Study Guide

Exam FM: Financial Mathematics
Society of Actuaries (SOA)

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Preliminary

Information

Objectives

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Learning outcomes

1.

Autres ressources

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- ›

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- ›

Notes sur les vidéos YouTube

Subjects of study

1 Time Value of Money (10%-15%)

Information

Objective

The Candidate will understand and be able to perform calculations relating to present value, current value, and accumulated value.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Interest rate (rate of interest) ;
- Simple interest ;
- Compound interest ;
- Accumulation function ;
- Future value ;
- Current value ;
- Present value ;
- Net present value ;
- Discount factor ;
- Discount rate (rate of discount) ;
- Convertible m-thly (. . . ?) ;
- Nominal rate ;
- Effective rate ;
- Inflation ;
- Real rate of interest ;
- Force of interest ;
- Equation of value.

b) Given any 3 of :

- Interest rate ;
- Period of time ;
- Present value ;
- Current value ;
- Future value,

calculate the remaining item using *simple* or *compound* interest ;

Solve time value of money equations involving variable force of interest ;

c) Given any 1 of :
   - Effective interest rate ;
   - Nominal interest rate convertible m-thly ;
   - Force of interst,

   calculate any of the other items ;
d) Write the equation of value given a set of cash flows and interest rate.

Related lessons ASM

Section 1 : Interest rates and Discount Rates
- 1a. Basic Concepts
- 1b. Why Do We Need a Force of Interest ?
- 1c. Defining the Force of Interest
- 1d. Finding the Fund in Terms of the Force of Interest
- 1e. The Simplest Case : A Constant Force of Interest
- 1f. Power Series
- 1g. The Variable Force of Interest Trap
- 1h. Equivalent Rates

Section 2 : Practical Applications
- 2a. Equations of Value, Time Value of Money, and Time Diagrams
- 2b. Unknown Time and Unknown Interest Rate

Résumés des chapitres

1a. Basic Concepts

-

Effective rate of interest

$a(t)$ **Accumulation function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of \$1.00 at time 0.

- $a(0) \equiv 1$.
- Generally **continuous** and **increasing**.

$a(t) - a(t-1)$ ***Amount* of growth** in the t^{th} year.

- a.k.a. the interest earned

$\frac{a(t)-a(t-1)}{a(t-1)}$ ***Rate* of growth** in the t^{th} year.

- a.k.a. effective rate of interest denoted i_t.

$A(t)$ **Amount function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of $\$k$ at time 0.

- $A(t) = ka(t)$.

i_t **Effective rate of interest** defined as the rate of growth based on the amount in the fund at the ***beginning*** of the year.

- $i_t = \frac{A(t)-A(t-1)}{A(t-1)}$.
- We deduce $A(t) = (1 + i_t)A(t-1)$.

Effective Rate of Discount

d_t **Effective rate of discount** defined as the rate of growth based on the amount in the fund at the ***end*** of the year.

- $d_t = \frac{A(t)-A(t-1)}{A(t)}$.
- Although we could get by without it, it's useful to determine the amount to pay today for a specified amount in the future.

Discounting Finding the price we'd be willing to pay for the promise to receive a future amount.

- a.k.a. finding the present value which is why $i = \frac{d}{1-d}$.
- $v = (1-d) = \frac{1}{1+i}$.
- $d = \frac{i}{1+i}$.

Nominal Rates of Interest

$i^{(m)}$ **Nominal** annual rate of of interest **compounded** m **times a year**.

$\frac{i^{(m)}}{m}$ **Effective** rate of of interest **for an** $m^{\mathbf{th}}$ **of a year**.

- Thus $(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m$.

1b. Why Do We Need a Force of Interest ?

- An effective rate of interest only gives information about the starting and ending values, but give no information about in between.
- Thus, the force of interest can give information at any given time about the rate of growth.
- The image of the four different fund's growth curves with the same starting and ending values is a perfect visualization.

1c. Defining the Force of Interest

- The derivative is divided by the amount function to obtain a rate of growth proportional to the amount invested.
- Two funds can have the same rate of change but different amounts originally invested.
- If one fund's growing with a smaller amount of money, then it's rate of change is actually less than the other.

TRAP If given the derivative of the accumulation function, $a'(t)$, use the property that the fund at the beginning is 1, $a(0) = 1$, to define the $+C$ when integrating for $a(t)$.

1d. Finding the Fund in Terms of the Force of Interest

- If we want to find the accumulation, or amount, function from the force of interest we inverse the equation.
- To do so, recall that $\frac{\partial}{\partial x}\ln(f(x)) = \frac{f'(x)}{f(x)}$.
- Also $\int_0^t \frac{\partial}{\partial r}\ln(a(r))dr = \ln(a(r))\Big|_0^t = \ln(a(t))$.

