i.e., the payments from time 11 to time 30, inclusive, or 30−10 payments.) The next change is an *increase of $4* (from $8 to $12) at time 19, so the adjustment term is $+4s_{\overline{12}|}$. Finally, payments *increase by $8* (from $12 to $20) at time 24, so the adjustment term is $+8s_{\overline{7}|}$. Putting it all together:

$$AV = 5s_{\overline{30}|} + 3s_{\overline{20}|} + 4s_{\overline{12}|} + 8s_{\overline{7}|}$$

3f. Perpetuities

Perpetuity-immediate $a_{\overline{\infty}|} = \frac{1}{i}$. One way to prove this is with the limit :

$$\lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty}\left(\frac{1-v^n}{i}\right) = \frac{1}{i}$$

Perpetuity-due $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$. One way to prove this is with the limit :

$$\lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \lim_{n\to\infty}\left(\frac{1-v^n}{d}\right) = \frac{1}{d}$$

Relationship The PV of a perpetuity-due exceeds that of a perpetuity-immediate by the payment of 1 at time 0 :

$$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|} = 1 + \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d}$$

$$\ddot{a}_{\overline{\infty}|} - a_{\overline{\infty}|} = \frac{1}{d} - \frac{1}{i} = 1$$

3g. The $a_{\overline{2n}|}/a_{\overline{n}|}$ Trick (and Variations)

$a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$ and can proven several ways :

Difference of squares :

$$a_{\overline{2n}|}/a_{\overline{n}|} = \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} = \frac{1-v^{2n}}{1-v^n} = \frac{(1-v^n)(1+v^n)}{1-v^n} = 1 + v^n$$

General reasoning :

$$a_{\overline{2n}|}/a_{\overline{n}|} = \frac{a_{\overline{n}|} + {}_{n|}a_{\overline{n}|}}{a_{\overline{n}|}} = \frac{a_{\overline{n}|} + v^n a_{\overline{n}|}}{a_{\overline{n}|}} = \frac{a_{\overline{n}|}(1+v^n)}{a_{\overline{n}|}} = (1+v^n)$$

This can be **generalized** :

$$a_{\overline{3n}|}/a_{\overline{n}|} = \frac{a_{\overline{n}|} + {}_{n|}a_{\overline{n}|} + {}_{2n|}a_{\overline{n}|}}{a_{\overline{n}|}} = \frac{a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}}{a_{\overline{n}|}}$$

$$= \frac{a_{\overline{n}|}(1+v^n+v^{2n})}{a_{\overline{n}|}} = (1+v^n+v^{2n})$$

3h. What If the Rate Is Unknown ?

If the PV, number, and amount of payments of an annuity are known, we can use the calculator to solve for the interest rate.

3i. What If the Rate Varies ?

Be careful not to mix up the PV and AV interest accumulation.
Also, split up annuities if the interest rate varies so as not to make mistakes.

4a. Annuities with "Off-Payments" Part I

off-payments Payments which are less or more frequent than the interest period.
For example :
- Payments of 1 at the end of each 5-year period over 40 years -> payments are *less frequent* than the interest period of one year.
- Payments of $\frac{1}{12}$ at the end of each month for 10 years -> payments are *more frequent* than the interest period of one year.

There are generally 2 approaches for handling these types of annuities

1. Use interest functions at the equivalent effective rate of interest for the **payment period**.
   - Method is generally easier for numerical answers (i.e., most of the time).
2. Use interest functions at the effective rate of interest **given in the problem**.
   - Method is generally easier for symbolic answers.

The first method is this subsection. For example, monthly payments of $\frac{1}{12}$ paid at the end of each month for 10 years at an effective rate of 5% per annum.

- First we find the equivalent rate :

$$\begin{aligned} 1 + j &= (1.05)^{1/12} \\ j &= 0.4074\% \end{aligned}$$

- Then the PV :

$$\begin{aligned} PV &= \frac{1}{12} a_{\overline{120}|j} \\ &= 7.8971 \end{aligned}$$

4b. Annuities with "Off-Payments" Part II

Fission method When payments are less frequent than the interest period.

Fusion method When payments are more frequent than the interest period.

Example of Fission method :

Payments of 1 every 5 years for 40 years at an annual effective rate of 5%. What annual payment R is equivalent to a payment of 1 every 5 years ?

$$Rs_{\overline{5|}} = 1 \qquad \Rightarrow \qquad R = \frac{1}{s_{\overline{5|}}}$$

So the PV becomes :

$$\left(\frac{1}{s_{\overline{5|}}}\right) a_{\overline{40|}}$$

Thus we have done « **fission** » by splitting up the payments into smaller payments.

It's very important to consider these 2 cases however :

- For payments which are in the beginning of the period, set up the equation as $P = R\ddot{a}_{\overline{n|}}$.
- For payments which are in the end of the period, set up the equation as $P = Rs_{\overline{n|}}$.

The fusion method leads to these formulas :

$$a^{(m)}_{\overline{n|}} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n|}} = s^{(m)}_{\overline{1|}} a_{\overline{n|}}$$

$$\ddot{a}^{(m)}_{\overline{n|}} = \frac{1-v^n}{d^{(m)}} = \frac{i}{d^{(m)}} a_{\overline{n|}} = \ddot{s}^{(m)}_{\overline{1|}} a_{\overline{n|}}$$

$$s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|} = s_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} s_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

The reasoning is we accumulate the payment over the m periods before treating it on an annual basis.
The same reasoning applies to perpetuities

$$a_{\overline{\infty}|}^{(m)} = \frac{1}{i^{(m)}} \qquad \ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}} \qquad a_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{m}$$

4c. Avoiding the m^{thly} Annuity Trap

- It's important not to forget to have the payment be on the base as the annuity.
- For example : payments of 100$ paid at the end of every month over 10 years at an effective annual rate of 5% means an annuity of $12 * 100 a_{\overline{n}|}^{(12)}$.

  So the annuity is compounded 12 times a year with the "yearly" payment of 1200$.
- So the payment, or coefficient of the $a_{\overline{n}|}$ term, is the ***sum* of the payments** in ***each* interest period**.

4d. Continuous Annuities

- $a_{\overline{n}|}^{(m)}$ always requires a **total payment of 1** each year, regardless of the value of m.
- The 1 is payable in $m$thly installments of $\frac{1}{m}$.
- Thus we obtain this result as m grows :

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \left(\frac{1-v^n}{i^{(m)}}\right) \qquad = \lim_{m\to\infty} \left(\frac{1-v^n}{\delta}\right) = \bar{a}_{\overline{n}|}$$

- We also obtain the relation $\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|}$.

4e."Double-Dots Cancel" (and so do "upper m's")

Given annuities being divided, we obtain :

$$\frac{\ddot{a}^{(m)}_{\overline{n}|}}{\ddot{a}^{(m)}_{\overline{p}|}} = \frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{p}|}} = \frac{a_{\overline{n}|}}{a_{\overline{p}|}} = \frac{a^{(m)}_{\overline{n}|}}{a^{(m)}_{\overline{p}|}} = \frac{1-v^n}{1-v^p}$$

4f. A Short Note on Remembering Annuity Formulas

PV all have $1-v^n$ as the numerator.

