

Part I

Introduction

Chapter 1

Governing equations

1.1 Particle description

Table 1.1: Various coordinates of classical mechanics.

Classical coordinates	$\mathbf{x}(t)$	$\mathbf{v}(t)$
Generalized coordinates	\mathbf{q}	$\dot{\mathbf{q}}$
Canonical coordinates	\mathbf{q}	\mathbf{p}
Time-dependent canonical coordinates	$\tilde{\mathbf{q}}(t)$	$\tilde{\mathbf{p}}(t)$

1.1.1 Lagrangian mechanics

- Define the position $\mathbf{x} = \mathbf{x}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ of a particle.
- Define the Lagrangian as $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where \mathbf{q} and $\dot{\mathbf{q}}$ are the generalized position and generalized velocity, respectively.
- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \right] = \left(\frac{\partial L}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}}. \quad (1.1)$$

- For example, the Lagrangian of a particle in an electromagnetic field where $\phi = \phi(\mathbf{q}, t)$ is the electric potential and $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$ is the magnetic potential, is

$$L = \frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e \phi. \quad (1.2)$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e \dot{q}_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \quad (1.3)$$

$$\frac{\partial L}{\partial \dot{q}_i} = m \dot{q}_i + e A_i \quad (1.4)$$

$$\begin{aligned} \frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \right] &= \frac{d}{dt} [mv_i + eA_i(\mathbf{x}, t)] \\ &= m \frac{dv_i}{dt} + ev_j \left(\frac{\partial A_i}{\partial q_j} \right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial A_i}{\partial t} \right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \quad (1.5)$$

Thus, the Euler-Lagrange equation becomes

$$m \frac{dv_i}{dt} = \left(-ev_j \frac{\partial A_i}{\partial q_j} - e \frac{\partial A_i}{\partial t} + ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.6)$$

In vector notation, this is written as

$$m \frac{d\mathbf{v}}{dt} = \left(-e\mathbf{v} \cdot \nabla_q \mathbf{A} - e \frac{\partial \mathbf{A}}{\partial t} + e \nabla_q (\mathbf{v} \cdot \mathbf{A}) - e \nabla_q \phi \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.7)$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}, \quad (1.8)$$

where $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$.

1.1.2 Hamiltonian mechanics

- Define the Hamiltonian as $H = H(\mathbf{q}, \mathbf{p}, t)$, where \mathbf{q} and \mathbf{p} are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.
- The Hamiltonian is obtained from the Lagrangian using

$$H = (\dot{\mathbf{q}} \cdot \mathbf{p} - L)_{\dot{\mathbf{q}}=\mathbf{f}(\mathbf{q}, \mathbf{p})}, \quad (1.9)$$

where the dependency of $\dot{\mathbf{q}}$ on \mathbf{q} and \mathbf{p} is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \quad (1.10)$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[\dot{q}_i p_i - \left(\frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e \phi \right) \right]_{\dot{\mathbf{q}}=\mathbf{f}(\mathbf{q}, \mathbf{p})}. \quad (1.11)$$

Evaluating eq. (1.10) gives $p_i = m \dot{q}_i + e A_i$, which allows us to express $\dot{\mathbf{q}}$ in terms of \mathbf{q} and \mathbf{p} as $\dot{q}_i = \frac{1}{m} (p_i - e A_i)$. Thus

$$\begin{aligned} H &= \frac{1}{m} (p_i - e A_i) p_i - \frac{1}{2m} (p_i - e A_i) (p_i - e A_i) - e \frac{1}{m} (p_i - e A_i) A_i + e \phi \\ &= \frac{1}{2m} (p_i - e A_i) (p_i - e A_i) + e \phi. \end{aligned} \quad (1.12)$$

- We introduce the variables $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$, which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \quad (1.13)$$

$$\tilde{\mathbf{p}} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \quad (1.14)$$

- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left(\frac{\partial H}{\partial p_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} \quad (1.15)$$

$$\frac{d\tilde{p}_i}{dt} = - \left(\frac{\partial H}{\partial q_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} \quad (1.16)$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x}, t) \quad (1.17)$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m \frac{dv_i}{dt} + ev_j \left(\frac{\partial A_j}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial A_i}{\partial t} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.18)$$

$$\begin{aligned} \frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i} \left[\frac{1}{2m} (p_j - eA_j)(p_j - eA_j) + e\phi \right] \\ &= \frac{1}{m} (p_j - eA_j) \frac{\partial}{\partial q_i} (p_j - eA_j) + e \frac{\partial \phi}{\partial q_i} \\ &= -\frac{e}{m} (p_j - eA_j) \frac{\partial A_j}{\partial q_i} + e \frac{\partial \phi}{\partial q_i} \end{aligned} \quad (1.19)$$

$$\begin{aligned} \left(\frac{\partial H}{\partial q_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m} [mv_j + eA_j(\mathbf{x}, t) - eA_j(\mathbf{x}, t)] \left(\frac{\partial A_j}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} \\ &= \left(-ev_j \frac{\partial A_j}{\partial q_i} + e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \quad (1.20)$$

Equation (1.16) thus leads to

$$m \frac{dv_i}{dt} = \left(-ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial A_i}{\partial t} + ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.21)$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m \frac{d\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}. \quad (1.22)$$

1.2 Kinetic description

We denote the distribution function for a species α as $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where \mathbf{r} and \mathbf{v} are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_\alpha d\mathbf{r} d\mathbf{v} = N_\alpha, \quad (1.23)$$

where N_α is the total number of particles corresponding to species α .

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution along with Maxwell's equations

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = C_\alpha + S_\alpha \quad (1.24)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \quad (1.25)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.26)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.27)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1.28)$$

$$\mathbf{J} = \sum_\alpha Z_\alpha e \int \mathbf{v} f_\alpha d\mathbf{v} \quad (1.29)$$

$$\rho_e = \sum_\alpha Z_\alpha e \int f_\alpha d\mathbf{v}. \quad (1.30)$$

In the above,

- m_α is the species mass
- e is the charge
- Z_α the charge number
- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ the charge current
- $\rho_e = \rho_e(\mathbf{r}, t)$ the charge density
- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the electric field
- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ the magnetic field.

