Blast

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1 Governing equations

We consider Lagrangian fluid particles, for which we define the position $\mathbf{x}^+ = \mathbf{x}^+(t, \mathbf{y})$, the determinant of the Jacobian $J^+ = J^+(t, \mathbf{y})$, the density $\rho^+ = \rho^+(t, \mathbf{y})$, the velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$, and the internal energy $e^+ = e^+(t, \mathbf{y})$. The Eulerian counterparts for the density, velocity, and internal energy are, respectively, $\rho = \rho(t, \mathbf{x})$, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, and $e = e(t, \mathbf{x})$. Also consider the volume Ω_0 as the set of all $\mathbf{y}$ vectors that make up the initial domain. The control volume $\Omega^+ = \Omega^+(t, \Omega_0)$ is then defined by

$$\Omega^+ = \{\mathbf{x}^+ : \mathbf{y} \in \Omega_0\}. \tag{1}$$

Note that $\Omega^+(0, \Omega_0) = \Omega_0$.

The governing equations for the Lagrangian fluid particles are derived in my hydrodynamics notes (see section on kinematics, Lagrangian governing equations, etc.). These are shown below

$$\frac{\partial \mathbf{x}^+}{\partial t} = \mathbf{u}^+, \tag{2}$$

$$\frac{\partial J^+}{\partial t} = J^+ \left(\nabla \cdot \mathbf{u}\right)_{\mathbf{x}=\mathbf{x}^+}, \tag{3}$$

$$\frac{\partial J^+ \rho^+}{\partial t} = 0, \tag{4}$$

$$\rho^+ \frac{\partial \mathbf{u}^+}{\partial t} = \left(\nabla \cdot \boldsymbol{\sigma}\right)_{\mathbf{x}=\mathbf{x}^+}, \tag{5}$$

$$\rho^+ \frac{\partial e^+}{\partial t} = \left(\boldsymbol{\sigma} : \nabla \mathbf{u}\right)_{\mathbf{x}=\mathbf{x}^+}. \tag{6}$$

A note on notation. The products that involve a tensor $\boldsymbol{\tau}$ can be expressed in Einstein notation as

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}, \tag{7}$$

$$\boldsymbol{\tau} \cdot \nabla \alpha = \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \tag{8}$$

$$\mathbf{f} \cdot \boldsymbol{\tau} \cdot \nabla \alpha = f_i \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \tag{9}$$

$$\boldsymbol{\tau} : \nabla \mathbf{f} = \tau_{ij} \frac{\partial f_i}{\partial x_j}. \tag{10}$$

where α is a scalar and $\mathbf{f}$ a vector. In these notes we'll mostly be using indices i and j for FE expansions, rather than for Einstein notation.

2   Finite element expansion

We introduce the coefficients $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_i(t)$, $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(t)$ and $\hat{e}_i = \hat{e}_i(t)$, as well as the Lagrangian basis functions $\phi_i^+ = \phi_i^+(\mathbf{y}) \in L^2$, and $w_i^+ = w_i^+(\mathbf{y}) \in H^1$. We note that $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{u}}_i$ are each vectors, e.g., the components of $\hat{\mathbf{u}}_i$ are $\hat{u}_{i,\alpha} = \hat{u}_{i,\alpha}(t)$ for $\alpha = x, y, z$. We also note that ϕ_i^+ and w_i^+ have Eulerian counterparts $\phi_i = \phi_i(t, \mathbf{x})$ and $w_i = w_i(t, \mathbf{x})$, respectively (see more details in section on finite elements in my notes for numerical methods). The coefficients are used in the following expansions

$$\mathbf{x}^+ = \sum_j^{N_w} \hat{\mathbf{x}}_j w_j^+, \tag{11}$$

$$\mathbf{u}^+ = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+, \tag{12}$$

$$e^+ = \sum_j^{N_\phi} \hat{e}_j \phi_j^+. \tag{13}$$

We note that the expansion coefficients are the same for the Lagrangian and Eulerian variables. For example, for the Eulerian velocity, we have

$$\mathbf{u} = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j. \tag{14}$$

3   Semi-discrete equations for $\mathbf{x}^+$ and $\mathbf{J}^+$

4   Semi-discrete equation for ρ^+

5   Semi-discrete equation for $\mathbf{u}^+$

We begin by multiplying both sides of eq. (5) by the basis functions for velocity, and integrating over all space, as follows

$$\int_{\Omega_0} \rho^+ \frac{\partial \mathbf{u}^+}{\partial t} w_i^+ \, dV_y = \int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} w_i^+ \, dV_y \tag{15}$$

showing that

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \sum_j^{N_w} \left(\frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \frac{\partial w_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_j^{N_w} \hat{\mathbf{u}}_j \nabla w_j \right) \\
&= \sum_j^{N_w} \left[\frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \left(\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j \right) \right] \\
&= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j.
\end{aligned} \tag{16}$$

