Fractional Replications

Consider a 2^8 experiment. There 8 main effects, 28 two-way interactions, and 56 three-way interactions. Thus the number of three-way and lower order effects is 92. A complete replicate of a 2^8 requires 256 runs. So the number of higher order effects that can be pooled to measure experimental error is 163.

Fractional Replications

Suppose we only ran half of the runs for a total of 128 runs. Can we still estimate all of the three –way and lower order effects? If we can, then we should still have 35 degrees of freedom for error from pooled higher order interaction effects.

Fractional Replications

To simplify the discussion, consider a 2^3 experiment in two blocks of size 4 where the three-way interaction effect is used to define the effects.

Suppose we only ran the runs in one of the blocks. What can we estimate?

A 2^3 experiment in two blocks with effect contrasts shown.

				Effects						
Block	***A***	***B***	***C***	***A***	***B***	***A*B***	***C***	***A*C***	***B*C***	***A*B*C***
2	0	0	0	-	-	+	-	+	+	-
1	1	0	0	+	-	-	-	-	+	+
1	0	1	0	-	+	-	-	+	-	+
2	1	1	0	+	+	+	-	-	-	-
1	0	0	1	-	-	+	+	-	-	+
2	1	0	1	+	-	-	+	+	-	-
2	0	1	1	-	+	-	+	-	+	-
1	1	1	1	+	+	+	+	+	+	+

A 2^3 experiment in two blocks with effect contrasts shown.

						Effects				
Block	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*
1	1	0	0	+	-	-	-	-	+	+
1	0	1	0	-	+	-	-	+	-	+
1	0	0	1	-	-	+	+	-	-	+
1	1	1	1	+	+	+	+	+	+	+

A 2^3 experiment in two blocks with effect contrasts shown.

						Effects				
Block	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*
1	1	0	0	+	-	-	-	-	+	+
1	0	1	0	-	+	-	-	+	-	+
1	0	0	1	-	-	+	+	-	-	+
1	1	1	1	+	+	+	+	+	+	+

A 2^3 experiment in two blocks with effect contrasts shown.

				Effects						
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*
a	1	0	0	+	-	-	-	-	+	+
b	0	1	0	-	+	-	-	+	-	+
c	0	0	1	-	-	+	+	-	-	+
abc	1	1	1	+	+	+	+	+	+	+

A 2^3 experiment in two blocks with effect contrasts shown.

				Effects						
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*
a	1	0	0	+	-	-	-	-	+	+
b	0	1	0	-	+	-	-	+	-	+
c	0	0	1	-	-	+	+	-	-	+
abc	1	1	1	+	+	+	+	+	+	+

Question: What does $a - b - c + abc$ estimate?

A 2^3 experiment in two blocks with effect contrasts shown.

				Effects						
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*
a	1	0	0	+	-	-	-	-	+	+
b	0	1	0	-	+	-	-	+	-	+
c	0	0	1	-	-	+	+	-	-	+
abc	1	1	1	+	+	+	+	+	+	+

Question: What does $a - b - c + abc$ estimate?

A 2^3 experiment in two blocks with effect contrasts shown.

				Effects						
Block	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*
1	1	0	0	+	-	-	-	-	+	+
1	0	1	0	-	+	-	-	+	-	+
1	0	0	1	-	-	+	+	-	-	+
1	1	1	1	+	+	+	+	+	+	+

That is: What does $a - b - c + abc$ represent in terms of the effects?

Consider the A Effect and the BC Effect.

Note that: $A = abc + ab + ac + a - bc - b - c - (1)$

and

$$BC = abc - ab - ac + a + bc - b - c + (1).$$

Therefore $A + BC = 2(abc + a - b - c)$

And

$$A + BC = 2(abc + a - b - c)$$

implies that

$$abc + a - b - c = (A + BC)/2$$

Thus

$$abc + a - b - c = (A + BC)/2$$

$$= A/2 \text{ if } BC = 0.$$

Thus if we knew that there was no two-way interaction between *A* and *B*, then we could get an estimate of *A* from only those runs in Block 1.

Note that

$$abc + a - b - c = (A + BC)/2 = A/2 \text{ if } BC = 0.$$

We say that *A* is *aliased* with *BC*.

Consider the B Effect and the AC Effect.

						Effects					
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*	*Total*
a	1	0	0	+	-	-	-	-	+	+	+
b	0	1	0	-	+	-	-	+	-	+	+
c	0	0	1	-	-	+	+	-	-	+	+
abc	1	1	1	+	+	+	+	+	+	+	+

One can show that

$$abc - c + b - a = (B + AC)/2\,,$$

$$abc + c - b - a = (C + AB)/2, \text{ and that}$$

$$abc + c + b + a = (Total + ABC)/2$$

Since

$$abc - c + b - a = (B + AC)/2\,,$$

$$abc + c - b - a = (C + AB)/2, \text{ and}$$

$$abc + c + b + a = (Total + ABC)/2$$

we say that *B* and *AC* are aliased, that *C* and *AB* are aliased, and that the *Total* and *ABC* are aliased.