The terms C_α and S_α represent collision and source terms.

If we express the collision term in the usual way, that is $C_\alpha = \sum_\beta C_{\alpha\beta}$, then we can make the following statements:

1. Conservation of particles:

$$\int C_{\alpha\alpha} d\mathbf{v} = 0 \quad \forall \alpha \quad \int C_{\alpha\beta} d\mathbf{v} = 0 \quad \forall \alpha, \beta | \beta \neq \alpha. \quad (1.31)$$

2. Conservation of momentum:

$$\int m_\alpha \mathbf{v} C_{\alpha\alpha} d\mathbf{v} = 0 \quad \forall \alpha \quad \sum_\alpha \sum_{\beta, \beta \neq \alpha} \int m_\alpha \mathbf{v} C_{\alpha\beta} d\mathbf{v} = 0. \quad (1.32)$$

3. Conservation of energy:

$$\int \frac{1}{2} m_\alpha v^2 C_{\alpha\alpha} d\mathbf{v} = 0 \quad \forall \alpha \quad \sum_\alpha \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2} m_\alpha v^2 C_{\alpha\beta} d\mathbf{v} = 0. \quad (1.33)$$

1.3 Fluid description

We now define the particle density $n_\alpha = n_\alpha(\mathbf{r}, t)$, the fluid velocity $\mathbf{u}_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$ and the fluid energy per unit mass $E_\alpha = E_\alpha(\mathbf{r}, t)$ as follows

$$n_\alpha = \int f_\alpha d\mathbf{v} \quad (1.34)$$

$$\mathbf{u}_\alpha = \frac{1}{n_\alpha} \int \mathbf{v} f_\alpha d\mathbf{v} \quad (1.35)$$

$$E_\alpha = \frac{1}{n_\alpha} \int \frac{1}{2} v^2 f_\alpha d\mathbf{v}. \quad (1.36)$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_\alpha}{\partial t} + \nabla \cdot (\mathbf{v} f_\alpha) + \nabla_v \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] = C_\alpha + S_\alpha \quad (1.37)$$

1.3.1 Mass

Integrating eq. (1.37) over all \mathbf{v} we obtain

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha \quad (1.38)$$

where

$$\hat{S}_\alpha = \int S_\alpha d\mathbf{v} \quad (1.39)$$

is an external source of mass.

1.3.2 Momentum

Multiplying eq. (1.37) by \mathbf{v} and then integrating over all \mathbf{v} leads to

$$\begin{aligned} \frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(\int \mathbf{v} \mathbf{v} f_\alpha d\mathbf{v} \right) + \int \nabla_v \cdot \left[\mathbf{v} \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] - \nabla_v \mathbf{v} \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] d\mathbf{v} = \\ \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \end{aligned} \quad (1.40)$$

We note that the third term in eq. (1.40) is zero since we are integrating over all space, and that $\nabla_v \mathbf{v}$ is the identity matrix. We thus have

$$\begin{aligned} \frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(\int \mathbf{v} \mathbf{v} f_\alpha d\mathbf{v} \right) - \frac{Z_\alpha e n_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \\ \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \end{aligned} \quad (1.41)$$

To proceed, we decompose \mathbf{v} into a mean and a fluctuation, that is, $\mathbf{v} = \mathbf{u}_\alpha + \mathbf{w}_\alpha$. Using this decomposition

$$\int \mathbf{v} \mathbf{v} f_\alpha d\mathbf{v} = \int (\mathbf{u}_\alpha \mathbf{u}_\alpha + 2\mathbf{u}_\alpha \mathbf{w}_\alpha + \mathbf{w}_\alpha \mathbf{w}_\alpha) f_\alpha d\mathbf{v} = n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v}. \quad (1.42)$$

Thus, eq. (1.41) becomes

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \quad (1.43)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \mathbf{w}_\alpha C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \quad (1.44)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (1.45)$$

where the stress tensor is

$$\boldsymbol{\sigma}_\alpha = - \int m_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v}, \quad (1.46)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_\alpha = \sum_{\beta, \beta \neq \alpha} \int m_\alpha \mathbf{w}_\alpha C_{\alpha\beta} d\mathbf{v}, \quad (1.47)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_\alpha = \int m_\alpha \mathbf{v} S_\alpha d\mathbf{v}. \quad (1.48)$$

The stress tensor is typically decomposed into isotropic p_α and anisotropic (shear) \mathbf{t}_α tensors as follows

$$\boldsymbol{\sigma}_\alpha = -p_\alpha \mathbf{I} + \mathbf{t}_\alpha, \quad (1.49)$$

where P_α is given by

$$p_\alpha = \frac{1}{3} \int m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v}. \quad (1.50)$$

Thus, conservation of momentum becomes

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha. \quad (1.51)$$

1.3.3 Energy

Multiplying eq. (1.37) by $\frac{1}{2}v^2$ and then integrating over all \mathbf{v} leads to

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_\alpha d\mathbf{v} \right] + \int \nabla_v \cdot \left[\frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] \\ - \nabla_v \left[\frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) \right] \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] d\mathbf{v} = \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (1.52)$$

We note that the third term above is zero since we are integrating over all space, and that $\nabla_v[1/2(\mathbf{v} \cdot \mathbf{v})] = \mathbf{v}$. Thus, we have

$$\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_\alpha d\mathbf{v} \right] - \frac{Z_\alpha e n_\alpha}{m_\alpha} \mathbf{E} \cdot \mathbf{u}_\alpha = \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \quad (1.53)$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_\alpha d\mathbf{v} = \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) (\mathbf{u}_\alpha + \mathbf{w}_\alpha) f_\alpha d\mathbf{v} = n_\alpha E_\alpha \mathbf{u}_\alpha + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \quad (1.54)$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{w}_\alpha f_\alpha d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha + 2\mathbf{u}_\alpha \cdot \mathbf{w}_\alpha + \mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \quad (1.55)$$

$$= \mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v}. \quad (1.56)$$

Using the expressions above, eq. (1.53) becomes

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot (n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} \mathbf{E} \cdot \mathbf{u}_\alpha = & -\nabla \cdot \left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \\ & + \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (1.57)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot (n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} \mathbf{E} \cdot \mathbf{u}_\alpha = & -\nabla \cdot \left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \\ & + \mathbf{u}_\alpha \cdot \sum_{\beta, \beta \neq \alpha} \int \mathbf{w}_\alpha C_{\alpha\beta} d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (1.58)$$