FV all have $(1+i)^n - 1$ as the numerator.

annuity *i*mmediate has i for ***i***mmediate $\frac{1-v^n}{i}$.

annuity *d*ue has d for ***d***ue $\frac{1-v^n}{d}$.

compounded m times a year changes i for $i^{(m)}$ or d for $d^{(m)}$.

compounded continuously changes i and d for δ.

4g. The $s_{\overline{n}|}$ Trap When Interest Variess

- If the force of interest is variable important not to fall into the trap of accumulating from 0.
- $a(t)$ is the accumulation from 0 so the AV at time 5 of a payment at time 4 is $\frac{a(5)}{a(4)}$ and not $a(1)$.

4h. Payments in Arithmetic Progression

P First payment.

Q Common difference.

We have $P \neq Q$ and these formulas don't have standard symbols but can treat any annuity in arithmetic progression.

PV of an annuity in arithmetic progession

$$A = Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}$$
$$\ddot{A} = P\ddot{a}_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{d}$$

AV of an annuity in arithmetic progession

$$S = (1+i)^n A = Ps_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{i}$$
$$\ddot{S} = (1+i)^n \ddot{A} = P\ddot{s}_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{d}$$

If we can memorize the first one, should be okay to deduce the rest.

If $P = Q = 1$, we have an **increasing annuity** which has a symbol.

$$(Ia)\overline{n}| = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$
$$(I\ddot{a})\overline{n}| = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$
$$(Is)\overline{n}| = (Ia)\overline{n}|(1+i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{i} \equiv \frac{s_{\overline{n+1}|} - (n+1)}{i}$$
$$(I\ddot{s})\overline{n}| = (I\ddot{a})\overline{n}|(1+i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

Relationship The AV of the increasing annuity is the sum of n annuities :

$$(Is)\overline{n}| = \sum_{t=1}^{n} s_{\overline{t}|} = \sum_{t=1}^{n} \frac{(1+i)^t - 1}{i} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

If $P = n$ and $Q = -1$, we have a **decreasing annuity** which has a symbol.

$$(Da)\overline{n}| = \frac{n - a_{\overline{n}|}}{i}$$

Relationship The PV of the decreasing annuity is the sum of n annuities :

$$(Da)_{\overline{n}|} = \sum_{t=1}^{n} a_{\overline{t}|} = \sum_{t=1}^{n} \left(\frac{1 - v^t}{i} \right) = \frac{n - a_{\overline{n}|}}{i}$$

Note We can combine level and increasing annuities (see ASM for examples).

Increasing Perpetuities where $P = Q = 1$:

$$(Ia)_{\overline{\infty}|} = \frac{1}{id} \equiv \frac{1}{i} + \frac{1}{i^2}$$

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

If $P \neq Q$:

$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

Increasing and then level perpetuity where payment is increasing for n years and remains at n therafter :

$$PV = (Ia)_{\overline{n}|} + v^n \left(\frac{n}{i} \right) = \frac{\ddot{a}_{\overline{n}|}}{i}$$

4i. Remembering Increasing Annuity Formulas

A few basic definitions :

I means the **annual rate** of payment increases once a year.
For example, the annual rate of payment is 1 in the first year, 2 in the second, etc.

$I^{(m)}$ means the **annual rate** of payment increases at m^{thly} intervals
For example, the annual rate of payment is $\frac{1}{m}$ in the first $\frac{1}{m}$-th

of a year, $\frac{2}{m}$ in the second $\frac{1}{m}$-th of a year, etc.

a means payments are made **annually**.

$a^{(m)}$ means payments are made $m$thly.

For example :

$(Ia)_{\overline{n}|}$ **annual rate** of payment of 1 in the first year, 2 in the second, etc.
The payments are made annually up to a payment of n in the nth year.

$(Ia)_{\overline{n}|}^{[(m)]}$ **annual rate** of payment of 1 in the first year, 2 in the second, etc.
The payments are made m-thly thus there are m payments of $\frac{1}{m}$ in the first year, m payments of $\frac{2}{m}$ in the second, etc.

$(I^{(m)}a)_{\overline{n}|}^{(m)}$ the **annual rate** of payment increases m-thly and payments are made in m-thly installments.
The payments at the end of each m-th of a year are $\frac{1}{m^2}, \frac{2}{m^2}, \ldots, \frac{mn}{m^2}$.
To remember, we can think of it as the « double m » symbol.

In brief :

$$
\begin{aligned}
(Ia)_{\overline{n}|} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \\
(Ia)_{\overline{n}|}^{[(m)]} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} \\
(I^{(m)}a)_{\overline{n}|}^{(m)} &= \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}} \\
(\bar{I}\bar{a})_{\overline{n}|} &= \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}
\end{aligned}
$$

NOTE Revise this section's table in the book (page 230) to understand logic completely.

4j. Payments in Geometric Progression

The PV of an annuity with a first payment of 1 and subsequent payments increasing by a factor of $(1+k)$ annually at an effective rate of interest of i is :

$$PV = \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}$$

Alternatively, we can define the new rate of interest i' as $i' = \frac{|i-k|}{1+k}$ (we insert absolute value as we subtract the larger from the smaller rate). With this rate, we can use normal formulas such as $a_{\overline{n}|}$.

4k. The Amazing Expanding Money Machine (Or Continouss Varying Annuities)

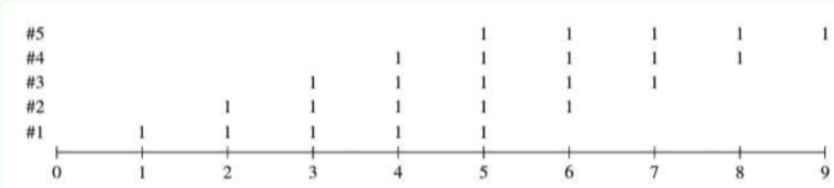
Chapter explains the intuition behind continuously paid annuities.

- For a the continuously increasing continuously paid annuity, $(\bar{I}a)_{\overline{n}|}$, we integrate 1 times v^t as the payments are linearly increasing ($\int_0^n v^t dt$).
- We can also integrate a function and have $PV = \int_0^n f(t)v^t dt$ if it is not constant.
- Then, if the interest varies, we have $PV = \int_0^n f(t)\mathrm{e}^{-\int_0^t \delta_r dr} dt$

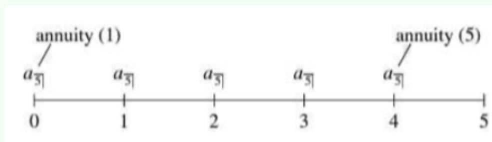
4l. A Short-Cut Method for the Palindromic Annuity

A palindrome is, for example, 1342431 where it is the same read backward or forward. This pattern can occur for annuities too.
The book takes for example a series of payments of $1, 2, 3, 4, 5, 4, 3, 2, 1$.

- This could be solved as $(Ia)_{\overline{5}|} + v^5(Da)_{\overline{4}|}$.
- However, there is a simpler method as this equation yields $a_{\overline{5}|} \cdot \ddot{a}_{\overline{5}|}$.
- First, we visualise the payments :



- Which we can rewrite as :



- Thus, $PV = a_{\overline{5}|}\ddot{a}_{\overline{5}|}$.
- If the original series is due, we have the annuity-due squared.