Force of interest

Force of interest the rate of growth at a point in time.

- a.k.a. finding the present value which is why $i = \frac{d}{1-d}$.
- $v = (1-d) = \frac{1}{1+i}$.
- $d = \frac{i}{1+i}$.

δ_t The **Force of interest** at time t.

- $\delta_t = \frac{A'(t)}{A(t)}$.
- $a(t) = \mathrm{e}^{\int_0^t \delta_r dr}$

1e. The Simplest Case : A Constant Force of Interest

-

Simple Force of interest

δ The constant force of interest.

- a.k.a. the nominal rate of interest compounded continuously.
- $\delta = \lim_{m\to\infty} i^{(m)} = i^{(\infty)} = \ln(1+i)$.
- $a(t) = \mathrm{e}^{\int_0^t \delta dr} = \mathrm{e}^{\delta t}$.

1f. Power Series

- Not really on past exams, section is « just in case ».

1g. The Variable Force of Interest Trap

- When we want the accumulated value of an amount not invested at the beginning, we integrate the force of interest over the respective integral.
- Alternatively, we can take the ratio of the accumulation function at both times.

Variable Force of interest

$$\begin{aligned} FV &= \mathrm{e}^{\int_{t_1}^{t_2} \delta_r dr} \\ &\equiv \frac{a(t_2)}{a(t_1)} \end{aligned}$$

1h. Equivalent Rates

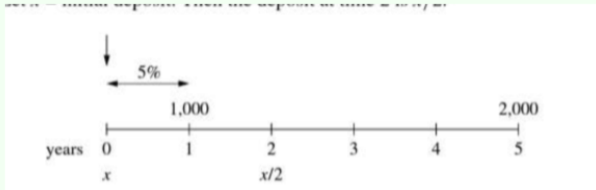
-

2a. Equations of Value, Time Value of Money, and Time Diagrams

Time value (equivalence principle) 1$ today is not equivalent to 1$ a year from now. However, 1$ today is equivalent to 1.05$ a year from now if the rate of interest is 5%.

Comparison date Date at which we solve the **equation of of value**.

- Important to use **time lines** to solve problems :



- We can treat a problem by **payment** or **interest** period.

2b. Unknown Time and Unknown Interest Rate

- Can approximate the time $\bar{t}$ by using a weighted average of the time of payment times the amount of payment divided by the total amount paid with the **method of equated time**. For example :

Time Due	Payment
1	5
3	1
10	15
Total Payments =	21

$$\bar{t} = \frac{1 * 5 + 3 * 1 + 10 * 15}{(5 + 1 + 15)}$$

2 Topic : Annuities / cash flows with non-contingent payments (exam weight)

Information

Objective

The Candidate will be able to calculate present value, current value, and accumulated value for sequences of non-contingent payments.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Annuity-immediate ;
- Annuity-due ;
- Perpetuity ;
- Payable m-thly or continously ;
- Level payment annuity ;
- Arithmetic increasing/decreasing annuity ;
- Geometric increasing/decreasing annuity ;
- Term of annuity ;

b) For each of the following types of annuity / cash flows, given sufficient information of :

- Immediate or due ;
- Present value ;
- Futur value ;
- Current value ;
- Interest rate ;
- Payment amount ;
- Term of annuity,

calculate any remaining item.

The types are :

- Level annuity, finite term ;

- Level perpetuity ;
- Non-level annuities / cash flows ;
  - Arithmetic progression, finite term and perpetuity ;
  - Geometric progression, finite term and perpetuity ;
  - Other non-level annuities / cash flows.

Related lessons ASM

Section 3 : Annuities

- 3a. The Geometric Series Trap
- 3b. Annuity-Immediate and Annuity-Due
- 3c. The Great Confusion : Annuity-Immediate and Annuity-Due
- 3d. Deferred Annuities
- 3e. A Short-Cut Method for Annuities with "Block" Payments
- 3f. Perpetuities
- 3g. The $a_{\overline{2n}|}/a_{\overline{n}|}$ Trick (and Variations)
- 3h. What If the Rate Is Unknown ?
- 3i. What If the Rate Varies ?