Multiplying by mass leads to the following equation

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = & \nabla \cdot (\mathbf{u}_\alpha \cdot \boldsymbol{\sigma}_\alpha) - \nabla \cdot \mathbf{q}_\alpha \\ & + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \end{aligned} \quad (1.59)$$

where heat flux due to random motion is

$$\mathbf{q}_\alpha = \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v}, \quad (1.60)$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_\alpha = \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) C_{\alpha\beta} d\mathbf{v}, \quad (1.61)$$

and the external source of energy is

$$\hat{Q}_\alpha = \int \frac{1}{2} m_\alpha (\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \quad (1.62)$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha &= \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha \\ &+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \end{aligned} \quad (1.63)$$

We also note that the energy $m_\alpha n_\alpha E_\alpha$ can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (1.42) one obtains

$$\begin{aligned} m_\alpha n_\alpha E_\alpha &= \int \frac{1}{2} m_\alpha (\mathbf{v} \cdot \mathbf{v}) f_\alpha d\mathbf{v} \\ &= \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v} + \frac{1}{2} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \\ &= \frac{3}{2} p_\alpha + \frac{1}{2} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \\ &= \frac{3}{2} p_\alpha + m_\alpha n_\alpha K_\alpha. \end{aligned} \quad (1.64)$$

where $K_\alpha = \frac{1}{2} \mathbf{u}_\alpha \cdot \mathbf{u}_\alpha$ is the kinetic energy of species α .

1.3.4 Kinetic and Internal Energies

The equation for the kinetic energy is obtained by dotting eq. (1.51) with \mathbf{u}_α . For this, we first show that

$$\mathbf{u}_\alpha \cdot \left[\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) \right] \quad (1.65)$$

$$= \mathbf{u}_\alpha \cdot \left\{ \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right\} \quad (1.66)$$

$$= \mathbf{u}_\alpha \cdot \left[m_\alpha \hat{S}_\alpha \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right] \quad (1.67)$$

$$= 2m_\alpha \hat{S}_\alpha K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \quad (1.68)$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \quad (1.69)$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K_\alpha \mathbf{u}_\alpha). \quad (1.70)$$

Thus, the equation for the turbulent kinetic energy is

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha &= \\ - \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \end{aligned} \quad (1.71)$$

Subtracting the above equation from eq. (1.63) leads to

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_\alpha \right) + \nabla \cdot \left(\frac{3}{2} p_\alpha \mathbf{u}_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \quad (1.72)$$

1.3.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha, \quad (1.73)$$

- Momentum

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (1.74)$$

- Total Energy

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha &= \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha \\ &+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \end{aligned} \quad (1.75)$$

- Kinetic Energy

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha &= \\ -\nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha &+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \end{aligned} \quad (1.76)$$

- Internal Energy

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_\alpha \right) + \nabla \cdot \left(\frac{3}{2} p_\alpha \mathbf{u}_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \quad (1.77)$$

Chapter 2

Fundamental concepts

2.1 The two-fluid model

This section focuses on a fluid model for plasmas that consist of electrons and a single species of ions.

2.1.1 General equations

The starting point are the multi-fluid conservation laws and the Maxwell equations. The assumptions are

1. There are two species: ions and electrons.
2. No sources (\hat{S}_α , $\hat{\mathbf{M}}_\alpha$, \hat{Q}_α).

Thus, the governing equations are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (2.1)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (2.2)$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Z e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i + \nabla \cdot \mathbf{t}_i + \mathbf{R}_i, \quad (2.3)$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e, \quad (2.4)$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_i \right) + \nabla \cdot \left(\frac{3}{2} p_i \mathbf{u}_i \right) = -p_i \nabla \cdot \mathbf{u}_i + \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \quad (2.5)$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_e \right) + \nabla \cdot \left(\frac{3}{2} p_e \mathbf{u}_e \right) = -p_e \nabla \cdot \mathbf{u}_e + \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \quad (2.6)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad (2.7)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.8)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.9)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (2.10)$$

$$\mathbf{J} = e(Zn_i\mathbf{u}_i - n_e\mathbf{u}_e), \quad (2.11)$$

$$\rho_e = e(Zn_i - n_e). \quad (2.12)$$

We'll also introduce the following simple equations of state

$$p_i = n_i k_B T_i, \quad (2.13)$$

$$p_e = n_e k_B T_e. \quad (2.14)$$

These equations correspond to eq. (2.22) in Freidberg's Ideal MHD book, but for ions that are not singly charged.

2.1.2 Isentropic plasmas

We now add a couple of extra assumptions

1. No stresses \mathbf{t}_α , heat flux \mathbf{q}_α , and collisions \mathbf{R}_α , Q_α . This in itself would make the flow isentropic.
2. The flow is not just isentropic but also homentropic. This allows for eqs. (2.19) and (2.20) to hold across all space, not just along streamlines.

