Plasma dynamics

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Particle description

Chapter 1

Classical Mechanics

Table 1.1: Various coordinates of classical mechanics.

Classical coordinates	$\mathbf{x}(t)$	$\mathbf{v}(t)$
Generalized coordinates	$\mathbf{q}$	$\dot{\mathbf{q}}$
Canonical coordinates	$\mathbf{q}$	$\mathbf{p}$
Time-dependent canonical coordinates	$\tilde{\mathbf{q}}(t)$	$\tilde{\mathbf{p}}(t)$

1.1 Lagrangian mechanics

- Define the position $\mathbf{x} = \mathbf{x}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ of a particle.
- Define the Lagrangian as $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the generalized position and generalized velocity, respectively.
- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] = \left(\frac{\partial L}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}. \tag{1.1}$$

- For example, the Lagrangian of a particle in an electromagnetic field where $\phi = \phi(\mathbf{q}, t)$ is the electric potential and $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$ is the magnetic potential, is

$$L = \frac{1}{2} m\dot{q}_i\dot{q}_i + e\dot{q}_i A_i - e\phi. \tag{1.2}$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e\dot{q}_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i} \tag{1.3}$$

$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i + eA_i \tag{1.4}$$

$$\begin{aligned}\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] &= \frac{d}{dt}\left[mv_i + eA_i(\mathbf{x}, t)\right] \\ &= m\frac{dv_i}{dt} + ev_j\left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}}.\end{aligned} \tag{1.5}$$

Thus, the Euler-Lagrange equation becomes

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.6}$$

In vector notation, this is written as

$$m\frac{d\mathbf{v}}{dt} = \left(-e\mathbf{v}\cdot\nabla_q\mathbf{A} - e\frac{\partial \mathbf{A}}{\partial t} + e\nabla_q(\mathbf{v}\cdot\mathbf{A}) - e\nabla_q\phi\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.7}$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}, \tag{1.8}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$.

1.2 Hamiltonian mechanics

- Define the Hamiltonian as $H = H(\mathbf{q}, \mathbf{p}, t)$, where $\mathbf{q}$ and $\mathbf{p}$ are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.
- The Hamiltonian is obtained from the Lagrangian using

$$H = \left(\dot{\mathbf{q}}\cdot\mathbf{p} - L\right)_{\dot{\mathbf{q}}=\mathbf{f}(\mathbf{q},\mathbf{p})}, \tag{1.9}$$

  where the dependency of $\dot{\mathbf{q}}$ on $\mathbf{q}$ and $\mathbf{p}$ is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \tag{1.10}$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[\dot{q}_i p_i - \left(\frac{1}{2}m\dot{q}_i\dot{q}_i + e\dot{q}_i A_i - e\phi\right)\right]_{\dot{\mathbf{q}}=f(\mathbf{q},\mathbf{p})}. \tag{1.11}$$

  Evaluating eq. (1.10) gives $p_i = m\dot{q}_i + eA_i$, which allows us to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$ and $\mathbf{p}$ as $\dot{q}_i = \frac{1}{m}(p_i - eA_i)$. Thus

$$\begin{aligned} H &= \frac{1}{m}(p_i - eA_i)p_i - \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) - e\frac{1}{m}(p_i - eA_i)A_i + e\phi \\ &= \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) + e\phi. \end{aligned} \tag{1.12}$$

- We introduce the variables $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$, which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \tag{1.13}$$

$$\tilde{\mathbf{p}} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}} \tag{1.14}$$

- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.15}$$

$$\frac{d\tilde{p}_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.16}$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x}, t) \tag{1.17}$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m\frac{dv_i}{dt} + ev_j \left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}} . \tag{1.18}$$

$$\begin{aligned}\frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i}\left[\frac{1}{2m}(p_j - eA_j)(p_j - eA_j) + e\phi\right] \\ &= \frac{1}{m}(p_j - eA_j)\frac{\partial}{\partial q_i}(p_j - eA_j) + e\frac{\partial \phi}{\partial q_i} \\ &= -\frac{e}{m}(p_j - eA_j)\frac{\partial A_j}{\partial q_i} + e\frac{\partial \phi}{\partial q_i}\end{aligned} \tag{1.19}$$

$$\begin{aligned}\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m}\left[mv_j + eA_j(\mathbf{x},t) - eA_j(\mathbf{x},t)\right]\left(\frac{\partial A_j}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} \\ &= \left(-ev_j\frac{\partial A_j}{\partial q_i} + e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} .\end{aligned} \tag{1.20}$$

Equation (1.16) thus leads to

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} . \tag{1.21}$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}} . \tag{1.22}$$

Chapter 2

Single-particle motion—guiding center theory

The motion of single particles is obtained by solving eq. (1.22), which we re-write below

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}. \tag{2.1}$$

In the above, $\mathbf{v} = \mathbf{v}(t)$ is the particle velocity, $\mathbf{x} = \mathbf{x}(t)$ the particle position, $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ the electric field, and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$ the magnetic field. In the subsections that follow, we will solve this equation of motion for simplified forms of $\mathbf{E}$ and $\mathbf{B}$. The solutions for the velocity vector will typically be of the form

$$\mathbf{v} = \mathbf{v}^{(c)} + \mathbf{v}^{(g)} + v^{||}\mathbf{b}, \tag{2.2}$$

where $\mathbf{v}^{(c)} = \mathbf{v}^{(c)}(t)$ is the gyromotion (cyclotron) velocity, $\mathbf{v}^{(g)} = \mathbf{v}^{(g)}(t)$ is the guiding center velocity, $v^{||} = v^{||}(t)$ is the parallel velocity. Not all of the velocities will always be present. $\mathbf{b} = \mathbf{B}/B$ is the unit magnetic field vector. The position of the particle is governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}. \tag{2.3}$$

Using eq. (2.2), we integrate the above to obtain

$$\int_0^t d\mathbf{x}(t') = \int_0^t \mathbf{v}^{(c)}(t')\,dt' + \int_0^t \mathbf{v}^{(g)}(t')\,dt' + \int_0^t v^{||}(t)\mathbf{b}\,dt'. \tag{2.4}$$

We introduce the positions $\mathbf{x}^{(c)} = \mathbf{x}^{(c)}(t)$, $\mathbf{x}^{(g)} = \mathbf{x}^{(g)}(t)$, and $\mathbf{x}^{||} = \mathbf{x}^{||}(t)$, which are defined as follows

$$\mathbf{x}^{(c)} = \int \mathbf{v}^{(c)}\,dt, \tag{2.5}$$

$$\mathbf{x}^{(g)} = \int \mathbf{v}^{(g)}\,dt, \tag{2.6}$$

$$\mathbf{x}^{||} = \int v^{||}\mathbf{b}\,dt. \tag{2.7}$$

Thus, eq. (2.4) is now re-written as

$$\mathbf{x}(t) - \mathbf{x}(0) = \mathbf{x}^{(c)}(t) - \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(t) - \mathbf{x}^{(g)}(0) - \mathbf{x}^{||}(t) - \mathbf{x}^{||}(0). \tag{2.8}$$

Without loss of generality, we will assume that the initial condition is as follows

$$\mathbf{x}(0) = \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(0) + \mathbf{x}^{||}(0). \tag{2.9}$$

Thus, the particle position is finally expressed as

$$\mathbf{x} = \mathbf{x}^{(c)} + \mathbf{x}^{(g)} + \mathbf{x}^{||}. \tag{2.10}$$

2.1 Uniform E and B fields

2.1.1 Only E field

Let's orient our coordinate system such that $\mathbf{E}$ points in the $\mathbf{e}_z$ direction. Thus, the equations of motion are

$$\begin{aligned}
\frac{dv_x}{dt} &= 0 \qquad & v_x(0) &= v_\perp \cos(\phi), \\
\frac{dv_y}{dt} &= 0 \qquad & v_y(0) &= v_\perp \sin(\phi), \\
\frac{dv_z}{dt} &= \frac{eE}{m} \qquad & v_z(0) &= v_{||}.
\end{aligned} \tag{2.11}$$

The solution of the above is

$$\begin{aligned}
v_x &= v_\perp \cos(\phi) \\
v_y &= v_\perp \sin(\phi) \\
v_z &= v_{||} + \frac{eE}{m}t.
\end{aligned} \tag{2.12}$$

2.1.2 Only B field

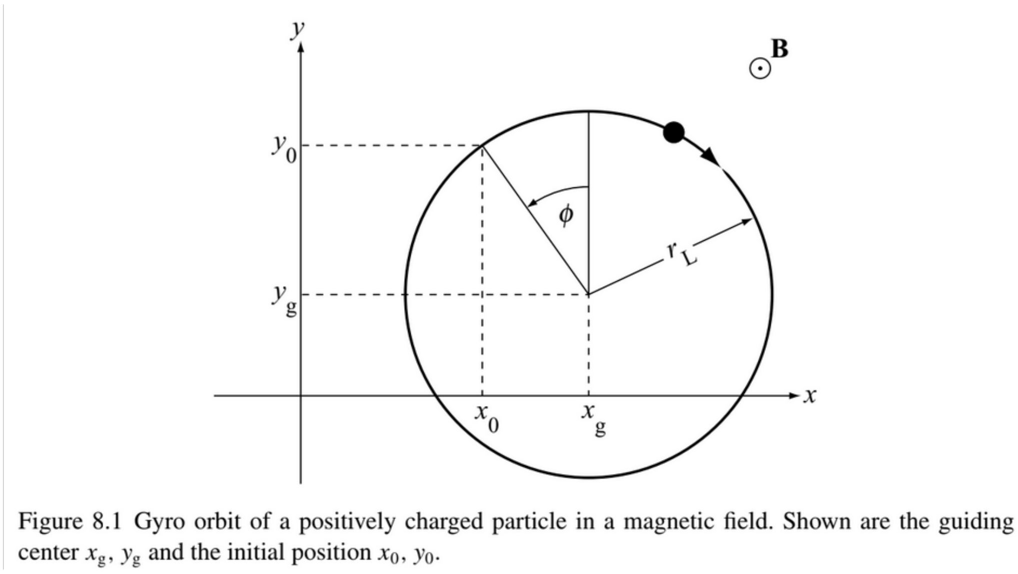

Figure 2.1: Coordinates for gyromotion (extracted from Plasma Physics and Fusion Energy, J. P. Freidberg).

Let's orient our coordinate system such that $\mathbf{B}$ points in the $\mathbf{e}_z$ direction. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB}{m}v_y \qquad v_x(0) = v_\perp \cos(\phi), \tag{2.13a}$$

$$\frac{dv_y}{dt} = -\frac{eB}{m}v_x \qquad v_y(0) = v_\perp \sin(\phi), \tag{2.13b}$$

$$\frac{dv_z}{dt} = 0 \qquad v_z(0) = v_{||}. \tag{2.13c}$$

The z component is decoupled from the rest and has a trivial solution. For the other two components, we begin by taking the time derivative of eq. (2.13b). Thus

$$\frac{d^2 v_y}{dt^2} = -\frac{eB}{m}\frac{dv_x}{dt} = -w_c^2 v_y, \tag{2.14}$$

where $w_c = |e|B/m$ is the gyro frequency. We know that the general solution to the above is $v_y = c_1 \cos(w_c t) + c_2 \sin(w_c t)$. If we use the ICs and assume ions, we have

$$v_y = -v_\perp \sin(w_c t - \phi). \tag{2.15}$$

Integrating eq. (2.13a) then gives

$$v_x = v_\perp \cos(w_c t - \phi). \tag{2.16}$$

The final solution, for either positive or negative charges, can be written as

$$\begin{aligned} v_x^{(c)} &= v_\perp \cos(w_c t \pm \phi) \\ v_y^{(c)} &= \pm v_\perp \sin(w_c t \pm \phi), \end{aligned} \tag{2.17}$$

where upper signs correspond to a negative charge. Integrating the equations above leads to

$$\begin{aligned} x^{(c)} &= r_L \sin(w_c t \pm \phi) \\ y^{(c)} &= \mp r_L \cos(w_c t \pm \phi). \end{aligned} \tag{2.18}$$

where $r_L = v_\perp / w_c$ is the gyro radius.

2.1.3 Both E and B fields

Let's orient our coordinate system such that $\mathbf{B}$ still points along $\mathbf{e}_z$. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x}{m} + \frac{eB}{m} v_y \qquad v_x(0) = v_\perp \cos(\phi) + \frac{E_y}{B}, \tag{2.19a}$$

$$\frac{dv_y}{dt} = \frac{eE_y}{m} - \frac{eB}{m} v_x \qquad v_y(0) = v_\perp \sin(\phi) - \frac{E_x}{B}, \tag{2.19b}$$

$$\frac{dv_z}{dt} = \frac{eE_{||}}{m} \qquad v_z(0) = v_{||}, \tag{2.19c}$$

where we have chosen the given initial conditions simply to facilitate the math. Again, the z component is decoupled from the rest and has the trivial solution $v_z = v_{||} + (eE_{||}/m)t$. Thus, eq. (2.2) for the x and y components are

$$\begin{aligned} v_x &= v_x^{(c)} + v_x^{(g)}, \\ v_y &= v_y^{(c)} + v_y^{(g)}. \end{aligned} \tag{2.20}$$

We assume $v_x^{(g)}$ and $v_y^{(g)}$ are time independent. Using eq. (2.20) in eq. (2.19) we obtain

$$\begin{aligned} 0 &= \frac{eE_x}{m} + \frac{eB}{m} v_y^{(g)} \\ 0 &= \frac{eE_y}{m} - \frac{eB}{m} v_x^{(g)}. \end{aligned} \tag{2.21}$$

Thus, $v_x^{(g)} = E_y/B$ and $v_y^{(g)} = -E_x/B$, which in vector notation can be expressed as

$$\mathbf{v}_E^{(g)} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \tag{2.22}$$

2.2 Non-uniform B field

2.2.1 Change in magnitude along perpendicular directions

The magnetic field still points in the $\mathbf{e}_z$ direction, but its magnitude changes in directions perpendicular to $\mathbf{e}_z$: $B = B(q_x, q_y)$. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(x,y)}{m}v_y \qquad v_x(0) = v_\perp \cos(\phi) - \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} \frac{1}{B(x^{(g)},y^{(g)})}, \tag{2.23a}$$

$$\frac{dv_y}{dt} = -\frac{eB(x,y)}{m}v_x \qquad v_y(0) = v_\perp \sin(\phi) + \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} \frac{1}{B(x^{(g)},y^{(g)})}, \tag{2.23b}$$

$$\frac{dv_z}{dt} = 0 \qquad v_z(0) = v_{||}. \tag{2.23c}$$

In the above, the x and y in $B(x, y)$ are the perpendicular components of the particle's position. The v_z component is decoupled from the rest and has a trivial solution. Thus, eqs. (2.2) and (2.10) for the x and y components are

$$v_x = v_x^{(c)} + v_x^{(g)}, \tag{2.24}$$

$$v_y = v_y^{(c)} + v_y^{(g)}, \tag{2.25}$$

$$x = x^{(c)} + x^{(g)}, \tag{2.26}$$

$$y = y^{(c)} + y^{(g)}. \tag{2.27}$$

We begin by employing a Taylor-series expansion for the magnetic field

$$B(x,y) = B(x^{(g)},y^{(g)}) + \left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)} + ... \tag{2.28}$$

Thus, eqs. (2.23a) and (2.23b) are now

$$\frac{dv_x}{dt} = \frac{eB(x^{(g)},y^{(g)})}{m}v_y + \frac{e}{m}\left(\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)}\right) v_y \tag{2.29}$$

$$\frac{dv_y}{dt} = -\frac{eB(x^{(g)},y^{(g)})}{m}v_x + \frac{e}{m}\left(\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)}\right) v_x. \tag{2.30}$$

As before, we assume $v_x^{(g)}$, $v_y^{(g)}$ are time independent. Also, for simplicity we assume ions only. Plugging in eqs. (2.24) and (2.25) into the above, we get

$$0 = B(x^{(g)},y^{(g)})v_y^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)}\right)\left(v_y^{(c)} + v_y^{(g)}\right), \tag{2.31}$$

$$0 = -B(x^{(g)},y^{(g)})v_x^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)}\right)\left(v_x^{(c)} + v_x^{(g)}\right). \tag{2.32}$$

We assume $v_x^{(g)} \ll v_x^{(c)}$ and $v_y^{(g)} \ll v_y^{(c)}$. Thus, the above becomes

$$0 = B(x^{(g)},y^{(g)})v_y^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)}\right) v_y^{(c)}, \tag{2.33}$$

$$0 = -B(x^{(g)}, y^{(g)})v_x^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} y^{(c)} \right) v_x^{(c)}. \tag{2.34}$$

We now use the definitions in eq. (2.17) and eq. (2.18). For example, with those definitions we can show that

$$\begin{aligned} x^{(c)}v_y^{(c)} &= [r_L \sin(w_c t - \phi)]\,[-v_\perp \sin(w_c t - \phi)] \\ &= -\frac{v_\perp^2}{w_c}\sin^2(w_c t - \phi) \\ &= -\frac{v_\perp^2}{2w_c}\{1 - \cos[2(w_c t - \phi)]\} \end{aligned} \tag{2.35}$$

Similar derivations can be carried out for $y^{(c)}v_y^{(c)}$, $x^{(c)}v_x^{(c)}$, and $y^{(c)}v_x^{(c)}$. Thus, eqs. (2.33) and (2.34) become

$$\begin{aligned} 0 = B(x^{(g)}, y^{(g)})v_y^{(g)} &- \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} \{1 - \cos[2(w_c t - \phi)]\} \\ &- \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} \sin[2(w_c t - \phi)], \end{aligned} \tag{2.36}$$

$$\begin{aligned} 0 = -B(x^{(g)}, y^{(g)})v_x^{(g)} &- \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} \sin[2(w_c t - \phi)] \\ &- \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} \{1 + \cos[2(w_c t - \phi)]\}. \end{aligned} \tag{2.37}$$

We neglect the oscillatory terms containing the sines and cosines—if it was not possible to neglect them, then the assumption that $v_x^{(g)}$, $v_y^{(g)}$ are time independent would not hold. Thus,

$$\begin{aligned} 0 &= B(x^{(g)}, y^{(g)})v_y^{(g)} - \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} \\ 0 &= -B(x^{(g)}, y^{(g)})v_x^{(g)} - \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}}. \end{aligned} \tag{2.38a}$$

Solving for the guiding center velocities, we finally have

$$\begin{aligned} v_x^{(g)} &= -\frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_y}\right|_{x^{(g)},y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})} \\ v_y^{(g)} &= \frac{v_\perp^2}{2w_c}\left.\frac{\partial B}{\partial q_x}\right|_{x^{(g)},y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}. \end{aligned} \tag{2.39}$$

In vector notation, this is written as

$$\mathbf{v}_{\nabla B}^{(g)} = \mp\frac{v_\perp^2}{2w_c}\frac{\mathbf{B}\times\nabla B}{B^2}. \tag{2.40}$$

In the above, the fields and w_c are evaluated at $(x^{(g)}, y^{(g)})$.

2.2.2 Change in magnitude along parallel directions

Ideally, one would introduce a gradient only in the direction parallel to the magnetic field, that is, one would have $\mathbf{B} = B(q_z)\mathbf{e}_z$. However, due to Gauss's law, this is too restrictive and instead we generalize and use $\mathbf{B} = B_x\mathbf{e}_x + B_z\mathbf{e}_z$, where $B_x = B_x(q_x, q_z)$ and $B_z = B_z(q_x, q_z)$. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{e}{m}v_y B_z(x,z), \tag{2.41}$$

$$\frac{dv_y}{dt} = -\frac{e}{m}[v_x B_z(x,z) - v_z B_x(x,z)], \tag{2.42}$$

$$\frac{dv_z}{dt} = -\frac{e}{m}v_y B_x(x,z). \tag{2.43}$$

However, the z direction no longer corresponds to the parallel direction, since the magnetic field also has a component along the x direction. To account for this, we will introduce a rotating reference frame, in which one of the axis will always be aligned with the magnetic field vector, and thus would denote the parallel direction. In the original static reference frame the unit vectors are $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and the velocity components are (v_x, v_y, v_x), whereas in this new rotating reference frame the unit vectors are $(\mathbf{e}_{\perp 1}, \mathbf{e}_{\perp 2}, \mathbf{b})$ and the velocity components are $(v_{\perp 1}, v_{\perp 2}, v_{||})$.

The rotating reference frame is described by the rotation matrix

$$\mathbf{Q}(t) = \begin{bmatrix} b_x & 0 & b_z \\ 0 & 1 & 0 \\ b_z & 0 & -b_x \end{bmatrix}, \tag{2.44}$$

where $b_x = b_x(t)$ and $b_y = b_y(t)$ are given by

$$b_x = \frac{B_x(x,z)}{B(x,z)} \qquad b_z = \frac{B_z(x,z)}{B(x,z)} \tag{2.45}$$

In the above, $B(x,z) = [B_x^2(x,z) + B_z^2(x,z)]^{1/2}$. As an example, the matrix above leads to the following transformations for the unit vectors and velocities in the rotating reference frame

$$\mathbf{b} = b_x\mathbf{e}_x + b_z\mathbf{e}_z \tag{2.46}$$

$$\mathbf{e}_{\perp 2} = \mathbf{e}_y \tag{2.47}$$

$$\mathbf{e}_{\perp 1} = b_z\mathbf{e}_x - b_x\mathbf{e}_z = \mathbf{e}_{\perp 2} \times \mathbf{b}, \tag{2.48}$$

$$v_{||} = b_x v_x + b_z v_z \tag{2.49}$$

$$v_{\perp 2} = v_y \tag{2.50}$$

$$v_{\perp 1} = b_z v_x - b_x v_z. \tag{2.51}$$

Using the transformation rule for the acceleration of a particle, but for some reason neglecting the coriollis and centrifugal forces, we obtain for the velocity derivatives

$$\frac{dv_{||}}{dt} = \frac{dv_x}{dt}b_x + \frac{dv_z}{dt}b_z - Kv_{\perp 1} \tag{2.52}$$

$$\frac{dv_{\perp 2}}{dt} = \frac{dv_y}{dt} \tag{2.53}$$

$$\frac{dv_{\perp 1}}{dt} = \frac{dv_x}{dt}b_z - \frac{dv_z}{dt}b_x + Kv_{||}, \tag{2.54}$$

where $K = K(t)$ is given by $K = b_x db_z/dt - b_z db_x/dt$. Using eqs. (2.41) to (2.43) in the above leads to

$$\frac{dv_{||}}{dt} = \frac{e}{m} v_y [B_z(x,z) b_x - B_x(x,z) b_z] - K v_{\perp 1} \tag{2.55}$$

$$\frac{dv_{\perp 2}}{dt} = -\frac{eB}{m}(v_x b_z - v_z b_x) \tag{2.56}$$

$$\frac{dv_{\perp 1}}{dt} = \frac{e}{m} v_y [B_z(x,z) b_z + B_x(x,z) b_x] + K v_{||} \tag{2.57}$$

Using the definitions for b_x and b_z in eq. (2.45), as well as the expressions for $v_{\perp 1}$, $v_{\perp 2}$ in eqs. (2.50) and (2.51), we get

$$\frac{dv_{||}}{dt} = -K v_{\perp 1}, \tag{2.58}$$

$$\frac{dv_{\perp 2}}{dt} = -w_c v_{\perp 1}, \tag{2.59}$$

$$\frac{dv_{\perp 1}}{dt} = w_c v_{\perp 2} + K v_{||}, \tag{2.60}$$

where $w_c = w_c(t)$ is given by $w_c = eB(x,z)/m$.