The finite element formulation of the momentum equation is thus

$$\int_\Omega \rho \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j w_i \, dV = -\int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \, dV \qquad \text{for } i = 1, ..., N_w. \tag{17}$$

The above is re-written as

$$\sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)} = -\int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \, dV \qquad \text{for } i = 1, ..., N_w. \tag{18}$$

where the mass bilinear form $m_{ij}^{(w)}$ is given by

$$m_{ij}^{(w)} = \int_\Omega \rho w_i w_j \, dV. \tag{19}$$

We now introduce the vector $\mathbf{U}$ whose components are $\hat{u}_i$. We also introduce the matrix $\mathbf{M}^{(w)}$ whose components are $m_{ij}^{(w)}$. Thus, the left-hand side of eq. (18) can be written as $\mathbf{M}^{(w)} \, d\mathbf{U}/dt$. We also introduce the vector bilinear form

$$\mathbf{f}_{ij} = \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \phi_j dV. \tag{20}$$

This is a *vector* bilinear form since $\mathbf{f}_{ij}$ has components $f_{ij,\alpha} = f_{ij,\alpha}(t)$, for $\alpha = x, y, z$, where α denotes the first index of $\boldsymbol{\sigma}$. We introduce the matrix $\mathbf{F}$, whose components are $\mathbf{f}_{ij}$. We also expand the field with constant value of one as follows

$$1 = \sum_i^{N_\phi} \hat{c}_i \phi_i. \tag{21}$$

If we define the vector $\mathbf{C}$ as that with components $\hat{c}_i$, we can show that

$$\mathbf{FC} = \sum_j^{N_\phi} \mathbf{f}_{ij} \hat{c}_j \qquad\qquad \text{for } i = 1, ..., N_w$$

$$= \sum_j^{N_\phi} \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV \hat{c}_j \qquad\qquad \text{for } i = 1, ..., N_w$$

$$= \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \left(\sum_j^{N_\phi} \hat{c}_j \phi_j \right) dV \qquad\qquad \text{for } i = 1, ..., N_w$$

$$= \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \, dV \qquad\qquad \text{for } i = 1, ..., N_w \tag{22}$$

The above is the negative of the right-hand side of eq. (18). Thus, combining all together we get

$$\mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} = -\mathbf{FC}. \tag{23}$$

We note that since both the Lagrangian and Eulerian velocities share the same coefficients $\mathbf{U}$, we now have a solution for both.

3

6   Semi-discrete equation for e^+

As with momentum conservation, we have

$$
\begin{aligned}
\frac{\partial \mathbf{e}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{e} &= \sum_j^{N_\phi} \left(\frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \frac{\partial \phi_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_j^{N_\phi} \hat{\mathbf{e}}_j \nabla \phi_j \right) \\
&= \sum_j^{N_\phi} \left[\frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \left(\frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j \right) \right] \\
&= \sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j .
\end{aligned}
\tag{24}
$$

For the right-hand side of the energy conservation equation, we have

$$
\boldsymbol{\sigma} : \nabla \mathbf{u} = \boldsymbol{\sigma} : \nabla \left(\sum_k^{N_w} \hat{\mathbf{u}}_k w_k \right) = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k .
\tag{25}
$$

The finite element formulation of the energy equation is thus

$$
\begin{aligned}
\int_\Omega \rho \sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j \phi_i \, dV &= \int_\Omega \left(\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k \right) \phi_i \, dV \qquad & \text{for } i = 1, ..., N_w \\
&= \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_k \phi_i \, dV \qquad & \text{for } i = 1, ..., N_w
\end{aligned}
\tag{26}
$$

The above is re-written as

$$
\sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} \qquad \text{for } i = 1, ..., N_w .
\tag{27}
$$

where the mass bilinear form m_{ij}^ϕ is given by

$$
m_{ij}^{(\phi)} = \int_\Omega \rho \phi_i \phi_j \, dV .
\tag{28}
$$

Note that in eq. (27) there is a dot product in the right-hand side, that is, the right-hand side expanded out is

$$
\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} = \sum_k^{N_w} \sum_{\alpha = x, y, z} \hat{u}_{k,\alpha} f_{ki,\alpha} .
\tag{29}
$$

We now introduce the vector $\mathbf{E}$ whose components are $\hat{e}_i$. We also introduce the matrix $\mathbf{M}^{(\phi)}$ whose components are $m_{ij}^{(\phi)}$. Thus, eq. (27) can be succinctly written as

$$
\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{U} .
\tag{30}
$$

Note again that on the right-hand side above there is a matrix-vector product *and* a dot product. We also note that since both the Lagrangian and Eulerian internal energies share the same coefficients $\mathbf{E}$, we now have a solution for both.