4m. The 0% Test : A Quick Check of Symbolic Answers

Trick to eliminate some of the answer choices when answers are in symbolic form.
Since answers should be correct for any rate of interest, they should be correct for 0% as well. With $i = 0$, we most notably get $a_{\overline{n}|} = s_{\overline{n}|} = n$.

NOTE revise this section to properly understand examples.

3 Topic : Loans (10%-20%)

Information

Objective

The Candidate will understand key concepts concerning loans and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Principal ;
   - Interest ;
   - Term of loan ;
   - Outstanding balance ;
   - Final payment ;
     - Drop payment ;
     - Baloon payment.
   - Amortization.

b) Calculate :
   - The missing item given any 4 of :
     - Term of loan ;
     - Interest rate ;
     - Payment amount ;
     - Payment period ;
     - Principal.
   - The outstanding balance at any point in time ;
   - The amount of interest and principal repayment in a given payment ;
   - Similar calculations to the above when refinancing is involved.

Related lessons ASM

Section 6 : Loans

- ??
- ??
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- ??

Chapter summaries

6a. Amortizing a Loan

Amortization is reducing the outstanding balance (B) of a loan by making payments (R) which are part interest (I) and part principal (P).

We're interested in distinguishing the interest and principal portions of each payment for many reasons :

- If you have a mortgage, only the interest portion of the payments are tax deductible.
- From the lender's point of view, the interest payments are his profit while the principal payments are only returning what he owns.

Components of amortizing

I_t **Interest paid** at the end of year t.

P_t **Principal repaid** at the end of year t.

- The payments are in geometric progression with a common ratio of $(1 + i)$.
  So, $P_t = P_1(1 + i)^{t-1}$ for example.

R_t **Loan payment** at the end of year t.

B_t **Outstanding loan** balance at the end of year t (just ***after*** the loan payment is made).

- Retrospective method : Looking backwards to the original loan amount, the loan payment and the principal repaid in that payment.

- Prospective method : Looking forward to the remaining payments.
  $B_t = PV(\text{remaining payments})$

Thus we get :

$$\begin{aligned} R_t &= P_t + I_t \\ I_t &= iB_{t-1} \\ P_t &= R_t - I_t \\ B_t &= B_{t-1} - P_t \\ B_n &= 0 \qquad\qquad\qquad\qquad B_0 = L \end{aligned}$$

An **amortization schedule** is usually used for organizing the recursive calculation :

Duration: t	Payment: R	Interest Paid: $I_t = iB_{t-1}$	Principal Repaid: $P_t = R - I_t$	Outstanding Principal: $B_t = B_{t-1} - P_t$
0				$a_{\overline{n}\rvert}$
1	1	$ia_{\overline{n}\rvert} = 1 - v^n$	v^n	$a_{\overline{n}\rvert} - v^n = a_{\overline{n-1}\rvert}$
2	1	$ia_{\overline{n-1}\rvert} = 1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-1}\rvert} - v^{n-1} = a_{\overline{n-2}\rvert}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$ia_{\overline{n-t+1}\rvert} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}\rvert} - v^{n-t+1} = a_{\overline{n-t}\rvert}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	$ia_{\overline{1}\rvert} = 1 - v$	v	$a_{\overline{1}\rvert} - v = 0$
Total	n	$n - a_{\overline{n}\rvert}$	$a_{\overline{n}\rvert}$	

Which lead to these formulas for a loan of $a_{\overline{n}\rvert}$:

$$I_t = 1 - v^{n-t+1} \qquad P_t = v^{n-t+1} \qquad B_t = a_{\overline{n-t}\rvert}$$

Also, we have the following interpretations for the totals :

Total Interest Paid (TI) is the sum of the n payments minus the original loan amount $a_{\overline{n}\rvert}$.

Total Principal Repaid (TP) is the original loan amount $a_{\overline{n}\rvert}$.

The amortization schedule can also be generalized for a loan of L, we simply multiply by a factor of $\frac{L}{a_{\overline{n}\rvert}}$ to cancel out the default loan of $a_{\overline{n}\rvert}$.

Note The amortization worksheet on the calculator can perform the calculations for us.

6b. Varying Series of Payments

Any series of payments whose prevent value is equal to the loan amount will repay it.
Important to remember that for any loan :

- The **interest paid** is the interest rate times the previous **loan balance**.
- The **principal repaid** is the loan payment minus the **interest paid**.
- The new **loan balance** is the previous balance minus the **principal repaid**

Finally, if a payment is not sufficient to repay the interest, we still write (in an amortization schedule) the full interest payment but the principal repaid *becomes negative* by the difference!

6c. Equal Principal Repayments (A Special Case of Varying Payments)

We can pay a **level amount of *principal*** and the interest on the outstanding balance. This is different than a level payment as a level payment includes the interest paid on the outstanding balance.

Note practice problems where split increasing into an annuity and an increasing / decreasing

6d. Final Payments (Balloon and Drop Payments)

There are at least 3 ways of repaying a loan by level payments, followed by an unequal final payment :

- A final payment *larger* than the level payments is a **balloon payment**.
- A final payment *smaller* than the level payments is a **drop payment**.

- A final payment *smaller* than the level payments made on a date in between regular payment dates *(not common practice)*.

4 Topic : Bonds (10%-20%)

Information

Objective

The Candidate will understand key concepts concerning bonds, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Price ;
   - Book value ;
   - Amortization of premium ;
   - Accumulation of discount ;
   - Redemption value ;
   - Par value / Face value ;
   - Yield rate ;
   - Coupon ;
   - Coupon rate ;
   - Term of bond ;
   - Callable / Non-callable.

b) Given sufficient partial information about the items listed below, calculate any of the remaining items :
   - Price, book value, amortization of premium, accumulation of discount ;
   - Redemption value, face value ;
   - Yield rate ;
   - Coupon, coupon rate ;
   - Term of bond, point in time that a bond has a given book value, amortization of premium, or accumulation of discount.

Related lessons ASM

Section 7 : Bonds

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Chapter summaries

7a. Bonds and Other Investments

Chapter will cover both how to determine the price of a bond to earn a give yield rate and how to determine the yield rate for a given price.

Bonds are a means of borrowing money where the lenders (or investors) receive interest payments ("coupons") for a fixed period of years (the term of the bond). At the end of the term, the lenders receive the original amount of the loan back.

The interest payments are called "coupons" because they used to be physical coupons attached to the bonds that people would redeem.

7b. Finding the Price of a Bond

Notation

P The **price** of the bond.

F The **face amount** (or **par value**).

- › The *par* or *face value* is the unit in which the bond is issued.

C The **redemption value**

- › By default, a bond is redeemable at par with $C = F$.

r The **coupon rate** per coupon payment period.

- › The amount of the coupon is Fr.

- The coupon rate is always given as an **annual** rate.
- Most bonds have semi-annual coupons.

g The "special" coupon rate used in mathematical formulas.

- Rate applied such that $Cg = Fr$.
- The coupon rate per unit of *redemption value* C.

n Number of remaining coupon **payments**.

i The effective rate of interest per coupon payment period.