Section 4 : Complex Annuities

- 4a. Annuities with "Off-Payments" Part I
- 4b. Annuities with "Off-Payments" Part II
- 4c. Avoiding the m^{thly} Annuity Trap
- 4d. Continuous Annuities
- 4e."Double-Dots Cancel" (and so do "upper m's")
- 4f. A Short Note on Remembering Annuity Formulas
- 4g. The $s_{\overline{n}|}$ Trap When Interest Variess
- 4h. Payments in Arithmetic Progression
- 4i. Remembering Increasing Annuity Formulas
- 4j. Payments in Geometric Progression
- 4k. The Amazing Expanding Money Machine (Or Continouss Varying Annuities)

- 4l. A Short-Cut Method for the Palindromic Annuity
- 4m. The 0% Test : A Quick Check of Symbolic Answers

Résumés des chapitres

3a. The Geometric Series Trap

Remember the formula for geometric series in words :

$$\begin{aligned} r^{10} + r^{20} + \cdots + r^{10n} &= r^{10}\frac{1-r^n}{1-r} \\ &= (\text{first term})\frac{1-(\text{ratio})^{\text{nb. of terms}}}{1-(\text{ratio})} \end{aligned}$$

-

3b. Annuity-Immediate and Annuity-Due

- Origin of the word : annu(us) latin for « yearly » ;
- Standard annuity formulas for $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$;
- Interesting to note the relations between them however.

Formulas

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= \frac{1-v^n}{1-v} \\ &= \frac{1-v^n}{d} \\ a_{\overline{n}|} &= v + v^2 + \cdots + v^{n-1} + v^n \\ &= v\left(\frac{1-v^n}{1-v}\right) \\ &= \frac{1-v^n}{i} \end{aligned}$$

$$
\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i) + \cdots + (1+i)^{n-1} + (1+i)^n \\
&= (1+i)\left(\frac{1-(1+i)^n}{1-(1+i)}\right) \\
&= \frac{(1+i)^n - 1}{d}
\end{aligned}
$$

$$
\begin{aligned}
s_{\overline{n}|} &= 1 + (1+i) + \cdots + (1+i)^{n-1} \\
&= \frac{1-(1+i)^n}{1-(1+i)} \\
&= \frac{(1+i)^n - 1}{i}
\end{aligned}
$$

Relations

$$
\begin{aligned}
\ddot{a}_{\overline{n}|} &= (1+i)a_{\overline{n}|} \\
&= a_{\overline{n-1}|} - 1
\end{aligned}
$$

$$
\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i)s_{\overline{n}|} \\
&= s_{\overline{n+1}|} - 1
\end{aligned}
$$

3c. The Great Confusion : Annuity-Immediate and Annuity-Due

- Defining whether annuities are due or immediate based on *when payments are made* is deceptive, it is more precise to define it based on the *valuation date.*

Annuity

An annuity is called an **annuity-immediate** if, in determining its present value, the valuation date is ***one period before*** the first payment (symbol $a_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its present value, the valuation date is ***on*** the date of the first payment (symbol $\bar{a}_{\overline{n}|}$).

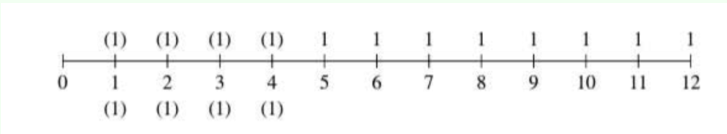
An annuity is called an **annuity-immediate** if, in determining its accumulated value, the valuation date is ***on*** the date of the last payment (symbol $s_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its accumulated value, the valuation date is ***one period after*** the date of the last payment (symbol $\bar{s}_{\overline{n}|}$).

- Important to distinguish dates in time from the number of payments.

  For example, if we're the 1st of January in 2000 and annual payments are made on the 1st of January from 2006 to 2010 then the AV on the date of the last deposit is $s_{\overline{5}|}$ and not $s_{\overline{10}|} - s_{\overline{5}|}$ nor $s_{\overline{2010}|}$, etc.
- Better to set up equations of value with annuity-immediate than annuity-due.

3d. Deferred Annuities

- An n-year annuity deferred r years ${}_{r|}a_n = v^r a_n$.
- Can interpret as "go to to time r and start paying what the symbol to the right says".
- Can also interpret by playing "Now you see it ..." and redefining ${}_{r|}a_n = a_{n+r} - a_r$:



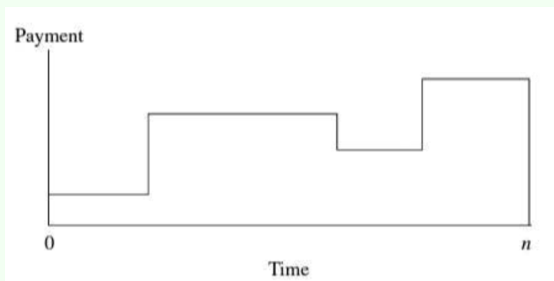
The value is thus ${}_{4|}a_8 = a_{12} - a_4$ in the example.

Deferred Annuities

$$ {}_{r|}a_n \equiv {}_{r+1|}\ddot{a}_n $$
$$ v^r a_n \equiv v^{r+1}\ddot{a}_n $$
$$ a_{n+r} - a_r $$

3e. A Short-Cut Method for Annuities with "Block" Payments

Block payments :



The long way is to calculate the payments by block and divide horizontally (first 8 payments, next 7 payments, etc.).