We now introduce a time transformation to simplify the equations above. To do so, we introduce the following variables

$$\hat{v}_{||} = \hat{v}_{||}(\tau) \qquad \hat{v}_{\perp 2} = \hat{v}_{\perp 2}(\tau) \qquad \hat{v}_{\perp 1} = \hat{v}_{\perp 1}(\tau) \tag{2.61}$$

$$\hat{x} = \hat{x}(\tau) \qquad \hat{z} = \hat{z}(\tau) \tag{2.62}$$

such that

$$v_{||} = \hat{v}_{||}(h(t)) \qquad v_{\perp 2} = \hat{v}_{\perp 2}(h(t)) \qquad v_{\perp 1} = \hat{v}_{\perp 1}(h(t)) \tag{2.63}$$

$$x = \hat{x}(h(t)) \qquad z = \hat{z}(h(t)). \tag{2.64}$$

The function $h(t)$ is given by

$$h(t) = \int_0^t w_c(t')dt'. \tag{2.65}$$

We also show that

$$b_x = \frac{B_x(x,z)}{B(x,z)} = \frac{B_x(\hat{x}(h(t)), \hat{z}(h(t)))}{B(\hat{x}(h(t)), \hat{z}(h(t)))}, \tag{2.66}$$

and thus

$$\frac{db_x}{dt} = \frac{dh(t)}{dt}\frac{d}{d\tau}\left[\frac{B_x(\hat{x},\hat{z})}{B(\hat{x},\hat{z})}\right]_{\tau=h(t)} = w_c \left.\frac{d\hat{b}_x}{d\tau}\right|_{\tau=h(t)}, \tag{2.67}$$

where $\hat{b}_x = \hat{b}_x(\tau)$ is given by $\hat{b}_x = B_x(\hat{x},\hat{z})/B(\hat{x},\hat{z})$. The analogous holds for b_z. This allows us to write

$$K = w_c \left(\hat{b}_x \frac{d\hat{b}_z}{d\tau} - \hat{b}_z \frac{d\hat{b}_x}{d\tau}\right)_{\tau=h(t)} = w_c \left.\hat{K}\right|_{\tau=h(t)}, \tag{2.68}$$

where $\hat{K} = \hat{K}(\tau)$ is given by $\hat{K} = \hat{b}_x d\hat{b}_z/d\tau - \hat{b}_z d\hat{b}_x/d\tau$. With these transformation, eqs. (2.58) to (2.60) are re-written as

$$\frac{d\hat{v}_{||}}{d\tau} = -\hat{K}\hat{v}_{\perp 1}, \tag{2.69}$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \tag{2.70}$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \hat{K}\hat{v}_{||}. \tag{2.71}$$

We now simplify B_z so that $B_z = B_z(q_z)$. To be consistent with Gauss's law, we require $B_x = B_x(q_x, q_y)$ where $B_x = -q_x dB_z/dq_z$. With these simplified forms, we have

$$\hat{K} = -\hat{b}_z^2 \frac{d}{d\tau}\left(\frac{\hat{b}_x}{\hat{b}_z}\right) \tag{2.72}$$

$$= -\frac{B_z^2(\hat{z})}{B^2(\hat{x},\hat{z})}\frac{d}{d\tau}\left(\frac{B_x(\hat{x},\hat{z})}{B_z(\hat{z})}\right) \tag{2.73}$$

$$= \frac{B_z^2(\hat{z})}{B^2(\hat{x},\hat{z})}\frac{d}{d\tau}\left[\hat{x}\left(\frac{1}{B_z}\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}\right]. \tag{2.74}$$

We now use the long-thin approximation. For this approximation, we assume that $B_x/B_z \ll 1$, and also that $\frac{1}{B_z}\frac{dB_z}{dq_z}$ changes very slowly. We thus have

$$\hat{K} \approx \frac{d\hat{x}}{d\tau}\left(\frac{1}{B_z}\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}. \tag{2.75}$$

Also, using the long-thin approximation in eqs. (2.49) and (2.51) allows us to write

$$v_{||} \approx v_z = \frac{dz}{dt} = \left(\frac{d\hat{z}}{d\tau}\right)_{\tau=h(t)} w_c \tag{2.76}$$

$$v_{\perp 1} \approx v_x = \frac{dx}{dt} = \left(\frac{d\hat{x}}{d\tau}\right)_{\tau=h(t)} w_c. \tag{2.77}$$

Evaluating the above at $t = h^{-1}(\tau)$, and defining $\hat{w}_c(\tau)$ from $w_c = \hat{w}_c(h(t))$, we obtain

$$\hat{v}_{||} \approx \frac{d\hat{z}}{d\tau}\hat{w}_c \tag{2.78}$$

$$\hat{v}_{\perp 1} \approx \frac{d\hat{x}}{d\tau}\hat{w}_c. \tag{2.79}$$

We also note that

$$\frac{dB_z(\hat{z})}{d\tau} = \left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}\frac{d\hat{z}}{d\tau}. \tag{2.80}$$

Using the expressions above in eq. (2.75), one can approximate $\hat{K}$ using either of the two forms below

$$\hat{K} \approx \frac{\hat{v}_{\perp 1}}{\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}} \approx \frac{\hat{v}_{\perp 1}}{\hat{v}_{||} B_z(\hat{z})}\frac{dB_z(\hat{z})}{d\tau}. \tag{2.81}$$

We thus write the governing equations for the velocities as

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_{\perp 1}^2}{\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}, \tag{2.82}$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \tag{2.83}$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \frac{\hat{v}_{\perp 1}}{B_z(\hat{z})}\frac{dB_z(\hat{z})}{d\tau}. \tag{2.84}$$

We now assume the solution for the perpendicular velocities is of the form

$$\hat{v}_{\perp 1} = \hat{v}_\perp \cos[\tau + \hat{\epsilon}] \tag{2.85}$$

$$\hat{v}_{\perp 2} = -\hat{v}_\perp \sin[\tau + \hat{\epsilon}], \tag{2.86}$$

where $\hat{v}_\perp = \hat{v}_\perp(\tau)$ and $\hat{\epsilon} = \hat{\epsilon}(\tau)$. Plugging these two assumed solutions into eqs. (2.83) and (2.84), and using some simple algebra, gives

$$\frac{d\hat{v}_\perp}{d\tau} = \frac{\hat{v}_\perp}{2B_z(\hat{z})}\frac{dB_z(\hat{z})}{d\tau}\left\{1 + \cos[2(\tau + \hat{\epsilon})]\right\}. \tag{2.87}$$

The above can be re-arranged and expressed as

$$\frac{d\ln\hat{\mu}}{d\tau} = \frac{d\ln B_z(\hat{z})}{d\tau}\cos[2(\tau + \hat{\epsilon})], \tag{2.88}$$

where $\hat{\mu} = \hat{\mu}(\tau)$ is the adiabatic invariant, and is given by

$$\hat{\mu} = \frac{m\hat{v}_\perp^2}{2B_z(\hat{z})}. \tag{2.89}$$

Integrating eq. (2.88) from τ_1 to τ_2 gives

$$\ln\hat{\mu}(\tau_2) - \ln\hat{\mu}(\tau_1) = \ln[B_z(\hat{z})]\cos[2(\tau + \hat{\epsilon})]|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2\ln[B_z(\hat{z})]\sin[2(\tau + \hat{\epsilon})]d\tau. \tag{2.90}$$

Picking τ_1 and τ_2 such that $[\tau_2 + \hat{\epsilon}(\tau_2)] - [\tau_1 + \hat{\epsilon}(\tau_1)] = 2\pi$, and assuming $B(\hat{z})$ doesn't change significantly from τ_1 to τ_2, gives $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$, that is, $\hat{\mu}$ is constant over one gyro-period. One can also define

$$\mu = \frac{mv_\perp^2}{2B_z(z)} \tag{2.91}$$

where $\mu = \mu(t)$ and $v_\perp = v_\perp(t)$. Given that $v_\perp = \hat{v}_\perp(h(t))$, we have $\mu = \hat{\mu}(h(t))$. Thus, $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ translates to $\mu(t_2) = \mu(t_1)$, where $t_2 = h^{-1}(\tau_2)$ and $t_1 = h^{-1}(\tau_1)$.

Finally, we focus not on the perpendicular velocities but the parallel velocity. Plugging-in the assumed solutions in the governing eq. (2.82) gives

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}\{1 + \cos[2(\tau + \hat{\epsilon})]\}. \tag{2.92}$$

We now average the above from τ_1 to τ_2 while assuming $B(\hat{z})$, $d\hat{v}_{||}/d\tau$ and $\hat{v}_\perp^2$ do not change significantly during that time scale. Note that since this is an average, we are not just integrating from τ_1 to τ_2, but we are also dividing by $\tau_2 - \tau_1$. After averaging, we obtain

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}. \tag{2.93}$$

or

$$m\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{\mu}}{\hat{w}_c}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}. \tag{2.94}$$

Converting back to time t gives

$$m\frac{dv_{||}}{dt} = -\mu\left(\frac{dB_z}{dq_z}\right)_{q_z=z}. \tag{2.95}$$

2.2.3 Change in direction

Rather than writing eq. (2.1) in terns if its components as done in previous sections, we leave the equation in vector form. Expressing the velocity as $\mathbf{v} = \mathbf{v}_\perp + v_{||}\mathbf{b}$ and assuming no electric field, we write eq. (2.1) as

$$\frac{d}{dt}(\mathbf{v}_\perp + v_{||}\mathbf{b}) = \mp w_c(\mathbf{v}_\perp + v_{||}\mathbf{b}) \times \mathbf{b}, \tag{2.96}$$

where upper sign corresponds to negative charge and lower sign to positive charge. For simplicity we will assume positively charged particles only. We then cross both sides of the above by $\mathbf{b}$, that is

$$\mathbf{b} \times \left\{ \left[\frac{d}{dt}(\mathbf{v}_\perp + v_{||}\mathbf{b}) - w_c(\mathbf{v}_\perp + v_{||}\mathbf{b}) \times \mathbf{b} \right] \times \mathbf{b} \right\} = 0. \tag{2.97}$$

The above is simplified using the following three manipulations

$$\begin{aligned} \mathbf{b} \times \left\{ \left[w_c(\mathbf{v}_\perp + v_{||}\mathbf{b}) \times \mathbf{b} \right] \times \mathbf{b} \right\} &= \mathbf{b} \times \{ [w_c \mathbf{v}_\perp \times \mathbf{b}] \times \mathbf{b} \} \\ &= -\mathbf{b} \times \{ w_c \mathbf{v}_\perp (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{b} \cdot w_c \mathbf{v}_\perp) \} \\ &= w_c \mathbf{v}_\perp \times \mathbf{b}. \end{aligned} \tag{2.98}$$

$$\mathbf{b} \times \left\{ \left[\frac{d\mathbf{v}_\perp}{dt} \right] \times \mathbf{b} \right\} = \frac{d\mathbf{v}_\perp}{dt}(\mathbf{b} \cdot \mathbf{b}) - \mathbf{b}\left(\mathbf{b} \cdot \frac{d\mathbf{v}_\perp}{dt} \right) = \left(\frac{d\mathbf{v}_\perp}{dt} \right)_\perp. \tag{2.99}$$

$$\begin{aligned} \mathbf{b} \times \left\{ \left[\frac{dv_{||}\mathbf{b}}{dt} \times \mathbf{b} \right] \right\} &= v_{||}\mathbf{b} \times \left\{ \left[\frac{d\mathbf{b}}{dt} \times \mathbf{b} \right] \right\} \\ &= v_{||} \left[\frac{d\mathbf{b}}{dt}(\mathbf{b} \cdot \mathbf{b}) - \mathbf{b}\left(\mathbf{b} \cdot \frac{d\mathbf{b}}{dt} \right) \right] \\ &= v_{||} \left[\frac{d\mathbf{b}}{dt} - \mathbf{b}\left(\frac{1}{2} \frac{d\mathbf{b} \cdot \mathbf{b}}{dt} \right) \right] \\ &= v_{||} \frac{d\mathbf{b}}{dt} \end{aligned} \tag{2.100}$$

Thus, we have

$$\left(\frac{d\mathbf{v}_\perp}{dt} \right)_\perp - w_c \mathbf{v}_\perp \times \mathbf{b} = -v_{||} \frac{d\mathbf{b}}{dt}. \tag{2.101}$$

As shown in Freidberg

$$\frac{d\mathbf{b}(\mathbf{x}(t))}{dt} = \frac{d\mathbf{x}(t)}{dt} \cdot \nabla \mathbf{b} = \mathbf{v} \cdot \nabla \mathbf{b} = \mathbf{v}_\perp \cdot \nabla \mathbf{b} + v_{||}\mathbf{b} \cdot \nabla \mathbf{b}, \tag{2.102}$$

where $\nabla \mathbf{b}$ is evaluated at $\mathbf{x} = \mathbf{x}(t)$. Thus, eq. (2.101) becomes

$$\left(\frac{d\mathbf{v}_\perp}{dt} \right)_\perp - w_c \mathbf{v}_\perp \times \mathbf{b} = -v_{||}\mathbf{v}_\perp \cdot \nabla \mathbf{b} - v_{||}^2 \mathbf{b} \cdot \nabla \mathbf{b}. \tag{2.103}$$

As was done for the other drifts, we assume the solution is of the form $\mathbf{v}_\perp = \mathbf{v}^{(c)} + \mathbf{v}^{(g)}$, where we assume again that $\mathbf{v}^{(g)}$ is time independent. The term $\mathbf{v}^{(c)}$ corresponds to gyromotion in a rotating reference frame, and is thus given by

$$\mathbf{v}^{(c)} = v_{\perp 1}^{(c)} \mathbf{e}_{\perp 1} + v_{\perp 2}^{(c)} \mathbf{e}_{\perp 2}, \tag{2.104}$$

where $\mathbf{e}_{\perp 1}$ and $\mathbf{e}_{\perp 2}$ are orthogonal to $\mathbf{b}$ and thus rotate in time. $v_{\perp 1}^{(c)}$ is given by eq. (2.16) and $v_{\perp 2}^{(c)}$ by eq. (2.15). We note that, in the non-rotating reference frame, $\mathbf{v}^{(c)}$ is expressed as $\mathbf{v}^{(c)} = v_x^{(c)}\mathbf{e}_x + v_y^{(c)}\mathbf{e}_y + v_z^{(c)}\mathbf{e}_z$. We now prove that $\mathbf{v}_\perp^{(c)}$ is the solution to the two terms on the left-hand side of eq. (2.103). To show this we first use the transformation rule for the acceleration of a particle in a rotating reference frame, but for some reason ignore the coriollis and centrifugal terms. Thus

$$\begin{aligned}\frac{d\mathbf{v}^{(c)}}{dt} &= \frac{dv_x^{(c)}}{dt}\mathbf{e}_x + \frac{dv_y^{(c)}}{dt}\mathbf{e}_y + \frac{dv_z^{(c)}}{dt}\mathbf{e}_z \\ &= \frac{dv_{\perp 1}^{(c)}}{dt}\mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt}\mathbf{e}_{\perp 2} + 2\Omega \times \mathbf{v}^{(c)}. \end{aligned} \tag{2.105}$$

We do not allow the rotating reference frame to rotate about the $\mathbf{b}$ axis. Thus, $\Omega = \Omega_{\perp 1}\mathbf{e}_{\perp 1} + \Omega_{\perp 2}\mathbf{e}_{\perp 2}$. Given that Ω and $\mathbf{v}^{(c)}$ are in the same plane, $\Omega \times \mathbf{v}^{(c)}$ must point in the $\mathbf{b}$ direction. Thus,

$$\left(\frac{d\mathbf{v}^{(c)}}{dt}\right)_\perp = \frac{dv_{\perp 1}^{(c)}}{dt}\mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt}\mathbf{e}_{\perp 2}. \tag{2.106}$$

This allows us to show that

$$\left(\frac{d\mathbf{v}^{(c)}}{dt}\right)_\perp - w_c\mathbf{v}^{(c)} \times \mathbf{b} = \frac{dv_{\perp 1}^{(c)}}{dt}\mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt}\mathbf{e}_{\perp 2} - w_c v_{\perp 2}^{(c)}\mathbf{e}_{\perp 1} + w_c v_{\perp 1}^{(c)}\mathbf{e}_{\perp 2} = 0. \tag{2.107}$$

We now plug in $\mathbf{v}_\perp = \mathbf{v}^{(c)} + \mathbf{v}^{(g)}$ in eq. (2.103) to obtain

$$-w_c\mathbf{v}^{(g)} \times \mathbf{b} = -v_{||}\mathbf{v}_\perp \cdot \nabla\mathbf{b} - v_{||}^2\mathbf{b}\cdot\nabla\mathbf{b}. \tag{2.108}$$

As explained in Freidberg, the term $v_{||}\mathbf{v}_\perp \cdot \nabla\mathbf{b}$ leads to small modifications of the gyro motion, but does not lead to a drift of the particles, and thus is ignored. Taking the cross product of eq. (2.108) with $\mathbf{b}$ finally gives the curvature drift

$$\mathbf{v}_\kappa^{(g)} = \pm\frac{v_{||}^2}{w_c}\frac{(\mathbf{b}\cdot\nabla\mathbf{b})\times\mathbf{B}}{B}. \tag{2.109}$$

We now show that, if we assume $\nabla\times\mathbf{B} = 0$, the grad-B drift

$$\mathbf{v}_{\nabla B}^{(g)} = \mp\frac{v_\perp^2}{2w_c}\frac{\mathbf{B}\times\nabla B}{B^2} \tag{2.110}$$

can be written in the same form as the curvature drift. We begin by showing that

$$\mathbf{B}\times\nabla B = \mathbf{B}\times\nabla(\mathbf{B}\cdot\mathbf{B})^{1/2} = \mathbf{B}\times\frac{1}{2B}\nabla(\mathbf{B}\cdot\mathbf{B}). \tag{2.111}$$

We now use the vector identity $\nabla(\mathbf{B}\cdot\mathbf{B}) = 2\mathbf{B}\times(\nabla\times\mathbf{B}) + 2\mathbf{B}\cdot\nabla\mathbf{B}$, and assume magnetic curl of zero to obtain

$$\begin{aligned}\mathbf{B}\times\nabla B &= \mathbf{B}\times\frac{1}{B}\mathbf{B}\cdot\nabla\mathbf{B} \\ &= \mathbf{B}\times\mathbf{b}\cdot\nabla(B\mathbf{b}) \\ &= \mathbf{B}\times(\mathbf{b}\cdot\nabla B)\,\mathbf{b} + \mathbf{B}\times B\,(\mathbf{b}\cdot\nabla\mathbf{b}) \\ &= -B\,(\mathbf{b}\cdot\nabla\mathbf{b})\times\mathbf{B}. \end{aligned} \tag{2.112}$$

Thus, the grab-B drift can be written as

$$\mathbf{v}_{\nabla B}^{(g)} = \pm\frac{v_\perp^2}{2w_c}\frac{(\mathbf{b}\cdot\nabla\mathbf{b})\times\mathbf{B}}{B}. \tag{2.113}$$

2.3 Non-uniform E field

2.4 Time-varying E field

Consider the scenario used in section 2.1.3, but with a time varying electric field. The equations of motion are

$$
\begin{aligned}
\frac{dv_x}{dt} &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y \qquad & v_x(0) &= v_\perp \cos(\phi) + \frac{E_y(t)}{B} + \frac{m}{eB^2}\frac{dE_x(t)}{dt}, \qquad (2.114a)\\
\frac{dv_y}{dt} &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x \qquad & v_y(0) &= v_\perp \sin(\phi) - \frac{E_x(t)}{B} + \frac{m}{eB^2}\frac{dE_y(t)}{dt}, \qquad (2.114b)\\
\frac{dv_z}{dt} &= \frac{eE_{||}(t)}{m} \qquad & v_z(0) &= v_{||}, \qquad (2.114c)
\end{aligned}
$$

where again we chose the initial conditions simply to be consistent with the solution that we'll derive. The parallel velocity is independent of the perpendicular velocities, and we won't worry about it for now. To solve for the perpendicular velocities, we again assume the general solution is

$$
\begin{aligned}
v_x &= v_x^{(c)} + v_x^{(g)}\\
v_y &= v_y^{(c)} + v_y^{(g)}
\end{aligned} \qquad (2.115)
$$

but now do not assume $v_x^{(g)}$ and $v_y^{(g)}$ are time independent. We expand $v_i^{(g)}$ as

$$
v_i^{(g)} = v_i^{(g,1)} + v_i^{(g,2)} + ..., \qquad (2.116)
$$

where $v_i^{(g,\alpha)} \sim \epsilon v_i^{(g,\alpha-1)}$, and the small parameter ϵ follows from assuming

$$
\frac{1}{v_i^{(g,\alpha)}}\frac{dv_i^{(g,\alpha)}}{dt} \sim \epsilon w_c. \qquad (2.117)
$$

That is, the time scale associated with the rate of change of all of the $v_i^{(g,\alpha)}$ components is much larger than the time scale of the gyromotion. In other words, we assume particles gyrate faster than how quickly their drift velocity changes. Using eq. (2.115) in eq. (2.114) leads to

$$
\begin{aligned}
\frac{dv_x^{(g,1)}}{dt} + \frac{dv_x^{(g,2)}}{dt} &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y^{(g,1)} + \frac{eB}{m}v_y^{(g,2)}\\
\frac{dv_y^{(g,1)}}{dt} + \frac{dv_y^{(g,2)}}{dt} &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x^{(g,1)} - \frac{eB}{m}v_x^{(g,2)}.
\end{aligned} \qquad (2.118)
$$

Collecting lowest order terms

$$
\begin{aligned}
0 &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y^{(g,1)}\\
0 &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x^{(g,1)},
\end{aligned} \qquad (2.119)
$$

and thus $v_x^{(g,1)} = E_y(t)/B$ and $v_y^{(g,1)} = -E_x(t)/B$, which in vector notation is

$$
\mathbf{v}^{(g,1)} = \frac{\mathbf{E}(t) \times \mathbf{B}}{B^2}. \qquad (2.120)
$$

Collecting first order terms

$$\frac{dv_x^{(g,1)}}{dt} = \frac{eB}{m} v_y^{(g,2)}$$
$$\frac{dv_y^{(g,1)}}{dt} = -\frac{eB}{m} v_x^{(g,2)}, \tag{2.121}$$

and thus $v_x^{(g,2)} = (m/eB^2)dE_x(t)/dt$ and $v_y^{(g,2)} = (m/eB^2)dE_y(t)/dt$, which in vector notation is

$$\mathbf{v}^{(g,2)} = \mp \frac{1}{w_c B} \frac{d\mathbf{E}_\perp}{dt}. \tag{2.122}$$

We note that, by looking at the solutions for $v_x^{(g,1)}$ and $v_y^{(g,1)}$, the assumption in eq. (2.117) is equivalent to stating that the electric field changes slowly.

2.5 Time-varying B field

Let's assume the magnetic field points in the z direction again. Using Faraday's law, we have

$$\left(\frac{\partial E_z}{\partial q_y} - \frac{\partial E_y}{\partial q_z}\right)\mathbf{e}_x - \left(\frac{\partial E_z}{\partial q_x} - \frac{\partial E_x}{\partial q_z}\right)\mathbf{e}_y + \left(\frac{\partial E_y}{\partial q_x} - \frac{\partial E_x}{\partial q_y}\right)\mathbf{e}_z = -\frac{\partial B}{\partial t}\mathbf{e}_z. \tag{2.123}$$

To satisfy the above, we set $E_z = 0$, and $E_x = E_x(q_x, q_y, t)$, $E_y = E_y(q_x, q_y, t)$. That is, a time varying magnetic field requires a time and spatially varying electric field.

We will further simplify our analysis by having $E_x = 0$ and $E_y = E_y(q_x, t)$. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(t)}{m} v_y, \tag{2.124}$$
$$\frac{dv_y}{dt} = \frac{eE_y(x,t)}{m} - \frac{eB(t)}{m} v_x, \tag{2.125}$$

with v_z constant. As done in previous sections, the velocities and positions are decomposed as follows

$$v_x = v_x^{(c)} + v_x^{(g)}, \tag{2.126}$$
$$v_y = v_y^{(c)} + v_y^{(g)}, \tag{2.127}$$
$$x = x^{(c)} + x^{(g)}, \tag{2.128}$$
$$y = y^{(c)} + y^{(g)}. \tag{2.129}$$

The electric field is then linearized using a Taylor-series expansion about the guiding center,

$$\frac{dv_x}{dt} = \frac{eB(t)}{m} v_y \tag{2.130}$$
$$\frac{dv_y}{dt} = \frac{e}{m}\left[E_y\left(x^{(g)}, t\right) + \left.\frac{\partial E_y}{\partial q_x}\right|_{x^{(g)}} x^{(c)}\right] - \frac{eB(t)}{m} v_x, \tag{2.131}$$

We assume positive ions for simplicity and re-write the above as

$$\frac{dv_x}{dt} = w_c v_y \tag{2.132}$$
$$\frac{dv_y}{dt} = \frac{w_c}{B(t)}\left[E_y\left(x^{(g)}, t\right) + \left.\frac{\partial E_y}{\partial q_x}\right|_{x^{(g)}} x^{(c)}\right] - w_c v_x. \tag{2.133}$$

where $w_c = w_c(t)$. We introduce new variables

$$\hat{v}_x = \hat{v}_x(\tau) \qquad \hat{v}_y = \hat{v}_y(\tau) \qquad \hat{x}^{(c)} = \hat{x}^{(c)}(\tau) \qquad \hat{x}^{(g)} = \hat{x}^{(g)}(\tau)$$
$$\hat{E}_y = \hat{E}_y(q_x, \tau) \qquad \hat{B} = \hat{B}(\tau) \tag{2.134}$$

such that

$$v_x(t) = \hat{v}_x(h(t)) \qquad v_y(t) = \hat{v}_y(h(t)) \qquad x^{(c)}(t) = \hat{x}^{(c)}(h(t)) \quad x^{(g)}(t) = \hat{x}^{(g)}(h(t))$$
$$E_y(q_x, t) = \hat{E}_y(q_x, h(t)) \qquad B(t) = \hat{B}(h(t)). \tag{2.135}$$

For the above

$$h(t) = \int_0^t w_c(t')\, dt'. \tag{2.136}$$

The equations of motion then become

$$\begin{aligned} \frac{d\hat{v}_x}{d\tau} &= \hat{v}_y \\ \frac{d\hat{v}_y}{d\tau} &= \frac{1}{\hat{B}(\tau)} \left[\hat{E}_y(\hat{x}^{(g)}, \tau) + \left. \frac{\partial \hat{E}_y}{\partial q_x} \right|_{\hat{x}^{(g)}} \hat{x}^{(c)} \right] - \hat{v}_x. \end{aligned} \tag{2.137}$$

For the gyromotion quantities, we'll assume they are of the following form,

$$\hat{v}_x^{(c)} = \hat{v}_\perp \cos(\tau + \hat{\epsilon}), \tag{2.138}$$
$$\hat{v}_y^{(c)} = -\hat{v}_\perp \sin(\tau + \hat{\epsilon}), \tag{2.139}$$
$$\hat{x}^{(c)} = \hat{r}_L \sin(\tau + \hat{\epsilon}), \tag{2.140}$$
$$\hat{y}^{(c)} = \hat{r}_L \cos(\tau + \hat{\epsilon}), \tag{2.141}$$

where $\hat{v}_\perp = \hat{v}_\perp(\tau)$, $\hat{\epsilon} = \hat{\epsilon}(\tau)$, $\hat{w}_c = \hat{w}_c(\tau) = e\hat{B}(\tau)/m$, and $\hat{r}_L = \hat{v}_\perp/\hat{w}_c$ are now time-dependent functions. Note that for this specific case, the τ-derivatives of the positions above are not equal to their respective velocities, and instead the relationship holds only to leading order. For the guiding center velocities, we'll guess a given form and then check if it satisfies the governing equations. Thus, we guess

$$\begin{aligned} \hat{v}_x^{(g)} &= \frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \\ \hat{v}_y^{(g)} &= \frac{d}{d\tau} \left(\frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \right). \end{aligned} \tag{2.142}$$

Plugging in all of these expressions in the evolution equations given by eq. (2.137), and using a bit of algebra, leads to

$$\frac{d \ln \hat{\mu}}{d\tau} = \frac{d \ln \hat{B}(\tau)}{d\tau} \cos[2(\tau + \hat{\epsilon})], \tag{2.143}$$

where $\hat{\mu} = \hat{\mu}(\tau)$ is given by

$$\hat{\mu} = \frac{m\hat{v}_\perp^2}{2\hat{B}(\tau)}. \tag{2.144}$$

Integrating over one gyro-period, i.e. from τ_1 to τ_2 such that $[\tau_2 + \epsilon(\tau_2)] - [\tau_1 + \epsilon(\tau_1)] = 2\pi$, gives

$$\ln \hat{\mu}(\tau_2) - \ln \hat{\mu}(\tau_1) = \ln[\hat{B}(\tau)] \cos[2(\tau + \hat{\epsilon})] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2 \ln[\hat{B}(\tau)] \sin[2(\tau + \hat{\epsilon})]\, d\tau. \tag{2.145}$$

Assuming $\hat{B}(\tau)$ doesn't change significantly from τ_1 to τ_2, then we have $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$, that is, $\hat{\mu}$ is constant over one gyro-period. On can also define

$$\mu = \frac{mv_\perp^2}{2B(t)} \tag{2.146}$$

where $\mu = \mu(t)$ and $v_\perp = v_\perp(t)$. Given that $v_\perp = \hat{v}_\perp(h(t))$, we have $\mu = \hat{\mu}(h(t))$. Thus, $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ translates to $\mu(t_2) = \mu(t_1)$, where $t_2 = h^{-1}(\tau_2)$ and $t_1 = h^{-1}(\tau_1)$.

