Notes calculatrice

- The option P/Y permits to set the payments per year and C/Y the number of time the interest rate is compounded per year.
- In this case, we set the rate to $i^{(m)}$, the number of periods to $n \times m$, the payment to monthly payment $\frac{P}{m}$.
- If we do 2ND xP/Y it multiplies the period by the number of payments per year.
- To calculate decreasing / increasing annuities just calculate each component one at a time.
- To calculate bonds, enter the coupon for the year (for example, if semi-annual sum both coupons).

Time Value of Money

Notation effective rate of interest

$a(t)$ **Accumulation function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of \$1 at time 0.

- $a(0) \equiv 1$.
- Generally **continuous** and **increasing**.

$A(t)$ **Amount function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of \$$k$ at time 0.

- $A(t) = ka(t)$.

i_t **Effective rate of interest** defined as the rate of growth based on the amount in the fund at the ***beginning*** of the year.

- $i_t = \frac{A(t)-A(t-1)}{A(t-1)}$.
- We deduce $A(t) = (1+i_t)A(t-1)$.

We then find :

$a(t) - a(t-1)$ ***Amount* of growth** in the t^{th} year.

- a.k.a. the interest earned

$\frac{a(t)-a(t-1)}{a(t-1)}$ ***Rate* of growth** in the t^{th} year.

- a.k.a. effective rate of interest denoted i_t.

Discounting is finding the price we'd be willing to pay for the promise to receive a future amount. That is to say, finding the present value which is why $i = \frac{d}{1-d}$.

Notation effective rate of discount

d_t **Effective rate of discount** defined as the rate of growth based on the amount in the fund at the ***end*** of the year.

- $d_t = \frac{A(t)-A(t-1)}{A(t)}$.
- Although we could get by without it, it's useful to determine the amount to pay today for a specified amount in the future.

We then find :

$$v = (1-d) = \frac{1}{1+i}$$

$$d = \frac{i}{1+i}$$

Notation nominal rates of interest

$i^{(m)}$ **Nominal** annual rate of of interest **compounded** m **times a year**.

$\frac{i^{(m)}}{m}$ **Effective** rate of of interest **for an** m^{th} **of a year**.

- Thus $(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m$.

The force of interest is the rate of growth at a specific point in time.

Notation force of interest

δ_t The **Force of interest** at time t.

- $\delta_t = \frac{A'(t)}{A(t)}$.

δ The **constant** force of interest.

- a.k.a. the nominal rate of interest compounded continuously.
- $\delta = \lim_{m\to\infty} i^{(m)} = i^{(\infty)} = \ln(1+i)$.

We then find :

$$a(t) = e^{\int_0^t \delta_r dr}$$

$$FV = e^{\int_{t_1}^{t_2} \delta_r dr} \equiv \frac{a(t_2)}{a(t_1)}$$

and for a constant force of interest :

$$a(t) = e^{\int_0^t \delta dr} = e^{\delta t}$$

Annuities / cash flows with non-contingent payments

Geometric series relation :

$$r^{10}+r^{20}+\cdots+r^{10n}=r^{10}\frac{1-r^n}{1-r}$$
$$=(\text{first term})\frac{1-(\text{ratio})^{\text{nb. of terms}}}{1-(\text{ratio})}$$

Annuity

An annuity is called an **annuity-immediate** if, in determining its present value, the valuation date is ***one period before*** the first payment (symbol $a_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its present value, the valuation date is ***on*** the date of the first payment (symbol $\ddot{a}_{\overline{n}|}$).

An annuity is called an **annuity-immediate** if, in determining its accumulated value, the valuation date is ***on*** the date of the last payment (symbol $s_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its accumulated value, the valuation date is ***one period after*** the date of the last payment (symbol $\ddot{s}_{\overline{n}|}$).

Standard Annuities :

$$\ddot{a}^{(m)}_{\overline{n}|}=\frac{1-v^n}{(i^{(m)}|d^{(m)})} \qquad \ddot{s}^{(m)}_{\overline{n}|}=\frac{(1+i)^n-1}{(i^{(m)}|d^{(m)})}$$

Annuities for Payments in Arithmetic Progression :

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|}=\frac{\ddot{a}^{(m)}_{\overline{n}|}-nv^n}{(i|d^{(m)})} \qquad (D^{(m)}\ddot{a})^{(m)}_{\overline{n}|}=\frac{n-a^{(m)}_{\overline{n}|}}{(i|d^{(m)})}$$

$$(I^{(m)}\ddot{s})^{(m)}_{\overline{n}|}=\frac{\ddot{s}^{(m)}_{\overline{n}|}-n}{(i|d^{(m)})} \qquad (D^{(m)}\ddot{s})^{(m)}_{\overline{n}|}=\frac{n(1+i)^n-s^{(m)}_{\overline{n}|}}{(i|d^{(m)})}$$

Annuities for Payments in Arithmetic Progression :

$$(\bar{I}\bar{s})_{\overline{n}|i}=\frac{\bar{s}_{\overline{n}|i}-n}{\delta} \qquad (\bar{D}\bar{s})_{\overline{n}|i}=\frac{ne^{\delta n}-\bar{s}_{\overline{n}|i}}{\delta}$$

$$(\bar{I}\bar{a})_{\overline{n}|i}=\frac{\bar{a}_{\overline{n}|i}-ne^{-\delta n}}{\delta} \qquad (\bar{D}\bar{a})_{\overline{n}|i}=\frac{n-\bar{a}_{\overline{n}|i}}{\delta}$$