PV of first 8 payments: $5a_{\overline{8|}}$

PV of next 7 payments: $12(a_{\overline{15|}} - a_{\overline{8|}})$

PV of next 7 payments: $10(a_{\overline{22|}} - a_{\overline{15|}})$

PV of next 6 payments: $15(a_{\overline{28|}} - a_{\overline{22|}})$

Adding up all the present values, combining terms and writing in descending order of the periods:

$$ PV = 15a_{\overline{28|}} - 5a_{\overline{22|}} + 2a_{\overline{15|}} - 7a_{\overline{8|}} $$

The short way starts from the end adding or decreasing annuities according to the change in payment amount.

We want the PV, so the comparison date is time 0. We start with the furthest payment ($15 at time 28) and immediately write $15a_{\overline{28}|}$. We move in closer from time 28 toward time 0 until there is a change. This occurs at time 22, when payments *decrease by $5* (from $15 to $10), so we write $-5a_{\overline{22}|}$. We move in closer, see another change at $t = 15$, when payments *increase by $2* (from $10 to $12) so we write $+2a_{\overline{15}|}$. Finally, the last change is at time 8, a *decrease of $7* (from $12 to $5), so we write $-7a_{\overline{8}|}$.

Putting all of this together, we have:

$$PV = 15a_{\overline{28}|} - 5a_{\overline{22}|} + 2a_{\overline{15}|} - 7a_{\overline{8}|}$$

The same idea is maintained for the AV :

Accumulated Value

The AV of annuities with block payments is obtained in much the same way as the PV. For example, consider the annuity just above. The comparison date is time 30 if we want the AV on the date of the last payment, so we start with the *furthest payment* from time 30, which is $5 at time 1. We immediately write $5s_{\overline{30}|}$. As we move toward the comparison date of time 30, we see that the first change is an *increase of $3* (from $5 to $8) at time 11, so our adjustment term is $+3s_{\overline{20}|}$. (20 is the number of payments that we must increase by $3, i.e., the payments from time 11 to time 30, inclusive, or 30−10 payments.) The next change is an *increase of $4* (from $8 to $12) at time 19, so the adjustment term is $+4s_{\overline{12}|}$. Finally, payments *increase by $8* (from $12 to $20) at time 24, so the adjustment term is $+8s_{\overline{7}|}$. Putting it all together:

$$AV = 5s_{\overline{30}|} + 3s_{\overline{20}|} + 4s_{\overline{12}|} + 8s_{\overline{7}|}$$

3f. Perpetuities

Perpetuity-immediate $a_{\overline{\infty}|} = \frac{1}{i}$. One way to prove this is with the limit :

$$\lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \left(\frac{1 - v^n}{i}\right) = \frac{1}{i}$$

Perpetuity-due $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$. One way to prove this is with the limit :

$$\lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \lim_{n\to\infty} \left(\frac{1 - v^n}{d}\right) = \frac{1}{d}$$

link The PV of a perpetuity-due exceeds that of a perpetuity-immediate by the payment of 1 at time 0 :

$$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|} = 1 + \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d}$$

$$\ddot{a}_{\overline{\infty}|} - a_{\overline{\infty}|} = \frac{1}{d} - \frac{1}{i} = 1$$

Relationship

3g. The $a_{\overline{2n}|}/a_{\overline{n}|}$ Trick (and Variations)

$a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$ and can proven several ways :

Difference of squares :

$$a_{\overline{2n}|}/a_{\overline{n}|} = \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} = \frac{1-v^{2n}}{1-v^n} = \frac{(1-v^n)(1+v^n)}{1-v^n} = 1 + v^n$$

General reasoning :

$$a_{\overline{2n}|}/a_{\overline{n}|} = \frac{a_{\overline{n}|} + {}_{n|}a_{\overline{n}|}}{a_{\overline{n}|}} = \frac{a_{\overline{n}|} + v^n a_{\overline{n}|}}{a_{\overline{n}|}} = \frac{a_{\overline{n}|}(1+v^n)}{a_{\overline{n}|}} = (1+v^n)$$

This can be **generalized** :

$$\begin{aligned} a_{\overline{3n}|}/a_{\overline{n}|} &= \frac{a_{\overline{n}|} + {}_{n|}a_{\overline{n}|} + {}_{2n|}a_{\overline{n}|}}{a_{\overline{n}|}} = \frac{a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}}{a_{\overline{n}|}} \\ &= \frac{a_{\overline{n}|}(1+v^n+v^{2n})}{a_{\overline{n}|}} = (1+v^n+v^{2n}) \end{aligned}$$

3h. What If the Rate Is Unknown ?

If the PV, number, and amount of payments of an annuity are known we can use the calculator tto solve for the interest rate.