As shown in the analysis above, for a time dependent magnetic field a drift of the following form is introduced

$$\hat{v}_y^{(g)} = \frac{d}{d\tau}\left(\frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)}\right). \tag{2.147}$$

Converting back to time t

$$v_y^{(g)} = \frac{1}{w_c}\frac{d}{dt}\left(\frac{E_y(x^{(g)}, t)}{B(t)}\right). \tag{2.148}$$

For the more general case where $E_x = E_x(q_x, q_y, t)$ and $E_y = E_y(q_x, q_y, t)$ then

$$\mathbf{v}_p^{(g)} = \mp\frac{1}{w_c}\frac{d}{dt}\left(\frac{\mathbf{E}_\perp}{B}\right), \tag{2.149}$$

where top sign is for electrons and bottom sign is for ions, and it is assumed that the electric field is evaluated at the guiding center. For an even more general case where the magnetic field does not necessarily point in one direction,

$$\mathbf{v}_p^{(g)} = \mp\frac{1}{w_c}\mathbf{b} \times \frac{d\mathbf{v}_E^{(g)}}{dt}. \tag{2.150}$$

Chapter 3

Plasma parameters, time scales, and length scales

A summary of fundamental time and length scales of plasmas in given in tables 3.1 and 3.2

Table 3.1: Plasma time scales, for either electrons ($\alpha = e$) or ions ($\alpha = i$).

Time scales	Formulas
Gyro period	$\tau_{c\alpha} = \dfrac{2\pi}{w_{c\alpha}} \quad w_{c\alpha} = \dfrac{q_\alpha B}{m_\alpha}$
Plasma period	$\tau_{p\alpha} = \dfrac{2\pi}{w_{p\alpha}} \quad w_{p\alpha} = \sqrt{\dfrac{n_\alpha q_\alpha^2}{m_\alpha \epsilon_0}}$

Table 3.2: Plasma length scales, for either electrons ($\alpha = e$) or ions ($\alpha = i$).

Length scales	Formulas
Gyro radius	$r_{c\alpha} = \dfrac{v_{T\alpha}}{w_{c\alpha}} = \dfrac{m v_{T\alpha}}{q_\alpha B}$
Debye length	$\lambda_{D\alpha} = \dfrac{v_{T\alpha}}{\sqrt{2} w_{p\alpha}} = \sqrt{\dfrac{\epsilon_0 k_B T_\alpha}{n_\alpha q_\alpha^2}}$
DeBroglie wave length	$\lambda_{B\alpha} = \dfrac{h}{\sqrt{\pi} m_\alpha v_{T\alpha}}$
Sphere radius	$a_\alpha = \left(\dfrac{3}{4\pi n_\alpha} \right)^{1/3}$

- Thermal velocity:

$$v_{T_\alpha} = \sqrt{\frac{2k_B T_\alpha}{m_\alpha}} \tag{3.1}$$

- Total Debye length:

$$\frac{1}{\lambda_D^2} = \sum_\alpha \frac{1}{\lambda_{D\alpha}^2}. \tag{3.2}$$

- Plasma parameter

$$\Lambda_\alpha = \frac{4}{3}\pi\lambda_{D\alpha}^3 n_\alpha \tag{3.3}$$

- Quantum plasma parameter

$$\chi_\alpha = \frac{4}{3}\pi\lambda_{B\alpha}^3 n_\alpha \tag{3.4}$$

- Coupling parameter:

$$\Gamma_\alpha = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha k_B T_\alpha} = \frac{1}{3}\Lambda_\alpha^{-2/3} \tag{3.5}$$

- Degeneracy parameter for electrons:

$$\Theta_e = \frac{k_B T_e}{E_{fe}} = \left(\frac{2^{10}\pi}{3^4}\right)^{1/3} \chi_e^{-2/3} \tag{3.6}$$

- Fermi energy for electrons:

$$E_{fe} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3} \tag{3.7}$$

Some notes on the coupling parameter We can define two types of Coulomb interactions: strong and weak. Strong Coulomb interactions are those for which the particle's Coulomb potential energy is larger than its kinetic energy, and viceversa for weak Coulomb interactions. Thus, we can also define two types of plasma regimes:

- Strongly-coupled plasmas: plasmas where the Coulomb interactions are mostly strong and thus drive the dynamics of its evolution. Coulomb interactions tend to be strong when the inter-particle distances are small, and thus this regime would be dominated by *short-range* interactions. These plasmas are also described as exhibiting *collisional* behavior, since a strong Coulomb interaction essentially means a collision has occurred.

- Weakly-coupled plasmas: plasmas where the Coulomb interactions are mostly weak, and as a result do not drive the dynamics of its evolution. The plasma dynamics are instead driven by *long-range* effects caused by smooth electromagnetic fields that result from integrating a large number of particles. These plasmas are also described as exhibiting *collective* behavior, since the long-range electromagnetic fields follow from the collective integration of many particles.

We describe an approximate Coulomb potential energy for particles in a plasma as

$$\mathrm{U} = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha}. \tag{3.8}$$

The impact parameter that has been used above is a_α, the sphere radius. This provides a decent measure on the average spacing between particles in a plasma. Since the volume of a single particle is $1/n_\alpha$, and if we assume that this volume is given by $4/3\pi a_\alpha^3$, then equating these two gives the expression for the sphere radius

$$a_\alpha = \left(\frac{3}{4\pi n_\alpha}\right)^{1/3}. \tag{3.9}$$

The kinetic energy of a particle in a plasma can be approximated by the thermal energy, thus

$$\mathrm{K} = \frac{1}{2} m_\alpha v_{T_\alpha}^2 = k_b T_\alpha. \tag{3.10}$$

The ratio of the particle's Coulomb potential energy and its kinetic energy is referred to as the coupling parameter Γ_α. That is

$$\Gamma_\alpha = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha k_b T_\alpha}. \tag{3.11}$$

$\Gamma_\alpha > 1$ denotes a strongly coupled plasma, and $\Gamma_\alpha < 1$ denotes a weakly coupled plasma.

As shown in eq. (3.5), the coupling and plasma parameters are inversely proportional to each other. Thus, $\Lambda_\alpha < 1$ implies strongly-coupled plasmas, and $\Lambda_\alpha > 1$ weakly-coupled plasmas. Since Λ_α represents the number of particles per Debye sphere, it is interesting to note that a large number of particles within such a sphere is needed to be in the weakly-coupled-plasma regime. However, this does not correspond to a plasma with large density, in fact, it corresponds to the opposite. The explicit n_α term in the definition $\Lambda_\alpha = (4/3)\pi\lambda_{D\alpha}^3 n_\alpha$ is dominated by the n_α in the denominator of $\lambda_{D\alpha}$. In other words, low plasma densities lead to large Debye spheres, which in turn leads to many particles per Debye sphere, and hence a weakly-coupled plasma.

Chapter 4

Single-particle motion—Collisions

4.1 Cross section

The cross section characterizes in a quantitative form the probability that two particles traveling towards each other will undergo an interaction (also sometimes referred to as a collision). For example, imagine an incident particle traveling towards a target particle. This target particle has a spherical force field, and it affects incident particles that come within this sphere. Projecting the spherical force field to a plane perpendicular to the velocity of the incident particle gives a circular cross section. If the incident particle path takes it within this cross section, then the incident particle feels the force field of the target particle, that is, they interact. If the incident particle path does not take it within the cross section, then the particles do not interact. This is an example of a finite cross section, there can also be infinite cross sections for which particles always interact, although the farther away they are the weaker the interaction (e.g. electromagnetic force fields).

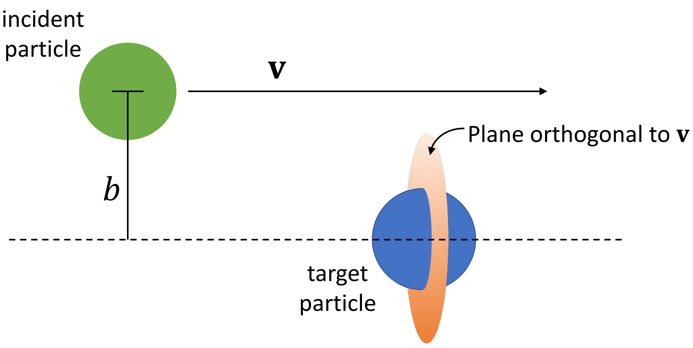

Figure 4.1: Cross section for particle interactions.

To quantify the above, the reader is referred to fig. 4.1. Imagine an incident particle traveling towards a stationary target particle with an impact parameter b and velocity $\mathbf{v}$ (if the target particle is not stationary, then $\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$, where $\mathbf{v}_1$ is the velocity of the target particle and $\mathbf{v}_2$ is the velocity of the incident particle.) As shown in fig. 4.1, the impact parameter is the perpendicular offset between the path of the incident particle, and the line parallel to the incident particle velocity that crosses the origin of the target particle (or the origin of the force field of the target particle). To determine the cross section, we ask the following question: for a particle with impact parameter b and velocity magnitude $v = |\mathbf{v}|$, does it interact with the target particle? Let's say it does not interact, then the infinitesimal surface located at b, i.e. $bdbd\phi$ for cylindrical coordinates, does not contribute to the cross section. If it does interact, then $bdbd\phi$ does contribute to the cross section. To obtain the total cross section $\sigma = \sigma(v)$, we sum over all infinitesimal areas $bdbd\phi$, but account whether a particle at a given b interacts or not with the target particle. This is expressed mathematically as

$$\sigma = \int_0^{2\pi} \int_0^{\infty} F(v, b)\, bdbd\phi. \tag{4.1}$$

In the above $F(v, b) = 1$ if an incident particle with impact parameter b and velocity v interacts with the target particle, and $F(v, b) = 0$ if it does not. However, in reality, $F(v, b)$ is not necessarily binary, and can take other values besides 0 and 1.

An example of $F(v, b)$ is that corresponding to particles that are hard spheres with radius R. For this case, $F(v, b) = H(2R - b)$, where H is the heaviside function. Thus,

$$\sigma = \int_0^{2\pi} \int_0^{\infty} H(2R - b)\, bdbd\phi = 2\pi \int_0^{2R} bdb = \pi(2R)^2, \tag{4.2}$$

as expected.

4.2 Mean free path, collision time, and collision frequency

The cross section then defines the mean free path λ_m, collision time τ_m and collision frequency ν_m. These are given by

$$\lambda_m = \frac{1}{n_1\sigma}, \tag{4.3}$$

$$\tau_m = \frac{\lambda_m}{v} = \frac{1}{n_1\sigma v}, \tag{4.4}$$

and

$$\nu_m = \frac{1}{\tau_m} = n_1\sigma v. \tag{4.5}$$

4.3 Coulomb scattering

4.3.1 Particle equations

Consider two particles, with positions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, velocities $\mathbf{v}_1 = \mathbf{v}_1(t)$ and $\mathbf{v}_2 = \mathbf{v}_2(t)$, charges q_1 and q_2, and masses m_1 and m_2, respectively. Their positions and velocities are governed by the following equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1, \tag{4.6}$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2, \tag{4.7}$$

$$m_1\frac{d\mathbf{v}_1}{dt} = -\frac{q_1q_2}{4\pi\epsilon}\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \tag{4.8}$$

$$m_2\frac{d\mathbf{v}_2}{dt} = -\frac{q_1q_2}{4\pi\epsilon}\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{4.9}$$

We note that the above system consists of twelve equations for twelve unknowns. We now introduce the center-of-mass position $\mathbf{R} = \mathbf{R}(t)$, the center-of-mass velocity $\mathbf{V} = \mathbf{V}(t)$, the shifted position $\mathbf{r} = \mathbf{r}(t)$ and the shifted velocity $\mathbf{v} = \mathbf{v}(t)$ as follows

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{4.10}$$

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \tag{4.11}$$

Thus, in terms of these new four variables, the particle equations can be written as

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}, \tag{4.12}$$

$$\frac{d\mathbf{V}}{dt} = 0, \tag{4.13}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \tag{4.14}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{\mathbf{r}}{r^3}, \tag{4.15}$$

where the reduced mass m_r is given by

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4.16}$$

The first two equations above give the trivial solution $\mathbf{V} =$ constant and $\mathbf{R} = \mathbf{R}(0) + \mathbf{V}t$. Thus, we have reduced the problem from twelve unknowns to six unknowns, namely $\mathbf{r}$ and $\mathbf{v}$.

4.3.2 Conservation of energy

Dotting eq. (4.15) by $\mathbf{v}$ gives

$$\begin{aligned}
\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\mathbf{v}\cdot\frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{d\mathbf{r}}{dt}\cdot\frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{2}\frac{dr^2}{dt}\frac{1}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}\frac{dr}{dt} \\
&= -\frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{d}{dt}\left(\frac{1}{r}\right).
\end{aligned} \tag{4.17}$$

For the left hand side above we have

$$\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{1}{2}\frac{dv^2}{dt}, \tag{4.18}$$

and thus we obtain the following expression for conservation of energy

$$\frac{d}{dt}\left(\frac{1}{2}m_r v^2 + \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r}\right) = 0. \tag{4.19}$$

4.3.3 Conservation of momentum

Crossing eq. (4.15) by $\mathbf{r}$ gives

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0, \tag{4.20}$$

and thus

$$\frac{d}{dt} \left[m_r \left(\mathbf{r} \times \mathbf{v} \right) \right] = 0. \tag{4.21}$$

That is, angular momentum is conserved. A consequence of this is that the vector $\mathbf{r} \times \mathbf{v}$ is always pointing in the same direction. Thus, if $\mathbf{r}(0)$ and $\mathbf{v}(0)$ form a plane, then $\mathbf{r}(t)$ and $\mathbf{v}(t)$ need to reside within that same plane for all times t so that $\mathbf{r}(t) \times \mathbf{v}(t)$ points in the same direction as $\mathbf{r}(0) \times \mathbf{v}(0)$. Therefore, the evolution of the position and velocity are confined to a plane and the problem can be reduced from six unknowns to four unknowns. This planar encounter is depicted in fig. 4.2.

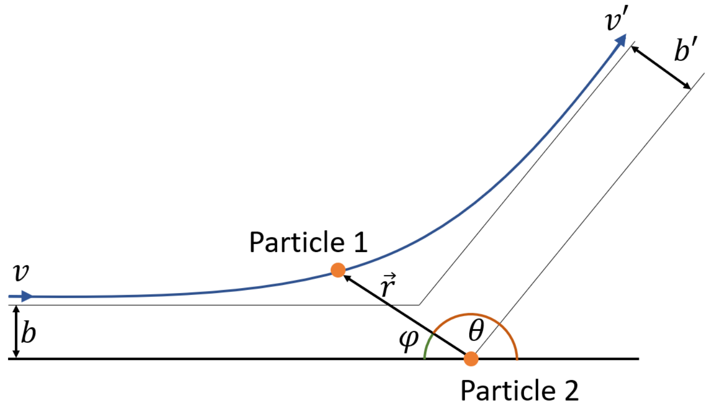

Figure 4.2: Depiction of Coulomb scattering.

4.3.4 Polar coordinates

We re-orient the plane of interaction (referred to above) so that it is orthogonal to the $\hat{\mathbf{z}}$ direction. Using polar coordinates, as shown in fig. 4.3, we get

$$r_x = r\cos\theta = r\cos(\pi - \varphi) = -r\cos\varphi, \tag{4.22}$$

$$r_y = r\sin\theta = r\sin(\pi - \varphi) = r\sin\varphi. \tag{4.23}$$

Also, since $\mathbf{r} = r\hat{\mathbf{r}}$, we have

$$\begin{aligned} \mathbf{v} =& \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}, \end{aligned} \tag{4.24}$$

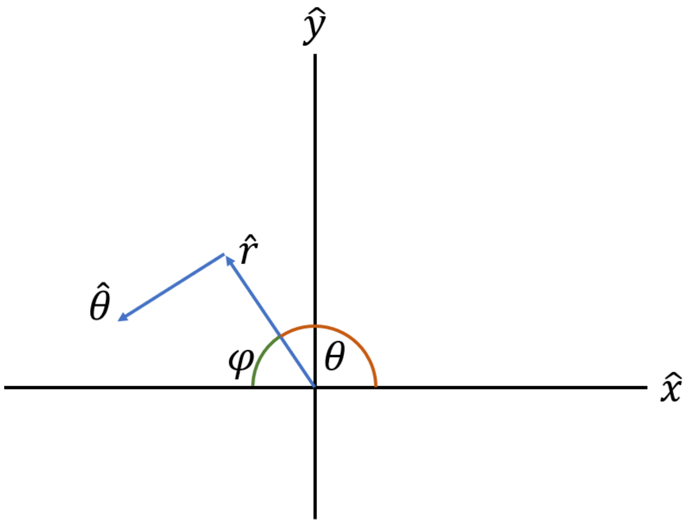

Figure 4.3: Polar coordinates in plane of interaction.

and

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}.\end{aligned} \tag{4.25}$$

4.3.5 The force equation

The radial component of eq. (4.15) thus becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}. \tag{4.26}$$

Since $\theta = \pi - \varphi$, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}. \tag{4.27}$$

4.3.6 The angular momentum equation

Using polar coordinates, we obtain

$$m_r\mathbf{r}\times\mathbf{v} = m_r r\hat{\mathbf{r}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right) = m_r r^2\frac{d\theta}{dt}\hat{\mathbf{z}} \tag{4.28}$$

Since angular momentum $m_r\mathbf{r}\times\mathbf{v}$ is conserved, we have

$$m_r r^2\frac{d\varphi}{dt} = L = constant. \tag{4.29}$$

We note here that L is positive since $d\varphi/dt$ is positive. Also, we have

$$m_r\mathbf{r}\times\mathbf{v} = -L\hat{\mathbf{z}}, \tag{4.30}$$

that is, the angular momentum is in the negative $\hat{\mathbf{z}}$ direction.

4.3.7 Particle trajectory

The goal is to find the radial position of the particle as a function of its angular orientation. That is, we want to find $\tilde{r} = \tilde{r}(\tilde{\varphi})$ such that

$$r(t) = \tilde{r}(\varphi(t)). \tag{4.31}$$

To simplify the math, we introduce $\tilde{u} = \tilde{u}(\tilde{\varphi})$ such that $\tilde{u} = 1/\tilde{r}$. Thus

$$\frac{d\tilde{u}}{d\tilde{\varphi}} = -\frac{1}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{\varphi}}, \tag{4.32}$$

or, after re-arranging

$$\frac{d\tilde{r}}{d\tilde{\varphi}} = -\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}. \tag{4.33}$$

We now proceed as follows. Taking the derivative of r, we get

$$\begin{aligned}
\frac{dr}{dt} &= \left(\frac{d\tilde{r}}{d\tilde{\varphi}}\right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} && [eq.\ (4.31)] \\
&= \left(-\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}\right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} && [eq.\ (4.33)] \\
&= \left(-\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}\right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} && [eq.\ (4.29)] \\
&= \left(-\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r\tilde{r}^2}\right)_{\tilde{\varphi}=\varphi(t)} && [eq.\ (4.31)] \\
&= \left(-\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r}\right)_{\tilde{\varphi}=\varphi(t)}
\end{aligned} \tag{4.34}$$

Taking the derivative of the above, we get

$$\begin{aligned}
\frac{d}{dt}\frac{dr}{dt} &= \left[\frac{d}{d\tilde{\varphi}}\left(-\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r}\right)\right]_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L}{m_r}\right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} && [eq.\ (4.29)] \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L}{m_r}\frac{L}{m_r\tilde{r}^2}\right)_{\tilde{\varphi}=\varphi(t)} && [eq.\ (4.31)] \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L^2\tilde{u}^2}{m_r^2}\right)_{\tilde{\varphi}=\varphi(t)}
\end{aligned} \tag{4.35}$$

Plugging the last relation into eq. (4.27) gives

$$\left[-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L^2\tilde{u}^2}{m_r^2} - \frac{1}{\tilde{u}}\left(\frac{L\tilde{u}^2}{m_r}\right)^2\right]_{\tilde{\varphi}=\varphi(t)} = \left(\frac{q_1q_2}{4\pi\epsilon_0 m_r}\tilde{u}^2\right)_{\tilde{\varphi}=\varphi(t)}, \tag{4.36}$$

which, upon re-arranging and dropping the $\varphi(t)$ dependance, becomes

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1q_2m_r}{4\pi\epsilon_0 L^2} \tag{4.37}$$

Using eq. (4.30), we have

$$\begin{aligned}
L &= -m_r(\mathbf{r} \times \mathbf{v}) \cdot \hat{\mathbf{z}} \\
&= -m_r[\sin(-\theta) r v_0 \hat{\mathbf{z}}] \cdot \hat{\mathbf{z}} \\
&= m_r \sin(\theta) r v_0 \\
&= m_r \sin(\pi - \varphi) r v_0 \\
&= m_r \sin\varphi r v_0 \\
&= m_r b v_0.
\end{aligned} \tag{4.38}$$

Thus, we write the evolution equation for $\tilde{u}$ as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2}{4\pi\epsilon_0 m_r b^2 v_0^2}. \tag{4.39}$$

Introducing the notation

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r v_0^2}, \tag{4.40}$$

the evolution equation for $\tilde{u}$ can be simply expressed as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{b_{90}}{b^2}. \tag{4.41}$$

The boundary conditions for eq. (4.41) are as follows

$$\text{as } \varphi(t) \to 0, \quad r(t) \to \infty \tag{4.42}$$

$$\text{as } \varphi(t) \to 0, \quad \frac{dr(t)}{dt} \to -v_0 \tag{4.43}$$

Given eq. (4.31), eq. (4.42) can only be satisfied if as $\tilde{\varphi} \to 0$, $\tilde{r} \to \infty$. Thus, we also have, as $\tilde{\varphi} \to 0$, $\tilde{u} \to 0$. Similarly, given eq. (4.34), eq. (4.43) can only be satisfied if as $\tilde{\varphi} \to 0$

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \to v_0. \tag{4.44}$$

Using eq. (4.38) we rewrite the above as

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \to \frac{1}{b}. \tag{4.45}$$

The general solution to eq. (4.41) is

$$\tilde{u} = A\cos\tilde{\varphi} + B\sin\tilde{\varphi} - \frac{b_{90}}{b^2}. \tag{4.46}$$

Applying the boundary conditions, we get

$$\tilde{u} = \frac{b_{90}}{b^2}\cos\tilde{\varphi} + \frac{1}{b}\sin\tilde{\varphi} - \frac{b_{90}}{b^2}, \tag{4.47}$$

which we finally re-write as

$$\frac{1}{\tilde{r}} = \frac{1}{b}\sin\tilde{\varphi} + \frac{b_{90}}{b^2}\left(\cos\tilde{\varphi} - 1\right). \tag{4.48}$$

Part II

Kinetic description

Chapter 5

Governing equations

We denote the distribution function for a species α as $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where $\mathbf{r}$ and $\mathbf{v}$ are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_\alpha \, d\mathbf{r} d\mathbf{v} = N_\alpha, \tag{5.1}$$

where N_α is the total number of particles corresponding to species α.

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution along with Maxwell's equations

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = C_\alpha + S_\alpha \tag{5.2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{5.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{5.6}$$

$$\mathbf{J} = \sum_\alpha Z_\alpha e \int \mathbf{v} f_\alpha \, d\mathbf{v} \tag{5.7}$$

$$\rho_e = \sum_\alpha Z_\alpha e \int f_\alpha \, d\mathbf{v}. \tag{5.8}$$

In the above,

- m_α is the species mass
- e is the charge
- Z_α the charge number
- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ the charge current
- $\rho_e = \rho_e(\mathbf{r}, t)$ the charge density
- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the electric field

- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ the magnetic field.

The terms C_α and S_α represent collision and source terms.

If we express the collision term in the usual way, that is $C_\alpha = \sum_\beta C_{\alpha\beta}$, then we can make the following statements:

1. Conservation of particles:

$$\int C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall\alpha \qquad\qquad \int C_{\alpha\beta}\, d\mathbf{v} = 0 \quad \forall\alpha, \beta|\beta \neq \alpha. \tag{5.9}$$

2. Conservation of momentum:

$$\int m_\alpha \mathbf{v} C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall\alpha \qquad\qquad \sum_\alpha \sum_{\beta, \beta\neq\alpha} \int m_\alpha \mathbf{v} C_{\alpha\beta} d\mathbf{v} = 0. \tag{5.10}$$

3. Conservation of energy:

$$\int \frac{1}{2} m_\alpha v^2 C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall\alpha \qquad\qquad \sum_\alpha \sum_{\beta, \beta\neq\alpha} \int \frac{1}{2} m_\alpha v^2 C_{\alpha\beta}\, d\mathbf{v} = 0. \tag{5.11}$$

5.1 Fluid equations

We now define the particle density $n_\alpha = n_\alpha(\mathbf{r}, t)$, the fluid velocity $\mathbf{u}_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$ and the fluid energy per unit mass $E_\alpha = E_\alpha(\mathbf{r}, t)$ as follows

$$n_\alpha = \int f_\alpha\, d\mathbf{v} \tag{5.12}$$

$$\mathbf{u}_\alpha = \frac{1}{n_\alpha} \int \mathbf{v} f_\alpha\, d\mathbf{v} \tag{5.13}$$

$$E_\alpha = \frac{1}{n_\alpha} \int \frac{1}{2} v^2 f_\alpha\, d\mathbf{v}. \tag{5.14}$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_\alpha}{\partial t} + \nabla \cdot (\mathbf{v} f_\alpha) + \nabla_v \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] = C_\alpha + S_\alpha \tag{5.15}$$

5.1.1 Mass

Integrating eq. (5.15) over all $\mathbf{v}$ we obtain

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha \tag{5.16}$$

where

$$\hat{S}_\alpha = \int S_\alpha\, d\mathbf{v} \tag{5.17}$$

is an external source of mass.