Alias Pairs

A, BC

B, AC

C, AB

ABC, Total

What if we used the treatment combinations in Block 2?

						Effects					
Block	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*	*Total*
2	0	0	0	-	-	+	-	+	+	-	+
2	1	1	0	+	+	+	-	-	-	-	+
2	1	0	1	+	-	-	+	+	-	-	+
2	0	1	1	-	+	-	+	-	+	-	+

Suppose we ran the four treatment combinations in Block 2.

						Effects					
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*	*Total*
(1)	0	0	0	-	-	+	-	+	+	-	+
ab	1	1	0	+	+	+	-	-	-	-	+
ac	1	0	1	+	-	-	+	+	-	-	+
bc	0	1	1	-	+	-	+	-	+	-	+

Here, one can show that

$$ab + ac - bc - (1) = (A - BC)/2.$$

So A and BC are still alias pairs.

Suppose we ran the four treatment combinations in Block 2.

						Effects					
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*	*Total*
(1)	0	0	0	-	-	+	-	+	+	-	+
ab	1	1	0	+	+	+	-	-	-	-	+
ac	1	0	1	+	-	-	+	+	-	-	+
bc	0	1	1	-	+	-	+	-	+	-	+

Here, one can show that

$$ab + bc - ac - (1) = (B - AC)/2.$$

So B and AC are still alias pairs.

Suppose we ran the four treatment combinations in Block 2.

						Effects					
Trt	*A*	*B*	*C*	*A*	*B*	*A*B*	*C*	*A*C*	*B*C*	*A*B*C*	*Total*
(1)	0	0	0	-	-	+	-	+	+	-	+
ab	1	1	0	+	+	+	-	-	-	-	+
ac	1	0	1	+	-	-	+	+	-	-	+
bc	0	1	1	-	+	-	+	-	+	-	+

Here, one can show that

$$ab - ac - bc + (1) = (C - AB)/2.$$

So *C* and *AB* are still alias pairs.

Alias Pairs are still:

A, BC

B, AC

C, AB

ABC, Total

Statistical Analysis of a ½ Replicate of a 2^3 Experiment

Trt	y	A	B	C	Total
a	8	+	-	-	+
b	11	-	+	-	+
c	12	-	-	+	+
abc	16	+	+	+	+
Effect Value		1	7	9	47
Std Effect		0.5	3.5	4.5	23.5

Trt	y	A	B	C	Total
a	8	+	-	-	+
b	11	-	+	-	+
c	12	-	-	+	+
abc	16	+	+	+	+
Effect Value		1	7	9	47
Std Effect		0.5	3.5	4.5	23.5
Effect		A +/- BC	B +/- AC	C +/- AB	Total +/- ABC

An Adjusted Yate's Approach

First list the runs for the first two factors *A* and *B* in our standard order.

Then add in a column for *C* so that

$A + B + C$ = Even or Odd depending on

the block selected.

Here is the block we selected.

Block	*A*	*B*	*C*
1	1	0	0
1	0	1	0
1	0	0	1
1	1	1	1

Here is the Standard Order For A and B.

Block	*A*	*B*	
1	0	0	
1	1	0	
1	0	1	
1	1	1	

An Adjusted Yate's Approach

After getting the standard order for A and B, we add a column for C such that A+B+C=odd.

A	*B*	*C*
0	0	1
1	0	0
0	1	0
1	1	1

Yate's Method (Adjusted)						
A	*B*	*C*	*y*	*Step 1*	*Step 2*	*Std Eff*
0	0	1	12	20	47	23.5
1	0	0	8	27	1	0.5
0	1	0	11	-4	7	3.5
1	1	1	16	5	9	4.5

Yate's Method (Adjusted)							
A	*B*	*C*	*y*	*Step 1*	*Step 2*	*Std Eff*	*Effect*
0	0	1	12	20	47	23.5	Total +/- ABC
1	0	0	8	27	1	0.5	A +/- BC
0	1	0	11	-4	7	3.5	B +/- AC
1	1	1	16	5	9	4.5	AB +/- C

Remark: Note that the alias of an effect is given by its generalized interaction with the effect used to define the ½ Replication.

Example:

$$A\,(ABC) = BC$$

$$B(ABC) = AC$$

$$C(ABC) = BC$$

$$(1)(ABC) = ABC$$

Added Factor Approach

Treatment Combinations		
A	***B***	
-	-	
+	-	
-	+	
+	+	

Added Factor Approach

Take $C = AB$.