3i. What If the Rate Varies ?

Be careful not to mix up the PV and AV interest accumulation.
Also, split up annuities if the interest rate varies so as not to make mistakes.

4a. Annuities with "Off-Payments" Part I

off-payments Payments which are less or more frequent than the interest period.
For example :

- Payments of 1 at the end of each 5-year period over 40 years -> payments are *less frequent* than the interest period of one year.
- Payments of $\frac{1}{12}$ at the end of each month for 10 years -> payments are *more frequent* than the interest period of one year.

There are generally 2 approaches for handling these types of annuities

1. Use interest functions at the equivalent effective rate of interest for the **payment period**.
   - Method is generally easier for numerical answers (i.e., most of the time).
2. Use interest functions at the effective rate of interest **given in the problem**.
   - Method is generally easier for symbolic answers.

The first method is this subsection. For example, monthly payments of $\frac{1}{12}$ paid at the end of each month for 10 years at an effective rate of 5% per annum.

- First we find the equivalent rate :

$$\begin{aligned} 1 + j &= (1.05)^{1/12} \\ j &= 0.4074\% \end{aligned}$$

- Then the PV :

$$\begin{aligned} PV &= \frac{1}{12} a_{\overline{120}|j} \\ &= 7.8971 \end{aligned}$$

4b. Annuities with "Off-Payments" Part II

Fission method When payments are less frequent than the interest period.

Fusion method When payments are more frequent than the interest period.

Example of Fission method :

Payments of 1 every 5 years for 40 years at an annual effective rate of 5%. What annual payment R is equivalent to a payment of 1 every 5 years ?

$$Rs_{\overline{5}|} = 1 \qquad \Rightarrow \qquad R = \frac{1}{s_{\overline{5}|}}$$

So the PV becomes :

$$\left(\frac{1}{s_{\overline{5}|}}\right) a_{\overline{40}|}$$

Thus we have done « **fission** » by splitting up the payments into smaller payments.

It's very important to consider these 2 cases however :

- For payments which are in the beginning of the period, set up the equation as $P = R\ddot{a}_{\overline{n}|}$.
- For payments which are in the end of the period, set up the equation as $P = Rs_{\overline{n}|}$.

The fusion method leads to these formulas :

$$a^{(m)}_{\overline{n}|} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|} = s^{(m)}_{\overline{1}|} a_{\overline{n}|}$$

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{d^{(m)}} = \frac{i}{d^{(m)}} a_{\overline{n}|} = \ddot{s}^{(m)}_{\overline{1}|} a_{\overline{n}|}$$

$$s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|} = s_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} s_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

The reasoning is we accumulate the payment over the m periods before treating it on an annual basis.
The same reasoning applies to perpetuities

$$a_{\overline{\infty}|}^{(m)} = \frac{1}{i^{(m)}} \qquad \ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}} \qquad a_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{m}$$

4c. Avoiding the m^{thly} Annuity Trap

- It's important not to forget to have the payment be on the base as the annuity.
- For example : payments of 100$ paid at the end of every month over 10 years at an effective annual rate of 5% means an annuity of $12 * 100 a_{\overline{n}|}^{(12)}$.

  So the annuity is compounded 12 times a year with the "yearly" payment of 1200$.
- So the payment, or coefficient of the $a_{\overline{n}|}$ term, is the ***sum* of the payments** in ***each* interest period**.

4d. Continuous Annuities

- $a_{\overline{n}|}^{(m)}$ always requires a **total payment of 1** each year, regardless of the value of m.
- The 1 is payable in $m$thly installments of $\frac{1}{m}$.
- Thus we obtain this result as m grows :

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \left(\frac{1-v^n}{i^{(m)}}\right) \qquad = \lim_{m\to\infty} \left(\frac{1-v^n}{\delta}\right) = \bar{a}_{\overline{n}|}$$

- We also obtain the relation $\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|}$.

4e."Double-Dots Cancel" (and so do "upper m's")

Given annuities being divided, we obtain :

$$\frac{\ddot{a}_{\overline{n}|}^{(m)}}{\ddot{a}_{\overline{p}|}^{(m)}} = \frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{p}|}} = \frac{a_{\overline{n}|}}{a_{\overline{p}|}} = \frac{a_{\overline{n}|}^{(m)}}{a_{\overline{p}|}^{(m)}} = \frac{1-v^n}{1-v^p}$$

4f. A Short Note on Remembering Annuity Formulas

PV all have $1 - v^n$ as the numerator.

FV all have $(1+i)^n - 1$ as the numerator.

annuity *i*mmediate has i for ***i***mmediate $\frac{1-v^n}{i}$.

annuity *d*ue has d for ***d***ue $\frac{1-v^n}{d}$.

compounded m times a year changes i for $i^{(m)}$ or d for $d^{(m)}$.

compounded continuously changes i and d for δ.

