The Compendium

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Sometimes, when people are either too lazy or negligent to help you, you've just gotta hunker down and help yourself!

The Misfit

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0.1 An Introduction

0.1.1 Inspiration

As a child, my favorite part of playing video games was figuring out how to beat the boss on a certain level. I would draw upon my knowledge of the different enemies I had faced up until that point and realize that the boss fought in a similar fashion—but with some new tweaks. I would then alter my already existing strategy to meet the boss's fighting style—and eventually defeat him.

Often times when learning a new subject—especially one whose matters build on the developments of another subject—it can be helpful to learn in a constructive fashion. Rather than jump into the more advanced subject off the bat, it might be more effective to identify the root of the new subject in the developments of the old. Trivially, this is natural to mathematics (as few subjects are independent of each other). However, with respect to pedagogy, it easy to find oneself "lost in the weeds," per se. Personally, if dependencies between matters are not elaborated upon, it becomes much harder to learn the current subject and anything that might build on it. I am writing this book to help myself identify these dependencies with greater ease. I hope it serves the same effect for you as well.

0.1.2 Representation Theory

So, what exactly is representation theory? To be curt, representation theory is the study of different algebraic structures *represented* as vector spaces. Specifically, mathematicians who study representation theory search for the appropriate *linear transformations* that preserve the patterns observed in an algebraic structure. Because the algebra of matrices is more intuitive to most mathematicians, it is valuable to be able to represent algebras using vector spaces. This book will explore the different liberties afforded by representations and how they can be used to standardize and in a way unify the fields of combinatorics, numbertheory, and algebra!

0.2 Groups and Group Homomorphisms

Let's start small and recapitulate a bit of algebra before we dive into playing around with representations. The simplest algebraic structure is the group, a set G paired with an operation $\otimes$ (often denoted $(G, \otimes)$). Recall, for a set to be a group with an operation $\otimes$, it must abide by the following properties: $\forall a, b, c \in G$

Definition 0.2.1

Group Properties

1. (Closure under $\otimes$) $a \otimes b \in G$
2. (Existence of an identity) $\exists e \in G \ni e \otimes a = a$
3. (Associativity) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

To construct our first representation, we need a group that can be mapped onto by G. Take the general linear group of order d, $GL_d(\mathbb{C})$[1] defined,

$$GL_d(\mathbb{C}) := \left\{ M \mid M := \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,d} \\ \vdots & \vdots & \vdots & \ddots \\ a_{d,1} & a_{d,2}, & \dots & a_{d,d} \end{pmatrix}, \text{ and } M \text{ is invertible with } a_{i,j} \in \mathbb{C}, \forall i, j \in [0, n] \right\}$$

Using $GL_d(\mathbb{C})$ is standard in representation theory when introducing the essence of a representation for three reasons:

1. Because its elements are *square*, $\forall A, B \in GL_d(\mathbb{C})$, $A \cdot B \in GL_d(\mathbb{C})$. (satisfying the closure condition for groups)
2. Note, $I_d := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots 0 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in GL_d(\mathbb{C})$ as $1, 0 \in \mathbb{C}$. So, $\forall A \in GL_d(\mathbb{C})$, $I_d \cdot A = A$.
   So, the identity element exists in $GL_d(\mathbb{C})$
3. Finally, it is a well known fact that matrix multiplication is associative[2]. So, $\forall A, B, C \in GL_d(\mathbb{C}), A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

[1] We could substitute any field in for $\mathbb{C}$, but $\mathbb{C}$ is the standard in the field and will be used here.
[2] But not necessarily commutative!

Essentially, the general linear group is special because it is easy to construct and (as is implied by the name) shares some of the same functions as a group![3] Now that we have two structures that are similar to each other, let us see if we can represent one using the other.

Let us now define a function $\mathcal{X} : G \to GL_d(\mathbb{C})$. $\mathcal{X}$ is known as the *character* of G and maps to the appropriate representations in $GL_d(\mathbb{C})$. Now, it should be known that the mapping itself is not invariant among all groups. Meaning, that there is no single definition for $\mathcal{X}$ which necessary applies to any two groups G and H. However, what is absolutely necessary of $\mathbb{X}$ is that it has the following properties:

Definition 0.2.2

Group Homomorphism

Let $\phi : G \to H$ be defined for groups $(G, \otimes_G)$ and $(H, \otimes_H)$. ϕ is a **group homomorphism** if,

1. $\forall a, b \in G, \phi(a \otimes_G b) = \phi(a) \otimes_H \phi(b)$.

2. For e_G the identity in G, $\phi(e_G) = e_H$.

Example 0.2.1

A Special Representation for C_4

Consider the following set and operation, $C_n := \{g^k | \forall k \in [0, n-1]\}$, $\otimes : C_n \times C_n \to C_n$, $g^i \otimes g^j = g^{i+j \text{ (mod n)}}$. The group $(C_n, \otimes)$ is called the cyclic group of order n and serves as an abstraction of the modulo arithmetic familiar to the integers[a] I will construct a character $\mathcal{X}$ that represents the elements of C_4 (the cyclic group of order 4) in the complex numbers.

Let $G := \{e, g, g^2, g^3\}$ and $H := \{1, i, -1, -i\}$. Consider, $\mathcal{X} : G \to H$, that maps $g^k \mapsto i^k \in H$. Recall, to show that $\mathcal{X}$ is a representation of G, we must show that it is a homomorphism between G and H. Note, for $g^m, g^n \in G$, $\mathcal{X}(g^m \otimes g^n) = \mathcal{X}(g^{m+n \text{ (mod 4)}}) = i^{m+n \text{ (mod 4)}}$. Note that $i^{m \text{ (mod 4)}} \cdot i^{n \text{ (mod 4)}} = i^{m+n \text{ (mod 4)}}$. So, $\mathcal{X}(g^m \otimes g^n) = i^m \cdot i^n$. Trivially, $\mathcal{X}(e) = 1$. So, $\mathcal{X}$ is a homomorphism between G and H and is thus a representation!

[a] i.e. addition in $\mathbb{Z}_n$.

The group homomorphism is mechanism that establishes a representation. It is the definition of the group homomorphism that establishes the representation of the elements

[3] It should be noted that the actual structure of the general linear group is that of a field (which affords us more liberties than are present in the otherwise limited structure of a group).

of G in H. In the next chapter we will explore how to represent a familiar object from combinatorics in $\mathbb{R}^n$.

0.3 Partitions and Permutations