5.1.2 Momentum

Multiplying eq. (5.15) by $\mathbf{v}$ and then integrating over all $\mathbf{v}$ leads to

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot\left(\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v}\right) + \int \nabla_v \cdot \left[\mathbf{v}\frac{Z_\alpha e}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_\alpha\right] - \nabla_v \mathbf{v}\cdot\left[\frac{Z_\alpha e}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_\alpha\right] d\mathbf{v} = \sum_{\beta,\beta\neq\alpha}\int \mathbf{v} C_{\alpha\beta}\, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.18)$$

We note that the third term in eq. (5.18) is zero since we are integrating over all space, and that $\nabla_v \mathbf{v}$ is the identity matrix. We thus have

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot\left(\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v}\right) - \frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B}) = \sum_{\beta,\beta\neq\alpha}\int \mathbf{v} C_{\alpha\beta}\, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.19)$$

To proceed, we decompose $\mathbf{v}$ into a mean and a fluctuation, that is, $\mathbf{v} = \mathbf{u}_\alpha + \mathbf{w}_\alpha$. Using this decomposition

$$\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v} = \int (\mathbf{u}_\alpha\mathbf{u}_\alpha + 2\mathbf{u}_\alpha\mathbf{w}_\alpha + \mathbf{w}_\alpha\mathbf{w}_\alpha) f_\alpha \, d\mathbf{v} = n_\alpha \mathbf{u}_\alpha\mathbf{u}_\alpha + \int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}. \quad (5.20)$$

Thus, eq. (5.19) becomes

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot(n_\alpha \mathbf{u}_\alpha\mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B}) = -\nabla\cdot\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta,\beta\neq\alpha}\int \mathbf{v} C_{\alpha\beta}\, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.21)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot(n_\alpha \mathbf{u}_\alpha\mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B}) = -\nabla\cdot\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta,\beta\neq\alpha}\int \mathbf{w}_\alpha C_{\alpha\beta}\, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.22)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha\mathbf{u}_\alpha) - Z_\alpha e n_\alpha(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B}) = \nabla\cdot\boldsymbol{\sigma}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (5.23)$$

where the stress tensor is

$$\boldsymbol{\sigma}_\alpha = -\int m_\alpha \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}, \quad (5.24)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_\alpha = \sum_{\beta,\beta\neq\alpha}\int m_\alpha \mathbf{w}_\alpha C_{\alpha\beta}\, d\mathbf{v}, \quad (5.25)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_\alpha = \int m_\alpha \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.26)$$

The stress tensor is typically decomposed into isotropic p_α and anisotropic (shear) $\mathbf{t}_\alpha$ tensors as follows

$$\boldsymbol{\sigma}_\alpha = -p_\alpha \mathbf{I} + \mathbf{t}_\alpha, \tag{5.27}$$

where P_α is given by

$$p_\alpha = \frac{1}{3}\int m_\alpha(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v}. \tag{5.28}$$

Thus, conservation of momentum becomes

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha. \tag{5.29}$$

5.1.3 Energy

Multiplying eq. (5.15) by $\frac{1}{2}v^2$ and then integrating over all $\mathbf{v}$ leads to

$$\begin{aligned}\frac{\partial n_\alpha E_\alpha}{\partial t} &+ \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v}\right] + \int \nabla_v \cdot \left[\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_\alpha\right] \\ &- \nabla_v \left[\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\right] \cdot \left[\frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_\alpha\right] d\mathbf{v} = \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta}\, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}.\end{aligned} \tag{5.30}$$

We note that the third term above is zero since we are integrating over all space, and that $\nabla_v[1/2(\mathbf{v}\cdot\mathbf{v})] = \mathbf{v}$. Thus, we have

$$\begin{aligned}\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v}\right] - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = \\ \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta}\, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}.\end{aligned} \tag{5.31}$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v} \;=\; \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}_\alpha + \mathbf{w}_\alpha) f_\alpha \, d\mathbf{v} \;=\; n_\alpha E_\alpha \mathbf{u}_\alpha + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \tag{5.32}$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha + 2\mathbf{u}_\alpha\cdot\mathbf{w}_\alpha + \mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \tag{5.33}$$

$$= \mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}. \tag{5.34}$$

Using the expressions above, eq. (5.31) becomes

$$\begin{aligned}\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla\cdot(n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = -\nabla\cdot\left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}\right) - \nabla\cdot \int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \\ + \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta}\, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}.\end{aligned} \tag{5.35}$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla\cdot(n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = -\nabla\cdot\left(\mathbf{u}_\alpha\cdot\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha\, d\mathbf{v}\right) - \nabla\cdot\int\frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha\, d\mathbf{v}$$
$$+\mathbf{u}_\alpha\cdot\sum_{\beta,\beta\neq\alpha}\int \mathbf{w}_\alpha C_{\alpha\beta}\, d\mathbf{v} + \sum_{\beta,\beta\neq\alpha}\int\frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)C_{\alpha\beta}\, d\mathbf{v} + \int\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})S_\alpha\, d\mathbf{v}. \quad (5.36)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha E_\alpha\mathbf{u}_\alpha) - Z_\alpha e n_\alpha\mathbf{E}\cdot\mathbf{u}_\alpha = \nabla\cdot(\mathbf{u}_\alpha\cdot\boldsymbol{\sigma}_\alpha) - \nabla\cdot\mathbf{q}_\alpha$$
$$+\mathbf{u}_\alpha\cdot\mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \quad (5.37)$$

where heat flux due to random motion is

$$\mathbf{q}_\alpha = \int\frac{1}{2}m_\alpha(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha\, d\mathbf{v}, \quad (5.38)$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_\alpha = \sum_{\beta,\beta\neq\alpha}\int\frac{1}{2}m_\alpha(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)C_{\alpha\beta}\, d\mathbf{v}, \quad (5.39)$$

and the external source of energy is

$$\hat{Q}_\alpha = \int\frac{1}{2}m_\alpha(\mathbf{v}\cdot\mathbf{v})S_\alpha\, d\mathbf{v}. \quad (5.40)$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha E_\alpha\mathbf{u}_\alpha + p_\alpha\mathbf{u}_\alpha) - Z_\alpha e n_\alpha\mathbf{E}\cdot\mathbf{u}_\alpha = \nabla\cdot(\mathbf{u}_\alpha\cdot\mathbf{t}_\alpha) - \nabla\cdot\mathbf{q}_\alpha$$
$$+\mathbf{u}_\alpha\cdot\mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \quad (5.41)$$

We also note that the energy $m_\alpha n_\alpha E_\alpha$ can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (5.20) one obtains

$$\begin{aligned} m_\alpha n_\alpha E_\alpha &= \int\frac{1}{2}m_\alpha(\mathbf{v}\cdot\mathbf{v})f_\alpha\, d\mathbf{v} \\ &= \int\frac{1}{2}m_\alpha(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)f_\alpha\, d\mathbf{v} + \frac{1}{2}m_\alpha n_\alpha(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha) \\ &= \frac{3}{2}P_\alpha + \frac{1}{2}m_\alpha n_\alpha(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha) \\ &= \frac{3}{2}P_\alpha + m_\alpha n_\alpha K_\alpha. \end{aligned} \quad (5.42)$$

where $K_\alpha = \frac{1}{2}\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha$ is the kinetic energy of species α.

5.1.4 Kinetic and Internal Energies

The equation for the kinetic energy is obtained by dotting eq. (5.29) with $\mathbf{u}_\alpha$. For this, we first show that

$$\mathbf{u}_\alpha \cdot \left[\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) \right] \tag{5.43}$$

$$= \mathbf{u}_\alpha \cdot \left\{ \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right\} \tag{5.44}$$

$$= \mathbf{u}_\alpha \cdot \left[m_\alpha \hat{S}_\alpha \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right] \tag{5.45}$$

$$= 2 m_\alpha \hat{S}_\alpha K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \tag{5.46}$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \tag{5.47}$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha). \tag{5.48}$$

Thus, the equation for the turbulent kinetic energy is

$$\frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha =$$
$$- \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \tag{5.49}$$

Subtracting the above equation from eq. (5.41) leads to

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_\alpha \right) + \nabla \cdot \left(\frac{3}{2} p_\alpha \mathbf{u}_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \tag{5.50}$$

5.1.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha, \tag{5.51}$$

- Momentum

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \tag{5.52}$$

- Total Energy

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha$$
$$+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \tag{5.53}$$

- Kinetic Energy

$$\frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha =$$
$$- \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \tag{5.54}$$

- Internal Energy

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha\right)+\nabla\cdot\left(\frac{3}{2}p_\alpha\mathbf{u}_\alpha\right) = -p_\alpha\nabla\cdot\mathbf{u}_\alpha+\mathbf{t}_\alpha : \nabla\mathbf{u}_\alpha-\nabla\cdot\mathbf{q}_\alpha+Q_\alpha+\hat{Q}_\alpha-\mathbf{u}_\alpha\cdot\hat{\mathbf{M}}_\alpha+m_\alpha K_\alpha\hat{S}_\alpha. \tag{5.55}$$

Chapter 6

Transport coefficients

Collison integral

$$\Omega_{\alpha\beta}^{(lk)} = \sqrt{\frac{k_B T}{2\pi M_{\alpha\beta}}} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} \, dg. \tag{6.1}$$

In the above $M_{\alpha\beta}$ is the reduced mass, given by

$$M_{\alpha\beta} = \frac{M_\alpha M_\beta}{M_\alpha + M_\beta}, \tag{6.2}$$

and $\phi_{\alpha\beta}^{(l)}$ is the collision cross section for a given velocity, and is computed as

$$\phi_{\alpha\beta}^{(l)} = 2\pi \int_0^\infty \left(1 - \cos^l \chi_{\alpha\beta}\right) b \, db. \tag{6.3}$$

The scattering angle $\chi_{\alpha\beta}$ is given by

$$\chi_{\alpha\beta} = \pi - 2 \int_{r_{\alpha\beta}^{\min}}^\infty \frac{b}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{\alpha\beta(r)}}{g^2 k_B T}\right]^{1/2}} \, dr. \tag{6.4}$$

For a Coulombic interaction between ions, we can define the natural scale fore the cross-sectional area as

$$\phi_{\alpha\beta}^{(0)} = \frac{\pi \left(Z_\alpha Z_\beta e^2\right)^2}{\left(2k_B T\right)^2}. \tag{6.5}$$

Given this definition, we express the collision integral as

$$\Omega_{\alpha\beta} = \sqrt{\frac{\pi}{M_{\alpha\beta}}} \frac{(Z_\alpha Z_\beta e^2)^2}{(2k_B T)^{3/2}} \mathcal{F}_{\alpha\beta}^{lk}, \tag{6.6}$$

where

$$\mathcal{F}_{\alpha\beta}^{(lk)} = \frac{1}{2\phi_0} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} \, dg \tag{6.7}$$

We note that $\mathcal{F}_{\alpha\beta}^{(lk)} = 4\mathcal{K}_{lk}(g_{\alpha\beta})$, where $\mathcal{K}_{lk}(g_{\alpha\beta})$ is the notation from the Stanton-Murillo paper.

Part III

Fluid description

Chapter 7

Magnetohydrodynamics

7.1 Two-fluid equations

The starting point are the multi-fluid conservation laws eqs. (5.51), (5.52) and (5.55) and the Maxwell equations eqs. (5.3) to (5.6). We assume there are two species: electrons and ions. Additionally, we assume no sources. Thus, the starting governing equations are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{7.1}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{7.2}$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i + \nabla \cdot \mathbf{t}_i + \mathbf{R}_i, \tag{7.3}$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + en_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e, \tag{7.4}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_i\right) + \nabla \cdot \left(\frac{3}{2}p_i \mathbf{u}_i\right) = -p_i \nabla \cdot \mathbf{u}_i + \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \tag{7.5}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_e\right) + \nabla \cdot \left(\frac{3}{2}p_e \mathbf{u}_e\right) = -p_e \nabla \cdot \mathbf{u}_e + \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \tag{7.6}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{7.7}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.9}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7.10}$$

$$\mathbf{J} = e(Zn_i \mathbf{u}_i - n_e \mathbf{u}_e) \tag{7.11}$$

$$\rho_e = e(Zn_i - n_e) \tag{7.12}$$

These equations correspond to eq. (2.22) in Freidberg's Ideal MHD book, but for ions that are not singly charged. For the sections below, however, we'll assume singly-charged ions.

7.2 Low-frequency, long-wavelength, asymptotic expansions

Two assumptions:

1. Transform full Maxwell's equations to low-frequency pre-Maxwell's equations. Formally achieved with $\epsilon_0 \to 0$. This has two consequences:

   - $\epsilon_0 \partial \mathbf{E}/\partial t \to 0$
     For this to be achieved it is required that $w/k \ll c$ and $V_{Ti}, V_{Te} \ll c$.
   - $\epsilon_0 \nabla \cdot \mathbf{E} \to 0$
     For this to be achieved it is required that $w \ll w_{pe}$ and $a \gg \lambda_D$.

2. Neglect electron inertia in the electron momentum equations. Formally achieved with $m_e \to 0$.

Due to the first assumption, the Maxwell equations eqs. (7.7) to (7.12) are now written as

$$n_i - n_e = 0 \tag{7.13}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.14}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.15}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{7.16}$$

7.3 Single-fluid equations

We define single-fluid variables as

$$\rho = m_i n_i + m_e n_e = m_i n \tag{7.17}$$

$$\mathbf{v} = \frac{m_i n_i \mathbf{u}_i + m_e n_e \mathbf{u}_e}{m_i n_i + m_e n_e} = \mathbf{u}_i \tag{7.18}$$

$$p = p_i + p_e = n(T_i + T_e) \tag{7.19}$$

$$T = \frac{T_i + T_e}{2}. \tag{7.20}$$

The two conservation of mass equations eqs. (7.1) and (7.2) will lead to two single-fluid equations. The first is obtained by multiplying eq. (7.1) by m_i, and the second is obtained by multiplying the ion and electron mass equations by e and then subtracting. The results are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7.21}$$

$$\nabla \cdot \mathbf{J} = 0. \tag{7.22}$$

Note that the second equation above is superfluous since it also follows from taking the divergence of eq. (7.16)

The two conservation of momentum equations will also lead to two single-fluid equations. The first is obtained by adding the ion and electron conservation of momentum equations to obtain

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u} \right) - \mathbf{J} \times \mathbf{B} + \nabla p = \nabla \cdot \mathbf{t}_i + \nabla \cdot \mathbf{t}_e. \tag{7.23}$$

For the second equation, we use $m_e \to 0$ and quasineutrality in the electron momentum equation to obtain

$$en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e, \tag{7.24}$$

Assuming quasi-neutrality, the definition of the current given in eq. (7.11) is now

$$\mathbf{J} = en(\mathbf{u}_i - \mathbf{u}_e), \tag{7.25}$$

which is also written as

$$\mathbf{J} = en(\mathbf{v} - \mathbf{u}_e). \tag{7.26}$$

Using the above in the electron continuity equation gives

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en}(\mathbf{J} \times \mathbf{B} - \nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e). \tag{7.27}$$

The two conservation of energy equations will also lead to two single-fluid equations. Each is evaluated using the single-fluid variables. As part of this derivation, we first rewrite the ion and electron internal energy eqs. (7.5) and (7.6) as

$$\frac{1}{\gamma - 1}\left(\frac{\partial p_i}{\partial t} + \mathbf{u}_i \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{u}_i\right) = \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \tag{7.28}$$

$$\frac{1}{\gamma - 1}\left(\frac{\partial p_e}{\partial t} + \mathbf{u}_e \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u}_e\right) = \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \tag{7.29}$$

where we have used $\gamma = 5/3$ (the ratio of specific heats for monoatomic systems). We then note that

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t} - \frac{1}{\rho}\nabla\rho \cdot \mathbf{v} = -\frac{\partial \ln \rho}{\partial t} - \nabla \ln \rho \cdot \mathbf{v}, \tag{7.30}$$

and thus

$$\gamma \nabla \cdot \mathbf{v} = -\frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial t} - \frac{1}{\rho^\gamma}\nabla \rho^\gamma \cdot \mathbf{v}. \tag{7.31}$$

The result above allows us to write

$$\begin{aligned}\frac{\partial p_\alpha}{\partial t} + \mathbf{v} \cdot \nabla p_\alpha + \gamma p_\alpha \nabla \cdot \mathbf{v} &= \frac{\partial p_\alpha}{\partial t} - p_\alpha \frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial t} + \mathbf{v} \cdot \nabla p_\alpha - p_\alpha \frac{1}{\rho^\gamma}\nabla \rho^\gamma \cdot \mathbf{v} \\ &= \rho^\gamma \left[\frac{\partial}{\partial t}\left(\frac{p_\alpha}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p_\alpha}{\rho^\gamma}\right)\right].\end{aligned} \tag{7.32}$$

Thus, the ion energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{p_i}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p_i}{\rho^\gamma}\right) = \frac{\gamma - 1}{\rho^\gamma}\left(\mathbf{t}_i : \nabla \mathbf{v} - \nabla \cdot \mathbf{q}_i + Q_i\right), \tag{7.33}$$

and the electron energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{p_e}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p_e}{\rho^\gamma}\right) = \frac{\gamma - 1}{\rho^\gamma}\left[\mathbf{t}_e : \nabla\left(\mathbf{v} - \frac{\mathbf{J}}{en}\right) - \nabla \cdot \mathbf{q}_e + Q_e\right] + \frac{1}{en}\mathbf{J} \cdot \nabla \left(\frac{p_e}{\rho^\gamma}\right). \tag{7.34}$$

7.4 Resisitive MHD

The electron collision term is modeled as

$$\mathbf{R}_e = m_e n_e \nu_{ei} \left(\mathbf{u}_i - \mathbf{u}_e\right). \tag{7.35}$$

Assuming quasi-neutrality, the expression for current in eq. (7.25) can be used to obtain

$$\mathbf{R}_e = \frac{m_e \nu_{ei}}{e} \mathbf{J}. \tag{7.36}$$

Neglecting all terms on the right-hand side of eq. (7.27) except for the electron collision term, we have

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en} \mathbf{R}_e. \tag{7.37}$$

Using eq. (7.36) in the above, we have

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{m_e \nu_{ei}}{e^2 n} \mathbf{J}, \tag{7.38}$$

which we re-write as

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}, \tag{7.39}$$

where

$$\eta = \frac{m_e \nu_{ei}}{e^2 n} \tag{7.40}$$

is the resistivity.

7.5 Ideal MHD

The ideal MHD equations are obtained by neglecting the right-hand sides of eqs. (7.23), (7.27), (7.33) and (7.34). Summing the two pressure equations, the resulting equations would be

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7.41}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{7.42}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \tag{7.43}$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = 0, \tag{7.44}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.45}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7.46}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{7.47}$$

Given the vector identity

$$\frac{1}{2} \nabla \left(B^2\right) = \mathbf{B} \times (\nabla \times \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{B}, \tag{7.48}$$

we can use Ampere's law to re-write the $\mathbf{J} \times \mathbf{B}$ term in the velocity equation as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} \left[(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla \left(B^2\right) \right]. \tag{7.49}$$

Similarly, given the vector identity

$$\nabla \times (\mathbf{B} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\, \mathbf{B} - (\mathbf{B} \cdot \nabla)\, \mathbf{v} + \mathbf{B}\, (\nabla \cdot \mathbf{v}) - \mathbf{v}\, (\nabla \cdot \mathbf{B})\,, \tag{7.50}$$

we can use Ohm's law to re-write the $\nabla \times \mathbf{E}$ term in Faraday's law as

$$\nabla \times \mathbf{E} = \nabla \times (-\mathbf{v} \times \mathbf{B}) = (\mathbf{v} \cdot \nabla)\, \mathbf{B} - (\mathbf{B} \cdot \nabla)\, \mathbf{v} + \mathbf{B}\, (\nabla \cdot \mathbf{v})\,. \tag{7.51}$$

Thus, the ideal MHD equations can be summarized as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7.52}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.53}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \frac{1}{\mu_0} \left[(\mathbf{B} \cdot \nabla)\, \mathbf{B} - \frac{1}{2} \nabla \left(B^2 \right) \right] \tag{7.54}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{B} = (\mathbf{B} \cdot \nabla)\, \mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) \tag{7.55}$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = 0, \tag{7.56}$$

If we assume incompressibility, then the above simplifies to

$$\nabla \cdot \mathbf{v} = 0, \tag{7.57}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.58}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \frac{1}{\mu_0} \left[(\mathbf{B} \cdot \nabla)\, \mathbf{B} - \frac{1}{2} \nabla \left(B^2 \right) \right] \tag{7.59}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{B} = (\mathbf{B} \cdot \nabla)\, \mathbf{v} \tag{7.60}$$

Chapter 8

Laser-plasma interactions

8.1 Governing equations

We'll start with the multi-fluid conservation laws described in section 7.1. We'll assume no shear stresses, collisions, and homentropic flows. Thus, the governing equations are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{8.1}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{8.2}$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i, \tag{8.3}$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + en_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e. \tag{8.4}$$

$$p_i = C_i n_i^{\gamma_i}, \tag{8.5}$$

$$p_e = C_e n_e^{\gamma_e}, \tag{8.6}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{8.7}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{8.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8.9}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{8.10}$$

$$\mathbf{J} = e(Zn_i \mathbf{u}_i - n_e \mathbf{u}_e) \tag{8.11}$$

$$\rho_e = e(Zn_i - n_e) \tag{8.12}$$

8.2 Electron-plasma and ion-acoustic waves

8.2.1 Linearization

The following decompositions will be used in the derivation of electron-plasma and ion-acoustic waves:

$$
\begin{aligned}
n_i &= n_{i0} + n_{i1}, \\
n_e &= n_{e0} + n_{e1}, \\
p_i &= p_{i0} + p_{i1}, \\
p_e &= p_{e0} + p_{e1}, \\
\mathbf{u}_i &= \mathbf{u}_{i0} + \mathbf{u}_{i1}, \\
\mathbf{u}_e &= \mathbf{u}_{e0} + \mathbf{u}_{e1}, \\
\mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_1, \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1.
\end{aligned} \tag{8.13}
$$

For these decompositions, we'll assume

1. Terms with a subscript 1 are small and thus products of two small quantities can be neglected.
2. $\mathbf{u}_{i0}$, $\mathbf{u}_{e0}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ are zero.
3. n_{i0}, n_{e0}, p_{i0}, and p_{e0} are uniform in space and time.

Using the variable decompositions in the electron density equation, we have

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \nabla \cdot \left[(n_{e0} + n_{e1}) (\mathbf{u}_{e0} + \mathbf{u}_{e1}) \right] = 0.$$

Using assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot (n_{e0} \mathbf{u}_{e1}) = 0. \tag{8.14}$$

Using the variable decompositions in the ion density equation, we have

$$\frac{\partial n_{i0} + n_{i1}}{\partial t} + \nabla \cdot \left[(n_{i0} + n_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1}) \right] = 0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i1}}{\partial t} + \nabla \cdot (n_{i0} \mathbf{u}_{i1}) = 0. \tag{8.15}$$

Using the variable decompositions in the electron momentum equation, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t} \left[m_e (n_{e0} + n_{e1}) (\mathbf{u}_{e0} + \mathbf{u}_{e1}) \right] + \nabla \cdot \left[m_e (n_{e0} + n_{e1}) (\mathbf{u}_{e0} + \mathbf{u}_{e1}) (\mathbf{u}_{e0} + \mathbf{u}_{e1}) \right] \\
&\qquad + e (n_{e0} + n_{e1}) \left[(\mathbf{E}_0 + \mathbf{E}_1) + (\mathbf{u}_{e0} + \mathbf{u}_{e1}) \times (\mathbf{B}_0 + \mathbf{B}_1) \right] = -\nabla (p_{e0} + p_{e1}).
\end{aligned}
$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e0} \mathbf{u}_{e1}}{\partial t} + \frac{e n_{e0}}{m_e} \mathbf{E}_1 = -\frac{1}{m_e} \nabla p_{e1}. \tag{8.16}$$

Using the variable decompositions in the ion momentum equation, we have

$$\frac{\partial}{\partial t}\left[m_i\left(n_{i0}+n_{i1}\right)\left(\mathbf{u}_{i0}+\mathbf{u}_{i1}\right)\right]+\nabla\cdot\left[m_i\left(n_{i0}+n_{i1}\right)\left(\mathbf{u}_{i0}+\mathbf{u}_{i1}\right)\left(\mathbf{u}_{i0}+\mathbf{u}_{i1}\right)\right]$$
$$-Ze\left(n_{i0}+n_{i1}\right)\left[\left(\mathbf{E}_0+\mathbf{E}_1\right)+\left(\mathbf{u}_{i0}+\mathbf{u}_{i1}\right)\times\left(\mathbf{B}_0+\mathbf{B}_1\right)\right]=-\nabla\left(p_{i0}+p_{i1}\right).$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i0}\mathbf{u}_{i1}}{\partial t}-\frac{Zen_{i0}}{m_i}\mathbf{E}_1=-\frac{1}{m_i}\nabla p_{i1}. \tag{8.17}$$

We'll often need to take the gradient of the ion and electron pressure. We begin by showing

$$\nabla p_\alpha=C_\alpha\gamma_\alpha n_\alpha^{\gamma_\alpha-1}\nabla n_\alpha=C_\alpha\gamma_\alpha\frac{n_\alpha^{\gamma_\alpha}}{n_\alpha}\nabla n_\alpha=\gamma_\alpha\frac{p_\alpha}{n_\alpha}\nabla n_\alpha,$$

or

$$n_\alpha\nabla p_\alpha=\gamma_\alpha p_\alpha\nabla n_\alpha,$$

where $\alpha=i,e$. Using the variable decompositions, we have

$$\left(n_{\alpha0}+n_{\alpha1}\right)\nabla\left(p_{\alpha0}+p_{\alpha1}\right)=\gamma_\alpha\left(p_{\alpha0}+p_{\alpha1}\right)\nabla\left(n_{\alpha0}+n_{\alpha1}\right).$$

Given the assumptions in items 1 and 3, the above simplifies to

$$n_{\alpha0}\nabla p_{\alpha1}=\gamma_\alpha p_{\alpha0}\nabla n_{\alpha1}. \tag{8.18}$$

Thus, for electrons we have

$$n_{e0}\nabla p_{e1}=\gamma_e p_{e0}\nabla n_{e1}, \tag{8.19}$$

and for ions

$$n_{i0}\nabla p_{i1}=\gamma_i p_{i0}\nabla n_{i1}. \tag{8.20}$$

An alternate derivation to obtain the previous results is to use an equation of state for the large pressure and density of the following form

$$p_{\alpha0}=n_{\alpha0}k_BT_\alpha, \tag{8.21}$$

where $\alpha=i,e$. Note that the temperature is one whole quantity, that is, it is not split into large and small terms. Let's assume that the large quantities also satisfy isentropic flow, that is $p_{\alpha0}=Cn_{\alpha0}^{\gamma_\alpha}$. Thus, we can show that

$$\begin{aligned}
p_{\alpha1}&=p_\alpha-p_{\alpha0}\\
&=Cn_\alpha^{\gamma_\alpha}-p_{\alpha0}\\
&=C(n_{\alpha0}+n_{\alpha1})^{\gamma_\alpha}-p_{\alpha0}\\
&=Cn_{\alpha0}^{\gamma_\alpha}\left(1+\frac{n_{\alpha1}}{n_{\alpha0}}\right)^{\gamma_\alpha}-p_{\alpha0}\\
&=p_{\alpha0}\left(1+\frac{n_{\alpha1}}{n_{\alpha0}}\right)^{\gamma_\alpha}-p_{\alpha0}.
\end{aligned}$$

Since $n_{\alpha1}/n_{\alpha0}$ is small, we can use the binomial series to proceed as follows

$$\begin{aligned}
p_{\alpha1}&=p_{\alpha0}\left(1+\gamma_\alpha\frac{n_{\alpha1}}{n_{\alpha0}}\right)-p_{\alpha0}\\
&=\gamma_\alpha\frac{p_{\alpha0}}{n_{\alpha0}}n_{\alpha1}.
\end{aligned}$$

With $p_{\alpha0}$ and $n_{\alpha0}$ constant, this then gives eq. (8.18). Finally, using eq. (8.21) in the above we get

$$p_{\alpha1}=\gamma_\alpha k_BT_\alpha n_{\alpha1}, \tag{8.22}$$

which can be interpreted as the equation of state for the small quantities.