4g. The $s_{\overline{n}|}$ Trap When Interest Variess

- If the force of interest is variable important not to fall into the trap of accumulating from 0.
- $a(t)$ is the accumulation from 0 so the AV at time 5 of a payment at time 4 is $\frac{a(5)}{a(4)}$ and not $a(1)$.

4h. Payments in Arithmetic Progression

P First payment.

Q Common difference.

We have $P \neq Q$ and these formulas don't have standard symbols but can treat any annuity in arithmetic progression.

PV of an annuity in arithmetic progession

$$A = Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}$$
$$\ddot{A} = P\ddot{a}_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{d}$$

AV of an annuity in arithmetic progession

$$S = (1+i)^n A = Ps_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{i}$$
$$\ddot{S} = (1+i)^n \ddot{A} = P\ddot{s}_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{d}$$

If we can memorize the first one, should be okay to deduce the rest.

If $P = Q = 1$, we have an **increasing annuity** which has a symbol.

$$(Ia)\overline{n}| = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$
$$(I\ddot{a})\overline{n}| = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$
$$(Is)\overline{n}| = (Ia)\overline{n}|(1+i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{i} \equiv \frac{s_{\overline{n+1}|} - (n+1)}{i}$$
$$(I\ddot{s})\overline{n}| = (I\ddot{a})\overline{n}|(1+i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

Relationship The AV of the increasing annuity is the sum of n annuities :

$$(Is)\overline{n}| = \sum_{t=1}^{n} s_{\overline{t}|} = \sum_{t=1}^{n} \frac{(1+i)^t - 1}{i} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

If $P = n$ and $Q = -1$, we have a **decreasing annuity** which has a symbol.

$$(Da)\overline{n}| = \frac{n - a_{\overline{n}|}}{i}$$

Relationship The PV of the decreasing annuity is the sum of n annuities :

$$(Da)_{\overline{n}|} = \sum_{t=1}^{n} a_{\overline{t}|} = \sum_{t=1}^{n} \left(\frac{1 - v^t}{i} \right) = \frac{n - a_{\overline{n}|}}{i}$$

Note We can combine level and increasing annuities (see ASM for examples).

Increasing Perpetuities where $P = Q = 1$:

$$(Ia)_{\overline{\infty}|} = \frac{1}{id} \equiv \frac{1}{i} + \frac{1}{i^2}$$

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

If $P \neq Q$:

$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

Increasing and then level perpetuity where payment is increasing for n years and remains at n therafter :

$$PV = (Ia)_{\overline{n}|} + v^n \left(\frac{n}{i} \right) = \frac{\ddot{a}_{\overline{n}|}}{i}$$

4i. Remembering Increasing Annuity Formulas

A few basic definitions :

I means the **annual rate** of payment increases once a year.
For example, the annual rate of payment is 1 in the first year, 2 in the second, etc.

$I^{(m)}$ means the **annual rate** of payment increases at m^{thly} intervals
For example, the annual rate of payment is $\frac{1}{m}$ in the first $\frac{1}{m}$-th

of a year, $\frac{2}{m}$ in the second $\frac{1}{m}$-th of a year, etc.

a means payments are made **annually**.

$a^{(m)}$ means payments are made $m$thly.

For example :

$(Ia)_{\overline{n}|}$ **annual rate** of payment of 1 in the first year, 2 in the second, etc.
The payments are made annually up to a payment of n in the nth year.

$(Ia)_{\overline{n}|}^{[(m)]}$ **annual rate** of payment of 1 in the first year, 2 in the second, etc.
The payments are made m-thly thus there are m payments of $\frac{1}{m}$ in the first year, m payments of $\frac{2}{m}$ in the second, etc.

$(I^{(m)}a)_{\overline{n}|}^{(m)}$ the **annual rate** of payment increases m-thly and payments are made in m-thly installments.
The payments at the end of each m-th of a year are $\frac{1}{m^2}, \frac{2}{m^2}, \ldots, \frac{mn}{m^2}$.
To remember, we can think of it as the « double m » symbol.

In brief :

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

$$(Ia)_{\overline{n}|}^{[(m)]} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}}$$

$$(I^{(m)}a)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}}$$

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

NOTE Revise this section's table in the book (page 230) to understand logic completely.

4j. Payments in Geometric Progression

The PV of an annuity with a first payment of 1 and subsequent payments increasing by a factor of $(1 + k)$ annually at an effective rate of interest of i is :

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i - k}$$

Alternatively, we can define the new rate of interest i' as $i' = \frac{|i-k|}{1+k}$ (we insert absolute value as we subtract the larger from the smaller rate). With this rate, we can use normal formulas such as $a_{\overline{n}|}$.

