1 Linear System of Equations

2 Matrix Algebra

3 Determinants

4 Vector Spaces

5 Eigenvalues and Eigenvectors

6 Orthogonality and Least Squares

6.1 Basic Definitions

- Imagine a subspace W of $\mathbb{R}^n$. The set of all vectors $\mathbf{z}$ that are orthogonal to all the vectors in W is called the **orthogonal complement** of W, and is denoted as W^+. (Note: this is an orthogonal complement, all complements need not be orthogonal).
- $(\text{Row}\, A)^\perp = \text{Null}\, A$ and $(\text{Col}\, A)^\perp = \text{Null}\, A^T$

6.2 Orthogonal Sets

- An **orthogonal set** is a set of vectors that are orthogonal with each other.
- Assume $0 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_p\mathbf{u}_p$. By multiplying by $\mathbf{u}_i$, we can show $c_i = 0$ for all i. Thus, the vectors in an orthogonal set are linearly independent, and therefore form an invertible matrix.
- Because they are linearly independent, they also form a basis for span$\{\mathbf{u}_i, ..., \mathbf{u}_p\}$.
- Let $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ be an orthogonal basis for W. For each $\mathbf{y}$ in W, we have: $\mathbf{y} = c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_\mathbf{p}$, where

$$c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$$

- An orthogonal set composed of unit vectors is called an **orthonormal set**.
- If an $m \times n$ matrix U has orthonormal columns, then $U^TU = I$.

- If such matrix is square, then $U^{-1} = U^T$ (which is the definition of an **orthogonal matrix**).

$$U \text{ is invertible} \to (U^T U)U^{-1} = IU^{-1} \to U^T = U^{-1}$$

- For an $m \times n$ matrix with orthonormal columns:

$$||U\mathbf{x}|| = ||\mathbf{x}|| \qquad \& \qquad (U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

6.3 Orthogonal Projections

- An **orthogonal projection** of $\mathbf{y}$ onto W is the component $\hat{\mathbf{y}}$ such that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ and $\mathbf{z}$ is orthogonal to W.

- The vector $\hat{\mathbf{y}}$ can be expressed in terms of any orthogonal basis $\{\mathbf{u}_1, ...\mathbf{u}_p\}$ of W as follows,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

- Let W be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ any vector in $\mathbb{R}^n$ and $\hat{\mathbf{y}}$ the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$ (or in other words, the best approximation to $\mathbf{y}$), because:

$$||\mathbf{y} - \hat{\mathbf{y}}|| < ||\mathbf{y} - \mathbf{v}|| \qquad \forall \mathbf{v} \in W \text{ distinct from } \hat{\mathbf{y}}$$

- If the orthogonal basis $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is orthonormal, then the matrix $U = [\mathbf{u}_1 \cdots \mathbf{u}_p]$ can be used to form the orthogonal projector $P = UU^T$, and

$$\hat{\mathbf{y}} = UU^T\mathbf{y}$$

7 Symmetric Matrices and Quadratic Forms