Thus, the governing equations are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (2.15)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (2.16)$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Z e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i, \quad (2.17)$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e, \quad (2.18)$$

$$p_i = C_i n_i^{\gamma_i}, \quad (2.19)$$

$$p_e = C_e n_e^{\gamma_e}, \quad (2.20)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad (2.21)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.22)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.23)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (2.24)$$

$$\mathbf{J} = e(Zn_i \mathbf{u}_i - n_e \mathbf{u}_e), \quad (2.25)$$

$$\rho_e = e(Zn_i - n_e), \quad (2.26)$$

$$p_i = n_i k_B T_i, \quad (2.27)$$

$$p_e = n_e k_B T_e. \quad (2.28)$$

2.2 Electron-plasma and ion-acoustic waves

2.2.1 Linearization

The following decompositions will be used in the derivation of electron-plasma and ion-acoustic waves:

$$\begin{aligned}
n_i &= n_{i0} + n_{i1}, \\
n_e &= n_{e0} + n_{e1}, \\
p_i &= p_{i0} + p_{i1}, \\
p_e &= p_{e0} + p_{e1}, \\
\mathbf{u}_i &= \mathbf{u}_{i0} + \mathbf{u}_{i1}, \\
\mathbf{u}_e &= \mathbf{u}_{e0} + \mathbf{u}_{e1}, \\
\mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_1, \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1.
\end{aligned} \tag{2.29}$$

For these decompositions, we'll assume

1. Terms with a subscript 1 are small and thus products of two small quantities can be neglected.
2. \mathbf{u}_{i0} , \mathbf{u}_{e0} , \mathbf{E}_0 , and \mathbf{B}_0 are zero.
3. n_{i0} , n_{e0} , p_{i0} , and p_{e0} are uniform in space and time.

Using the variable decompositions in the evolution equation for electron density eq. (2.16), we have

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \nabla \cdot [(n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] = 0.$$

Using assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot (n_{e0}\mathbf{u}_{e1}) = 0. \tag{2.30}$$

Using the variable decompositions in the evolution equation for ion density eq. (2.15), we have

$$\frac{\partial n_{i0} + n_{i1}}{\partial t} + \nabla \cdot [(n_{i0} + n_{i1})(\mathbf{u}_{i0} + \mathbf{u}_{i1})] = 0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i1}}{\partial t} + \nabla \cdot (n_{i0}\mathbf{u}_{i1}) = 0. \tag{2.31}$$

Using the variable decompositions in the momentum equation for electrons eq. (2.18), we have

$$\begin{aligned}
&\frac{\partial}{\partial t} [m_e (n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] + \nabla \cdot [m_e (n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] \\
&\quad + e (n_{e0} + n_{e1}) [(\mathbf{E}_0 + \mathbf{E}_1) + (\mathbf{u}_{e0} + \mathbf{u}_{e1}) \times (\mathbf{B}_0 + \mathbf{B}_1)] = -\nabla (p_{e0} + p_{e1}).
\end{aligned}$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e0}\mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e}\mathbf{E}_1 = -\frac{1}{m_e}\nabla p_{e1}. \tag{2.32}$$

Using the variable decompositions in the momentum equation for ions eq. (2.17), we have

$$\begin{aligned} \frac{\partial}{\partial t} [m_i (n_{i0} + n_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1})] + \nabla \cdot [m_i (n_{i0} + n_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1})] \\ - Ze (n_{i0} + n_{i1}) [(\mathbf{E}_0 + \mathbf{E}_1) + (\mathbf{u}_{i0} + \mathbf{u}_{i1}) \times (\mathbf{B}_0 + \mathbf{B}_1)] = -\nabla (p_{i0} + p_{i1}). \end{aligned}$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i0} \mathbf{u}_{i1}}{\partial t} - \frac{Zen_{i0}}{m_i} \mathbf{E}_1 = -\frac{1}{m_i} \nabla p_{i1}. \quad (2.33)$$

We'll often need to take the gradient of the ion and electron pressure. Given $p_\alpha = C_\alpha n_\alpha^{\gamma_\alpha}$, where $\alpha = i, e$, we have

$$\nabla p_\alpha = C_\alpha \gamma_\alpha n_\alpha^{\gamma_\alpha - 1} \nabla n_\alpha = C_\alpha \gamma_\alpha \frac{n_\alpha^{\gamma_\alpha}}{n_\alpha} \nabla n_\alpha = \gamma_\alpha \frac{p_\alpha}{n_\alpha} \nabla n_\alpha,$$

Using the equation of state $p_\alpha = n_\alpha k_B T_\alpha$, the above becomes

$$\nabla p_\alpha = \gamma_\alpha k_B T_\alpha \nabla n_\alpha. \quad (2.34)$$

Given the assumption in item 3, the above simplifies to

$$\nabla p_{\alpha 1} = \gamma_\alpha k_B T_\alpha \nabla n_{\alpha 1}. \quad (2.35)$$

2.2.2 Electron Plasma Waves

On top of the assumptions in section 2.2.1, we'll assume

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.
2. Uniform ion density, $n_{i1} = 0$.

Combining Equation (2.32) with eq. (2.35) gives

$$\frac{\partial n_{e0} \mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e} \mathbf{E}_1 = -\frac{\gamma_e k_B T_e}{m_e} \nabla n_{e1}. \quad (2.36)$$

Taking the time derivative of eq. (2.30) and using eq. (2.36) leads to the the wave equation for electron density

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{en_{e0}}{m_e} \nabla \cdot \mathbf{E}_1 = \frac{\gamma_e k_B T_e}{m_e} \nabla^2 n_{e1}. \quad (2.37)$$

For electron plasma waves, we'll assume that n_i varies in space and time so slowly that it can be assumed to be constant. That is, we assume $n_{i1} = 0$. Thus, Gauss's law now takes the form

$$\nabla \cdot \mathbf{E}_1 = \frac{e}{\epsilon_0} (Zn_{i0} - n_{e0} - n_{e1}).$$

Using the quasi-neutrality assumption ($Zn_{i0} = n_{e0}$)

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_{e1}. \quad (2.38)$$

Plugging the above in the electron wave equation we obtain

$$\frac{\partial^2 n_{e1}}{\partial t^2} + \frac{e^2 n_{e0}}{m_e \epsilon_0} n_{e1} = \frac{\gamma_e k_B T_e}{m_e} \nabla^2 n_{e1},$$

We now introduce the electron plasma frequency

$$w_{pe} = \left(\frac{n_{e0} e^2}{m_e \epsilon_0} \right)^{1/2}, \quad (2.39)$$

and the thermal velocity

$$v_{T\alpha} = \sqrt{\frac{k_B T_\alpha}{m_\alpha}}. \quad (2.40)$$

These two allow us to write the equation for n_{e1} as

$$\frac{\partial^2 n_{e1}}{\partial t^2} + w_{pe}^2 n_{e1} - \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0. \quad (2.41)$$

Assuming a mode of the form $n_{e1} = \tilde{n}_{e1} \exp(-i\omega t)$, where $\tilde{n}_{e1} = \tilde{n}_{e1}(\mathbf{x})$, gives the following

$$(w^2 - w_{pe}^2) n_{e1} + \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0. \quad (2.42)$$

If we assume $w_{pe} < w$, then $\tilde{n}_{e1} = \hat{n}_{e1} \exp(i\mathbf{k}_e \cdot \mathbf{x})$ is a solution to eq. (2.42). Plugging in this expression for \tilde{n}_{e1} in eq. (2.42) finally gives

$$w^2 - w_{pe}^2 - \gamma_e v_{Te}^2 k_e^2 = 0. \quad (2.43)$$

The above is the dispersion relation for electron plasma waves.