4k. The Amazing Expanding Money Machine (Or Continouss Varying Annuities)

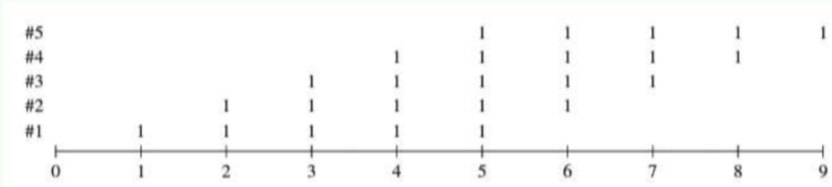
Chapter explains the intuition behind continuously paid annuities.

- For a the continuously increasing continuously paid annuity, $(\bar{I}a)_{\overline{n}|}$, we integrate 1 times v^t as the payments are linearly increasing $(\int_0^n v^t dt)$.
- We can also integrate a function and have $PV = \int_0^n f(t)v^t dt$ if it is not constant.
- Then, if the interest varies, we have $PV = \int_0^n f(t)\mathrm{e}^{-\int_0^t \delta_r dr} dt$

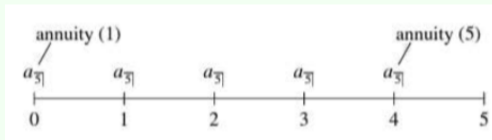
4l. A Short-Cut Method for the Palindromic Annuity

A palindrome is, for example, 1342431 where it is the same read backward or forward. This pattern can occur for annuities too.
The book takes for example a series of payments of $1, 2, 3, 4, 5, 4, 3, 2, 1$.

- This could be solved as $(Ia)_{\overline{5}|} + v^5(Da)_{\overline{4}|}$.
- However, there is a simpler method as this equation yields $a_{\overline{5}|} \cdot \ddot{a}_{\overline{5}|}$.
- First, we visualise the payments :



- Which we can rewrite as :



- Thus, $PV = a_{\overline{5}|}\ddot{a}_{\overline{5}|}$.
- If the original series is due, we have the annuity-due squared.

4m. The 0% Test : A Quick Check of Symbolic Answers

Trick to eliminate some of the answer choices when answers are in symbolic form.
Since answers should be correct for any rate of interest, they should be correct for 0% as well. With $i = 0$, we most notably get $a_{\overline{n}|} = s_{\overline{n}|} = n$.

NOTE revise this section to properly understand examples.

3 Topic : Loans (10%-20%)

Information

Objective

The Candidate will understand key concepts concerning loans and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Principal ;
- Interest ;
- Term of loan ;
- Outstanding balance ;
- Final payment ;
  - Drop payment ;
  - Baloon payment.
- Amortization.

b) Calculate :

- The missing item given any 4 of :
  - Term of loan ;
  - Interest rate ;
  - Payment amount ;
  - Payment period ;
  - Principal.
- The outstanding balance at any point in time ;
- The amount of interest and principal repayment in a given payment ;
- Similar calculations to the above when refinancing is involved.

Related lessons ASM

Section 6 : Loans

- XX. Title-of-ASM-chapter

Vidéos YouTube

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Résumés des chapitres

XX. Title-of-ASM-chapter

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Notes sur les vidéos YouTube

4 Topic : Bonds (10%-20%)

Information

Objective

The Candidate will understand key concepts concerning bonds, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Price ;
- Book value ;
- Amortization of premium ;
- Accumulation of discount ;
- Redemption value ;
- Par value / Face value ;
- Yield rate ;
- Coupon ;
- Coupon rate ;
- Term of bond ;
- Callable / Non-callable.

b) Given sufficient partial information about the items listed below, calculate any of the remaining items :

- Price, book value, amortization of premium, accumulation of discount ;
- Redemption value, face value ;
- Yield rate ;
- Coupon, coupon rate ;
- Term of bond, point in time that a bond has a given book value, amortization of premium, or accumulation of discount.

Related lessons ASM

Section 7 : Bonds

- XX. Title-of-ASM-chapter

Vidéos YouTube

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Résumés des chapitres

XX. Title-of-ASM-chapter

-

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5 Topic : General Cash Flows and Portfolios (15%-20%)

Information

Objective

The Candidate will understand key concepts concerning yield curves, rates of return, and measures of duration and convexity, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Yield rate / rate of return ;
- Dollar-weighted rate of return ;
- Time-weighted rate of return ;
- Current value ;
- Duration (Macaulay and modified) ;
- Convexity (Macaulay and modified) ;
- Portfolio ;
- Spot rate ;
- Forward rate ;
- Yield Curve ;
- Stock price ;
- Stock dividend ;

b) Calculate :

- The dollar-weighted and time-weighted rate of return ;
- The duration and convexity of a set of cash flows ;
- Either Macaulay or modified duration given the other ;
- The approximate change in present value due to a change in interest rate,

  - Using 1st-order linear approximation based on modified duration ;
  - Using 1st-order approximation based on Macaulay duration.
- The price of a stock using the dividend discount model ;
- The present value of a set of cash flows, using a yield curve developed from forward and spot rates.