8.2.2 Electron Plasma Waves

On top of the assumptions in section 8.2.1, we'll assume

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.

2. Uniform ion density, $n_{i1} = 0$.

Combining Equation (8.16) with eq. (8.19) gives

$$\frac{\partial n_{e0}\mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e}\mathbf{E}_1 = -\frac{\gamma_e p_{e0}}{n_{e0}m_e}\nabla n_{e1}. \tag{8.23}$$

Taking the time derivative of eq. (8.14) and using eq. (8.23) leads to the the wave equation for electron density

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{en_{e0}}{m_e}\nabla\cdot\mathbf{E}_1 = \frac{\gamma_e p_{e0}}{n_{e0}m_e}\nabla^2 n_{e1}. \tag{8.24}$$

For electron plasma waves, we'll assume that n_i varies in space and time so slowly that it can be assumed to be constant. That is, we assume $n_{i1} = 0$. Thus, Gauss's law now takes the form

$$\nabla\cdot\mathbf{E}_1 = \frac{e}{\epsilon_0}\left(Zn_{i0} - n_{e0} - n_{e1}\right).$$

Using the quasi-neutrality assumption ($Zn_{i0} = n_{e0}$)

$$\nabla\cdot\mathbf{E}_1 = -\frac{e}{\epsilon_0}n_{e1}. \tag{8.25}$$

Plugging the above in the electron wave equation we obtain

$$\frac{\partial^2 n_{e1}}{\partial t^2} + \frac{e^2 n_{e0}}{m_e\epsilon_0}n_{e1} = \frac{\gamma_e p_{e0}}{n_{e0}m_e}\nabla^2 n_{e1},$$

or

$$\frac{\partial^2 n_{e1}}{\partial t^2} + w_{pe}^2 n_{e1} - \frac{\gamma_e p_{e0}}{n_{e0}m_e}\nabla^2 n_{e1} = 0. \tag{8.26}$$

Assuming a mode of the form $n_{e1} = \hat{n}_{e1}\exp[i\left(\mathbf{k}_e\cdot\mathbf{x} - wt\right)]$, where $\mathbf{k}_e$ is the wave vector of the electron-plasma wave, gives the following dispersion relation

$$w^2 - w_{pe}^2 - \frac{\gamma_e p_{e0}}{n_{e0}m_e}k_e^2 = 0. \tag{8.27}$$

If we define the thermal velocity without the factor of two, that is, $v_{T\alpha} = \sqrt{k_B T_\alpha/m_\alpha}$, then we have

$$w^2 - w_{pe}^2 - \gamma_e k_e^2 v_{T_{e0}}^2 = 0. \tag{8.28}$$

8.2.3 Ion Acoustic Waves

On top of the assumptions in section 8.2.1, we'll assume

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.

2. Approximate quasi-neutrality for the fluctuations, $Zn_{i1} \approx n_{e1}$.

3. Negligible electron mass, $m_e \to 0$.

Combining eq. (8.17) with eq. (8.20) gives

$$\frac{\partial n_{i0}\mathbf{u}_{i1}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_1 = -\frac{\gamma_i p_{i0}}{n_{i0}m_i}\nabla n_{i1}. \tag{8.29}$$

Taking the time derivative of eq. (8.15) and using eq. (8.29) leads to the the wave equation for ion density

$$\frac{\partial^2 n_{i1}}{\partial t^2} + \frac{Zen_{i0}}{m_i}\nabla\cdot\mathbf{E}_1 = \frac{\gamma_i p_{i0}}{n_{i0}m_i}\nabla^2 n_{i1}. \tag{8.30}$$

For this case, we assume that the mass of the electron, which is significantly smaller than that of the ions, is negligible. Thus, eq. (8.16) simplifies to

$$en_{e0}\mathbf{E}_1 = -\frac{\gamma_e p_{e0}}{n_{e0}}\nabla n_{e1}. \tag{8.31}$$

Plugging in the above in the ion wave equation we obtain

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{Zn_{i0}}{n_{e0}}\frac{\gamma_e p_{e0}}{n_{e0}m_i}\nabla^2 n_{e1} + \frac{\gamma_i p_{i0}}{n_{i0}m_i}\nabla^2 n_{i1}.$$

Due to quasi-neutrality, we have $Zn_{i0} = n_{e0}$ and $Zn_{i1} \approx n_{e1}$, which gives

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{1}{m_i}\left(\frac{Z\gamma_e p_{e0}}{n_{e0}} + \frac{\gamma_i p_{i0}}{n_{i0}}\right)\nabla^2 n_{i1}.$$

Since $p_{i0}/n_{i0} = k_B T_i$ and $p_{e0}/n_{e0} = k_B T_e$, we finally have

$$\frac{\partial^2 n_{i1}}{\partial t^2} - \left(\frac{Z\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}\right)\nabla^2 n_{i1} = 0. \tag{8.32}$$

Assuming a mode of the form $n_{i1} = \hat{n}_{i1}\exp[i(\mathbf{k}_i\cdot\mathbf{x} - wt)]$, where $\mathbf{k}_i$ is the wave vector of the ion-acoustic wave, we obtain the following dispersion relation

$$w^2 - k_i^2 v_s^2 = 0, \tag{8.33}$$

where

$$v_s = \sqrt{\frac{Z\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}}. \tag{8.34}$$

8.3 Longitudinal and transverse waves

We begin with the Helmholtz decomposition

$$\mathbf{F} = \mathbf{F}_l + \mathbf{F}_t, \tag{8.35}$$

where $\mathbf{F}_l$ is the longitudinal component and $\mathbf{F}_t$ the transverse component. These are defined by

$$\nabla\times\mathbf{F}_l = 0, \tag{8.36}$$
$$\nabla\cdot\mathbf{F}_t = 0. \tag{8.37}$$

Assume the vector functions under consideration are of the following form

$$\mathbf{F}(\mathbf{x},t) = \hat{\mathbf{F}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - wt\right)\right], \tag{8.38}$$

$$\mathbf{F}_l(\mathbf{x},t) = \hat{\mathbf{F}}_l \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - wt\right)\right], \tag{8.39}$$

$$\mathbf{F}_t(\mathbf{x},t) = \hat{\mathbf{F}}_t \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - wt\right)\right]. \tag{8.40}$$

These are the so-called plane waves. Note that $\hat{\mathbf{F}}$, $\hat{\mathbf{F}}_l$, and $\hat{\mathbf{F}}_t$ are complex vectors also, where the real and complex components point in the same direction. Also note that different waves will wave different wave vectors and frequencies. For example, we'll use $\mathbf{k}_e$ for electron-plasma waves, $\mathbf{k}_i$ for ion-acoustic waves, $\mathbf{k}_L$ for laser waves and $\mathbf{k}_s$ for scattered waves. The same applies for the frequencies w_e, w_i, w_L, w_s.

Equations (8.36) and (8.37) now translate to

$$\mathbf{k}\times\hat{\mathbf{F}}_l = 0 \tag{8.41}$$

$$\mathbf{k}\cdot\hat{\mathbf{F}}_t = 0 \tag{8.42}$$

The first says $\hat{\mathbf{F}}_l$ is parallel to $\mathbf{k}$ and the second says $\hat{\mathbf{F}}_t$ is orthogonal to $\mathbf{k}$. Thus, $\hat{\mathbf{F}}_l\cdot\hat{\mathbf{F}}_t = 0$. We will often have situations where $\nabla\times\mathbf{F} = \nabla\cdot\mathbf{F} = 0$, which by its own does not imply $\mathbf{F} = 0$. However, due to the wave form of $\mathbf{F}$ (eq. (8.38)), the previous expressions translate to $\mathbf{k}\times\hat{\mathbf{F}} = \mathbf{k}\cdot\hat{\mathbf{F}} = 0$. The latter equality states that $\mathbf{k}$ and $\hat{\mathbf{F}}$ are orthogonal, that is, the angle between them is 90°. The former equality leads to $|\hat{\mathbf{F}}|\sin(90°) = 0$, which in turn means $\hat{\mathbf{F}} = 0$. As a result, the entire vector $\mathbf{F}$ is equal to zero. In other words,

$$\nabla\times\mathbf{F} = \nabla\cdot\mathbf{F} = 0 \rightarrow \mathbf{F} = 0. \tag{8.43}$$

8.3.1 Electron-plasma and ion-acoustic waves

For both electron-plasma and ion-acoustic waves we can assume the magnetic field does not change. Thus, Faraday's law gives

$$\nabla\times\mathbf{E} = \nabla\times\mathbf{E}_t = 0. \tag{8.44}$$

By definition, $\nabla\cdot\mathbf{E}_t = 0$. Thus, using eq. (8.43) we get $\mathbf{E}_t = 0$, that is, $\mathbf{E} = \mathbf{E}_l$.

For electron-plasma waves, we can write eq. (8.23) in spectral form to obtain

$$-iwn_{e0}\hat{\mathbf{u}}_{e1} + \frac{en_{e0}}{m_e}\hat{\mathbf{E}}_{1,l} = -\mathbf{k}_e\frac{\gamma_e p_{e0}}{n_{e0}m_e}\hat{n}_{e1}. \tag{8.45}$$

Since the second term on the left-hand side and the term on the right-hand side point along $\mathbf{k}_e$, $\hat{\mathbf{u}}_{e1}$ also points along $\mathbf{k}_e$, that is, $\hat{\mathbf{u}}_{e1} = \hat{\mathbf{u}}_{e1,l}$.

For ion-acoustic waves, we can write eq. (8.29) in spectral to obtain

$$-iwn_{i0}\hat{\mathbf{u}}_{i1} - \frac{Zen_{i0}}{m_i}\hat{\mathbf{E}}_{1,l} = -\mathbf{k}_i\frac{\gamma_i p_{i0}}{n_{i0}m_i}\hat{n}_{i1}. \tag{8.46}$$

Since the second term on the left-hand side and the term on the right-hand side point along $\mathbf{k}_i$, $\hat{\mathbf{u}}_{i1}$ also points along $\mathbf{k}_i$, that is, $\hat{\mathbf{u}}_{i1} = \hat{\mathbf{u}}_{i1,l}$. Finally, we note that the electric field being purely longitudinal is in agreement with eq. (8.31).

8.4 Electromagnetic waves in plasmas

In the Appendix we analyzed electromagnetic waves in vacuum, that is, for cases where $\rho_e = \mathbf{J} = 0$. In this section we relax both of these assumptions. Consider the electric and magnetic fields as well as the scalar and vector potentials, which satisfy

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \tag{8.47}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{8.48}$$

For the above, we choose $\nabla \cdot \mathbf{A} = 0$. Using the fact that the magnetic field is solenoidal, we have

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{B}_l + \nabla \cdot \mathbf{B}_t = \nabla \cdot \mathbf{B}_l = 0.$$

However, by definition, $\nabla \times \mathbf{B}_l = 0$ as well. Thus, using eq. (8.43), we have $\mathbf{B}_l = 0$. The same argument applies to the vector potential, and thus $\mathbf{A}_l = 0$. For the electric field, we have

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}_l + \nabla \cdot \mathbf{E}_t = \nabla \cdot \mathbf{E}_l = \nabla \cdot (-\nabla\phi)\,,$$

where we used eq. (8.47) for the last equality. In other words, we have

$$\nabla \cdot (\mathbf{E}_l + \nabla\phi) = 0.$$

By definition, we also have

$$\nabla \times (\mathbf{E}_l + \nabla\phi) = 0.$$

Thus, using eq. (8.43), we have $\mathbf{E}_l = -\nabla\phi$. A similar argument can be used to show $\mathbf{E}_t = -\partial\mathbf{A}/\partial t$. Our goal in this section will be to determine equations for $\mathbf{E}_l$, $\mathbf{E}_t$ and $\mathbf{B}$.

We'll begin with the conservation of charge equation

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

which we re-write as

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_l = 0,$$

Using Poisson's equation $\nabla^2\phi = -\rho_e/\epsilon_0$ in the above, we get

$$\frac{\partial}{\partial t}\left(-\epsilon_0\nabla^2\phi\right) + \nabla \cdot \mathbf{J}_l = 0,$$

or

$$\nabla \cdot \left(\frac{\partial \nabla\phi}{\partial t} - \frac{1}{\epsilon_0}\mathbf{J}_l\right) = 0.$$

However, by definition, we also have

$$\nabla \times \left(\frac{\partial \nabla\phi}{\partial t} - \frac{1}{\epsilon_0}\mathbf{J}_l\right) = 0.$$

Using eq. (8.43), we conclude

$$\frac{\partial \nabla\phi}{\partial t} = \frac{1}{\epsilon_0}\mathbf{J}_l. \tag{8.49}$$

This gives the equation for $\mathbf{E}_l$, namely,

$$\frac{\partial^2 \mathbf{E}_l}{\partial t^2} + \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}_l}{\partial t} = 0. \tag{8.50}$$

Both $\mathbf{E}_t$ and $\mathbf{B}$ can be extracted from $\mathbf{A}$, so now we proceed to find an equation for the transverse vector potential. Ampere's law with Maxwell's correction gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

The above is re-written as

$$\nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \left(-\frac{\partial \nabla \phi}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right),$$

which gives

$$\frac{\partial^2 \mathbf{A}}{\partial t} - \frac{1}{\mu_0 \epsilon_0} \nabla^2 \mathbf{A} = \frac{1}{\epsilon_0} \mathbf{J} - \frac{\partial \nabla \phi}{\partial t},$$

or

$$\frac{\partial^2 \mathbf{A}}{\partial t} - c_0^2 \nabla^2 \mathbf{A} = \frac{1}{\epsilon_0} \mathbf{J} - \frac{\partial \nabla \phi}{\partial t},$$

where $c_0 = 1/\sqrt{\mu_0 \epsilon_0}$. Expanding the current density as $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$, and using eq. (8.49), we get

$$\frac{\partial^2 \mathbf{A}}{\partial t} - c_0^2 \nabla^2 \mathbf{A} = \frac{1}{\epsilon_0} \mathbf{J}_t. \tag{8.51}$$

Using the functional form in eq. (8.38) for $\mathbf{A}$ and $\mathbf{J}_t$ gives

$$-w^2 \hat{\mathbf{A}} + k^2 c_0^2 \hat{\mathbf{A}} = \frac{1}{\epsilon_0} \hat{\mathbf{J}}_t. \tag{8.52}$$

That is, $\mathbf{A}$ and $\mathbf{J}_t$ point in the same direction.

Taking the time derivative of eq. (8.51) gives the equation for $\mathbf{E}_t$, that is

$$\frac{\partial^2 \mathbf{E}_t}{\partial t^2} - c_0^2 \nabla^2 \mathbf{E}_t + \frac{1}{\epsilon_0} \frac{\partial \mathbf{J}_t}{\partial t} = 0. \tag{8.53}$$

Taking the curl of eq. (8.51) gives the equation for $\mathbf{B}$, that is

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - c_0^2 \nabla^2 \mathbf{B} - \frac{1}{\epsilon_0} \nabla \times \mathbf{J}_t = 0. \tag{8.54}$$

Using the functional form in eq. (8.38) for $\mathbf{E}_t$, $\mathbf{B}$ and $\mathbf{J}_t$ gives

$$-w^2 \hat{\mathbf{E}}_t + k^2 c_0^2 \hat{\mathbf{E}}_t - \frac{iw}{\epsilon_0} \hat{\mathbf{J}}_t = 0, \tag{8.55}$$

$$-w^2 \mathbf{B} + k^2 c_0^2 \mathbf{B} - \frac{i}{\epsilon_0} \mathbf{k} \times \mathbf{J}_t = 0. \tag{8.56}$$

That is, $\mathbf{E}_t$ points in the same direction as $\mathbf{J}_t$, which as shown before points in the same direction as $\mathbf{A}$. Additionally, $\mathbf{B}$ points in the direction of $\mathbf{k} \times \mathbf{J}_t$, that is, it is orthogonal to $\mathbf{E}_t$.

We briefly note that taking the curl of eq. (8.9), and using eq. (8.10), gives the wave equation for the total electric field $\mathbf{E}$, that is

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c_0^2 \nabla^2 \mathbf{E} + c_0^2 \nabla(\nabla \cdot \mathbf{E}) + \frac{1}{\epsilon_0} \frac{\partial \mathbf{J}}{\partial t} = 0. \tag{8.57}$$

The above can be considered as the sum of the following three equations

$$\begin{aligned} \frac{\partial^2 \mathbf{E}_l}{\partial t^2} + \frac{1}{\epsilon_0} \frac{\partial \mathbf{J}_l}{\partial t} &= 0, \\ \frac{\partial^2 \mathbf{E}_t}{\partial t^2} - c_0^2 \nabla^2 \mathbf{E}_t + \frac{1}{\epsilon_0} \frac{\partial \mathbf{J}_t}{\partial t} &= 0, \\ -c_0^2 \nabla^2 \mathbf{E}_l + c_0^2 \nabla(\nabla \cdot \mathbf{E}_l) &= 0. \end{aligned}$$

The first is the equation for the longitudinal electric field, that is eq. (8.50). The second is the equation for the transverse electric field, that is eq. (8.53). The third equation above follows from the vector identity $\nabla \times (\nabla \times \mathbf{F}) = -\nabla^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F})$ and the fact that $\nabla \times \mathbf{E}_l = 0$.

It will often be the case that transverse waves will oscillate at such a fast rate that the ions, which have a large inertia, will be unable to react quickly enough. Thus, we can assume $\mathbf{u}_{i,t} = 0$. Given the definition of the current density in eq. (8.11), the transverse current density is expressed as $\mathbf{J}_t = e\left(Zn_i\mathbf{u}_{i,t} - n_e\mathbf{u}_{e,t}\right)$, which now simplifies to

$$\mathbf{J}_t = -en_e\mathbf{u}_{e,t}. \tag{8.58}$$

Thus, the transverse electron velocity $\mathbf{u}_{e,t}$ points in the same direction as $\mathbf{J}_t$, which is the same direction as $\mathbf{E}_t$ and $\mathbf{A}$. The next section focuses on deriving an expression for $\mathbf{u}_{e,t}$.

We begin with eq. (8.4), the electron momentum equation, which, due to the electron continuity equation, can be written as

$$m_e n_e \frac{\partial \mathbf{u}_e}{\partial t} + m_e n_e \mathbf{u}_e \cdot \nabla \mathbf{u}_e + en_e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) = -\nabla p_e,$$

or

$$\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e + \frac{e}{m_e}\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) = -\frac{1}{n_e m_e}\nabla p_e,$$

Using the scalar and vector potentials we have

$$\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e + \frac{e}{m_e}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_e \times (\nabla \times \mathbf{A})\right] = -\frac{1}{n_e m_e}\nabla p_e.$$

Using the vector identity $\nabla\left(F^2/2\right) = \mathbf{F} \times (\nabla \times \mathbf{F}) + \mathbf{F} \cdot \nabla \mathbf{F}$, we write the above as

$$\frac{\partial \mathbf{u}_e}{\partial t} - \mathbf{u}_e \times (\nabla \times \mathbf{u}_e) + \nabla\left(\frac{u_e^2}{2}\right) + \frac{e}{m_e}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_e \times (\nabla \times \mathbf{A})\right] = -\frac{1}{n_e m_e}\nabla p_e, \tag{8.59}$$

which is equivalent to

$$\frac{\partial \mathbf{u}_e}{\partial t} - \mathbf{u}_e \times (\nabla \times \mathbf{u}_{e,t}) + \nabla\left(\frac{u_e^2}{2}\right) + \frac{e}{m_e}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_e \times (\nabla \times \mathbf{A})\right] = -\frac{1}{n_e m_e}\nabla p_e. \tag{8.60}$$

We'll now introduce a more specific coordinate system. We'll be dealing with at most three waves at a time, a laser wave, a scattered wave, and a plasma wave (either electron-plasma or ion-acoustic wave). We'll assume all three of these waves lie on a so-called base plane. That is, $\mathbf{k}_e$ (or $\mathbf{k}_i$), $\mathbf{k}_L$, and $\mathbf{k}_s$ are all on this plane. We now choose the main transverse direction, that is, the direction of $\mathbf{u}_{e,t}$, $\mathbf{J}_t$, $\mathbf{E}_t$, and $\mathbf{A}$ to be the direction orthogonal to this plane, so that these vectors are orthogonal to any $\mathbf{k}$. As an aside, we note that the longitudinal and transverse components of the electron velocity can belong to different waves. That is

$$\mathbf{u}_{e,l} = \hat{\mathbf{u}}_{e,l}\exp\left[i(\mathbf{k}_p \cdot \mathbf{x} - w_p t)\right] \tag{8.61}$$

$$\mathbf{u}_{e,t} = \hat{\mathbf{u}}_{e,t}\exp\left[i(\mathbf{k}_q \cdot \mathbf{x} - w_q t)\right]. \tag{8.62}$$

Note that since all wave vectors point along the base plane, u_e^2, ϕ and p_e will only vary along that plane, and thus their gradients will be confined to that plane. Thus, the component of eq. (8.60) along the main transverse direction is

$$\frac{\partial \mathbf{u}_{e,t}}{\partial t} - \mathbf{u}_{e,l} \times (\nabla \times \mathbf{u}_{e,t}) + \frac{e}{m_e}\left[-\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_{e,l} \times (\nabla \times \mathbf{A})\right] = 0. \tag{8.63}$$

Using $c = w/k$, we show the following scalings

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial \mathbf{u}_{e,t}}{\partial t} &= -\frac{iw\mathbf{u}_{e,t}}{c^2} \sim i\frac{\mathbf{u}_{e,t}}{c}k, \\
\frac{1}{c^2}\mathbf{u}_{e,l} \times (\nabla \times \mathbf{u}_{e,t}) &= \frac{i\mathbf{u}_{e,l} \times (\mathbf{k} \times \mathbf{u}_{e,t})}{c^2} \sim i\frac{\mathbf{u}_{e,l}}{c}\frac{\mathbf{u}_{e,t}}{c}k, \\
\frac{1}{c^2}\frac{\partial \mathbf{A}}{\partial t} &= -\frac{iw\mathbf{A}}{c^2} \sim i\frac{\mathbf{A}}{c}k, \\
\frac{1}{c^2}\mathbf{u}_{e,l} \times (\nabla \times \mathbf{A}) &= \frac{i\mathbf{u}_{e,l} \times (\mathbf{k} \times \mathbf{A})}{c^2} \sim i\frac{\mathbf{u}_{e,l}}{c}\frac{\mathbf{A}}{c}k.
\end{aligned} \tag{8.64}$$

Thus, assuming $\mathbf{u}_{e,l} \ll c$, the terms involving the double cross product are smaller than those involving the time derivative. As a result, eq. (8.63) becomes

$$\frac{\partial \mathbf{u}_{e,t}}{\partial t} - \frac{e}{m_e}\frac{\partial \mathbf{A}}{\partial t} = 0. \tag{8.65}$$

The above is equivalent to

$$-iw\mathbf{u}_{e,t} + iw\frac{e\mathbf{A}}{m_e} = 0, \tag{8.66}$$

which upon re-arranging gives

$$\mathbf{u}_{e,t} = \frac{e\mathbf{A}}{m_e}. \tag{8.67}$$

Using both the transverse current given by eq. (8.58) and the transverse velocity given by eq. (8.67), eq. (8.51) can be re-written as

$$\frac{\partial^2 \mathbf{A}}{\partial t} - c_0^2\nabla^2\mathbf{A} = -\frac{en_e}{\epsilon_0}\mathbf{u}_{e,t} = -\frac{e^2 n_e}{\epsilon_0 m_e}\mathbf{A}.$$

We now use the decomposition $n_e = n_{e0} + n_{e1}$, where n_{e0} is time independent. The above becomes

$$\frac{\partial^2 \mathbf{A}}{\partial t} + w_{pe}^2\mathbf{A} - c_0^2\nabla^2\mathbf{A} = -\frac{e^2 n_{e1}}{\epsilon_0 m_e}\mathbf{A}, \tag{8.68}$$

where $w_{pe}^2 = e^2 n_{e0}/m_e\epsilon_0$.