Treatment Combinations ($C = AB$)		
A	*B*	*C*
-	-	+
+	-	-
-	+	-
+	+	+

Take $C = - AB$.

Treatment Combinations ($C = -AB$)		
A	*B*	*C*
-	-	-
+	-	+
-	+	+
+	+	-

Now consider a ½ replicate of a 2^4 factorial experiment with factors *A, B, C, & D*. Also suppose we choose the treatment combinations where *A+B+C+D* is even/odd.

The alias pairs are:

A* +/- *BCD ***B* +/- *ACD***

C* +/- *ABD ***D* +/- *ABC***

AB* +/- *CD ***AC* +/- *BD***

AD* +/- *BC ***Total* +/- *ABCD***

A 1/2 Replicate of a 2^4 Experiment

Trt	*y*		
(1)	4		
ab	9		
ac	6		
ad	12		
bc	11		
bd	8		
cd	5		
abcd	10		

A 1/2 Replicate of a 2^4 Experiment					
Trt	*y*	*A +/- BCD*	*B +/- ACD*	*C +/- ABD*	*D +/- ABC*
(1)	4	-1	-1	-1	-1
ab	9	1	1	-1	-1
ac	6	1	-1	1	-1
ad	12	1	-1	-1	1
bc	11	-1	1	1	-1
bd	8	-1	1	-1	1
cd	5	-1	-1	1	1
abcd	10	1	1	1	1
Contrast Value		9	11	-1	5
Std Effect		3.182	3.889	-0.354	1.768

A 1/2 Replicate of a 2^4 Experiment				
Trt	*y*	*AB +/- CD*	*AC +/- BD*	*AD +/- BC*
(1)	4	1	1	1
ab	9	1	-1	-1
ac	6	-1	1	-1
ad	12	-1	-1	1
bc	11	-1	-1	1
bd	8	-1	1	-1
cd	5	1	-1	-1
abcd	10	1	1	1
Contrast Value		-9	-9	9
Std Effect		-3.182	-3.182	3.182

An Adjusted Yate's Analysis

Runs from a 2^3 Exp
(1)
a
b
ab
c
ac
bc
abc

An Adjusted Yate's Analysis

Runs from a 2^3 Exp	Runs in 1/2 Rep
(1)	(1)
a	*ad*
b	*bd*
ab	*ab*
c	*cd*
ac	*ac*
bc	*bc*
abc	*abcd*

An Adjusted Yate's Analysis

Runs from a 2^3 Exp	Runs in 1/2 Rep	y
(1)	(1)	4
a	*ad*	12
b	*bd*	8
ab	*ab*	9
c	*cd*	5
ac	*ac*	6
bc	*bc*	11
abc	*abcd*	10

An Adjusted Yate's Analysis

		y	Step 1	Step 2	Step 3
(1)	(1)	4	16	33	65
a	*ad*	12	17	32	9
b	*bd*	8	11	9	11
ab	*ab*	9	21	0	-9
c	*cd*	5	8	1	-1
ac	*ac*	6	1	10	-9
bc	*bc*	11	1	-7	9
abc	*abcd*	10	-1	-2	5

		y	Step 3	Std Eff
(1)	(1)	4	65	
a	*ad*	12	9	3.182
b	*bd*	8	11	3.8891
ab	*ab*	9	-9	-3.182
c	*cd*	5	-1	-0.354
ac	*ac*	6	-9	-3.182
bc	*bc*	11	9	3.182
abc	*abcd*	10	5	1.7678

		y	Step 3	Std Eff	Effect
(1)	(1)	4	65		
a	*ad*	12	9	3.182	*A +/- BCD*
b	*bd*	8	11	3.8891	*B +/- ACD*
ab	*ab*	9	-9	-3.182	*AB +/- CD*
c	*cd*	5	-1	-0.354	*C +/- ABD*
ac	*ac*	6	-9	-3.182	*AC +/- BD*
bc	*bc*	11	9	3.182	*BC +/- AD*
abc	*abcd*	10	5	1.7678	*ABC +/- D*

ANOVA METHOD

The ANOVA method can be used to get the mean squares for each alias pair. When doing so, one must include only one of each alias pair in the analysis model.

SAS ANOVA Analysis

A SAS ANOVA analysis for the ½ replicate of our 2^4 factorial design where the treatment combinations utilized in the study are identified by the evenness or oddness of the sum of all four factor levels can be obtained from:

```
DATA; INPUT A B C D Y; LINES;
 0 0 0 0  4
 1 1 0 0  9
 1 0 1 0  6
 1 0 0 1 12
 0 1 1 0 11
 0 1 0 1  8
 0 0 1 1  5
 1 1 1 1 10
```

```
PROC PRINT;
  RUN;
 PROC ANOVA;
  CLASSES A B C D;
  MODEL Y = A B C D A*B A*C A*D;
  RUN;
```

See ST722_9_1.sas

Source	DF	Anova SS	Mean Square	F Value	Pr > F
A	1	10.12500000	10.12500000	.	.
B	1	15.12500000	15.12500000	.	.
C	1	0.12500000	0.12500000	.	.
D	1	3.12500000	3.12500000	.	.
A*B	1	10.12500000	10.12500000	.	.
A*C	1	10.12500000	10.12500000	.	.
A*D	1	10.12500000	10.12500000	.	.