- It is the "yield-to-maturity" for a bond selling at price P. Thus, unlike r which is a fixed feature of the bond, i will vary according to the price P.
- The interest rate i such that $P = PV(\text{bond payments})$.

Several formulas can be derived for bonds, the textbook covers three :

The Basic Formula The price P of a bond to yield an effective rate i is the PV of the bond payments at that rate ; that is, the PV of the coupons Fr plus the PV of the redemption value C.

$$P = Fra_{\overline{n|}} + Cv^n \equiv Cga_{\overline{n|}} + Cv^n$$

The Premium / Discount Formula Rewrite the annuity formula $a_{\overline{n|}} = \frac{1-v^n}{i}$ to isolate $v^n = 1 - ia_{\overline{n|}}$ and replace it in the formula :

$$P = C + (Fr - Ci)a_{\overline{n|}} \equiv C + (Cg - Ci)a_{\overline{n|}}$$

This formula is very useful mathematically and can be used to, amongst other things, determine the premium or discount paid for a bond :

- If $Fr > Ci$ then $P > C$ and the bond is at a premium as the coupons are more profitable than the yield rate.
- If $Fr < Ci$ then $P > C$ and the bond is at a discount as the coupons are less profitable than the yield rate.

The Makeham Formula Useful for a group of bonds with the same coupon rate but redeemable in installments *(staggered redemption dates)*. Serial bonds are no longer on the syllabus, however.

7c. Premium and Discount

Condition	Implication	Bond is purchased at a..	equal to
$g > i$	$P > C$	premium	$P - C = (Cg - Ci)a_{\overline{n}\vert}$
$g < i$	$P < C$	discount	$C - P = (Ci - Cg)a_{\overline{n}\vert}$

Bond terminology

- **Book value** instead of *outstanding loan balance*.
- **Amortization of premium** instead of *principal repaid*.
  - "Writing down" occurs when the payment covers more than the interest and repays some of the principal as well.
  - The amount is $P_t = Fr - I_t$.
- **Accumulation of discount**.
  - "Writing up" occurs when the payment covers less than the interest and does not repay the principal.
  - The amount is $P_t = I_t - Fr$.
- **Coupon** instead of *payment amount*.

Bond amortization schedule

Period t	Coupon	Interest Earned $I_t = iB_{t-1}$	Amount for Amortization of Premium P_t=Coupon$-I_t$	Book Value $B_t = B_{t-1} - P_t$
0				1,074.04
1	40	32.22	7.78	1,066.26
2	40	31.99	8.01	1,058.25
3	40	31.75	8.25	1,050.00
Totals	120	95.96	24.04	

		Interest Earned	Amount for Accumulation of Discount	Book Value
Period t	Coupon	$I_t = iB_{t-1}$	$P_t = I_t - \text{Coupon}$	$B_t = B_{t-1} + P_t$
0				1,015.96
1	40	50.80	10.80	1,026.76
2	40	51.34	11.34	1,038.10
3	40	51.90	11.90	1,050.00
Totals	120	154.04	34.04	

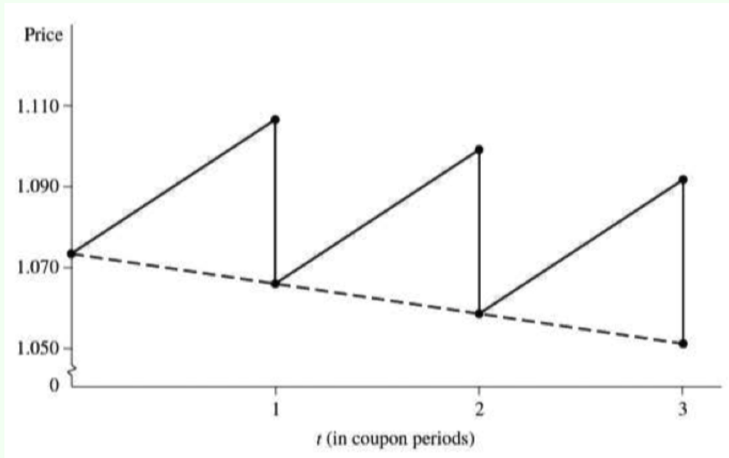
Note The amortization worksheet on the calculator can calculate values for us using the TVM values.

7d. Price Between Coupon Dates

2 types of prices between coupon dates

1. The price actually paid for the bond on the date of purchase (a.k.a. settlement date).
   - Many names : "full price", "dirty price", "flat price", "price-plus-accrued".
   - $B_{t+k} = B_t(1+i)^k, 0 \le k \le 1$.
   - This is the money that actually changes hands when the bond is sold.
2. The price quoted for the bond in the financial press.
   - Many names : "clean price", "market price", "price".
   - $B_{t+k} - kFr = B_t(1+i)^k - kFr, 0 \le k \le 1$.
   - The practice is to quote this price which *excludes the value of the coupon* which has *accrued* by the date of purchase.
   - The *accrued coupon* is commonly called the ***accrued interest***.

In practice, the price excluding accrued interest is quoted because the full price has a saw-toothed progression :



There are 2 methods of counting the days for the fraction $k = \frac{\text{days since last coupon on purchase date}}{\text{days between coupons}}$:

1. The "actual/actual" method.
   - This method is used for government bonds.
   - The actual number of days is used for both the numerator and denominator of k.
2. The "30/360" method.
   - This method is used for corporate and municipal bonds.
   - Each month is considered to have 30 days and a year assumed to have 360 to calculate k.

7e. Determination of Yield Rates

It is likely we'll have to calculate a yield rate from the price of a bond and the remainder of the information. This cannot be done by hand and must be calculated with the financial calculator.
However, there is an approximation, which is not on the syllabus, to approximate the yield rate—the **Bond Salesman's Method**.

$$
\begin{aligned}
\text{Total interest} &= nCg + C - P \\
\begin{matrix}\text{Average interest}\\ \text{per period}\end{matrix} &= \frac{nCg + C - P}{n} \\
\text{Average investment} &= \frac{P + C}{2} \\
\begin{matrix}\text{Approximate yield}\\ \text{rate per period } i\end{matrix} &= \frac{nCg + C - P}{\frac{n}{2}(P + C)}
\end{aligned}
$$

7f. Callable Bonds

Callable bonds

- The borrower (issuer) can redeem (a.k.a. call back or buy back) the bond prior to maturity.
- There is generally a **call date** prior to which the issuer cannot call back the bond.
- The **call price** is the redemption price the issuer must pay the lender if the bond is called.
  This price may differ from the regular redemption price C.

Context

- Calling back the bond is advantageous for the issuer in case there is a decline in interest rates after issue—they could issue new bonds with lower coupons.
- The difficulty with callable bonds for the investor is :
  - The uncertainty of the term.
  - The difficulty in determining the yield rate relative to the selling price—if there is a decrease in the interest rate, the investor has to invest the redemption value for a lower return.

Thus, to price these bonds we have to find the worst possible date of redemption at which we ensure a minimal return.

- For a bond selling at a **P**remium, the **E**arliest redemption date is the **W**orst one (**PEW** to remember).
- For a bond selling at a discount, the latest redemption date is the worst one.