8.5 Electromagnetic waves in a stable plasma

We start with eq. (8.68), but focus on the stable-plasma case, that is, $n_{e1} = 0$. Thus, we have

$$\frac{\partial^2 \mathbf{A}}{\partial t} + w_{pe}^2\mathbf{A} - c_0^2\nabla^2\mathbf{A} = 0. \tag{8.69}$$

We note that n_{e0} is only time independent, that is, it is still allowed to vary across space. As a result, w_{pe}^2 is also allowed to vary across space. We'll now use the functional form in eq. (8.38) for $\mathbf{A}$. However, we'll express the vector potential as $\mathbf{A} = \tilde{\mathbf{A}}\exp(-iwt)$, where $\tilde{\mathbf{A}} = \hat{\mathbf{A}}\exp(i\mathbf{k}\cdot\mathbf{x})$. Thus, eq. (8.69) gives

$$-w^2\mathbf{A} + w_{pe}^2\mathbf{A} - c_0^2\nabla^2\mathbf{A} = 0. \tag{8.70}$$

We re-write the above as

$$\frac{w^2}{c_0^2}\mathbf{A} - \frac{w^2}{c_0^2}\frac{w_{pe}^2}{w^2}\mathbf{A} + \nabla^2\mathbf{A} = 0. \tag{8.71}$$

Defining $\epsilon = 1 - w_{pe}^2/w^2$, we ultimately get

$$\frac{w^2}{c_0^2}\epsilon\mathbf{A} + \nabla^2\mathbf{A} = 0. \tag{8.72}$$

Taking the time derivative of eq. (8.69) gives the equation for $\mathbf{E}_t$, that is

$$\frac{\partial^2\mathbf{E}_t}{\partial t^2} + w_{pe}^2\mathbf{E}_t - c_0^2\nabla^2\mathbf{E}_t = 0. \tag{8.73}$$

Using $\mathbf{E}_t = \tilde{\mathbf{E}}_t \exp(-iwt)$, where $\tilde{\mathbf{E}}_t = \hat{\mathbf{E}}_t \exp(i\mathbf{k}\cdot\mathbf{x})$, we get

$$-w^2\mathbf{E}_t + w_{pe}^2\mathbf{E}_t - c_0^2\nabla^2\mathbf{E}_t = 0. \tag{8.74}$$

We re-write the above as

$$\frac{w^2}{c_0^2}\mathbf{E}_t - \frac{w^2}{c_0^2}\frac{w_{pe}^2}{w^2}\mathbf{E}_t + \nabla^2\mathbf{E}_t = 0, \tag{8.75}$$

which becomes

$$\frac{w^2}{c_0^2}\epsilon\mathbf{E}_t + \nabla^2\mathbf{E}_t = 0. \tag{8.76}$$

Taking the curl of eq. (8.68) gives the equation for $\mathbf{B}$, that is

$$\frac{\partial^2\mathbf{B}}{\partial t^2} + w_{pe}^2\mathbf{B} - c_0^2\nabla^2\mathbf{B} + \nabla w_{pe}^2 \times \mathbf{A} = 0. \tag{8.77}$$

Using $\mathbf{B} = \tilde{\mathbf{B}} \exp(-iwt)$, where $\tilde{\mathbf{B}} = \hat{\mathbf{B}} \exp(i\mathbf{k}\cdot\mathbf{x})$, we get

$$-w^2\mathbf{B} + w_{pe}^2\mathbf{B} - c_0^2\nabla^2\mathbf{B} + \nabla w_{pe}^2 \times \mathbf{A} = 0. \tag{8.78}$$

We re-write the above as

$$\frac{w^2}{c_0^2}\mathbf{B} - \frac{w^2}{c_0^2}\frac{w_{pe}^2}{w^2}\mathbf{B} + \nabla^2\mathbf{B} - \frac{w^2}{c_0^2}\nabla\left(\frac{w_{pe}^2}{w^2}\right) \times \mathbf{A} = 0, \tag{8.79}$$

which becomes

$$\frac{w^2}{c_0^2}\epsilon\mathbf{B} + \nabla^2\mathbf{B} + \frac{w^2}{c_0^2}\nabla\epsilon \times \mathbf{A} = 0. \tag{8.80}$$

Using eq. (8.72) we get

$$\frac{w^2}{c_0^2}\epsilon\mathbf{B} + \nabla^2\mathbf{B} - \frac{1}{\epsilon}\nabla\epsilon \times \nabla^2\mathbf{A} = 0. \tag{8.81}$$

The vector identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$ gives $\nabla^2\mathbf{A} = -\nabla \times (\nabla \times \mathbf{A}) = -\nabla \times \mathbf{B}$. Thus, we finally get

$$\frac{w^2}{c_0^2}\epsilon\mathbf{B} + \nabla^2\mathbf{B} + \frac{1}{\epsilon}\nabla\epsilon \times (\nabla \times \mathbf{B}) = 0. \tag{8.82}$$

8.6 Stimulated Raman and Brillouin instabilities

8.6.1 Linearization

The following decompositions will be used in the derivation of stimulated Raman and Brillouin instabilities:

$$
\begin{aligned}
n_i &= n_{i0} + n_{i1}, \\
n_e &= n_{e0} + n_{e1}, \\
p_i &= p_{i0} + p_{i1}, \\
p_e &= p_{e0} + p_{e1}, \\
\mathbf{u}_{i,l} &= \mathbf{u}_{i0,l} + \mathbf{u}_{i1,l}, \\
\mathbf{u}_{e,l} &= \mathbf{u}_{e0,l} + \mathbf{u}_{e1,l}, \\
\mathbf{E}_l &= \mathbf{E}_{0,l} + \mathbf{E}_{1,l}, \\
\mathbf{A} &= \mathbf{A}_L + \mathbf{A}_s.
\end{aligned}
\tag{8.83}
$$

For these decompositions, we'll assume

1. Terms with a subscript 1 are small and thus products of two small quantities can be neglected.
2. $\mathbf{u}_{i0,l}$, $\mathbf{u}_{e0,l}$, and $\mathbf{E}_{0,l}$,are zero.
3. n_{i0}, n_{e0}, p_{i0}, and p_{e0} are uniform in space and time.

Thus, unlike the previous section, we do not assume the plasma is stable, that is, we assume fluctuations such as n_{e1} are small but non zero. $\mathbf{A}_L$ is the vector potential associated with the laser light, and $\mathbf{A}_s$ is the potential associated with the scattered light. For linearization purposes, we'll assume $\mathbf{A}_s$ is small.

Using the decomposition for $\mathbf{A}$, eq. (8.68) is written as

$$\frac{\partial^2 \mathbf{A}_L}{\partial t} + \frac{\partial^2 \mathbf{A}_s}{\partial t} + w_{pe}^2 \mathbf{A}_L + w_{pe}^2 \mathbf{A}_s - c_0^2 \nabla^2 \mathbf{A}_L - c_0^2 \nabla^2 \mathbf{A}_s = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_L - \frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_s, \tag{8.84}$$

Dropping products of small quantities we have

$$\frac{\partial^2 \mathbf{A}_L}{\partial t} + \frac{\partial^2 \mathbf{A}_s}{\partial t} + w_{pe}^2 \mathbf{A}_L + w_{pe}^2 \mathbf{A}_s - c_0^2 \nabla^2 \mathbf{A}_L - c_0^2 \nabla^2 \mathbf{A}_s = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_L. \tag{8.85}$$

We'll assume the laser light is stable, that is, it satisfies eq. (8.69), which we re-write below

$$\frac{\partial^2 \mathbf{A}_L}{\partial t} + w_{pe}^2 \mathbf{A}_L - c_0^2 \nabla^2 \mathbf{A}_L = 0. \tag{8.86}$$

Thus, eq. (8.85) becomes

$$\frac{\partial^2 \mathbf{A}_s}{\partial t} + w_{pe}^2 \mathbf{A}_s - c_0^2 \nabla^2 \mathbf{A}_s = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_L. \tag{8.87}$$

The above shows that the fluctuating n_{e1} couples with the laser light to serve as a source for the scattered light.

The electron density equation is now written as

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \nabla \cdot \left[(n_{e0} + n_{e1}) \left(\mathbf{u}_{e,t} + \mathbf{u}_{e0,l} + \mathbf{u}_{e1,l} \right) \right] = 0,$$

which, since $\mathbf{u}_{e,t}$ is transverse, can be written as

$$\frac{\partial n_{e0}+n_{e1}}{\partial t}+\mathbf{u}_{e,t}\cdot\nabla\left(n_{e0}+n_{e1}\right)+\nabla\cdot\left[\left(n_{e0}+n_{e1}\right)\left(\mathbf{u}_{e0,l}+\mathbf{u}_{e1,l}\right)\right]=0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e1}}{\partial t}+\mathbf{u}_{e,t}\cdot\nabla n_{e1}+\nabla\cdot\left(n_{e0}\mathbf{u}_{e1,l}\right)=0.$$

Since $\mathbf{u}_{e,t}$ and ∇n_{e1} are orthogonal, we finally have

$$\frac{\partial n_{e1}}{\partial t}+\nabla\cdot\left(n_{e0}\mathbf{u}_{e1,l}\right)=0. \tag{8.88}$$

The ion density equation is now written as

$$\frac{\partial n_{i0}+n_{i1}}{\partial t}+\nabla\cdot\left[\left(n_{i0}+n_{i1}\right)\left(\mathbf{u}_{i,t}+\mathbf{u}_{i0,l}+\mathbf{u}_{i1,l}\right)\right]=0,$$

As stated in section 8.4, it is often the case that transverse waves oscillate at such a fast rate that the ions, which have large inertia, are unable to react on comparable time scales. Thus, we can assume $\mathbf{u}_{i,t}=0$,

$$\frac{\partial n_{i0}+n_{i1}}{\partial t}+\nabla\cdot\left[\left(n_{i0}+n_{i1}\right)\left(\mathbf{u}_{i0,l}+\mathbf{u}_{i1,l}\right)\right]=0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i1}}{\partial t}+\nabla\cdot\left(n_{i0}\mathbf{u}_{i1,l}\right)=0. \tag{8.89}$$

Consider the electron momentum equation. Subtracting eq. (8.63) from eq. (8.60) gives

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t}-\mathbf{u}_{e,t}\times\left(\nabla\times\mathbf{u}_{e,t}\right)+\nabla\left(\frac{u_e^2}{2}\right)+\frac{e}{m_e}\left[-\nabla\phi+\mathbf{u}_{e,t}\times\left(\nabla\times\mathbf{A}\right)\right]=-\frac{1}{n_e m_e}\nabla p_e. \tag{8.90}$$

Since $\mathbf{u}_{e,t}=e\mathbf{A}/m_e$, the above simplifies to

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t}+\nabla\left(\frac{u_e^2}{2}\right)-\frac{e}{m_e}\nabla\phi=-\frac{1}{n_e m_e}\nabla p_e,$$

or

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t}+\nabla\left(\frac{u_e^2}{2}\right)+\frac{e}{m_e}\mathbf{E}_l=-\frac{1}{n_e m_e}\nabla p_e.$$

Since $\mathbf{u}_{e,l}$ and $\mathbf{u}_{e,t}$ are orthogonal $u_e^2=\mathbf{u}_e\cdot\mathbf{u}_e=u_{e,l}^2+u_{e,t}^2$. The electron momentum equation is then

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t}+\nabla\left(\frac{u_{e,l}^2+u_{e,t}^2}{2}\right)+\frac{e}{m_e}\mathbf{E}_l=-\frac{1}{n_e m_e}\nabla p_e.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e0}\mathbf{u}_{e1,l}}{\partial t}+n_{e0}\nabla\left(\frac{u_{e,t}^2}{2}\right)+\frac{e n_{e0}}{m_e}\mathbf{E}_{1,l}=-\frac{1}{m_e}\nabla p_{e1}.$$

For the transverse electron velocity we have

$$u_{e,t}^2=\left(\frac{e\mathbf{A}}{m_e}\right)\cdot\left(\frac{e\mathbf{A}}{m_e}\right)=\frac{e^2}{m_e^2}\left(\mathbf{A}_L\cdot\mathbf{A}_L+2\mathbf{A}_L\cdot\mathbf{A}_s+\mathbf{A}_s\cdot\mathbf{A}_s\right).$$

Since the product of small quantities can be neglected, the $\mathbf{A}_s \cdot \mathbf{A}_s$ term is dropped. We'll also ignore the $\mathbf{A}_L \cdot \mathbf{A}_L$ term, given that $\mathbf{A}_L$ is stable and thus its magnitude does not play a critical role in the growth of the instabilities. Thus, the electron momentum equation becomes

$$\frac{\partial n_{e0}\mathbf{u}_{e1,l}}{\partial t} + \frac{e^2 n_{e0}}{m_e^2}\nabla\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) + \frac{e n_{e0}}{m_e}\mathbf{E}_{1,l} = -\frac{1}{m_e}\nabla p_{e1}. \tag{8.91}$$

Consider now the ion momentum equation, given by eq. (8.3), which we re-write below as

$$\frac{\partial n_i \mathbf{u}_i}{\partial t} + \nabla \cdot \left(n_i \mathbf{u}_i \mathbf{u}_i\right) - \frac{Zen_i}{m_i}\left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right) = -\frac{1}{m_i}\nabla p_i,$$

The longitudinal component of the above is

$$\frac{\partial n_i \mathbf{u}_{i,l}}{\partial t} + \left[\nabla \cdot \left(n_i \mathbf{u}_i \mathbf{u}_i\right)\right]_l - \frac{Zen_i}{m_i}\left(\mathbf{E}_l + \mathbf{u}_{i,t} \times \mathbf{B}\right) = -\frac{1}{m_i}\nabla p_i,$$

where $[\cdot]_l$ denotes longitudinal component. Since $\mathbf{u}_{i,t} = 0$, we have

$$\frac{\partial n_i \mathbf{u}_{i,l}}{\partial t} + \left[\nabla \cdot \left(n_i \mathbf{u}_i \mathbf{u}_i\right)\right]_l - \frac{Zen_i}{m_i}\mathbf{E}_l = -\frac{1}{m_i}\nabla p_i.$$

Using the variable decompositions, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\left(n_{i0} + n_{i1}\right)\left(\mathbf{u}_{i0,l} + \mathbf{u}_{i1,l}\right)\right] \\
&\qquad + \left\{\nabla \cdot \left[\left(n_{i0} + n_{i1}\right)\left(\mathbf{u}_{i,t} + \mathbf{u}_{i0,l} + \mathbf{u}_{i1,l}\right)\left(\mathbf{u}_{i,t} + \mathbf{u}_{i0,l} + \mathbf{u}_{i1,l}\right)\right]\right\}_l \\
&\qquad\qquad - \frac{Ze}{m_i}\left(n_{i0} + n_{i1}\right)\left(\mathbf{E}_{0,l} + \mathbf{E}_{1,l}\right) = -\frac{1}{m_i}\nabla\left(p_{i0} + p_{i1}\right).
\end{aligned}$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i0}\mathbf{u}_{i1,l}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_{1,l} = -\frac{1}{m_i}\nabla p_{i1}. \tag{8.92}$$

8.6.2 Stimulated Raman Scattering

We employ the same assumptions as for the electron-plasma waves, that is

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.
2. Uniform ion density, $n_{i1} = 0$.

Combining eq. (8.91) with eq. (8.19) gives

$$\frac{\partial n_{e0}\mathbf{u}_{e1,l}}{\partial t} + \frac{e^2 n_{e0}}{m_e^2}\nabla\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) + \frac{e n_{e0}}{m_e}\mathbf{E}_{1,l} = -\frac{\gamma_e p_{e0}}{n_{e0} m_e}\nabla n_{e1}. \tag{8.93}$$

Taking the time derivative of eq. (8.88) and using eq. (8.93) leads to the wave equation for electron density

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{e^2 n_{e0}}{m_e^2}\nabla^2\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) - \frac{e n_{e0}}{m_e}\nabla \cdot \mathbf{E}_{1,l} = \frac{\gamma_e p_{e0}}{n_{e0} m_e}\nabla^2 n_{e1}.$$

As before, using eq. (8.25) we obtain

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{e^2 n_{e0}}{m_e^2}\nabla^2\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) + \frac{e^2 n_{e0}}{m_e \epsilon_0} n_{e1} = \frac{\gamma_e p_{e0}}{n_{e0} m_e}\nabla^2 n_{e1}.$$

or

$$\frac{\partial^2 n_{e1}}{\partial t^2} + w_{pe}^2 n_{e1} - \frac{\gamma_e p_{e0}}{n_{e0} m_e}\nabla^2 n_{e1} = \frac{e^2 n_{e0}}{m_e^2}\nabla^2\left(\mathbf{A}_L \cdot \mathbf{A}_s\right). \tag{8.94}$$

Thus, the scattered laser light $\mathbf{A}_s$ couples with the laser light to serve as a source for the electron-plasma wave.

8.6.3 Stimulated Brillouin Scattering

We employ the same assumptions as for the ion-acoustic waves, that is

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.
2. Approximate quasi-neutrality for the fluctuations, $Zn_{i1} \approx n_{e1}$.
3. Negligible electron mass, $m_e \to 0$.

Combining eq. (8.92) with eq. (8.20) gives

$$\frac{\partial n_{i0}\mathbf{u}_{i1,l}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_{1,l} = -\frac{\gamma_i p_{i0}}{n_{i0}m_i}\nabla n_{i1}. \tag{8.95}$$

Taking the time derivative of eq. (8.89) and using eq. (8.95) leads to the wave equation for ion density

$$\frac{\partial^2 n_{i1}}{\partial t^2} + \frac{Zen_{i0}}{m_i}\nabla\cdot\mathbf{E}_{1,l} = \frac{\gamma_i p_{i0}}{n_{i0}m_i}\nabla^2 n_{i1}. \tag{8.96}$$

For this case, we assume that the mass of the electron, which is significantly smaller than that of the ions, is negligible. Thus, eq. (8.91) simplifies to

$$\frac{e^2 n_{e0}}{m_e}\nabla\left(\mathbf{A}_L\cdot\mathbf{A}_s\right) + en_{e0}\mathbf{E}_{1,l} = -\frac{\gamma_e p_{e0}}{n_{e0}}\nabla n_{e1}. \tag{8.97}$$

Plugging in the above in the ion wave equation we obtain

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{Zn_{i0}}{n_{e0}}\frac{\gamma_e p_{e0}}{n_{e0}m_i}\nabla^2 n_{e1} + \frac{\gamma_i p_{i0}}{n_{i0}m_i}\nabla^2 n_{i1} + \frac{Ze^2 n_{i0}}{m_i m_e}\nabla^2\left(\mathbf{A}_L\cdot\mathbf{A}_s\right). \tag{8.98}$$

Due to quasi-neutrality, we have $Zn_{i0} = n_{e0}$ and $Zn_{i1} \approx n_{e1}$, which gives

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{1}{m_i}\left(\frac{Z\gamma_e p_{e0}}{n_{e0}} + \frac{\gamma_i p_{i0}}{n_{i0}}\right)\nabla^2 n_{i1} + \frac{Ze^2 n_{i0}}{m_i m_e}\nabla^2\left(\mathbf{A}_L\cdot\mathbf{A}_s\right). \tag{8.99}$$

Since $p_{i0}/n_{i0} = k_B T_{i0}$ and $p_{e0}/n_{e0} = k_B T_{e0}$, we finally have

$$\frac{\partial^2 n_{i1}}{\partial t^2} - \left(\frac{Z\gamma_e k_B T_{e0} + \gamma_i k_B T_{i0}}{m_i}\right)\nabla^2 n_{i1} = \frac{Ze^2 n_{i0}}{m_i m_e}\nabla^2\left(\mathbf{A}_L\cdot\mathbf{A}_s\right). \tag{8.100}$$

Thus, the scattered laser light $\mathbf{A}_s$ couples with the laser light to serve as a source for the ion-acoustic wave.

Chapter 9

Instabilities

9.1 Linear stability analysis

Consider the following system of PDEs for a two-dimensional problem

$$\frac{\partial w}{\partial t} = \frac{\partial \phi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial w}{\partial y}, \tag{9.1}$$

$$w = \nabla^2 \phi. \tag{9.2}$$

In the above, $\phi = \phi(x, y, t)$ is the electrostatic potential and $w = w(x, y, t)$ the vorticity.

The analysis begins by splitting the variables into equilibrium and fluctuating components, namely

$$\phi = \phi_0 + \phi_1, \tag{9.3}$$

$$w = w_0 + w_1. \tag{9.4}$$

For the above, $\phi_0 = \phi_0(x)$, $w_0 = w_0(x)$, and $\phi_1 = \phi_1(x, y, t)$, $w_1 = w_1(x, y, t)$. We now introduce a Fourier series decomposition for the fluctuating variables, and focus on a single Fourier mode as follows

$$\phi_1(x, y, t) = F(x, k_y, t)e^{ik_y y} = \tilde{\phi}(x, k_y)e^{\gamma t + ik_y y}, \tag{9.5}$$

$$w_1(x, y, t) = G(x, k_y, t)e^{ik_y y} = \tilde{w}(x, k_y)e^{\gamma t + ik_y y}. \tag{9.6}$$

In the above, γ, which can be complex, is the growth rate factor, and $\tilde{\phi} = \tilde{\phi}(x, k_y)$, $\tilde{w} = \tilde{w}(x, k_y)$ are the remaining part of the Fourier coefficient.

We then plug in the decompositions for ϕ and w in the governing PDEs. Collecting the lowest order terms leads to equations for the equilibrium solution. For example, the Poisson equation to lowest order is

$$w_0 = \nabla^2 \phi_0 = \frac{\partial^2 \phi_0}{\partial x^2}. \tag{9.7}$$

Combining terms up to next order gives

$$\frac{\partial w_1}{\partial t} = \frac{\partial \phi_0}{\partial y}\frac{\partial w_1}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial w_0}{\partial x} - \frac{\partial \phi_0}{\partial x}\frac{\partial w_1}{\partial y} - \frac{\partial \phi_1}{\partial x}\frac{\partial w_0}{\partial y}, \tag{9.8}$$

$$w_1 = \nabla^2 \phi_1. \tag{9.9}$$

Using the expression for ϕ_1 in the Poisson equation above leads to

$$w_1 = \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi}\right) e^{\gamma t + ik_y y}, \tag{9.10}$$

or

$$\tilde{w} = \frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi}. \tag{9.11}$$

We'll now evaluate each of the terms in eq. (9.8).

$$\begin{aligned}
\frac{\partial w_1}{\partial t} &= \gamma \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) e^{\gamma t + i k_y y}, \\
\frac{\partial \phi_0}{\partial y} \frac{\partial w_1}{\partial x} &= 0, \\
\frac{\partial \phi_1}{\partial y} \frac{\partial w_0}{\partial x} &= i k_y \tilde{\phi} e^{\gamma t + i k_y y} \frac{\partial^3 \phi_0}{\partial x^3}, \\
\frac{\partial \phi_0}{\partial x} \frac{\partial w_1}{\partial y} &= \frac{\partial \phi_0}{\partial x} i k_y \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) e^{\gamma t + i k_y y}, \\
\frac{\partial \phi_1}{\partial x} \frac{\partial w_0}{\partial y} &= 0.
\end{aligned} \tag{9.12}$$

Combining all of the above, we obtain

$$\gamma \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) = i k_y \tilde{\phi} \frac{\partial^3 \phi_0}{\partial x^3} - \frac{\partial \phi_0}{\partial x} i k_y \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right). \tag{9.13}$$

This is re-written as

$$\gamma \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) = -i k_y \left[\frac{\partial \phi_0}{\partial x} \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) - \frac{\partial^3 \phi_0}{\partial x^3} \tilde{\phi} \right]. \tag{9.14}$$

Chapter 10

Simple models

10.1 Hasegawa-Mima

References for this model can be found in Hasegawa and Mima [1977], Horton and Hasegawa [1994].

10.1.1 Assumptions

1. Singly-charged ions.
2. No shear stresses, collisions, or sources.
3. Cold ion approximation, i.e. $T_e \gg T_i$ and thus $\nabla p_i \approx 0$, Hasegawa and Mima [1977].
4. Isothermal electron fluid, i.e. T_e is constant.
5. Electrostatic field, i.e. $\mathbf{E} = -\nabla\phi$.
6. Magnetic field is constant.
7. Neglect parallel ion velocity, i.e. $u_{i,||} \approx 0$, Hasegawa and Mima [1977].
8. Quasi-neutrality, i.e. $n_i \approx n_e$.
9. Adiabatic electrons, i.e. $n_e = n_0 \exp(e\phi/T_e)$, where $n_0 = n_0(x_1)$.

10.1.2 Derivation

Using the assumptions in items 1 and 2, the momentum eq. (5.52) for ions becomes

$$m_i n_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right) = e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i. \tag{10.1}$$

Using the assumptions in items 3 and 5, the above becomes

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{e}{m_i} \nabla\phi + \frac{e}{m_i} \mathbf{u}_i \times \mathbf{B}. \tag{10.2}$$

Introduce the coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and assume $\mathbf{B}$ points in the $\mathbf{e}_3$ direction. Defining the perpendicular velocity as $\mathbf{u}_{i,\perp} = [u_{i,1}, u_{i,2}, 0]^T$ and the perpendicular gradient as $\nabla_\perp = [\partial_1, \partial_2, 0]^T$, we have

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_{i,\perp} = -\frac{e}{m_i} \nabla_\perp \phi + \frac{e}{m_i} \mathbf{u}_i \times \mathbf{B}. \tag{10.3}$$

Using the assumption in item 7 and noting that $\mathbf{u}_i \times \mathbf{B} = \mathbf{u}_{i,\perp} \times \mathbf{B}$, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i}\nabla_\perp \phi + \frac{e}{m_i}\mathbf{u}_{i,\perp} \times \mathbf{B}. \tag{10.4}$$

We now introduce the scalings for a characteristic frequency w and length scale r

$$\frac{w}{w_{c,i}} \sim \epsilon \qquad \frac{r_s}{r} \sim \epsilon, \tag{10.5}$$

where $w_{c,i} = eB/m_i$ is the cyclotron frequency, $r_s = v_s/w_{c,i}$ is a reference length scale, and $v_s = \sqrt{T_e/m_i}$ a reference velocity scale. Given these variables, we assume

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} \sim \mathbf{u}_{i,\perp} w \qquad \nabla_\perp \mathbf{u}_{i,\perp} \sim \frac{\mathbf{u}_{i,\perp}}{r} \qquad \mathbf{E} \sim \mathbf{u}_{i,\perp} B. \tag{10.6}$$

Finally, we introduce the decomposition $\mathbf{u}_{i,\perp} = \mathbf{u}_{i,\perp}^{(0)} + \mathbf{u}_{i,\perp}^{(1)}$, where $\mathbf{u}_{i,\perp}^{(0)} \sim v_s$ and $\mathbf{u}_{i,\perp}^{(1)} \sim \epsilon v_s$. We use this decomposition in eq. (10.4) and then divide the PDE by $w_{c,i} v_s$. The order of each element in the resulting equation is as follows

1. $\frac{\partial \mathbf{u}_{i,\perp}^{(0)}}{\partial t} \sim \epsilon$.
2. $\frac{\partial \mathbf{u}_{i,\perp}^{(1)}}{\partial t} \sim \epsilon^2$.
3. $\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon$.
4. $\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^2$.
5. $\mathbf{u}_{i,\perp}^{(1)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon^2$.
6. $\mathbf{u}_{i,\perp}^{(1)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^3$.
7. $-\frac{e}{m_i}\nabla_\perp \phi \sim 1$.
8. $\frac{e}{m_i}\mathbf{u}_{i,\perp}^{(0)} \times \mathbf{B} \sim 1$.
9. $\frac{e}{m_i}\mathbf{u}_{i,\perp}^{(1)} \times \mathbf{B} \sim \epsilon$.

Combining the first order terms we obtain

$$0 = -\nabla_\perp \phi + \mathbf{u}_{i,\perp}^{(0)} \times \mathbf{B}, \tag{10.7}$$

which, upon crossing by $\mathbf{B}$, gives

$$\mathbf{u}_{i,\perp}^{(0)} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B}. \tag{10.8}$$

Combining the terms of order ϵ we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}^{(0)}}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} = \frac{e}{m_i}\mathbf{u}_{i,\perp}^{(1)} \times \mathbf{B}, \tag{10.9}$$

which, upon crossing by $\mathbf{B}$, gives

$$\mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B}\left[\frac{\partial \nabla_\perp \phi}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\nabla_\perp \phi\right)\right]. \tag{10.10}$$

The above is the polarization drift, cf. eq. (2.122). The velocity given by eq. (10.8) is referred to as the $E \times B$ drift, and the velocity given by eq. (10.10) as the polarization drift.