2.2.3 Ion Acoustic Waves

On top of the assumptions in section 2.2.1, we'll assume

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.
2. Approximate quasi-neutrality for the fluctuations, $Zn_{i1} \approx n_{e1}$.
3. Negligible electron mass, $m_e \rightarrow 0$.

Combining eq. (2.33) with eq. (2.35) gives

$$\frac{\partial n_{i0} \mathbf{u}_{i1}}{\partial t} - \frac{Z n_{i0}}{m_i} \mathbf{E}_1 = -\frac{\gamma_i k_B T_i}{m_i} \nabla n_{i1}. \quad (2.44)$$

Taking the time derivative of eq. (2.31) and using eq. (2.44) leads to the wave equation for ion density

$$\frac{\partial^2 n_{i1}}{\partial t^2} + \frac{Z n_{i0}}{m_i} \nabla \cdot \mathbf{E}_1 = \frac{\gamma_i k_B T_i}{m_i} \nabla^2 n_{i1}. \quad (2.45)$$

For this case, we assume that the mass of the electron, which is significantly smaller than that of the ions, is negligible. Thus, eq. (2.32) simplifies to

$$e n_{e0} \mathbf{E}_1 = -\gamma_e k_B T_e \nabla n_{e1}. \quad (2.46)$$

Using the above in the ion wave equation we obtain

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{Z n_{i0}}{n_{e0}} \frac{\gamma_e k_B T_e}{m_i} \nabla^2 n_{e1} + \frac{\gamma_i k_B T_i}{m_i} \nabla^2 n_{i1}.$$

Due to quasi-neutrality, we have $Zn_{i0} = n_{e0}$ and $Zn_{i1} \approx n_{e1}$, which gives

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \left(\frac{Z \gamma_e k_B T_e}{m_i} + \frac{\gamma_i k_B T_i}{m_i} \right) \nabla^2 n_{i1}.$$

We now introduce the following velocity

$$v_s = \sqrt{\frac{Z\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}}, \quad (2.47)$$

which allows us to write the equation for n_{i1} as

$$\frac{\partial^2 n_{i1}}{\partial t^2} - v_s^2 \nabla^2 n_{i1} = 0. \quad (2.48)$$

Assuming a mode of the form $n_{i1} = \tilde{n}_{i1} \exp(-i\omega t)$, where $\tilde{n}_{i1} = \tilde{n}_{i1}(\mathbf{x})$, gives the following

$$\omega^2 n_{i1} + v_s^2 \nabla^2 n_{i1} = 0. \quad (2.49)$$

Since ω^2 is always positive, $\tilde{n}_{i1} = \hat{n}_{i1} \exp(i\mathbf{k}_i \cdot \mathbf{x})$ is a solution to eq. (2.49). Plugging in this expression for \tilde{n}_{i1} in eq. (2.49) finally gives

$$\omega^2 - k_i^2 v_s^2 = 0. \quad (2.50)$$

The above is the dispersion relation for ion-acoustic waves.

2.3 Plasma parameters

2.3.1 Debye length

To describe the Debye length we'll use the governing equations from section 2.1.1 as well as the following assumptions

1. Stationary plasma $\mathbf{u}_i = \mathbf{u}_e = 0$.
2. No shear stresses \mathbf{t}_α , heat flux \mathbf{q}_α , and collisions $\mathbf{R}_\alpha, Q_\alpha$.
3. Effects of magnetic fields can be neglected, i.e. $\mathbf{B} = 0$.
4. Constant temperatures T_i, T_e .

Thus we have

$$\begin{aligned} -Zen_i \mathbf{E} &= -\nabla p_i, \\ en_e \mathbf{E} &= -\nabla p_e, \\ \nabla \cdot \mathbf{E} &= \frac{\rho_e}{\epsilon_0}, \\ \nabla \times \mathbf{E} &= 0, \\ \rho_e &= e(Zn_i - n_e), \\ p_i &= n_i k_B T_i, \\ p_e &= n_e k_B T_e. \end{aligned}$$

We can simplify the above to obtain

$$\begin{aligned} Zen_i \nabla \phi &= -k_B T_i \nabla n_i, \\ -en_e \nabla \phi &= -k_B T_e \nabla n_e, \\ -\nabla^2 \phi &= \frac{e}{\epsilon_0} (Zn_i - n_e). \end{aligned}$$

The physical domain will be one dimensional with length L , namely, $[-L/2, L/2]$. Thus, we have

$$Zen_i \frac{d\phi}{dx} = -k_B T_i \frac{dn_i}{dx}, \quad (2.51)$$

$$-en_e \frac{d\phi}{dx} = -k_B T_e \frac{dn_e}{dx}, \quad (2.52)$$

$$-\frac{d^2\phi}{dx^2} = \frac{e}{\epsilon_0}(Zn_i - n_e). \quad (2.53)$$

A voltage V is applied across this domain and thus the boundary condition for the electric potential becomes $\phi(-L/2) - \phi(L/2) = V$. We'll also enforce the condition $\phi(0) = 0$ to fix the arbitrary constant of the potential.

The simplest case to consider is the one for which the one-dimensional domain is empty. Gauss's law then gives

$$\frac{d^2\phi}{dx^2} = 0,$$

which, upon applying the boundary conditions gives the trivial solution $\phi = -(V/L)x$ and a constant electric field $E = V/L$ pointing from left to right.