Annuities for Payments in Arithmetic Progression :

$$\ddot{a}_{\overline{n}|i^R}=\frac{1-(1+i^R)^{-n}}{\left(\frac{i^R}{1+i^R}\right)}\frac{1}{1+r} \qquad s_{\overline{n}|i^R}=\frac{(1+i)^n-(1+r)^n}{i-r}$$

Relations between types of Annuities :

$$\ddot{a}_{\overline{n}|}=(1+i)a_{\overline{n}|}$$
$$=a_{\overline{n-1}|}-1$$
$$\ddot{s}_{\overline{n}|}=(1+i)s_{\overline{n}|}$$
$$=s_{\overline{n+1}|}-1$$

Relations for deferred Annuities :

$${}_{r|}a_n\equiv{}_{r+1|}\ddot{a}_n$$
$${}_{r|}a_n=a_{n+r}-a_r$$
$${}_{r|}a_n=v^r a_n$$
$$\equiv v^{r+1}\ddot{a}_n$$

Other relations :

$$a_{\overline{2n}|}/a_{\overline{n}|}=(1+v^n)$$
$$a_{\overline{3n}|}/a_{\overline{n}|}=(1+v^n+v^{2n})$$

Perpetuities :

$$\ddot{a}_{\overline{\infty}|}=\frac{1}{(i|d)}$$
$$(I\ddot{a})_{\overline{\infty}|}=\frac{1}{(i|d)d}$$

Payments in arithmetic progression :
For a first payment of P and a common difference of Q,

$$PV=Pa_{\overline{n}|}+Q\frac{a_{\overline{n}|}-nv^n}{i}$$

Bonds

Callable Bond

Bond in which the borrower (issuer) may redeem the bond at a certain **call price** before it has matured. Generally, the bond cannot be redeemed before a certain **call date**.

- The **call premium** is the difference between the *call price* and the *redemption value* at maturity.

Context

- When interest rates drop, the bond issuer can redeem the bonds and reinvest the call price into bonds with a lower yield rate.
- To compensate for the higher risk, callable bonds typically offer a *higher yield rate*.

To price callable bonds, we find the worst possible date of redemption at which we ensure a minimal return :

- For bonds selling at a **P**remium, the **E**arliest redemption date is the **W**orst one (**PEW**).
- For bonds selling at a discount, the latest redemption date is the worst one.

General Cash Flows and Portfolios

Stocks

Price of a share of Stock

Notation

D First dividend.

k Annual rate of increase, $\boxed{k < i}$.

Level dividends $\boxed{P = \frac{D}{i}}$.

Increasing dividends $\boxed{P = \frac{D}{i-k}}$.

Duration and Convexity

$P(i)$ PV of the CFs at interest rate i.

$$P(i) = \sum_{t=0}^{n} (A_t v^t)$$

Duration

$D_{\textbf{mac}}(i)$ Macaulay duration (PV sensitivity) at interest rate i (at force of interest δ).

$$D_{\text{mac}}(i) = \frac{\sum_{t=0}^{n} (t)(A_t v^t)}{P(i)} = \frac{-P'(\delta)}{P(\delta)}$$

$D_{\textbf{mod}}(i)$ Modified duration (PV sensitivity) at interest rate i.

$$D_{\text{mod}}(i) = \frac{\sum_{t=0}^{n} (t)(A_t v^{t+1})}{P(i)} = \frac{-P'(i)}{P(i)}$$

$D_{\text{mod}}(i) = v D_{\text{mac}}(i)$.

For a portfolio with n components of duration D_i and price P_i, $D(ptf) = \frac{\sum_{i=1}^{n} D_i P_i}{\sum_{i=1}^{n} P_i}$.

Approximations

Notation

i_0 Original interest rate used to calculate the PV of the cash flows.

i_n New interest rate for which we want to approximate the change in Pv.

First-Order Modified Price Approximation $\boxed{P(i) \approx P(i_0)[1 - (i - i_0) D_{\text{mod}}(i_0)]}$.

First-Order Macaulay Price Approximation $\boxed{P(i) \approx P(i_0) \left(\frac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)}}$.

Convexity

$C_{\textbf{mac}}(i)$ Macaulay convexity at interest rate i.

$$C_{\text{mac}}(i) = \frac{\sum_{t=0}^{n} (t^2)(A_t v^t)}{P(i)} = \frac{P''(\delta)}{P(\delta)}$$

$C_{\textbf{mod}}(i)$ Modified convexity at interest rate i.

$$C_{\text{mod}}(i) = \frac{\sum_{t=0}^{n} (t)(t+1)(A_t v^{t+2})}{P(i)} = \frac{P''(i)}{P(i)}$$

$C_{\text{mod}}(i) = v^2 \left(C_{\text{mac}}(i) + D_{\text{mac}}(i)\right)$

Immunization

<table>
<tr><th>Redington</th><th>Full</th></tr>
<tr><td colspan="2">$PV_{\text{A}} = PV_{\text{L}}$</td></tr>
<tr><td colspan="2">$MacD_{\text{A}} = MacD_{\text{L}}$
or
$P'_{\text{A}} = P'_{\text{L}}$</td></tr>
<tr><td>$C_{\text{A}} = C_{\text{L}}$
or
$P''_{\text{A}} > P''_{\text{L}}$</td><td>There has to be an A CF before and after each L CF</td></tr>
<tr><td>Immunizes against small changes in i</td><td>Immunizes against any change in i</td></tr>
</table>