Related lessons ASM

Section 5 :

- 5a
- 5b
- 5c
- 5d
- 5e
- 5f

Section 8 :

- 8a
- 8b
- 8c

Section 10 :

- **??**
- **??**
- **??**
- **??**
- **??**
- **??**
- **??**

Section 11 :

- **??**

Vidéos YouTube

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Résumés des chapitres

5a.

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5b.

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5c.

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5d.

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5e.

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5f.

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8a.

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8b.

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8c.

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6 Topic : Immunization (10%-15%)

Information

Objective

The Candidate will understand key concepts concerning cash flow matching and immunization, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :
   - Cash flow matching ;
   - Immunization (including full immunization) ;
   - Redington immunization.

b) Construct an investment portfolio to :
   - Redington immunize a set of liability cash flows ;
   - Fully immunize a set of liability cash flows ;
   - Exactly match a set of liability cash flows.

Related lessons ASM

Section 10 : Duration, Convexity, and Immunization

- 10h. Redington Immunization
- 10i. Full Immunization
- 10j. A Note on Rebalancing
- 10k. Immunization by Exact Matching ("Dedication")

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Résumés des chapitres

10h. Redington Immunization

›

10i. Full Immunization

›

10j. A Note on Rebalancing

›

10k. Immunization by Exact Matching ("Dedication")

›

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7 Topic : Interest Rate Swaps (0-10%)

Information

Objective

The Candidate will understand key concepts concerning interest rate swaps, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Swap rate ;
- Swap term (tenor) ;
- Notional amount ;
- Market value of a swap ;
- Settlement dates ;
- Settlement period ;
- Counterparties ;
- Deferred swap ;
- Amortizing swap ;
- Accreting swap ;
- Interest rate swap net payments.

b) Given sufficient information, calculate :

- The market value ;
- Notional amount ;
- Spot rates or swap rate,
- deferred or otherwise ;
- with either constant or varying notional amount.

of an interest rate swap

Related lessons ASM

Section 11 : Interest Rate Swaps

- 11b. What is an Interest Rate Swap ?

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Résumés des chapitres

11b. What is an Interest Rate Swap ?

›

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8 Topic : Determinants of Interest Rates (0-10%)

Information

Objective

The Candidate will understand key concepts concerning the determinants of interest rates, the components of interest, and how to perform related calculations.

Learning outcomes

The candidate will be able to :

a) Define and recognize the *definitions* of the following terms :

- Real risk-free rate ;
- Inflation rate ;
- Default risk premium ;
- Liquidity premium ;
- Maturity risk premium.

b) Explain how the components of interest rates apply in various contexts, such as :

- Commercial loans ;
- Mortgages ;
- Credit cards ;
- Bonds ;
- Government securities.

c) Explain the **roles** of the Federal Reserve and the FOMC in carrying out *fiscal* policy and *monetary* policy and the **tools** used thereby including :

- Targeting the federal funds rate ;
- Setting reserve requirements ;

- Setting the discount rate.

d) Explain the theories of why interest rates differ by term, including :
   - Liquidity preference (opportunity cost) ;
   - Expectations ;
   - Preferred habitat ;
   - Market segmentation.

e) Explain how interest rates differ from one country to another (e.g., U.S. vs. Canada) ;

f) In the context of loans with and without inflation protection :
   - **Identify** the *real* interest and the *nominal* interest rate ;
   - **Calculate** the effect of changes in inflation on loans with inflation protection.

Related lessons ASM

Section 9 : Determinants of Interest Rates

- 9a. What is Interest ?
- 9b. Quotation Bases for Interest Rates
- 9c. Components of the Interest Rate : No Inflation or Default Risk
- 9d. Components of the Interest Rate : no Inflation but with Default Risk
- 9e. Components of the Interest Rate : Known Inflation
- 9f. Components of the Interest Rate : Uncertain Inflationo
- 9g. Savings and Lending Interest Rates
- 9h. Government and Corporate Bonds
- 9i. The Role of Central Banks

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Résumés des chapitres

9a. What is Interest ?

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9b. Quotation Bases for Interest Rates

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9c. Components of the Interest Rate : No Inflation or Default Risk

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9d. Components of the Interest Rate : no Inflation but with Default Risk

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9e. Components of the Interest Rate : Known Inflation

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9f. Components of the Interest Rate : Uncertain Inflationo

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9g. Savings and Lending Interest Rates

›

9h. Government and Corporate Bonds

›

9i. The Role of Central Banks

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