Using the assumption in item 2, the continuity equation for ions is

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0. \tag{10.11}$$

Using the assumption in item 7 the above becomes

$$\frac{\partial n_i}{\partial t} + \nabla_\perp \cdot (n_i \mathbf{u}_{i,\perp}) = 0, \tag{10.12}$$

or

$$\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp n_i + n_i \nabla_\perp \cdot \mathbf{u}_{i,\perp} = 0, \tag{10.13}$$

One of the main components of the derivation of the Hasegawa-Mima equation is the assumption that advection is governed by the lowest-order velocity only; that is, by $\mathbf{u}_{i,\perp}^{(0)}$ and not $\mathbf{u}_{i,\perp}^{(1)}$. Thus, the above is written as

$$\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp n_i + n_i \nabla_\perp \cdot \mathbf{u}_{i,\perp} = 0. \tag{10.14}$$

We note that $\mathbf{u}_{i,\perp}^{(0)}$ is divergence free, and thus we have

$$\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp n_i + n_i \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{10.15}$$

We divide by n_i to express the density in terms of its logarithm

$$\frac{\partial \ln n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln n_i + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{10.16}$$

We now use the assumptions in items 8 and 9 to obtain

$$\ln n_i = \ln \left[n_0 \exp\left(\frac{e\phi}{T_e} \right) \right] = \ln n_0 + \frac{e\phi}{T_e}. \tag{10.17}$$

Taking into account the fact that n_0 is time independent, the continuity equation becomes

$$\frac{\partial}{\partial t}\left(\frac{e\phi}{T_e} \right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left[\ln n_0 + \frac{e\phi}{T_e} \right] + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{10.18}$$

Since $\mathbf{u}_{i,\perp}^{(0)}$ and $\nabla_\perp \phi$ are orthogonal, the above simplifies to

$$\frac{\partial}{\partial t}\left(\frac{e\phi}{T_e} \right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln n_0 + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0, \tag{10.19}$$

which we re-write as

$$\frac{\partial}{\partial t}\left(\frac{e\phi}{T_e} \right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln \left(\frac{n_0}{w_{c,i}} \right) + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{10.20}$$

Given the definition of the polarization drift, we have

$$\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i} B} \left\{ \frac{\partial \nabla_\perp^2 \phi}{\partial t} + \nabla_\perp \cdot \left[\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp (\nabla_\perp \phi) \right] \right\}. \tag{10.21}$$

The second term above is best computed using tensor notation, and we'll use u_j to denote the components of $\mathbf{u}_{i,\perp}^{(0)}$. Thus,

$$\frac{\partial}{\partial x_i}\left[\left(u_j\frac{\partial}{\partial x_j}\right)\frac{\partial \phi}{\partial x_i}\right]=\frac{\partial u_j}{\partial x_i}\frac{\partial^2\phi}{\partial x_j\partial x_i}+u_j\frac{\partial}{\partial x_j}\left(\frac{\partial^2\phi}{\partial x_i\partial x_i}\right). \tag{10.22}$$

Using the definition of $\mathbf{u}_{i,\perp}^{(0)}$, the first term on the right-hand side above can be expressed as

$$\begin{aligned}\frac{\partial u_j}{\partial x_i}\frac{\partial^2\phi}{\partial x_j\partial x_i} &= -\frac{1}{B^2}\epsilon_{jpq}\frac{\partial^2\phi}{\partial x_p\partial x_i}B_q\frac{\partial^2\phi}{\partial x_j\partial x_i}\\ &= -\frac{1}{B^2}\epsilon_{qjp}\frac{\partial^2\phi}{\partial x_j\partial x_i}\frac{\partial^2\phi}{\partial x_p\partial x_i}B_q\\ &= -\frac{1}{B^2}\epsilon_{qjp}\left(\frac{\partial^2\phi}{\partial x_j\partial x_1}\frac{\partial^2\phi}{\partial x_p\partial x_1}+\frac{\partial^2\phi}{\partial x_j\partial x_2}\frac{\partial^2\phi}{\partial x_p\partial x_2}\right)B_q.\end{aligned} \tag{10.23}$$

Since $\epsilon_{qjp}\partial_j a\partial_p a\to\nabla a\times\nabla a=0$ for any scalar a, the term above is identically zero. Thus, we have

$$\nabla_\perp\cdot\mathbf{u}_{i,\perp}^{(1)}=-\frac{1}{w_{c,i}B}\left[\frac{\partial\nabla_\perp^2\phi}{\partial t}+\mathbf{u}_{i,\perp}^{(0)}\cdot\nabla_\perp\left(\nabla_\perp^2\phi\right)\right], \tag{10.24}$$

and eq. (10.20) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\frac{e\phi}{T_e}\right)+\mathbf{u}_{i,\perp}^{(0)}\cdot\nabla_\perp\left[\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\ln\left(\frac{n_0}{w_{c,i}}\right)\right]=0. \tag{10.25}$$

Plugging in for $\mathbf{u}_{i,\perp}^{(0)}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\frac{e\phi}{T_e}\right)-\left(\nabla_\perp\phi\times\frac{\mathbf{b}}{B}\right)\cdot\nabla_\perp\left[\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\ln\left(\frac{n_0}{w_{c,i}}\right)\right]=0. \tag{10.26}$$

We now introduce the normalizations

$$\phi(t,\mathbf{x})=\frac{T_e}{e}\hat{\phi}(\hat{t},\hat{\mathbf{x}})\qquad n_0(x_1)=\hat{n}_0(\hat{x}_1), \tag{10.27}$$

where $\hat{t}=tw_{c,i}$ and $\hat{\mathbf{x}}=\mathbf{x}/r_s$. Neglecting the hat notation for the sake of simplicity, eq. (10.26) finally becomes

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2\phi-\phi\right)-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp\left[\nabla_\perp^2\phi-\ln\left(\frac{n_0}{w_{c,i}}\right)\right]=0. \tag{10.28}$$

Using the following expansion

$$\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp=\frac{\partial\phi}{\partial x_2}\frac{\partial}{\partial x_1}-\frac{\partial\phi}{\partial x_1}\frac{\partial}{\partial x_2}, \tag{10.29}$$

The Hasegawa-Mima equation can be written as

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2\phi-\phi\right)-\frac{\partial\phi}{\partial x_2}\frac{\partial\nabla_\perp^2\phi}{\partial x_1}+\frac{\partial\phi}{\partial x_1}\frac{\partial\nabla_\perp^2\phi}{\partial x_2}+\beta\frac{\partial\phi}{\partial x_2}=0, \tag{10.30}$$

where

$$\beta=\frac{\partial}{\partial x_1}\ln\left(\frac{n_0}{w_{c,i}}\right). \tag{10.31}$$

10.1.3 Spectral space

In this section we derive the equation for the Fourier coefficient $\hat{\phi}_{\mathbf{n}} = \hat{\phi}(t)_{\mathbf{n}}$, which relates to the potential through the following

$$\phi(t,x) = \sum_{\mathbf{n}=-\infty}^{\infty} \hat{\phi}_{\mathbf{n}}(t) e^{i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}, \tag{10.32}$$

$$\hat{\phi}_{\mathbf{n}}(t) = \frac{1}{L^2}\int_{L^2} \phi(t,\mathbf{x}) e^{-i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}\, d\mathbf{x}. \tag{10.33}$$

We introduce the operator $\mathcal{F}\{\}_{\mathbf{n}}$, which is defined by

$$\mathcal{F}\{\phi(t,\mathbf{x})\}_{\mathbf{n}} = \frac{1}{L^2}\int_{L^2} \phi(t,\mathbf{x}) e^{-i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}\, d\mathbf{x}. \tag{10.34}$$

The equation for $\hat{\phi}_{\mathbf{n}}$ is obtained by applying this operator to eq. (10.28). Thus, the time derivative term in that equation becomes

$$\mathcal{F}\left\{\frac{\partial}{\partial t}\left(\nabla_\perp^2 - \phi\right)\right\}_{\mathbf{n}} = \frac{\partial}{\partial t}\mathcal{F}\left\{\nabla_\perp^2\phi - \phi\right\}_{\mathbf{n}} = \frac{\partial}{\partial t}\left(-k_{\mathbf{n}}^2\hat{\phi}_{\mathbf{n}} - \hat{\phi}_{\mathbf{n}}\right) = -\left(1+k_{\mathbf{n}}^2\right)\frac{\partial\hat{\phi}_{\mathbf{n}}}{\partial t}. \tag{10.35}$$

We assume $\nabla \ln(n_o/w_{ci})$ is constant in space. Thus, the term containing the inhomogeneity becomes

$$\begin{aligned}
&\mathcal{F}\left\{(\nabla_\perp\phi\times\mathbf{b})\cdot\nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right)\right\}_{\mathbf{n}} \\
&\qquad = \left(\mathcal{F}\left\{\nabla_\perp\phi\right\}_{\mathbf{n}}\times\mathbf{b}\right)\cdot\nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right) = i\left(\mathbf{k}_{\mathbf{n}}\times\mathbf{b}\right)\cdot\nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right)\hat{\phi}_{\mathbf{n}}.
\end{aligned} \tag{10.36}$$

The remaining term is computed as follows

$$\begin{aligned}
&\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}} \\
&\quad = \mathcal{F}\left\{-\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left[\hat{\phi}_{\mathbf{n}'} i\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{b}\right)e^{i\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{x}}\right]\cdot\left[\hat{\phi}_{\mathbf{n}''}\left(-ik_{\mathbf{n}''}^2\mathbf{k}_{\mathbf{n}''}\right)e^{i\mathbf{k}_{\mathbf{n}''}\cdot\mathbf{x}}\right]\right\}_{\mathbf{n}} \\
&\quad = -\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{b}\right)\cdot\mathbf{k}_{\mathbf{n}''}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\mathcal{F}\left\{e^{i\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{x}}e^{i\mathbf{k}_{\mathbf{n}''}\cdot\mathbf{x}}\right\} \\
&\quad = \sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''}
\end{aligned} \tag{10.37}$$

Since $\mathbf{n}'$ and $\mathbf{n}''$ are just symbolic variables for the summation, we can write the above as follows

$$\begin{aligned}
&\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}} \\
&\quad =\frac{1}{2}\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''} \\
&\qquad +\frac{1}{2}\sum_{\mathbf{n}''=-\infty}^{\infty}\sum_{\mathbf{n}'=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}''}\times\mathbf{k}_{\mathbf{n}'}\right)\cdot\mathbf{b}k_{\mathbf{n}'}^2\hat{\phi}_{\mathbf{n}''}\hat{\phi}_{\mathbf{n}'}\delta_{\mathbf{n},\mathbf{n}''+\mathbf{n}'} \\
&\quad = \sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\frac{1}{2}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''}
\end{aligned} \tag{10.38}$$

Thus, we finally have

$$\mathcal{F}\left\{-\left(\nabla_{\perp}\phi \times \mathbf{b}\right) \cdot \nabla_{\perp}^{3}\phi\right\}_{\mathbf{n}} = \sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''} \left(\mathbf{k}_{\mathbf{n}'} \times \mathbf{k}_{\mathbf{n}''}\right) \cdot \mathbf{b} \left(k_{\mathbf{n}''}^{2} - k_{\mathbf{n}'}^{2}\right) \hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''} \tag{10.39}$$

Combining the results above, we obtain

$$\frac{\partial \hat{\phi}_{\mathbf{n}}}{\partial t} + i w_{\mathbf{n}} \hat{\phi}_{\mathbf{n}} = \sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''} \Lambda_{\mathbf{n}',\mathbf{n}''}^{\mathbf{n}} \hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}, \tag{10.40}$$

where

$$w_{\mathbf{n}} = -\frac{\left(\mathbf{k}_{\mathbf{n}} \times \mathbf{b}\right)}{1+k_{\mathbf{n}}^{2}} \cdot \nabla_{\perp} \ln\left(\frac{n_o}{w_{ci}}\right), \tag{10.41}$$

and

$$\Lambda_{\mathbf{n}',\mathbf{n}''}^{\mathbf{n}} = \frac{1}{2}\frac{\left(\mathbf{k}_{\mathbf{n}'} \times \mathbf{k}_{\mathbf{n}''}\right) \cdot \mathbf{b} \left(k_{\mathbf{n}''}^{2} - k_{\mathbf{n}'}^{2}\right)}{1+k_{\mathbf{n}}^{2}}. \tag{10.42}$$

Note that $w_{\mathbf{n}}$ can also be written as

$$w_{\mathbf{n}} = -\frac{k_{2,\mathbf{n}}\mathbf{e}_1 - k_{1,\mathbf{n}}\mathbf{e}_2}{1+k_{\mathbf{n}}^{2}} \cdot \beta\mathbf{e}_1 = -\frac{k_{2,\mathbf{n}}\beta}{1+k_{\mathbf{n}}^{2}}. \tag{10.43}$$

10.2 Hasegawa-Wakatani

References for this model can be found in Wakatani and Hasegawa [1984], Hasegawa and Wakatani [1987].

10.2.1 Assumptions

1. Singly-charged ions.
2. No shear stresses in the electron momentum equation, no collisions in the ion momentum equation, no sources.
3. Cold ion approximation, i.e. $T_e \gg T_i$ and thus $\nabla p_i \approx 0$.
4. Isothermal electron fluid, i.e. T_e is constant.
5. Electrostatic field, i.e. $\mathbf{E} = -\nabla\phi$.
6. Magnetic field is constant.
7. Neglect parallel ion velocity, i.e. $u_{i,||} \approx 0$.
8. Quasi-neutrality, i.e. $n_i \approx n_e$.
9. $n_i = n_0 + n'$, where $n_0 = n_0(x_1)$ and n' is smaller than n_0.
10. Perpendicular components of the ion shear-stress term are modeled as $(\nabla \cdot \mathbf{t}_i)_{\perp}/(m_i n_i) = \nu\nabla_{\perp}^{2}\mathbf{u}_{i,\perp}$.
11. Assume infinitesimally small electron mass, i.e. $m_e \to 0$.

10.2.2 Derivation

Using the assumptions in items 1 and 2, the momentum eq. (5.52) for ions becomes

$$m_i n_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right) = e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i + \nabla \cdot \mathbf{t}. \tag{10.44}$$

Using the assumptions in items 3 and 5, the above becomes

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{e}{m_i} \nabla \phi + \frac{e}{m_i} \mathbf{u}_i \times \mathbf{B} + \frac{\nabla \cdot \mathbf{t}_i}{m_i n_i}. \tag{10.45}$$

As before, introduce the coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and assume $\mathbf{B}$ points in the $\mathbf{e}_3$ direction. Defining the perpendicular velocity as $\mathbf{u}_{i,\perp} = [u_{i,1}, u_{i,2}, 0]^T$, the perpendicular gradient as $\nabla_\perp = [\partial_1, \partial_2, 0]^T$, and the perpendicular shear stress as $(\nabla \cdot \mathbf{t}_i)_\perp = [(\nabla \cdot \mathbf{t})_1, (\nabla \cdot \mathbf{t})_2, 0]^T$, we have

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_{i,\perp} = -\frac{e}{m_i} \nabla_\perp \phi + \frac{e}{m_i} \mathbf{u}_i \times \mathbf{B} + \frac{(\nabla \cdot \mathbf{t}_i)_\perp}{m_i n_i}. \tag{10.46}$$

Using the assumption in item 7 and noting that $\mathbf{u}_i \times \mathbf{B} = \mathbf{u}_{i,\perp} \times \mathbf{B}$, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i} \nabla_\perp \phi + \frac{e}{m_i} \mathbf{u}_{i,\perp} \times \mathbf{B} + \frac{(\nabla \cdot \mathbf{t}_i)_\perp}{m_i n_i}. \tag{10.47}$$

Finally, using the assumption in item 10, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i} \nabla_\perp \phi + \frac{e}{m_i} \mathbf{u}_{i,\perp} \times \mathbf{B} + \nu \nabla_\perp^2 \mathbf{u}_{i,\perp}. \tag{10.48}$$

The same scaling analysis performed for the derivation of the Hasegawa Mima equation is now applied. The only new term in eq. (10.48) is the viscous term. We note that the kinematic viscosity ν scales as

$$\nu \sim r^2 w. \tag{10.49}$$

Thus, the viscous stress term leads to the following scalings

1. $\nu \nabla_\perp^2 \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon$
2. $\nu \nabla_\perp^2 \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^2$

As before, the first order terms lead to the $E \times B$ drift

$$\mathbf{u}_{i,\perp}^{(0)} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B}. \tag{10.50}$$

However, the equation for terms of order ϵ now contains the viscous term as shown below

$$\frac{\partial \mathbf{u}_{i,\perp}^{(0)}}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} = \frac{e}{m_i} \mathbf{u}_{i,\perp}^{(1)} \times \mathbf{B} + \nu \nabla_\perp^2 \mathbf{u}_{i,\perp}^{(0)}. \tag{10.51}$$

Upon crossing by $\mathbf{B}$, the above gives

$$\mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i} B} \left[\frac{\partial \nabla_\perp \phi}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp (\nabla_\perp \phi) \right] + \frac{\nu}{w_{c,i} B} \nabla_\perp^2 (\nabla_\perp \phi). \tag{10.52}$$

That is, an additional viscous term appears in the polarization drift.

As shown in the derivation of the Hasegawa-Mima equation, the continuity equation for ions can be expressed in the form of eq. (10.16), which is repeated below for convenience

$$\frac{\partial \ln n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln n_i + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{10.53}$$

Given the assumption in item 9, the natural logarithm of density is re-written as follows

$$\ln n_i = \ln\left(n_0 + n'\right) = \ln\left[n_0\left(1 + \frac{n'}{n_0}\right)\right] = \ln n_0 + \ln\left(1 + \frac{n'}{n_0}\right). \tag{10.54}$$

We now introduce $n = n'/n_0$, which is small due to the assumption in item 9. Thus, a Taylor series expansion would allow us to write

$$\ln n_i = \ln n_0 + n. \tag{10.55}$$

Since n_0 is time independent (assumption in item 9), the continuity equation becomes

$$\frac{\partial n}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\ln n_0 + n\right) + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{10.56}$$

Using the same derivation for eq. (10.24), we now have

$$\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B}\left[\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\nabla_\perp^2 \phi\right)\right] + \frac{\nu}{w_{c,i}B}\nabla_\perp^4 \phi. \tag{10.57}$$

Plugging this in eq. (10.56), we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - n\right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - n - \ln n_0\right) - \frac{\nu}{w_{c,i}B}\nabla_\perp^4 \phi = 0. \tag{10.58}$$

Plugging in for $\mathbf{u}_{i,\perp}^{(0)}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - n\right) - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B}\right) \cdot \nabla_\perp \left(\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - n - \ln n_0\right) - \frac{\nu}{w_{c,i}B}\nabla_\perp^4 \phi = 0. \tag{10.59}$$

Using the assumptions in items 1 and 2, the momentum eq. (5.52) for electrons becomes

$$m_e n_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right) = -e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e + \mathbf{R}_e. \tag{10.60}$$

Using the assumptions in items 4, 5 and 11, the above simplifies to

$$0 = -e n_e (-\nabla \phi + \mathbf{u}_e \times \mathbf{B}) - T_e \nabla n_e + \mathbf{R}_e. \tag{10.61}$$

Dividing by $-en_e$, we get

$$0 = -\nabla \phi + \mathbf{u}_e \times \mathbf{B} + \frac{T_e}{e}\nabla \ln n_e - \frac{1}{e n_e}\mathbf{R}_e. \tag{10.62}$$

We now focus on the component of the equation above that is parallel to $\mathbf{B}$, that is

$$0 = -\nabla_{||}\phi + \frac{T_e}{e}\nabla_{||} \ln n_e - \frac{1}{e n_e} R_{e,||}. \tag{10.63}$$

Using quasineutrality to replace n_e by n_i, and plugging in eq. (10.55) for n_i gives

$$0 = -\nabla_{||}\phi + \frac{T_e}{e}\nabla_{||}(\ln n_0 + n) - \frac{1}{en_e}R_{e,||}. \tag{10.64}$$

We note that the gradient of $\ln n_0$ above is zero since n_0 does not vary along the direction of the magnetic field. The definition of the electron collision term is given by eq. (7.36), that is, $\mathbf{R}_e = (m_e\nu_{ei}/e)\mathbf{J}$. Using the definition of the resistivity given by eq. (7.40) ($\eta = m_e\nu_{ei}/e^2n_e$), we get $\mathbf{R}_e = en_e\eta\mathbf{J}$. Thus, we now have

$$0 = -\nabla_{||}\phi + \frac{T_e}{e}\nabla_{||}n - \eta J_{||}, \tag{10.65}$$

which, upon re-arranging, gives

$$J_{||} = \frac{T_e}{e\eta}\nabla_{||}\left(n - \frac{e\phi}{T_e}\right). \tag{10.66}$$

The perpendicular component of eq. (10.62) is as follows

$$0 = -\nabla_{\perp}\phi + \mathbf{u}_e \times \mathbf{B} + \frac{T_e}{e}\nabla_{\perp}\ln n_e - \frac{1}{en_e}\mathbf{R}_{e,\perp}. \tag{10.67}$$

Since $\mathbf{u}_e \times \mathbf{B} = \mathbf{u}_{e,\perp} \times \mathbf{B}$ we have

$$0 = -\nabla_{\perp}\phi + \mathbf{u}_{e,\perp} \times \mathbf{B} + \frac{T_e}{e}\nabla_{\perp}\ln n_e - \frac{1}{en_e}\mathbf{R}_{e,\perp}. \tag{10.68}$$

Again, using the definition of the electron collision term, we get

$$0 = -\nabla_{\perp}\phi + \mathbf{u}_{e,\perp} \times \mathbf{B} + \frac{T_e}{e}\nabla_{\perp}\ln n_e - \eta\mathbf{J}_{\perp}. \tag{10.69}$$

Crossing the above by $\mathbf{B}$ gives

$$\mathbf{u}_{e,\perp} = -\nabla_{\perp}\phi \times \frac{\mathbf{b}}{B} - \frac{T_e}{en_eB}\mathbf{b} \times \nabla_{\perp}n_e + \frac{\eta}{B}\mathbf{b} \times \mathbf{J}_{\perp}. \tag{10.70}$$

Typically the last term on the right-hand side above is significantly smaller, and thus it can be neglected. The electron velocity is thus

$$\mathbf{u}_{e,\perp} = -\nabla_{\perp}\phi \times \frac{\mathbf{b}}{B} - \frac{T_e}{en_eB}\mathbf{b} \times \nabla_{\perp}n_e. \tag{10.71}$$

The continuity equation for electrons is as follows

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\mathbf{u}_e) = 0. \tag{10.72}$$

We split the convection term in the above into the perpendicular and parallel components

$$\frac{\partial n_e}{\partial t} + \nabla_{\perp} \cdot (n_e\mathbf{u}_{e,\perp}) + \nabla_{||} \cdot (n_e u_{e,||}) = 0. \tag{10.73}$$

Given the assumption in item 7, we have $J_{||} = en_e(u_{i,||} - u_{e,||}) = -en_eu_{e,||}$. Thus, the above becomes

$$\frac{\partial n_e}{\partial t} + \nabla_{\perp} \cdot (n_e\mathbf{u}_{e,\perp}) = \frac{1}{e}\nabla_{||}J_{||}. \tag{10.74}$$

Using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$ we show that

$$\nabla_\perp \cdot (\mathbf{b} \times \nabla_\perp n_e) = \nabla_\perp n_e \cdot (\nabla_\perp \times \mathbf{b}) - \mathbf{b} \cdot (\nabla_\perp \times \nabla_\perp n_e) = 0. \tag{10.75}$$

As a result, the second term on the right-hand side of eq. (10.71) does not contribute to $\nabla_\perp \cdot (n_e \mathbf{u}_{e,\perp})$. The electron continuity equation then becomes

$$\frac{\partial n_e}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp n_e = \frac{1}{e} \nabla_{||} J_{||}. \tag{10.76}$$

Dividing by n_e and using quasi-neutrality

$$\frac{\partial \ln n_i}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp \ln n_i = \frac{1}{e n_i} \nabla_{||} J_{||}. \tag{10.77}$$

Using the expression for $\ln n_i$ in eq. (10.55), we get

$$\frac{\partial n}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp (\ln n_0 + n) = \frac{1}{e n_i} \nabla_{||} J_{||}. \tag{10.78}$$

Finally, using the assumption in item 9, we neglect n' in the denominator of the right-hand side, and obtain

$$\frac{\partial n}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp (n + \ln n_0) = \frac{1}{e n_0} \nabla_{||} J_{||}. \tag{10.79}$$

Combining eqs. (10.59), (10.66) and (10.79) leads to the dimensional form of the Hasegawa-Wakatani model

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{w_{c,i} B} \nabla_\perp^2 \phi \right) - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp \left(\frac{1}{w_{c,i} B} \nabla_\perp^2 \phi \right) \\ = \frac{T_e}{e^2 n_0 \eta} \nabla_{||}^2 \left(n - \frac{e\phi}{T_e} \right) + \frac{\nu}{w_{c,i} B} \nabla_\perp^4 \phi. \end{aligned} \tag{10.80}$$

$$\frac{\partial n}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp (n + \ln n_0) = \frac{T_e}{e^2 n_0 \eta} \nabla_{||}^2 \left(n - \frac{e\phi}{T_e} \right). \tag{10.81}$$

It is quite common to replace the parallel-gradient operator $\nabla_{||}$ by a coefficient, say $1/l^2$.

We now introduce the following non-dimensionalization

$$\phi(t, x_1, x_2, x_3) = \frac{T_e}{e} \hat{\phi}(\hat{t}, \hat{x}_1, \hat{x}_2, x_3) \tag{10.82}$$

$$n_0(x_1) = \hat{n}_0(\hat{x}_1) \tag{10.83}$$

$$n(t, x_1, x_2, x_3) = \hat{n}(\hat{t}, \hat{x}_1, \hat{x}_2, x_3), \tag{10.84}$$

where $\hat{t} = t w_{c,i}$, $\hat{x}_1 = x_1/r_s$, and $\hat{x}_2 = x_2/r_s$. Neglecting the hat notation for the sake of simplicity, the Hasegawa-Wakatani model in non-dimensional form is written as

$$\frac{\partial \nabla_\perp^2 \phi}{\partial t} - (\nabla_\perp \phi \times \mathbf{b}) \cdot \nabla_\perp \left(\nabla_\perp^2 \phi \right) = c_1 (\phi - n) + c_2 \nabla_\perp^4 \phi, \tag{10.85}$$

$$\frac{\partial n}{\partial t} - (\nabla_\perp \phi \times \mathbf{b}) \cdot \nabla_\perp (n + \ln n_0) = c_1 (\phi - n), \tag{10.86}$$

where

$$c_1 = -\frac{T_e}{e^2 n_0 \eta w_{c,i}} \nabla_{||}^2 \qquad c_2 = \frac{\nu}{w_{c,i} r_s^2}. \tag{10.87}$$

If $\nabla_{||}$ is replaced by $1/l^2$, then the c_1 operator is simply a coefficient.