Now let's fill up the physical space with a plasma consisting of electrons and ions. Integrating eqs. (2.51) and (2.52) we obtain

$$\begin{aligned} \ln n_i &= -\frac{Ze}{k_B T_i} \phi + C_1, \\ \ln n_e &= \frac{e}{k_B T_e} \phi + C_2, \end{aligned}$$

which we re-write as

$$\begin{aligned} n_i &= D_1 \exp\left(-\frac{Ze\phi}{k_B T_i}\right), \\ n_e &= D_2 \exp\left(\frac{e\phi}{k_B T_e}\right). \end{aligned}$$

We introduce n_{i0} and n_{e0} as the densities at $x = 0$. Since $\phi = 0$ at $x = 0$, we have

$$\begin{aligned} n_i &= n_{i0} \exp\left(-\frac{Ze\phi}{k_B T_i}\right), \\ n_e &= n_{e0} \exp\left(\frac{e\phi}{k_B T_e}\right). \end{aligned}$$

A condition for the densities at the center is that they satisfy quasi-neutrality, that is, $Zn_{i0} = n_{e0}$. Note that this does not mean $Zn_i = n_e$. We'll assume the terms inside the exponential are small, so that we can use $\exp \alpha = 1 + \alpha$. Thus we have

$$\begin{aligned} n_i &= n_{i0} - \frac{n_{i0} Ze\phi}{k_B T_i}, \\ n_e &= n_{e0} + \frac{n_{e0} e\phi}{k_B T_e}. \end{aligned}$$

Plugin this into eq. (2.53) gives

$$\frac{d^2\phi}{dx^2} - \left(\frac{n_{i0} Z^2 e^2}{\epsilon_0 k_B T_i} + \frac{n_{e0} e^2}{\epsilon_0 k_B T_e} \right) \phi = 0. \quad (2.54)$$

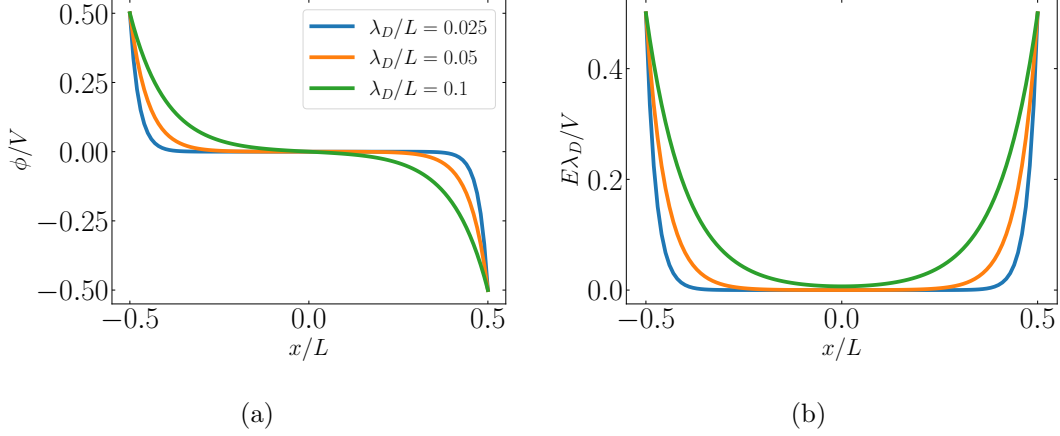


Figure 2.1: Electric potential and field for various values of λ_D/L .

We now introduce the Debye length for a particular species α as

$$\lambda_{D\alpha} = \left(\frac{\epsilon_0 k_B T_\alpha}{n_{\alpha 0} Z^2 e^2} \right)^{1/2}, \quad (2.55)$$

and the total Debye length λ_D as

$$\frac{1}{\lambda_D} = \sum_{\alpha} \frac{1}{\lambda_{D\alpha}}. \quad (2.56)$$

Using this, we can re-write eq. (2.54) simply as

$$\frac{d^2 \phi}{dx^2} - \frac{1}{\lambda_D^2} \phi = 0. \quad (2.57)$$

The solution to the above that also conforms to the boundary conditions is

$$\phi = -\frac{V}{2} \frac{\sinh\left(\frac{x}{\lambda_D}\right)}{\sinh\left(\frac{L}{2\lambda_D}\right)}. \quad (2.58)$$

A plot of ϕ and the corresponding E is shown in fig. 2.1 for various values of λ_D/L . As the figure shows, the presence of a plasma shields out the electric field, in contrast to the vacuum case where a constant electric field is present across the domain. Given a fixed domain size L , varying the parameter λ_D determines how well the plasma shields away the electric field.

We note that λ_{De} , w_{pe} , and v_{Te} are all related to each other as shown below

$$\lambda_{De} w_{pe} = \left(\frac{\epsilon_0 k_B T_e}{n_{e0} e^2} \right)^{1/2} \left(\frac{n_{e0} e^2}{m_e \epsilon_0} \right)^{1/2} = \left(\frac{k_B T_e}{m_e} \right)^{1/2} = v_{Te}. \quad (2.59)$$

2.3.2 Plasma frequency

At the end of section 2.2.2 we assumed $w_{pe} < w$ in eq. (2.42) to obtain a dispersion relation for electron-plasma waves. We now consider the cases $w_{pe} > w$ and $w_{pe} = w$ to better understand the role played by the plasma frequency w_{pe} .

