Exterior calculus

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Contents

1 m-forms

- $T_p\mathbb{R}^n$: the set of all n-dimensional vectors whose origin is at point p.
- A 1-form is a linear form[1] $w : T_p\mathbb{R}^n \to \mathbb{R}$.
- A 1-form belongs to the dual space of $T_p\mathbb{R}^n$.
- $dx(\mathbf{v})$ is a 1-form that grabs the first component of a vector $\mathbf{v}$. $dy(\mathbf{v})$ is a 1-form that grabs the second component of a vector $\mathbf{v}$, and so on.
- An example of a 1-form $w : T_p\mathbb{R}^2 \to \mathbb{R}$ would be $w(\mathbf{v}) = adx(\mathbf{v}) + bdy(\mathbf{v})$, or simply $w = adx + bdy$.
- A general 1-form $w : T_p\mathbb{R}^n \to \mathbb{R}$ is expressed as follows: $w = a_1 dx_1 + ... + a_n dx_n$.
- The exterior product $w_1 \wedge w_2$ of two 1-forms w_1 and w_2 is defined as

$$w_1 \wedge w_2(\mathbf{v}_1, \mathbf{v}_2) = \begin{vmatrix} w_1(\mathbf{v}_1) & w_2(\mathbf{v}_1) \\ w_1(\mathbf{v}_2) & w_2(\mathbf{v}_2) \end{vmatrix} \tag{1}$$

- A 2-form is an anti-symmetric bilinear form[1] $T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ that is defined as the exterior product $w_1 \wedge w_2$.
- $w_1 \wedge w_2 = -w_2 \wedge w_1$, and thus $w_1 \wedge w_1 = 0$.
- The exterior product $w_1 \wedge ... \wedge w_m$ of n 1-forms w_i is defined as

$$w_1 \wedge ... \wedge w_m(\mathbf{v}_1, ..., \mathbf{v}_m) = \begin{vmatrix} w_1(\mathbf{v}_1) & \cdots & w_2(\mathbf{v}_1) \\ \vdots & \ddots & \vdots \\ w_1(\mathbf{v}_2) & \cdots & w_2(\mathbf{v}_2) \end{vmatrix} \tag{2}$$

[1] See my notes on functional analysis for a definition of a linear, bilinear, and multilinear form.

- An m-form is an anti-symmetric multi-linear form[1] $w : (T_p\mathbb{R}^n)^m \to \mathbb{R}$ that is defined as the exterior product $w_1 \wedge ... \wedge w_m$.
- Examples of m-forms for $n = 4$ are the following:
  - 1-form: $dx_1 \qquad dx_2 \qquad dx_3 \qquad dx_4$
  - 2-form: $dx_1 \wedge dx_2 \qquad dx_1 \wedge dx_3 \qquad dx_1 \wedge dx_4 \qquad dx_2 \wedge dx_3 \qquad dx_2 \wedge dx_4 \qquad dx_3 \wedge dx_4$
  - 3-form: $dx_1 \wedge dx_2 \wedge dx_3 \qquad dx_1 \wedge dx_2 \wedge dx_4 \qquad dx_1 \wedge dx_3 \wedge dx_4 \qquad dx_2 \wedge dx_3 \wedge dx_4$
  - 4-form: $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$
- With the examples above as reference, it is clear to see that every m-form on $T_p\mathcal{R}^n$ can be written as

$$w = \sum_{1 \leq i_1 < i_2 < ... < i_m \leq n} a_{i_1, i_2, ..., i_m} dx_{i_1} \wedge dx_{i_2} \wedge ... dx_{i_m} \tag{3}$$

- If α is a k-form and β and l-form, then $\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta$.
- $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$.
- The dimensions of the space of m-forms on $T_p\mathbb{R}^n$ is

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{4}$$

  For the m-form examples above with $n = 4$, we get
  - 1-form: $4!/1!(3)! = 4$
  - 2-form: $4!/2!(2)! = 6$
  - 3-form: $4!/3!(1)! = 4$
  - 4-form: $4!/4!(0)! = 1$

2 Differential m-forms