10.2.3 Relationship to other models

The Hasegawa-Wakatani eqs. (10.85) and (10.86) contain two limits. Assuming c_1 is a coefficient rather than an operator, one of the limits is obtained by letting $c_1 = 0$. Then, the ϕ and n equations are decoupled, and the ϕ equation corresponds to the third (and only non-zero) component of the 2D Navier-Stokes equations for vorticity

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{u} \cdot \nabla_\perp \mathbf{w} = \nu \nabla_\perp^2 \mathbf{w}, \tag{10.88}$$

where

$$\mathbf{u} = \nabla_\perp \times (-\phi \mathbf{b}) = -\nabla_\perp \phi \times \mathbf{b}, \tag{10.89}$$

and

$$\mathbf{w} = \nabla_\perp \times \mathbf{u} = \nabla_\perp^2 \phi \mathbf{b}. \tag{10.90}$$

If, on the other hand, $c_1 \to \infty$, then dividing eq. (10.86) by c_1 shows that $n = \phi$. Subtracting eq. (10.86) from eq. (10.85) and assuming $c_2 = 0$ one obtains

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2 \phi - \phi\right) - \left(\nabla_\perp \phi \times \mathbf{b}\right) \cdot \nabla_\perp \left(\nabla_\perp^2 \phi - \ln n_0\right) = 0. \tag{10.91}$$

The above is the Hasegawa-Mima equation.

Part IV

Appendices

Appendix A

Electromagnetism

This appendix first focuses on electrostatics and magnetostatics, which can be understood as follows

$$\begin{aligned}\text{stationary charges} &\rightarrow \text{constant electric fields} = \text{electrostatics}\\ \text{stationary currents} &\rightarrow \text{constant magnetic fields} = \text{magnetostatics.}\end{aligned} \tag{A.1}$$

A.1 Electrostatics

- Coulomb's Law

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}} \tag{A.2}$$

- Electric Field $\mathbf{E}$ derived from $\mathbf{F} = Q\mathbf{E}$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{A.3}$$

- If there are multiple point charges

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i \tag{A.4}$$

- **Charge distributions and fields:** if the charges are so small and so numerous that they can be described using a continuous distribution (i.e. $q_i \rightarrow dq = \rho d\tau$, where ρ is a charge density and $d\tau$ and infinitesimal volume)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{r^2}\hat{\mathbf{r}}d\tau' \tag{A.5}$$

If the charge distribution is localized to a surface or a line, then the analogous of the above is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{r}')}{r^2}\hat{\mathbf{r}}da' \qquad \text{or} \qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{r^2}\hat{\mathbf{r}}dl' \tag{A.6}$$

Taking the divergence and curl of eq. (A.5):

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{A.7}$$

$$\nabla\times\mathbf{E} = 0 \tag{A.8}$$

- **Fields and potentials**
  Since $\nabla \times \mathbf{E} = 0$ we have
$$\mathbf{E} = -\boldsymbol{\nabla} V. \tag{A.9}$$
  where V is the electric potential. Fundamental theorem of calculus can be used to express the potential $V(\mathbf{r})$ as
$$V(\mathbf{r}) - V(\mathcal{O}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{A.10}$$
  where $\mathcal{O}$ is the reference point, at which one usually defines $V(\mathcal{O}) = 0$ (e.g. sea-level as the altitude at which height is equal to zero).

- **Charge distributions and potentials**
  Divergence of eq. (A.9) gives
$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{A.11}$$
  whose solution is
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{r} d\tau'. \tag{A.12}$$

- Define potential energy U as the negative of the work required to move charge Q from $\mathbf{a}$ to $\mathbf{b}$.
$$U = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})] \tag{A.13}$$
  If the reference point is infinity, then $U(\mathbf{r}) = QV(\mathbf{r})$.

- Potential energy of a set of charges q_i
$$U = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{A.14}$$
  where $V(\mathbf{r}_i)$ is the potential due to all charges except the one at $\mathbf{r}_i$. The continuous form is
$$U = \frac{1}{2}\int \rho V \, d\tau = \frac{\epsilon_0}{2}\int E^2 \, d\tau \tag{A.15}$$
  were now V represents the potential due to all charges. Thus, if ρ is such that it defines a set of point charges (e.g. $\delta(\mathbf{r})$), then eq. (A.15) would be equal to eq. (A.14) plus the additional terms corresponding to $i = j$. Those additional terms correspond to the energy required to create point charges, which is infinity.

- Electrostatic conductors: materials whose charges are free to move but are in a state of electrostatic equilibrium. $\mathbf{E} = 0$ inside, since if it were not, then charges would move and the material would not be in electrostatic equilibrium. As a consequence, $\rho = 0$ inside, all the charge is on the surface, and $\mathbf{E}$ is perpendicular to the outer surface.

- If there is a cavity within the conductor, and within the cavity a charge q, an amount $-q$ of charge will reside in the inner surface, and an amount q on the outer surface, and that configuration will lead to $\mathbf{E} = 0$ inside the conductor.

- Faraday cage: if there are no charges within such cavity, then $\mathbf{E} = 0$ within the cavity as well, regardless of how many charges are outside the conductor. If $\mathbf{E}$ was not zero inside the cavity, then its field lines would start and end on the cavity walls. Letting the field lines be part of a closed loop, the rest of which is inside the conductor, then the line integral along the closed loop would be positive, in violation of $\nabla \times \mathbf{E} = 0$.

- A capacitor consists of two conductors, one with charge Q and the other with charge $-Q$. The constant of proportionality between Q and the voltage difference between the two conductors is the capacitance $C = Q/V$. The energy stored in a capacitor is $W = \frac{1}{2}CV^2$.

A.2 Magnetostatiscs

- Lorentz force law: $\mathbf{F} = Q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$
- Given the charge densities λ, σ, and ρ
  - Current [Amperes]: the amount of charge that passes a point in a small amount of time.
  $$\mathbf{I} = \lambda \mathbf{v} \tag{A.16}$$
  - Surface current density: the amount of charge that passes a line in a small amount of time.
  $$\mathbf{K} = \sigma \mathbf{v} \tag{A.17}$$
  - Volume current density: the amount of charge that passes an area in a small amount of time.
  $$\mathbf{J} = \rho \mathbf{v} \tag{A.18}$$
- Magnetic component of Lorentz force
$$\mathbf{F}_{\text{mag}} = \int \mathbf{I} \times \mathbf{B}\, dl = \int \mathbf{K} \times \mathbf{B}\, da = \int \mathbf{J} \times \mathbf{B}\, d\tau \tag{A.19}$$
- Conservation of current
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{A.20}$$
- **Charge currents and fields**
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, dl' \tag{A.21}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, da' \tag{A.22}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, d\tau' \tag{A.23}$$

  Taking the divergence and curl of eq. (A.23):
$$\nabla \cdot \mathbf{B} = 0 \tag{A.24}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{A.25}$$
- A steady straight-line current leads to a circular magnetic field around it. A steady circular current leads to a straight magnetic field line along the axis of the circle.
- **Fields and potentials**
  Since $\nabla \cdot \mathbf{B} = 0$ we have
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{A.26}$$
  where $\mathbf{A}$ is the magnetic vector potential.

- **Charge currents and potentials**
  The magnetic field is not altered if a function whose curl vanishes (that is $\nabla\lambda$) is added to **A**. Thus, λ can be picked to make **A** divergence-less. Taking the curl of **B** then leads to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \tag{A.27}$$

  whose solution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau'. \tag{A.28}$$

A.3 Electric Fields in Matter

A.4 Magnetic Fields in Matter

A.5 Electrodynamics

A.5.1 Ohm's Law

- Ohm's law refers to the proportionality between the force per unit charge applied to charged elements and the resulting volume current that occurs. That is,

$$\mathbf{J} = \sigma \mathbf{f}, \tag{A.29}$$

  where **f** is the force per unit charge, and the proportionality σ is the conductivity. If one neglects the magnetic contribution to **f**, which is typically done for non-plasmas, then

$$\mathbf{J} = \sigma \mathbf{E}. \tag{A.30}$$

  For steady currents ($\partial\rho/\partial t = 0$) and uniform conductivity

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = 0 \tag{A.31}$$

  and thus, the charge density is zero. This is similar to a conductor, but now we have charges moving.

- Similarly, given an applied voltage, a current will result. The constant of proportionality R, known as the resistance, is given by

$$V = IR. \tag{A.32}$$

A.5.2 Electromagnetic induction

- Defined the electromotive force (emf) as

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \tag{A.33}$$

- The universal flux rule states: whenever the magnetic flux through a loop

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \tag{A.34}$$

  changes, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{A.35}$$

  will appear in the loop. This can occur in two ways:

1. Magnetic field doesn't change, loop changes:
   For example, a loop of wire is pulled to the right through a constant magnetic field. In this case the emf is magnetic.
2. Magnetic field changes, loop doesn't change:
   There is a stationary loop (any loop, not necessarily a physical loop of wire), and the magnetic field through it changes. In this case, the **changing magnetic field induces an electric field** and thus the emf is electric. Using eq. (A.35) we get **Faraday's law**
   $$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}, \tag{A.36}$$
   which, in differential form is
   $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{A.37}$$

- Lenz's law: Nature abhors a change in flux. Thus, as the magnetic flux changes and it induces an electric field over a loop, the resulting current goes in a direction such that it would create an opposing flux that tries to cancel the original change in magnetic flux.

- Mutual inductance:
  If there is a steady current going through a wire loop, this will create a magnetic field and thus a magnetic flux through another wire loop close by. The constant of proportionality between the flux through the second loop and the current in the first is the mutual inductance M. That is
  $$\Phi_2 = M_{21} I \tag{A.38}$$
  Note: If I ran the same current on loop two, then the flux in loop one would be $\Phi_1 = M_{12} I$. However, it can be shown that $M_{21} = M_{12}$ and thus $\Phi_1 = \Phi_2$.

  Now, imagine the current in loop one changes in time. The magnetic field associated with that current changes in time, and thus the magnetic flux through loop two changes as well. That is,
  $$\Phi_2(t) = M I_1(t). \tag{A.39}$$
  Due to Faraday's law an induced emf would be created in the second loop,
  $$\mathcal{E}_2(t) = -M \frac{dI(t)}{dt}. \tag{A.40}$$
  This emf creates a current $I_2(t)$ in the second loop.

- Self inductance:
  The changing magnetic field associated with the changing current in loop one also creates a changing flux within this loop. This is given by
  $$\Phi_1(t) = L I_1(t), \tag{A.41}$$
  where L is the self-inductance. Again, the changing flux leads to an emf within loop one, called the back emf
  $$\mathcal{E}(t) = -L \frac{dI(t)}{dt}. \tag{A.42}$$
  This emf drives a new current in loop one that opposes the original current change.

- The energy stored in magnetic fields is given by
  $$W = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{J} \, d\tau = \frac{1}{2\mu_0} \int B^2 \, d\tau. \tag{A.43}$$

- Ampere's law eq. (A.25) was derived using assumptions of magnetostatics. Maxwell extended Ampere's law to work for magnetodynamics, so that the divergence of eq. (A.25) would actually give zero on both sides. Thus, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{A.44}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{A.45}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{A.46}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{A.47}$$

- As shown earlier, Faraday's law indicates that a changing magnetic field induces an electric field. Maxwell's correction to Ampere's law then indicates that a changing electric field induces a magnetic field.

A.6 Conservation Laws

A.6.1 Conservation of energy

- Suppose you assemble a distribution of charges and currents, which at time t produce fields $\mathbf{E}$ and $\mathbf{B}$.

- The potential energy of the system, as shown in previous sections, would be

$$U_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right). \tag{A.48}$$

- The question then arises, how much energy would be transferred to the charges as these charges are allowed to move?

- Label U_{mech} as the energy density gained by these charges as they are allowed to move.

- The evolution of U_{mech} is then

$$\frac{\partial U_{mech}}{\partial t} = -\frac{\partial U_{em}}{\partial t} - \nabla \cdot \mathbf{S}. \tag{A.49}$$

  In the above $\mathbf{S}$ is the pointing vector and is defined as

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}). \tag{A.50}$$

  It represents the flux of energy across space.

- As the evolution equation above shows, a decreasing potential energy in the electromagnetic field constitutes a transfer of this lost energy to that of the charges.

- In integral form, we write the above as

$$\frac{d}{dt}\int_V U_{mech}\, d\tau = -\frac{d}{dt}\int_V U_{em}\, d\tau - \oint \mathbf{S} \cdot d\mathbf{a}. \tag{A.51}$$

A.6.2 Conservation of momentum

- We now ask, how much momentum would be transferred to the charges as these are allowed to move?
- Label P_{mech} as the momentum density gained by the charges as they move around.
- Label P_{em} as the momentum density stored in the electromagnetic fields themselves. This is defined as
$$P_{em} = \mu_0 \epsilon_0 \mathbf{S}. \tag{A.52}$$
- The evolution of P_{mech} is then
$$\frac{\partial P_{mech}}{\partial t} = -\frac{\partial P_{em}}{\partial t} + \nabla \cdot \mathbf{T}. \tag{A.53}$$
In the above $\mathbf{T}$ is the Maxwell stress tensor and is defined as
$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right). \tag{A.54}$$
It represents the flux of momentum across space.
- As the evolution equation above shows, a decreasing momentum in the electromagnetic field constitutes a transfer of this lost momentum to that of the charges.
- In integral form, we write the above as
$$\frac{d}{dt} \int_V P_{mech}\, d\tau = -\frac{d}{dt} \int_V P_{em}\, d\tau + \oint \mathbf{T} \cdot d\mathbf{a}. \tag{A.55}$$

A.7 Electromagnetic waves

A.7.1 Simple waves

- The simplest kind of waves can be written as
$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t + \phi \right) \tag{A.56}$$
where
$$\begin{aligned} A &: \text{magnitude} \\ \lambda &: \text{wavelength} \\ T &: \text{period} \\ \phi &: \text{phase constant} \end{aligned}$$
Thus, as x goes from zero to λ, for example, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in space. Similarly, as t goes from zero to T, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in time.
- Defining the wavevector and angular frequency as
$$k = \frac{2\pi}{\lambda} \qquad w = \frac{2\pi}{T}, \tag{A.57}$$
then
$$u(x,t) = A \sin(kx - wt + \phi), \tag{A.58}$$

- The frequency ν is the inverse of the period, $\nu = 1/T$.
- By inspecting the form of the simple sinusoidal wave above, it is clear that the velocity of the wave is

$$v = \frac{w}{k} = \frac{\lambda}{T} = \lambda\nu. \tag{A.59}$$

- A general wave can be Fourier decomposed as follows

$$u(x,t) = \sum_n \hat{u}_n e^{i(k_n z - wt + \phi)}, \tag{A.60}$$

where $k_n = 2\pi n/L$. The above is often re-written as

$$u(x,t) = \sum_n \tilde{u}_n e^{i(k_n z - wt)}, \tag{A.61}$$

where $\tilde{u}_n = \hat{u}_n e^{i\phi}$.

- For the more general three-dimensional case, a wave is decomposed as follows

$$\mathbf{u}(x,t) = \sum_{\mathbf{n}} \tilde{\mathbf{u}}_{\mathbf{n}} e^{i(\mathbf{k_n}\cdot\mathbf{x} - wt)}, \tag{A.62}$$

where $\mathbf{k_n} = 2\pi\mathbf{n}/L$ and $\mathbf{n} = [n_1, n_2, n_3]$.

- A plane wave is one for which the only existing $\mathbf{k_n}$'s point along a single direction. Without loss of generality, we can assume this direction is the z direction and thus write

$$\mathbf{u}(x,t) = \sum_{n_3} \tilde{\mathbf{u}}_{n_3} e^{i\left(k_{n_3}\cdot z - wt\right)}, \tag{A.63}$$

A.7.2 Electromagnetic waves in vacuum

- The application of eq. (A.62) to electric and magnetic fields gives

$$\mathbf{E} = \sum_{\mathbf{n}} \tilde{\mathbf{E}}_{\mathbf{n}} e^{i(\mathbf{k_n}\cdot\mathbf{x} - wt)}, \tag{A.64}$$

and

$$\mathbf{B} = \sum_{\mathbf{n}} \tilde{\mathbf{B}}_{\mathbf{n}} e^{i(\mathbf{k_n}\cdot\mathbf{x} - wt)}. \tag{A.65}$$

- For $\rho = \mathbf{J} = 0$, Maxwell's equations can be combined to give the wave equations for $\mathbf{E}$ and $\mathbf{B}$, that is,

$$\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{1}{\epsilon_0\mu_0}\nabla^2\mathbf{E} = 0, \tag{A.66}$$

$$\frac{\partial^2\mathbf{B}}{\partial t^2} - \frac{1}{\epsilon_0\mu_0}\nabla^2\mathbf{B} = 0. \tag{A.67}$$

The speed of electromagnetic waves is thus $c = 1/\sqrt{\epsilon_0\mu_0}$.

- Using eq. (A.64) in $\nabla\cdot\mathbf{E} = 0$ gives $\mathbf{k}_n\cdot\tilde{\mathbf{E}}_{\mathbf{n}} = 0$. That is, the $\mathbf{E}$ field is orthogonal to the direction of propagation of the mode.
- Using eq. (A.65) in $\nabla\cdot\mathbf{B} = 0$ gives $\mathbf{k}_n\cdot\tilde{\mathbf{B}}_{\mathbf{n}} = 0$. That is, the $\mathbf{B}$ field is orthogonal to the direction of propagation of the mode.
- Using eqs. (A.64) and (A.65) in $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ gives $\mathbf{k_n}\times\tilde{\mathbf{E}}_{\mathbf{n}} = w\tilde{\mathbf{B}}_{\mathbf{n}}$. That is, the $\mathbf{B}$ field is orthogonal to the $\mathbf{E}$ field.

Appendix B

Nuclear Fusion

B.1 Basic definitions

- Atomic number (Z): # of protons
- Mass number (A): # of protons + # of neutrons
- Atomic mass (m_a): mass of a particular isotope of an element.
- Relative atomic mass (A_r): (defined for an element only)(previously referred to as atomic weight). Average of the atomic masses of all the different isotopes in a *sample*, with each isotope's contribution to the average being its abundance within the sample (as a percentage).
- Standard Atomic Weight (A_r^o): (defined for an element only). Average of the atomic masses of all the different isotopes in *planet earth*, with each isotope's contribution to the average being tis abundance in earth (as a percentage).
- Atomic mass unit (u): unit of mass, equivalent to $\frac{1}{12}$ the mass of a carbon-12 atom. That is

$$1u = \frac{m_c}{12}. \tag{B.1}$$

  where m_c is the mass of a carbon-12 atom, in grams. Think of u as similar to a microgram.

- Mole: # of elementary entities equal to # of atoms in 12 grams of carbon-12. That is,

$$1mol = \frac{12g}{m_c} \tag{B.2}$$

  Using eq. (B.1), we get

$$1u = \frac{1}{mol}g. \tag{B.3}$$

  The value of the mole is $6.02214086 \times 10^{23}$.

- Molar mass (M):
  - If it is an atom (e.g. Carbon, C), then it is its atomic weight, but one uses eq. (B.3) to express the value in g/mol.
  - If it is a compound (e.g. Methane, CH_4), simply add up the atomic weights of each atom in the molecule, and again, express the result in g/mol.

- If it is a mixture (e.g. air, $N_2, O_2, Ar, CO_2, ...$), then it is the weighted average of the atomic weights of the constituents, and the result again is expressed in g/mol.

- Avogadro's number (N_a): a conversion factor so that things can be measured in terms of moles.

$$N_a = \frac{6.02214086 \times 10^{23}}{mol} \tag{B.4}$$

B.2 The fusion reaction

- The fundamental relation for nuclear reactions is $E = mc^2$. A mass m can be transformed into energy E, and viceversa. Two examples for m are the following:
  - Defect mass (Dm): the difference in mass between the atom and the sum of its constituents,

$$Dm = Nm_n + Zm_p - m_a. \tag{B.5}$$

For carbon

$$Dm = 6 \times 1.008664u + 6 \times 1.007276u - 12u = 0.09564u. \tag{B.6}$$

For fluorine

$$Dm = 10 \times 1.008664u + 9 \times 1.007276u - 18.998403u = 0.154u. \tag{B.7}$$

The binding energy is then the energy corresponding to the mass defect as given by $E = (Dm)c^2$.

  - Mass change of a fusion reaction:

$$m = \text{mass of particles before reaction} - \text{mass of particles after reaction} \tag{B.8}$$

Consider the DT reaction as an example, then we have

$$m = 2.013553u\ (D) + 3.015501u\ (T) - 4.001503u\ (\alpha) - 1.008665u\ (n) = 0.018886u \tag{B.9}$$

The above mass translates to $E_f = mc^2 = 17.6MeV$.

- Momentum conservation:
Lets assume the particles before a fusion reaction move sufficiently slow that their velocities can be neglected. Conservation of momentum thus gives

$$0 = m_1 v_1 + m_2 v_2, \tag{B.10}$$

where m_1, m_2, v_1, v_2 are the mass and velocity of particles after the reaction.

- Energy conservation:
Energy is not conserved since some of the mass is converted to energy. The energy balance can be written as $E_{after} - E_{before} = E_f$. Assuming again that the particles before a fusion reaction move sufficiently slow, then

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = E_f, \tag{B.11}$$

where E_f is obtained from Einstein's equation.

B.3 Fusion power density

The fusion power density S_f is the fusion energy produced per unit volume per unit time. Label the energy generated by each fusion collision between particles 1 and 2 by E_f, and the number of those fusion collisions per unit volume per unit time (also known as reaction rate) as R_{12}. Then the fusion power density is given by

$$S_f = E_f R_{12}. \tag{B.12}$$

We note that E_f is an energy released by the reaction (it can either be the total energy, the energy carried out by the alpha particles only, the energy carried out by the neutrons only, etc.).

The reaction rate between two distinct particle is given by

$$R_{12} = n_1 n_2 \langle \sigma v \rangle, \tag{B.13}$$

where n_1 and n_2 are the number densities of particles 1 and 2, respectively. The expected value $\langle \sigma v \rangle$ is given by

$$\langle \sigma v \rangle = \frac{1}{n_1 n_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) \sigma(v) v \, d\mathbf{v}_1 d\mathbf{v}_2. \tag{B.14}$$

Thus, the fusion power density can be expressed as

$$S_f = E_f \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) \sigma(v) v \, d\mathbf{v}_1 d\mathbf{v}_2. \tag{B.15}$$

Using the definition of the cross-section eq. (4.1), the above becomes

$$S_f = E_f \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\infty} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) F(v,b) vb \, db d\phi d\mathbf{v}_1 d\mathbf{v}_2. \tag{B.16}$$

For cases in which we are not interested in the energy generated by the collision, but instead on some other physical property associated with the collision (for example change in momentum rather than change in energy) then the above needs to be generalized. Thus, we would use

$$S = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\infty} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) E(v,b) F(v,b) vb \, db d\phi d\mathbf{v}_1 d\mathbf{v}_2, \tag{B.17}$$

where $E(v,b)$ is the physical property associated with the collision.

Appendix C

Lagrangian and Eulerian PDFs

C.1 Eulerian PDF

Consider an Eulerian velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. The Eulerian PDF $f = f(\mathbf{V}; \mathbf{x}, t)$ gives the probability that the velocity field will have a value of $\mathbf{V}$ at location $\mathbf{x}$ and at time t. We'll also introduce the fine-grained Eulerian PDF $f' = f'(\mathbf{V}; \mathbf{x}, t)$, which is defined as

$$f'(\mathbf{V}; \mathbf{x}, t) = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}). \tag{C.1}$$

Note: a delta function of a 3D argument means the following $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$. The Eulerian PDF can be obtained from the fine-grained Eulerian PDF using

$$f(\mathbf{V}; \mathbf{x}, t) = \langle f'(\mathbf{V}; \mathbf{x}, t)\rangle. \tag{C.2}$$

The proof is as follows,

$$\begin{aligned} \langle f'(\mathbf{V}; \mathbf{x}, t)\rangle &= \langle \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V})\rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) f(\mathbf{V}'; \mathbf{x}, t) d\mathbf{V}' \\ &= f(\mathbf{V}; \mathbf{x}, t). \end{aligned} \tag{C.3}$$

C.2 Lagrangian PDF

Consider a Lagrangian particle with velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$ and position $\mathbf{x}^+(t, \mathbf{y})$. The Lagrangian PDF $f_L = f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$ gives the probability that the particle that started at location $\mathbf{y}$ at the reference time t_0 will have a velocity $\mathbf{V}$ and position $\mathbf{x}$ at time t. We'll also introduce the fine-grained Eulerian PDF $f'_L = f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$, which is defined as

$$f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}). \tag{C.4}$$

Note: a delta function of a 3D argument means the following $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$. The Lagrangian PDF can be obtained from the fine-grained Lagrangian PDF using

$$f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\rangle. \tag{C.5}$$

The proof is as follows,

$$\begin{aligned} \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\rangle &= \langle \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})\rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V})\delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{V}', \mathbf{x}'; t|\mathbf{y}) d\mathbf{V}' d\mathbf{x}' \\ &= f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}). \end{aligned} \tag{C.6}$$

C.3 Relation between Lagrangian and Eulerian PDFs

As a quick side note, we mention that the inverse of $\mathbf{x}^+$ is $\mathbf{y}^+ = \mathbf{y}^+(t, \mathbf{z})$, which gives the initial location of a fluid particle that at time t is located at position $\mathbf{z}$. Thus, $\mathbf{x}^+(t, \mathbf{y}^+(t, \mathbf{z})) = \mathbf{z}$.

We begin as follows

$$\begin{aligned}\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} &= \int \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})\, d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})\, d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})|\det D\mathbf{x}^+|\, d\mathbf{y},\end{aligned} \tag{C.7}$$

where we have introduced $|\det D\mathbf{x}^+|$, which is the absolute value of the determinant of the Jacobean $\partial\mathbf{x}^+/\partial\mathbf{y}$, and is equal to one for incompressible flows. Using integration by substitution we obtain

$$\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} = \int \delta(\mathbf{u}(\mathbf{z}, t) - \mathbf{V})\delta(\mathbf{z} - \mathbf{x})\, d\mathbf{z} = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \tag{C.8}$$

Given the definition of $f'(\mathbf{V}; \mathbf{x}, t)$, we have

$$\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} = f'(\mathbf{V}; \mathbf{x}, t). \tag{C.9}$$

Taking the expectation of the above we obtain

$$\int f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} = f(\mathbf{V}; \mathbf{x}, t). \tag{C.10}$$

A summary of all of the relations derived thus far is given by the following graph

$$\begin{tikzcd}
\text{Eulerian fine-grained PDF} \arrow[d, "\text{eq. (C.2)}"] & \text{Lagrangian fine-grained PDF} \arrow[l, "\text{eq. (C.9)}"'] \arrow[d, "\text{eq. (C.5)}"] \\
\text{Eulerian PDF} & \text{Lagrangian PDF} \arrow[l, "\text{eq. (C.10)}"']
\end{tikzcd}$$

C.4 Evolution equation for fine-grained Eulerian PDF

C.5 Evolution equation for fine-grained Lagrangian PDF

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