Radiation

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Chapter 1

Introduction

1.1 Definitions

Consider an infinitesimal amount of energy dE which is the energy at location $\mathbf{x}$ and time t with frequencies in the infinitesimal range $d\nu$ about the frequency ν and flowing in the direction of the solid angle $d\mathbf{\Omega}$ about the vector $\mathbf{\Omega}$ and passing through an infinitesimal area $d\mathbf{A}$ with unit normal $\mathbf{A}$. We express this energy in terms of a distribution $I_\nu = I_\nu(\mathbf{x}, t, \nu, \mathbf{\Omega})$ as follows

$$dE = I_\nu(\mathbf{\Omega} \cdot \mathbf{A}) dt d\nu d\Omega dA. \tag{1.1}$$

I_ν is referred to as the spectral radiance, or spectral specific intensity. Various additional radiation quantities can be defined in terms of I_ν, as shown in table 1.1.

Any quantity dependent on ν can be integrated over all frequencies to obtain a total value. For example, for the spectral radiance/spectral specific intensity, we have

$$I = \int_0^\infty I_\nu \, d\nu. \tag{1.2}$$

In the above, $I = I(\mathbf{x}, t, \mathbf{\Omega})$ is the radiance, or specific intensity.

	spectral	
	definition	units
Spectral radiance / Spectral specific intensity	I_ν	$\left[\frac{\mathrm{J}}{\mathrm{s\cdot m^2\cdot sr\cdot Hz}}\right]$
0^{th} moment	$J_\nu = \dfrac{1}{4\pi}\displaystyle\int_{4\pi} I_\nu \, d\Omega$	$\left[\frac{\mathrm{J}}{\mathrm{s\cdot m^2\cdot Hz}}\right]$
1^{st} moment	$\mathbf{H}_\nu = \dfrac{1}{4\pi}\displaystyle\int_{4\pi} I_\nu \boldsymbol{\Omega} \, d\Omega = \dfrac{\mathbf{F}_\nu}{4\pi}$	$\left[\frac{\mathrm{J}}{\mathrm{s\cdot m^2\cdot Hz}}\right]$
2^{nd} moment	$\mathbf{K}_\nu = \dfrac{1}{4\pi}\displaystyle\int_{4\pi} I_\nu \boldsymbol{\Omega}\boldsymbol{\Omega} \, d\Omega = \dfrac{c}{4\pi}\mathbf{P}_\nu$	$\left[\frac{\mathrm{J}}{\mathrm{s\cdot m^2\cdot Hz}}\right]$
Spectral radiant energy density	$E_\nu = \dfrac{1}{c}\displaystyle\int_{4\pi} I_\nu \, d\Omega = \dfrac{4\pi}{c} J_\nu$	$\left[\frac{\mathrm{J}}{\mathrm{m^3\cdot Hz}}\right]$
One-sided spectral radiant energy flux	$S_\nu^{\mathbf{A}} = \displaystyle\int_{\boldsymbol{\Omega}\cdot\mathbf{A}>0} I_\nu (\boldsymbol{\Omega}\cdot\mathbf{A}) \, d\Omega$	$\left[\frac{\mathrm{J}}{\mathrm{s\cdot m^2\cdot Hz}}\right]$

Table 1.1: Radiation quantities. In the above $\mathbf{F}_\nu$ is the radiation flux and $\mathbf{P}_\nu$ the radiation pressure tensor.

Chapter 2

Thermal radiation

For blackbody radiation we have

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1}. \tag{2.1}$$

Consider the identity

$$\int_0^\infty \frac{x^3}{\exp(yx)-1}\,dx = \frac{1}{15}\left(\frac{\pi}{y}\right)^4. \tag{2.2}$$

Using the above to integrate over all frequencies, we get

$$I = \frac{2h}{c^2}\frac{1}{15}\left(\frac{\pi kT}{h}\right)^4. \tag{2.3}$$

Defining the Stefan-Boltzmann constant as

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3}, \tag{2.4}$$

we have

$$I = \frac{1}{\pi}\sigma T^4. \tag{2.5}$$

In this case I_ν is isotropic, that is, it is independent of the direction $\mathbf{\Omega}$. Thus $J_\nu = I_\nu$ and therefore $E_\nu = (4\pi/c)I_\nu$. Integrating over all frequencies leads to $E = (4\pi/c)I$.

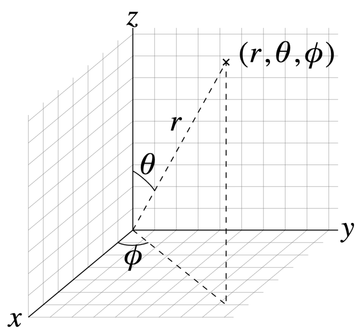

Figure 2.1: Spherical coordinates from Wikipedia.

For the one-sided spectral radiant energy flux, we make reference to the diagram for spherical coordinates in fig. 2.1. Let's assume $\mathbf{A} = \mathbf{z}$ without loss of generality. Then, we have

$$S_\nu^{\hat{\mathbf{z}}} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_\nu \cos\theta\, d\Omega = I_\nu \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \cos\theta \sin\theta\, d\theta d\phi = \pi I_\nu. \tag{2.6}$$

	total	spectral
Radiance / Specific intensity	$I = \frac{1}{\pi}\sigma T^4$	$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}$
Radiant energy density	$E = \frac{4}{c}\sigma T^4$	$E_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT) - 1}$
One-sided radiant energy flux	$S^{\hat{\mathbf{z}}} = \sigma T^4$	$S^{\hat{\mathbf{z}}}_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}$

Table 2.1: Radiation quantities for a blackbody spectrum

Similarly as before, integrating over all frequencies leads to $S^{\hat{\mathbf{z}}} = \pi I$. These and the other relations derived above are shown in table 2.1.