We first consider the case of $w_{pe} > w$. Plugging in eq. (2.38) in eq. (2.42) and using the definition of the electric potential, we have

$$(w^2 - w_{pe}^2) \nabla^2 \phi_1 + \gamma_e v_{Te}^2 \nabla^4 \phi_1 = 0. \quad (2.60)$$

We'll use the same setting as that for the derivation of the Debye length. The only difference is that the potential drop across the domain now oscillates with a frequency of w , that is $\phi_1(-L/2) - \phi_1(L/2) = V \exp(-iwt)$. Simplifying due to the one-dimensional domain and rearranging, eq. (2.60) becomes

$$\frac{d^4 \phi_1}{dx^4} - \frac{w_{pe}^2 - w^2}{\gamma_e v_{Te}^2} \frac{d^2 \phi_1}{dx^2} = 0.$$

Using eq. (2.59) the above becomes

$$\frac{d^4 \phi_1}{dx^4} - \frac{1}{\gamma_e \lambda_{De}^2} \left(1 - \frac{w^2}{w_{pe}^2} \right) \frac{d^2 \phi_1}{dx^2} = 0,$$

which we re-write as

$$\frac{d^4 \phi_1}{dx^4} - \frac{1}{\hat{\lambda}_{De}^2} \frac{d^2 \phi_1}{dx^2} = 0,$$

where

$$\hat{\lambda}_{De}^2 = \gamma_e \lambda_{De}^2 \left(1 - \frac{w^2}{w_{pe}^2} \right)^{-1}.$$

Integrating the fourth-order PDE above twice, we get

$$\frac{d^2 \phi_1}{dx^2} - \frac{1}{\hat{\lambda}_{De}^2} \phi_1 + C_1 x + C_2 = 0$$

Since to obtain eq. (2.42) we assumed $n_{e1} = \tilde{n}_{e1} \exp(-iwt)$, where $\tilde{n}_{e1} = \tilde{n}_{e1}(\mathbf{x})$, we'll also assume $\phi_1 = \tilde{\phi}_1 \exp(-iwt)$, where $\tilde{\phi}_1 = \tilde{\phi}_1(\mathbf{x})$. Plugging this into the ODE above we get

$$\left(\frac{d^2 \tilde{\phi}_1}{dx^2} - \frac{1}{\hat{\lambda}_{De}^2} \tilde{\phi}_1 \right) + \left(\frac{C_1 x + C_2}{\exp(-iwt)} \right) = 0$$

Note that the first term in parenthesis above does not depend on time, and thus the second term in parenthesis should not do so either. This can only be accomplished if we set $C_1 = C_2 = 0$. Thus we finally have

$$\frac{d^2 \tilde{\phi}_1}{dx^2} - \frac{1}{\hat{\lambda}_{De}^2} \tilde{\phi}_1 = 0. \quad (2.61)$$

The solution to the above that also conforms to the boundary conditions is ...

It is often important to know the plasma density at which the electron plasma frequency w_{pe} equals the external frequency w . This density is referred to as the critical density n_{cr} . Equating the electron plasma frequency with the external frequency we get

$$\frac{n_{cr} e^2}{m_e \epsilon_0} = w^2,$$

which leads to

$$n_{cr} = \frac{m_e \epsilon_0 w^2}{e^2} \quad (2.62)$$

The above can be re-written as

$$n_{cr} = \frac{m_e \epsilon_0 (2\pi\nu)^2}{e^2} = \frac{4\pi^2 m_e \epsilon_0 c^2}{e^2} \frac{1}{\lambda^2} = 1.115 \times 10^{27} \frac{1}{\lambda_{\mu m}^2},$$

where $\lambda_{\mu m}$ is in units of microns and n_{cr} in units of $\#/\text{m}^3$.

2.3.3 The coupling parameter

Coulomb interactions are those which occur when two charge particles head towards each other. We can define two types of Coulomb interactions: strong and weak. Strong Coulomb interactions are those for which the particle's Coulomb potential energy is larger than its kinetic energy, and viceversa for weak Coulomb interactions. Thus, we can also define two types of plasma regimes:

- Strongly-coupled plasmas: plasmas where the Coulomb interactions are mostly strong and thus drive the dynamics of its evolution. Coulomb interactions tend to be strong when the inter-particle distances are small, and thus this regime would be dominated by *short-range* interactions. These plasmas are also described as exhibiting *collisional* behavior, since a strong Coulomb interaction essentially means a collision has occurred.
- Weakly-coupled plasmas: plasmas where the Coulomb interactions are mostly weak, and as a result do not drive the dynamics of its evolution. The plasma dynamics are instead driven by *long-range* effects caused by smooth electromagnetic fields that result from integrating a large number of particles. These plasmas are also described as exhibiting *collective* behavior, since the long-range electromagnetic fields follow from the collective integration of many particles.

We describe an approximate Coulomb potential energy for particles in a plasma as

$$U = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha}. \quad (2.63)$$

The impact parameter that has been used above is a_α , the sphere radius. This provides a decent measure on the average spacing between particles in a plasma. Since the volume of a single particle is $1/n_\alpha$, and if we assume that this volume is given by $4/3\pi a_\alpha^3$, then equating these two gives the expression for the sphere radius

$$a_\alpha = \left(\frac{3}{4\pi n_\alpha} \right)^{1/3}. \quad (2.64)$$

The thermal velocity of a particle is given by

$$v_{T_\alpha} = \sqrt{\frac{k_B T_\alpha}{m_\alpha}} \quad (2.65)$$

A measure of the kinetic energy of a particle is given in terms of the thermal velocity as shown below

$$K = m_\alpha v_{T_\alpha}^2 = k_B T_\alpha \quad (2.66)$$

The ratio of the particle's Coulomb potential energy and its kinetic energy is referred to as the coupling parameter Γ_α . That is

$$\Gamma_\alpha = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha k_B T_\alpha}. \quad (2.67)$$

$\Gamma_\alpha > 1$ denotes a strongly coupled plasma, and $\Gamma_\alpha < 1$ denotes a weakly coupled plasma.

2.3.4 The plasma parameter

The standard plasma parameter Λ_α is defined as

$$\Lambda_\alpha = \frac{4}{3}\pi\lambda_{D\alpha}^3 n_\alpha. \quad (2.68)$$

There is a one to one relationship between the coupling parameter and the standard plasma parameter. Simple algebra shows that

$$\Gamma_\alpha = (1/3)\Lambda_\alpha^{-2/3}. \quad (2.69)$$

Thus, the coupling and plasma parameters are inversely proportional to each other. $\Lambda_\alpha < 1$ implies strongly-coupled plasmas, and $\Lambda_\alpha > 1$ weakly-coupled plasmas. Since Λ_α represents the number of particles per Debye sphere, it is interesting to see that a large number of particles within such a sphere is needed to be in the weakly-coupled-plasma regime. However, this does not correspond to a plasma with large density, in fact, it corresponds to the opposite. The explicit n_α term in the definition $\Lambda_\alpha = (4/3)\pi\lambda_{D\alpha}^3 n_\alpha$ is dominated by the n_α in the denominator of $\lambda_{D\alpha}$. In other words, low plasma densities lead to large Debye spheres, which in turn leads to many particles per Debye sphere, and hence a weakly-coupled plasma.