5 Topic : General Cash Flows and Portfolios (15%-20%)

Information

Objective

The Candidate will understand key concepts concerning yield curves, rates of return, and measures of duration and convexity, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Yield rate / rate of return ;
   - Dollar-weighted rate of return ;
   - Time-weighted rate of return ;
   - Current value ;
   - Duration (Macaulay and modified) ;
   - Convexity (Macaulay and modified) ;
   - Portfolio ;
   - Spot rate ;
   - Forward rate ;
   - Yield Curve ;
   - Stock price ;
   - Stock dividend.

b) Calculate :
   - The dollar-weighted and time-weighted rate of return ;
   - The duration and convexity of a set of cash flows ;

- Either Macaulay or modified duration given the other ;
- The approximate change in present value due to a change in interest rate using a 1st-order :
  - Linear approximation based on modified duration ;
  - Approximation based on Macaulay duration.
- The price of a stock using the dividend discount model ;
- The present value of a set of cash flows, using a yield curve developed from forward and spot rates.

Related lessons ASM

Section 5 :

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Section 8 :

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Section 10 :

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Section 11 :

- ??

Chapter summaries

5a. Net Present Value and Internal Rate of Return

Net Present Value (NPV) Method consists of comparing the NPV of the cash inflows and outflows.

A_t Cash inflow at time t ("**A**ssets").

L_t Cash outflow at time t ("**L**iabilities").

i The rate we believe could be earned on alternative investments with the same degree of risk.

- a.k.a. the **cost of capital**, opportunity cost of capital or the interest preference rate.

$P(i)$ NPV at rate i.

- This notation is used as the NPV is a function of the interest rate i.

$$P(i) = \sum A_t v^t - \sum L_t v^t = \sum (A_t - L_t) v^t$$

Yield Rate or Internal Rate of Return (IRR) Because $P(i)$ can be negative or positive depending on the interest rate, we deduce there must be a rate such that it is equal to 0. In fact, the solution such that $P(i) = 0$ is the IRR :

$$P(i) = \sum (A_t - L_t) v^t = 0$$

Note This can be solved by the calculator.

- Access the cashflows worksheet through CF.
- Cash flows are the `C01` options.
- The `F01` options are the frequency for the payments.
- Press the NPV button after cashflows are entered, enter the interest rate, then go down to the `NPV` field and hit CPT.
- To obtain the IRR of a series of cashflows, press the IRR button followed by CPT after entering the series.

5b. Is the Yield Rate Unique ?

For most transactions there is one positive solutions however in the theory there are as many as the degree of the polynomial in the equation $P(i)$ (i.e. the largest exponent). Situations where it is not unique are often somewhat artificial but they can happen.

This sections highlights the limitations of the yield rate as an answer to « Is this a good investment ? » and that there are other methods which are not part of the syllabus to answer it.

5c. Reinvestment Rates

Two general scenarios :

1. Suppose we have 1 in a fund crediting interest at rate i.
   Annually, we withdraw the interest and reinvest it into another fund earning i'. Their accumulated value at the end of n years is :

   $$AV = 1 + i s_{\overline{n}|i'}$$

2. Suppose that we deposit 1 in a fund crediting interest at rate i at the beginning of each year.
   Annually, we withdraw the interest and reinvest it into another fund earning i'. Their accumulated value at the end of n years is :

   $$AV = n + i(Is)\overline{n}|i'$$

5d. Yield Rate Earned by a Fund

A Amount in the fund at the beginning of the year.

B Amount in the fund at the end of the year.

C_t Deposit in or withdrawal from (positive or negative) the fund at time t.

C Net sum of the deposits / withdrawals.

$$C = \sum C_t$$

I Amount of interest earned in the period.

$$I = B - A - C$$

With this we approximate the yield rate as :

$$i \approx \frac{2I}{A + B - I}$$

5e. Dollar-Weighted and Time-Weighted Interest Rates

Dollar-weighted consists of isolating i from the cashflows with the assumption of simple interest. **Time-weighted** ignores how much time each transaction spent accumulating and multiplies balances. It is easy to make a mistake and best practice to draw a table.

See Coaching Actuaries' formula sheet for a good visualisation.

5f. Portfolio Methods and Investment Year Methods

Portfolio Method The yield rate is proportional to the amount members invested regardless of the date they invested.

- May seem unfair, but if there are losses the new members are the losers.
- This is how savings accounts, TFSAs, etc. operate.

Investment Year Method (IYM) The yield rate is proportional to the date members invested.

- There are several methods, but this is no longer on the syllabus.
- This is how a Certificate of Deposit (CD) operates.

Feature	Bond	Preferred Stock	Common Stock
Nature of the Security	Loan to a company or governmental unit.	Capital investment in a corporation, with no guarantee of a return.	Same as preferred stock.
Relation of the Investor to the Company	**Creditor** of (or lender to) the company.	**Part-owner** of the company, with the right to vote for the Board of Directors and on other corporate matters.	Same as preferred stock.
Term of the Investment	Usually fixed at issue although some may be called before.	Generally has no maturity date.	Same as preferred stock.
Investment Returns	Periodic coupons throughout the term plus the face amount on maturity.	Fixed periodic dividends.	Periodic dividends at the company's discretion.
Price	Varies with current interest rates. Default risk may also affect the price.	Varies with current interest rates as the dividends are a fixed amount.	Very volatile.
Degree of Security	First in degree of security as payments on debt are made before issuing dividends.	Preferred stock dividends are paid before common stock dividends can be.	Last in degree of security.

8a. Bonds Preferred Stock and Common Stock

A company can raise money with :

Debt Bonds are a form of debt.

Equity Stocks are a form of equity.

There are a few types of bonds :

Junk bonds Often used as a way to finance the takeover of a one corporation by another and are much riskier than investment grade bonds.

Investment grade bonds Bonds with a very low risk of default.

Zero-coupon bonds Bond without coupons in which the investor receives a single lump sum payment at maturity.

8b. Price of a Share of Stock

The price of a typical preferred stock (or perpetual bond) is the PV of a perpetuity of the dividends (or coupons) :

$$P = \frac{Fr}{i}$$

With the **dividend discount model**, or **the discounted cash flow technique**, if the dividends D for a common stock increase by a constant ratio of $(1 + k)$ we obtain :

$$P = \frac{D}{i - k}$$

However this method is *theoretical* as the future dividends of common stock are difficult to predict.

8c. Other Financial Instruments

Sector funds Mutual funds with investments in a particular sector.

Index funds No active management of the fund, it reproduces the results of the index it follows.

Load Commission to buy shares, but there are many **no-load** mutual funds as well.

Money Market Mutual Funds Invest in very safe, short-term debt instruments (for example, T-bills).

- Interest rates frequently change.

Certificate of Deposit Deposit in a bank for a fixed term with a guaranteed interest rate.

- There is usually an interest penalty for an early withdrawal.
- Interest rates frequently change.

10a. Duration of a Single Cash Flow

Fixed-income investments Investments that pay **fixed** (*known*) amounts of future income.

- e.g., bond coupons, mortgage payments, etc.
- Opposite of investments with **unknown** future amounts.
  e.g., dividends.

