Geometry

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Contents

1 Linear forms

- A linear form (or linear functional) is a mapping $w : V \to \mathbb{R}$ such that

$$w(\mathbf{v}_1 + \mathbf{v}_2) = w(\mathbf{v}_1) + w(\mathbf{v}_2) \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V, \tag{1}$$

$$w(\alpha \mathbf{v}) = \alpha w(\mathbf{v}) \qquad \forall \alpha \in \mathbb{R}, \mathbf{v} \in V. \tag{2}$$

- A bilinear form (or bilinear functional) is a mapping $w : V \times U \to \mathbb{R}$ such that

$$w(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{u}) = w(\mathbf{v}_1, \mathbf{u}) + w(\mathbf{v}_2, \mathbf{u}) \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V, \mathbf{u} \in U, \tag{3}$$

$$w(\mathbf{v}, \mathbf{u}_1 + \mathbf{u}_2) = w(\mathbf{v}, \mathbf{u}_1) + w(\mathbf{v}, \mathbf{u}_2) \qquad \forall \mathbf{v} \in V, \mathbf{u}_1, \mathbf{u}_2 \in U, \tag{4}$$

$$w(\alpha \mathbf{v}, \mathbf{u}) = \alpha w(\mathbf{v}, \mathbf{u}) \qquad \forall \alpha \in \mathbb{R}, \mathbf{v} \in V, \mathbf{u} \in U, \tag{5}$$

$$w(\mathbf{v}, \alpha \mathbf{u}) = \alpha w(\mathbf{v}, \mathbf{u}) \qquad \forall \alpha \in \mathbb{R}, \mathbf{v} \in V, \mathbf{u} \in U, \tag{6}$$

- A multi-linear form (or multi-linear functional) is a mapping $w : V^{(1)} \times ...V^{(n)} \to \mathbb{R}$ such that

$$w(\mathbf{v}^{(1)}, ..., \mathbf{v}_1^{(i)} + \mathbf{v}_2^{(i)}, ..., \mathbf{v}^{(n)}) = w(\mathbf{v}^{(1)}, ..., \mathbf{v}_1^{(i)}, ..., \mathbf{v}^{(n)}) + w(\mathbf{v}^{(1)}, ..., \mathbf{v}_2^{(i)}, ..., \mathbf{v}^{(n)}), \tag{7}$$

$$w(\mathbf{v}^{(1)}, ..., \alpha \mathbf{v}^{(i)}, ..., \mathbf{v}^{(n)}) = \alpha w(\mathbf{v}^{(1)}, ..., \mathbf{v}^{(i)}, ..., \mathbf{v}^{(n)}), \tag{8}$$

  $\forall i$, $\forall \alpha \in \mathbb{R}$, $\forall \mathbf{v}^{(1)} \in V^{(1)}$, ... , $\forall \mathbf{v}^{(n)} \in V^{(n)}$, and $\forall \mathbf{v}_1^{(i)}, \mathbf{v}_2^{(i)} \in V^{(i)}$.

2 k-forms

- $T_p\mathbb{R}^n$: the set of all n-dimensional vectors whose origin is at point p.
- A 1-form is a linear form $w : T_p\mathbb{R}^n \to \mathbb{R}$.
- A 1-form belongs to the dual space of $T_p\mathbb{R}^n$.
- $dx(\mathbf{v})$ is a 1-form that grabs the first component of a vector $\mathbf{v}$. $dy(\mathbf{v})$ is a 1-form that grabs the second component of a vector $\mathbf{v}$, and so on.
- An example of a 1-form $w : T_p\mathbb{R}^2 \to \mathbb{R}$ would be $w(\mathbf{v}) = adx(\mathbf{v}) + bdy(\mathbf{v})$, or simply $w = adx + bdy$.
- A general 1-form $w : T_p\mathbb{R}^n \to \mathbb{R}$ is expressed as follows: $w = a_1dx_1 + ... + a_ndx_n$.
- The exterior product $w_1 \wedge w_2$ of two 1-forms w_1 and w_2 is defined as

$$w_1 \wedge w_2(\mathbf{v}_1, \mathbf{v}_2) = \begin{vmatrix} w_1(\mathbf{v}_1) & w_2(\mathbf{v}_1) \\ w_1(\mathbf{v}_2) & w_2(\mathbf{v}_2) \end{vmatrix} \tag{9}$$

- A 2-form is an anti-symmetric bilinear form $T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ that is defined as the exterior product $w_1 \wedge w_2$.
- $w_1 \wedge w_2 = -w_2 \wedge w_1$, and thus $w_1 \wedge w_1 = 0$.
- The exterior product $w_1 \wedge ... \wedge w_m$ of n 1-forms w_i is defined as

$$w_1 \wedge ... \wedge w_m(\mathbf{v}_1, ..., \mathbf{v}_m) = \begin{vmatrix} w_1(\mathbf{v}_1) & \cdots & w_2(\mathbf{v}_1) \\ \vdots & \ddots & \vdots \\ w_1(\mathbf{v}_2) & \cdots & w_2(\mathbf{v}_2) \end{vmatrix} \tag{10}$$

- An m-form is an anti-symmetric multi-linear form $w : (T_p\mathbb{R}^n)^m \to \mathbb{R}$ that is defined as the exterior product $w_1 \wedge ... \wedge w_m$.