Alternate Form of the Design ($D=ABC$)

Runs	1/2 Rep	
A B C	*A B C D*	y
- - -	- - - -	4
+ - -	+ - - +	12
- + -	- + - +	8
+ + -	+ + - -	9
- - +	- - + +	5
+ - +	+ - + -	6
- + +	- + + -	11
+ + +	+ + + +	10

A ½ Replicate of a 2^5.

Consider the data in Table 7.5, and suppose that we take a ½ replicate of this data by choosing the combinations where W+M+T+C+P is even.

Table 9.4 Data from a ½ Replicate of a 2^5.

Obs	W	M	T	C	P	QUALITY
1	0	0	0	0	0	4.8
2	1	1	0	0	0	2.2
3	1	0	1	0	0	4.2
4	0	1	1	0	0	3.0
5	1	0	0	1	0	2.2
6	0	1	0	1	0	8.4
7	0	0	1	1	0	5.3
8	1	1	1	1	0	8.9

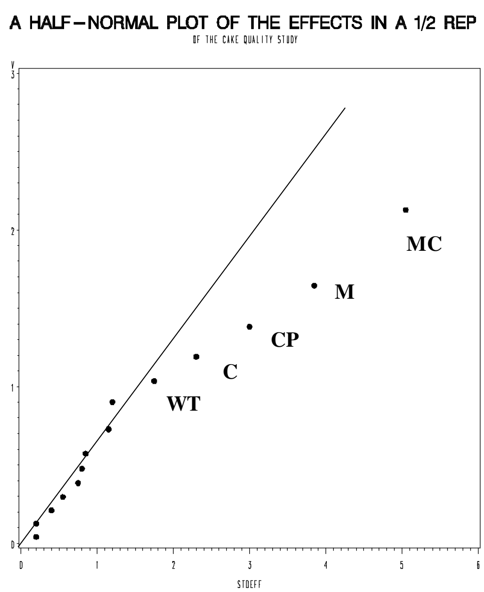
Rest of the data for Table 9.4.

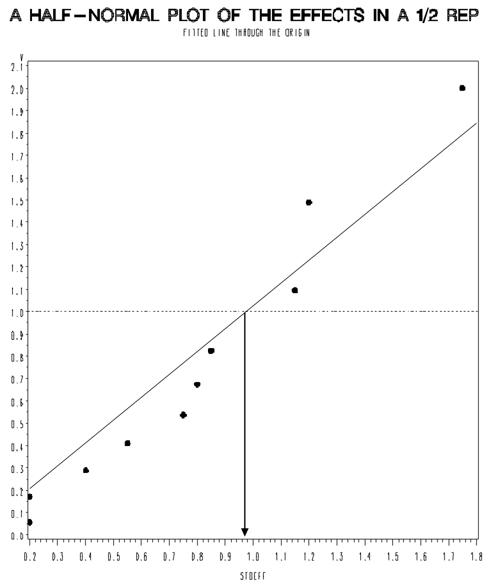
Obs	W	M	T	C	P	QUALITY
9	1	0	0	0	1	5.0
10	0	1	0	0	1	5.8
11	0	0	1	0	1	4.6
12	1	1	1	0	1	5.2
13	0	0	0	1	1	2.9
14	1	1	0	1	1	6.6
15	1	0	1	1	1	2.7
16	0	1	1	1	1	7.0

The following analysis is given in ST722_9_2.sas

Table 9.5 Effect Mean Squares

Source	DF	Anova SS	Mean Square	F Value	Pr > F
W	1	1.44000000	1.44000000	.	.
M	1	14.82250000	14.82250000	.	.
W*M	1	0.30250000	0.30250000	.	.
T	1	0.56250000	0.56250000	.	.
W*T	1	3.06250000	3.06250000	.	.
M*T	1	0.04000000	0.04000000	.	.
C	1	5.29000000	5.29000000	.	.
W*C	1	0.16000000	0.16000000	.	.
M*C	1	25.50250000	25.50250000	.	.
T*C	1	1.32250000	1.32250000	.	.
P	1	0.04000000	0.04000000	.	.
W*P	1	0.64000000	0.64000000	.	.
M*P	1	0.72250000	0.72250000	.	.
T*P	1	1.32250000	1.32250000	.	.
C*P	1	9.00000000	9.00000000	.	.





You can now begin with Assignment 6.