- Given that the *future* income is fixed, the *present* value of the investments depends on *current* interest rates.
  Thereby, companies need tools to measure the impact that changes in interest rates have on asset values.
  e.g., duration, convexity, etc.
- An issue arising from this is that a financial institution **guarantees** *future* payments.
  However, its ability to fulfill them depends on the value of the investment it holds.

Immunization Protection against the effect of future changes in interest rates.

- The institution tries to "immunize" itself from the impact of changes in interest rates.

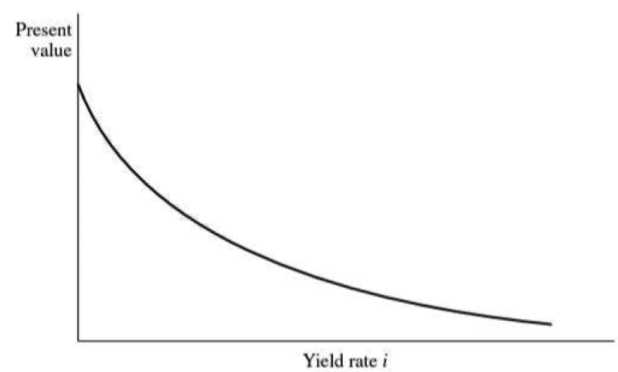
Price sensitivity of a bond to a change in the interest rate.

- i.e., the percentage change in price.
- The greater the term of the bond, the more sensitive its price.

Duration The time remaining to a single cash inflow or outflow.

Given the shape of the price/yield curve, a decrease in the interest rate will lead to a larger difference in price than an increase.

The price/yield curve for a zero coupon bond (the graph of the price as the interest rate changes from zero to infinity) looks like this:



10b. Macaulay Duration

Macaulay duration Weigh the time at which each cash flow occurs by its *present value*.

- The greater the Macaulay duration $D_{\text{mac}}(i)$, the greater the sensitivity of the PV of the CFs to a change in the interest rate i.
- There is no standard notation for the duration, but the study note uses $D_{\text{mac}}(i)$.
- It is often just called ***duration***.
- The special case where $i = 0$ is duration by the method of equated time with $D_{\text{mac}}(0) = \frac{\sum_{t=0}^{n}(A_t)(t)}{\sum_{t=0}^{n}(A_t)}$.
- For a single CF, the duration is simply the time remaining to the CF.
  For example, the duration of a zero-coupon bond of 1 000$ is

$D_{\text{mac}}(i) = \frac{1000v^n n}{1000v^n} = n.$

So we get :

$P(i)$ PV of the CFs at interest rate i.

$$P(i) = \sum_{t=0}^{n}(A_t v^t)$$

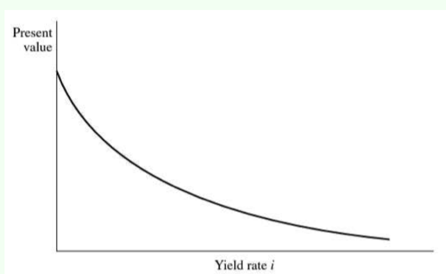
$D_{\textbf{mac}}(i)$ Macaulay duration at interest rate i.

$$D_{\text{mac}}(i) = \frac{\sum_{t=0}^{n}(t)(A_t v^t)}{P(i)}$$

10c. Macaulay Duration as a Measure of Price Sensitivity

The Macaulay Duration can also be defined as a measure of sensitivity of the PV of the CFs to a change in the interest rate. To define it mathematically, there are 2 considerations :

- The slope of the PV/yield curve is negative and as such we multiply the derivative by -1 :



- Price sensitivity is a *relative* measure.
  To obtain the rate of change in the PV as a percentage of the PV, we divide the derivative of the PV of the CFs by the PV itself.

Thus, PV sensitivity $= \frac{-P'(i)}{P(i)}$.
This is actually the **modified duration** and deriving by the *force of*

interest leads to the Macaulary Duration :

$$\text{PV sensitivity (with respect to } \delta) = \frac{-P'(\delta)}{P(\delta)}$$
$$= D_{\text{mac}}(i)$$

10d. Modified Duration

As noted above, we get :

$$\text{PV sensitivity (with respect to } i) = \frac{-P'(i)}{P(i)} = D_{\text{mod}}(i)$$

And we find :

$$D_{\text{mod}}(i) = \frac{\sum_{t=0}^{n}(t)(A_t v^{t+1})}{P(i)} = vD_{\text{mac}}(i)$$

Notes

- To remember that the modified duration adds a v to the (Macaulay) duration, note that « modified duration » is the same as saying « **v**ariation of duration ».
- Modified duration is sometimes referred to as **volatility**.

First-Order approximations

i_0 Initial interest rate used to calculate the PV of the CFs.

i New interest rate for which we want to approximate the change in the PV of the CFs.

- They are « first-order » approximations because they correspond to the terms of the Taylor series expansion of $P(i_0 + \Delta i_0$ up to the first power of Δi_0.

First-Order modified approximation :

$$P(i) \approx P(i_0)[1 - (i - i_0)D_{\text{mod}}(i_0)]$$

- The approximate change in PV due to a change in interest rate from i_0 to i reduces to $(i - i_0)P'(i)$.
- Because the slope of the PV is always negative, $P'(i)$ is always negative.
  Therefore, an *increase* in the interest rate implies that $(i - i_0) > 0$ which leads to a *decrease* in the PV of the CFs and vice-versa.

First-Order Macaulay approximation :

$$P(i) \approx P(i_0) \left(\frac{1 + i_0}{1 + i} \right)^{D_{\text{mac}}(i_0)}$$

Study Note

Cash Flows

Cash Flow (a, t) An **amount** a $(a \in \mathbb{R})$ at **time** t $(t \in \mathbb{R}^+)$.

Cash Flow Series Sequence of cash flows (a_k, t_k) $(k \in \mathbb{N})$.

i Periodic effective interest rate (same period as the CFs).

$P(i)$ Present value of the CF series as a function of the interest rate :

$$P(i) = \sum_{k \in \mathbb{N}} \left(a_k (1 + i)^{-t_k} \right) = \sum_{k=1}^{n} \left(a_k (1 + i)^{-t_k} \right)$$

- We implicitly assume that the sum converges and as such suppose that N is a finite set of the form $\{1, \ldots, n\}$.
- Because the formulas for duration imply division by $P(i)$, we assume that $P(i) \neq 0$.

Note

- First-Order modified approximation is $\leq$
  First-Order Macaulay approximation $\leq$
  Actual PV (for positive CFs).

10e. Duration of a Portfolio

To determine the duration of a portfolio of bonds, we can take a weighted average of the duration of each bond.
The weight is the PV of the bond's payments. In other words, the price of the bond at the interest rate for which we are calculating the duration.

$$D(\text{ptf.}) = \frac{D_1 P_1 + \cdots + D_n P_n}{P_1 + \cdots + P_n}$$

10f. Change in Duration As Time Goes By

10g. Convexity

11a. Spot Rates and Forward Rates

6 Topic : Immunization (10%-15%)

Information

Objective

The Candidate will understand key concepts concerning cash flow matching and immunization, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Cash flow matching ;
   - Immunization (including full immunization) ;
   - Redington immunization.

b) Construct an investment portfolio to :
   - Redington immunize a set of liability cash flows ;
   - Fully immunize a set of liability cash flows ;
   - Exactly match a set of liability cash flows.

