Time Value of Money

Notation effective rate of interest

$a(t)$ **Accumulation function** defined as the *Accumulated Value* (*AV*) of the fund at time t of an initial investment of \$1 at time 0.

- $a(0) \equiv 1$.
- Generally **continuous** and **increasing**.

$A(t)$ **Amount function** defined as the Accumulated Value (AV) of the fund at time t of an initial investment of \$$k$ at time 0.

- $A(t) = ka(t)$.

i_t **Effective rate of interest** defined as the rate of growth based on the amount in the fund at the ***beginning*** of the year.

- $i_t = \frac{A(t)-A(t-1)}{A(t-1)}$.
- We deduce $A(t) = (1+i_t)A(t-1)$.

We then find :

$a(t) - a(t-1)$ ***Amount* of growth** in the t^{th} year.

- a.k.a. the interest earned

$\frac{a(t)-a(t-1)}{a(t-1)}$ ***Rate* of growth** in the t^{th} year.

- a.k.a. effective rate of interest denoted i_t.

Discounting is finding the price we'd be willing to pay for the promise to receive a future amount. That is to say, finding the present value which is why $i = \frac{d}{1-d}$.

Notation effective rate of discount

d_t **Effective rate of discount** defined as the rate of growth based on the amount in the fund at the ***end*** of the year.

- $d_t = \frac{A(t)-A(t-1)}{A(t)}$.
- Although we could get by without it, it's useful to determine the amount to pay today for a specified amount in the future.

We then find :

$$v = (1-d) = \frac{1}{1+i}$$

$$d = \frac{i}{1+i}$$

Notation nominal rates of interest

$i^{(m)}$ **Nominal** annual rate of of interest **compounded** m **times a year**.

$\frac{i^{(m)}}{m}$ **Effective** rate of of interest **for an** m^{th} **of a year**.

- Thus $(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m$.

The force of interest is the rate of growth at a specific point in time.

Notation force of interest

δ_t The **Force of interest** at time t.

- $\delta_t = \frac{A'(t)}{A(t)}$.

δ The **constant** force of interest.

- a.k.a. the nominal rate of interest compounded continuously.
- $\delta = \lim_{m\to\infty} i^{(m)} = i^{(\infty)} = \ln(1+i)$.

We then find :

$$a(t) = e^{\int_0^t \delta_r dr}$$

$$FV = e^{\int_{t_1}^{t_2} \delta_r dr} \equiv \frac{a(t_2)}{a(t_1)}$$

and for a constant force of interest :

$$a(t) = e^{\int_0^t \delta dr} = e^{\delta t}$$

Annuities / cash flows with non-contingent payments

Geometric series relation :

$$\begin{aligned} r^{10}+r^{20}+\cdots+r^{10n} &= r^{10}\frac{1-r^n}{1-r} \\ &= (\text{first term})\frac{1-(\text{ratio})^{\text{nb. of terms}}}{1-(\text{ratio})} \end{aligned}$$

Annuity

An annuity is called an **annuity-immediate** if, in determining its present value, the valuation date is ***one period before*** the first payment (symbol $a_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its present value, the valuation date is ***on*** the date of the first payment (symbol $\ddot{a}_{\overline{n}|}$).

An annuity is called an **annuity-immediate** if, in determining its accumulated value, the valuation date is ***on*** the date of the last payment (symbol $s_{\overline{n}|}$). An annuity is called an **annuity-due** if, in determining its accumulated value, the valuation date is ***one period after*** the date of the last payment (symbol $\ddot{s}_{\overline{n}|}$).

Standard Annuities :

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{(i^{(m)}|d^{(m)})} \qquad \ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n-1}{(i^{(m)}|d^{(m)})}$$

Annuities for Payments in Arithmetic Progression :

$$(I^{(m)}\ddot{a})^{(m)}_{\overline{n}|} = \frac{\ddot{a}^{(m)}_{\overline{n}|}-nv^n}{(i|d^{(m)})} \qquad (D^{(m)}\ddot{a})^{(m)}_{\overline{n}|} = \frac{n-a^{(m)}_{\overline{n}|}}{(i|d^{(m)})}$$

$$(I^{(m)}\ddot{s})^{(m)}_{\overline{n}|} = \frac{\ddot{s}^{(m)}_{\overline{n}|}-n}{(i|d^{(m)})} \qquad (D^{(m)}\ddot{s})^{(m)}_{\overline{n}|} = \frac{n(1+i)^n-s^{(m)}_{\overline{n}|}}{(i|d^{(m)})}$$

Annuities for Payments in Arithmetic Progression :

$$(\bar{I}\bar{s})_{\overline{n}|i} = \frac{\bar{s}_{\overline{n}|i}-n}{\delta} \qquad (\bar{D}\bar{s})_{\overline{n}|i} = \frac{ne^{\delta n}-\bar{s}_{\overline{n}|i}}{\delta}$$

$$(\bar{I}\bar{a})_{\overline{n}|i} = \frac{\bar{a}_{\overline{n}|i}-ne^{-\delta n}}{\delta} \qquad (\bar{D}\bar{a})_{\overline{n}|i} = \frac{n-\bar{a}_{\overline{n}|i}}{\delta}$$

Annuities for Payments in Arithmetic Progression :

$$\ddot{a}_{\overline{n}|i^R} = \frac{1-(1+i^R)^{-n}}{\left(\frac{i^R}{1+i^R}\right)}\frac{1}{1+r} \qquad s_{\overline{n}|i^R} = \frac{(1+i)^n-(1+r)^n}{i-r}$$

Relations between types of Annuities :

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= (1+i)a_{\overline{n}|} \\ &= a_{\overline{n-1}|}-1 \\ \ddot{s}_{\overline{n}|} &= (1+i)s_{\overline{n}|} \\ &= s_{\overline{n+1}|}-1 \end{aligned}$$

Relations for deferred Annuities :

$$\begin{aligned} {}_{r|}a_n &\equiv {}_{r+1|}\ddot{a}_n \\ {}_{r|}a_n &= a_{n+r}-a_r \\ {}_{r|}a_n &= v^r a_n \\ &\equiv v^{r+1}\ddot{a}_n \end{aligned}$$

Other relations :

$$a_{\overline{2n}|}/a_{\overline{n}|} = (1+v^n)$$

$$a_{\overline{3n}|}/a_{\overline{n}|} = (1+v^n+v^{2n})$$

Perpetuities :

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{(i|d)}$$

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{(i|d)d}$$

Payments in arithmetic progression :
For a first payment of P and a common difference of Q,

$$PV = Pa_{\overline{n}|}+Q\frac{a_{\overline{n}|}-nv^n}{i}$$