Related lessons ASM

Section 10 : Duration, Convexity, and Immunization

- ??
- ??
- ??
- ??

Chapter summaries

10h. Redington Immunization

›

10i. Full Immunization

›

10j. A Note on Rebalancing

›

10k. Immunization by Exact Matching ("Dedication")

›

Notes sur les vidéos YouTube

7 Topic : Interest Rate Swaps (0-10%)

Information

Objective

The Candidate will understand key concepts concerning interest rate swaps, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Swap rate ;
   - Swap term (tenor) ;
   - Notional amount ;
   - Market value of a swap ;
   - Settlement dates ;
   - Settlement period ;
   - Counterparties ;
   - Deferred swap ;
   - Amortizing swap ;
   - Accreting swap ;
   - Interest rate swap net payments.

b) Given sufficient information, calculate :
   - The market value ;
   - Notional amount ;
   - Spot rates or swap rate,

   of an interest rate swap
   - deferred or otherwise ;
   - with either constant or varying notional amount.

Related lessons ASM

Section 11 : Interest Rate Swaps

> ??

Chapter summaries

11b. What is an Interest Rate Swap ?

8 Topic : Determinants of Interest Rates (0-10%)

Information

Objective

The Candidate will understand key concepts concerning the determinants of interest rates, the components of interest, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Real risk-free rate ;
   - Inflation rate ;
   - Default risk premium ;
   - Liquidity premium ;
   - Maturity risk premium.

b) Explain how the components of interest rates apply in various contexts, such as :
   - Commercial loans ;
   - Mortgages ;
   - Credit cards ;
   - Bonds ;
   - Government securities.

c) Explain the **roles** of the Federal Reserve and the FOMC in carrying out *fiscal* policy and *monetary* policy and the **tools** used thereby including :
   - Targeting the federal funds rate ;

- Setting reserve requirements ;
- Setting the discount rate.

d) Explain the theories of why interest rates differ by term, including :
  - Liquidity preference (opportunity cost) ;
  - Expectations ;
  - Preferred habitat ;
  - Market segmentation.

e) Explain how interest rates differ from one country to another (e.g., U.S. vs. Canada) ;

f) In the context of loans with and without inflation protection :
  - **Identify** the *real* interest and the *nominal* interest rate ;
  - **Calculate** the effect of changes in inflation on loans with inflation protection.

Related lessons ASM

Section 9 : Determinants of Interest Rates

- **??**
- **??**
- **??**
- **??**
- **??**
- **??**
- **??**
- **??**
- **??**

Chapter summaries

9a. What is Interest ?

From an economic perspective, can view interest in one of two ways :

- Compensation for deferring consumption (e.g., putting money in a savings account rather than spending it).
- Cost of consuming resources which aren't available (e.g., using a credit card to buy something).

People's choices will vary based on this consumption ; if interest rates (the compensation) are high, more people will tend to save rather than spend. The demand curve is based on this.

9b. Quotation Bases for Interest Rates

d quoted rate.

N Number of days to maturity.

P Price at issue.

C Maturity value.

I Amount of interest earned $= C - P$

Quoted rate of US T-bills $\frac{360}{N} \times \frac{C-P}{C}$.

- This rate is very simple to calculate and in the 1930s was much easier.
- Thus, even if the compound rate $P_t = P_0(1+i)^t$ is a more accurate representation of the rate of growth, the quoted rate is used.

Quoted rate of Canadian Government T-Bills is $\frac{365}{N} \times \frac{C-P}{P}$.

- Thus we compare to the price and assume 365 days instead of 360.

To minimize mistakes, it is good practice to convert all interest rates to a common basis.

Effective interest rate The rate i defined by $P_t = P_0(1+i)^t$.

Effective per annum interest rate Special case of the above when t is measured in years.

- Note this is not the **constant** rate and as such a function of t shouldn't be powered to an exposant but multiplied.

Continuously compounded rate The rate r defined by $P_t = P_0 e^{rt}$.

Continuously compounded per annum rate Special case of the above when t is measured in years.

- Note this is not the **constant** rate and as such a function of t shouldn't be integrated but summed.

Unlike effective rates, continuous ones are additive and easier to work with.

9c. Components of the Interest Rate : No Inflation or Default Risk

Market segmentation theory

- People interested in borrowing money for the short and long term are different.
  - In the short-term, people may have a cash shortfall but expect to earn enough money to repay the loan by the end of the year.
  - In the long-term, people may be starting a business and expect to repay the loan in a few years.
- The same idea holds for the lenders.
  - In the short-term, people are saving money for a short-term goal and need access to their money relatively soon.
  - In the long-term, people may be saving for retirement and don't mind not having immediate access to their money for a long time.

The opportunity cost, or liquidity preference, theory

- Ceteris Paribus, lenders tend to prefer to lend money for shorter terms.
- Thus, borrows generally have to pay a higher rate of interest, not just higher compensation, to persuade lenders to lend money for longer.

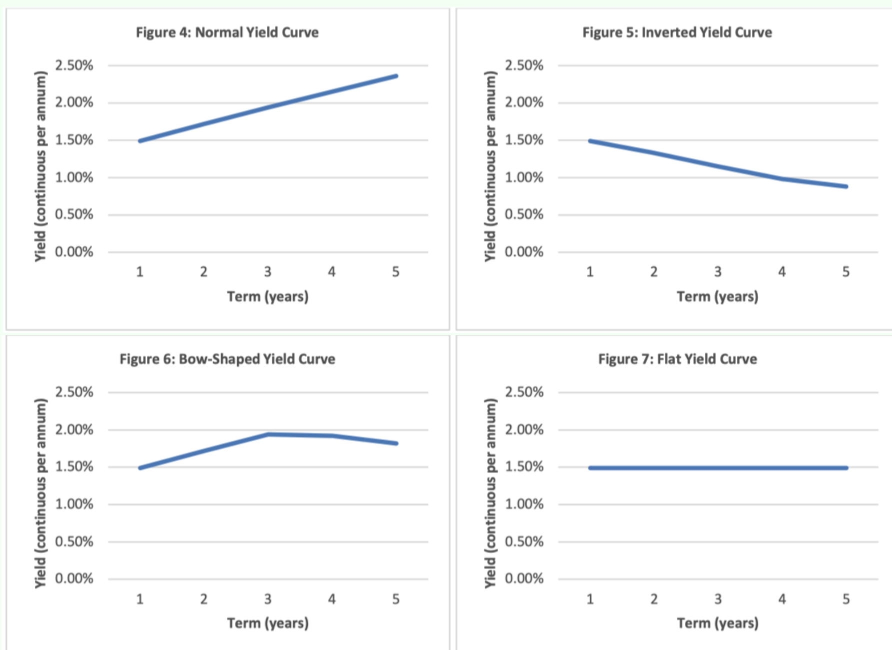
Expectations theory

- Interest rates on longer term loans provide information on the future expected rate of shorter term loans.

Preferred habitat theory

- Similarly to the market segmentation theory, differences in rates are due to differences in the pools of lenders / borrowers.
- However, lenders / borrowers are **not rigidly segmented** by term.

Yield Curve Collection of interest rate values for the different terms (*spot rates*)



9d. Components of the Interest Rate : no Inflation but with Default Risk

Notation for problems involving defaults (amounts)

x Amount received from every borrower (assuming no defaults).

y Amount received from every borrower who does not default (assuming defaults at rate q).

q Rate of default, expressed either as a decimal or a percentage of all borrowers.

p Fraction of the recoveries received from those who default at the end of the loan term.

when there is default with

no recovery The lender requires :

$$x = (1-q)y$$

partial recovery of p The lender requires :

$$x = (1-q)y + qpy \equiv [1-(1-p)q]y$$

Notation for problems involving defaults (rates)

R Rate on the loan with default risk.

r Rate on the (same term) loan without default risk.

- In other words, the *compensation for deferred risk.*

s When continuous it is the difference between the two rates, when effective it is the *compensation for default risk.*

- When rates are continuous per annum, we can decompose them as $R = r + s$.
- When rates are effective per annum, we can decompose them as $1 + R = (1+r)(1+s)$.
- When it is effective, we can approximate $R \approx r + s$. In this case, the difference between the exact and approximate R is the error in ignoring the rs term.

9e. Components of the Interest Rate : Known Inflation

Inflation Tendency for prices to increase over time.

- Usually tracked with a specific basket of goods and services.

- There are *consumer* price indexes (CPI) and *producer* price indexes (PPI) for **domestic** products.

Inflation Rate Change in the price-level of an index.

- Generally compounded per annum.
- Provide information on the *general* tendency for prices to increase.

- Inflation thus leads to lenders requiring a higher interest rate from borrowers to compensate for the **loss in purchasing power** over the term of the loan.
- This is balanced by borrowers generally having a higher future income.

R Equilibrium interest rate.

- Generally continuous per annum.
- Accounts for deferred spending and default risk.

i Certain rate of inflation per year (in both wages and price).

- Generally continuous per annum.
- Have to be careful not to get confused with i representing inflation.

R^* Equilibrium rate in a world of certain inflation (also considers default risk).

$$R^* = r + s + i, \text{continuous per annum}$$
$$1 + R^* = (1 + r)(1 + s)(1 + i), \text{effective per annum}$$

9f. Components of the Interest Rate : Uncertain Inflationo

Loans with inflation protection preserve the lender's purchasing power but have uncertainty into the amount the borrow must ultimately repay. A borrower may be willing to pay the lender additional interest to avoid the uncertainty of payment. In this **loan**

without inflation protection, the *lender assumes the risk* of the purchasing power not being preserved and sets higher repayment levels.

Notation

r Compensation for deferred consumption.

c Cost of inflation protection.

s The **credit spread**, or spread for credit risk.

- These vary according to the terms and a collection of spreads is a **spread curve**.
- In practice, this is calculated by the difference in the observed rates on the loan with and without default risk :

$$s = R^* - R_2$$

i_a Actual (realized) rate of inflation.

i_e Compensation for expected inflation.

- When there is certain inflation, i.e. $s = 0$, then i_e is simply the expected rate of inflation i.

i_u Compensation for unexpected inflation.

- When inflation rates are stable i_u, will be fairly small. When they're volatile, i_u could be quite large.

R_1 Quoted rate, or **real interest rate**, on a loan **with inflation protection** but no default risk.

$$R_1 = r - c$$

$R_1^{(a)}$ Rate actually realized (loan with inflation protection).

$$R_1^{(a)} = r - c + i_a$$

R_2 Quoted rate, or **nominal interest rate**, on a loan **without inflation protection** and no default risk.

$$R_2 = r + i_e + i_u$$

R^* The interest rate on a loan **with risk of default**.

$$R^* = r + s + i_e + i_u$$

- Although r, c, i_e, and i_u cannot be observed individually, only the rates R_1 and R_2.
- That said, we can observe $R_2 - R_1 = i_e + i_u - c$ which are the compensation and costs for inflation.
- Financial commentators therefore often say the difference provides an indication of the inflation rate expected by the market. However, this only holds when the market is stable and i_u is small.
  Thereby, the difference tends to overestimate the market's expectation of future inflation.

Note Practice some problems with inflation indexes to ensure I know how to apply the theory.

9g. Savings and Lending Interest Rates

Financial intermediaries Third party who matches up borrowers and lenders.

- In retail markets these are banks as well as savings and loan companies.
- They accept deposits from the general public and in turn lend to individuals and businesses.
- They must therefore charge a rate greater than that paid on the deposits to cover **overhead costs** and **losses on defaults**.
- Thereby, the failure of a bank can impact not only depositors, but also the economy by stopping transactions with debit and credit cards.

Shadow banking Lenders who do not accept deposits from the general but instead directly from investors.

- Therefore, they're not regulated as banks.

- Their risk and growing importance is considerable (see 2008's crash).

Financial technology companies (fintech) Alternative payment providers which operate outside the traditional banking system.

- E.g. PayPal, Apple Pay, Bitcoin, etc.

Banks have 2 typical savings products :

Savings accounts Completely liquid.

Certificates of deposit (CD) Illiquid for a fixed period of time.

The rate of these can be influenced by :

- The overhead cost of the bank.
  - This is the main factor.
  - This is why online banks are cheaper, and banks in different regions may have different rates.
- Banks looking to grow may offer a higher rate on their savings products.
- The credit rating of a bank can also affect it's interest rate with riskier ones having to offer a greater rate.

Banks have 3 basic types of lending products :

Secured loans Secured by property (collateral) such as mortgages, auto loans, or home equity loans.

Unsecured loans Not secured by any property such as credit cards.

Guaranteed loans Guaranteed by a third party (e.g. the government) such as student loans.

A bank will charge the **prime rate** to its best and most creditworthy customer

9h. Government and Corporate Bonds

Most of the section is covered in the table from section 8a Zero-coupon bonds, aka strip bonds as their creation is analogous to "stripping" the coupons from a bond. Can determine the yield curve for an entity which issues bonds infrequently—the zero-coupon yield curve. The higher the coupon the lower the yield as the coupons will represent a larger proportion of the revenue from the bond and aren't reinvested into its value.

US Treasury securities are considered among the safest bonds in the world and are virtually free of default risk. 2 basic types : T-Bills, and T-Bonds (notes) with a maturity greater than one year. There are 2 types of Treasury notes : **nominal return bonds** with a maturity amount fixed at issue, and the more recent **inflation-indexed bonds** (or **real return bonds**). The US government is well situated to take the risk of higher than expected inflation as it's returns are directly based on salaries.

States and local governments can issue **revenue bonds** which are backed by revenues from a dedicated source (e.g. a toll or a fee on hot water), and **general obligation bonds** backed by general taxing. These are however not free of default risk (see Detroit) but are potentially tax-free or tax-preferred.

Note The study note describes the terminology and terms relative to call and put options, bid-ask spreads, etc.

Note practice calculating bonds and rates

9i. The Role of Central Banks

The **central bank** has two main functions :

1. Facilitate the operation of a country's payment system.
2. Act as a lender of last resort for banks.
   - Usually however banks turn to each other for assistance.

- They borrow from the central bank at the **discount rate** set by the FOMC.