2.3.5 Electron degeneracy

- DeBroglie wavelength

$$\lambda_{B\alpha} = \frac{h}{\sqrt{2\pi m_\alpha v_{T\alpha}}} \quad (2.70)$$

- Quantum plasma parameter

$$\chi_\alpha = \frac{4}{3}\pi\lambda_{B\alpha}^3 n_\alpha \quad (2.71)$$

- Fermi energy:

$$E_{f\alpha} = \frac{\hbar^2}{2m_\alpha} (3\pi^2 n_\alpha)^{2/3} \quad (2.72)$$

- The Fermi energy can be used to define the Fermi temperature $T_{f\alpha}$, Fermi velocity $v_{f\alpha}$, Fermi momentum $p_{f\alpha}$, and Fermi wavevector $k_{f\alpha}$

$$E_{f\alpha} = k_B T_{f\alpha} = \frac{1}{2} m_\alpha v_{f\alpha}^2 = \frac{p_{f\alpha}^2}{2m_\alpha} = \frac{(\hbar k_{f\alpha})^2}{2m_\alpha} \quad (2.73)$$

- Degeneracy parameter:

$$\Theta_\alpha = \frac{k_B T_\alpha}{E_{f\alpha}} = \left(\frac{2^{10}\pi}{3^4} \right)^{1/3} \chi_\alpha^{-2/3} \quad (2.74)$$

Chapter 3

Collisions

3.1 Cross section

3.1.1 General definition

Two particles traveling towards each other can undergo an interaction. Types of interactions include Coulomb collisions between two charged particles, fusion reactions between ions, and photon-matter phenomena such as Compton scattering, the photoelectric effect, and pair production.

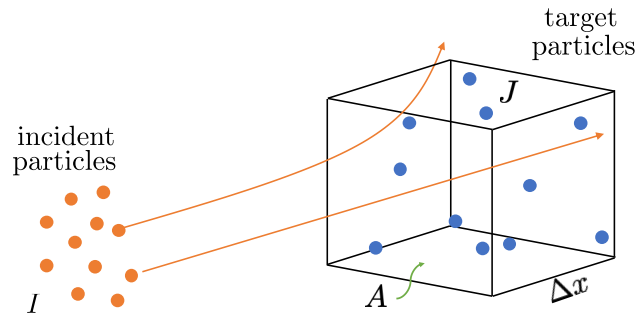


Figure 3.1: Interaction of incident and target particles.

To define the cross section, we'll consider I incident particles heading towards J stationary target particles (see fig. 3.1). Not all of the incident particles will interact with the target particles, some will instead continue to travel in a uniform trajectory. The number of incident particles that do interact with the target particles is labeled as N . The cross section σ is then a constant of proportionality defined by the following equation

$$N = \sigma I n_A. \quad (3.1)$$

In the above, n_A is the areal number density, that is, $n_A = J/A = n\Delta x$, where n is the volume number density.

3.1.2 Differential cross section

Consider the scattering of two particles: an incident and a target particle. If we fix the reference frame to follow the target particle, then the scattering can be depicted as shown in fig. 3.2. The

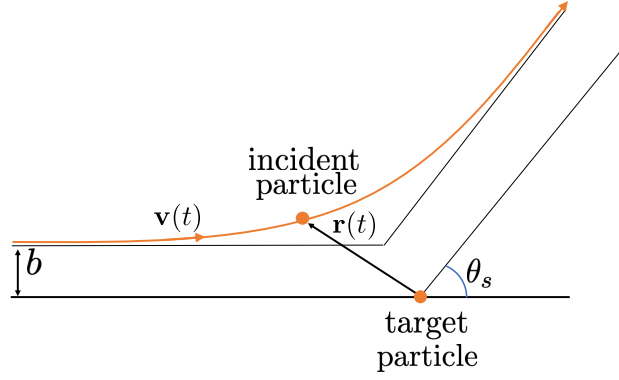


Figure 3.2: Depiction of particle scattering.

displacement parameter is labelled as b , and the scattering angle as θ_s . For three dimensional scattering, the encounter is as shown in fig. 3.3. Note that in that figure the incident particle starts within the $x - z$ plane, and after scattering the particle is confined to a plane that is tilted an angle ϕ_s from the $x - z$ plane.

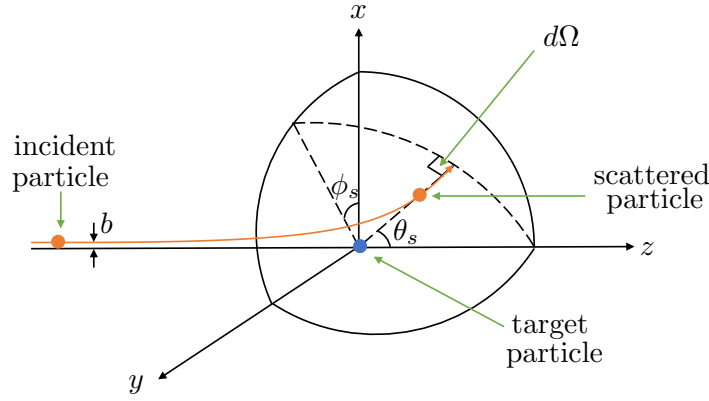


Figure 3.3: Depiction of particle scattering in 3D.

From the entire set of incident particles N that interact with the target particles, one can define an infinitesimal subset $N_{\theta,\phi}d\Omega$ as the number of particles scattered into an infinitesimal solid angle $d\Omega = \sin\theta_s d\theta_s d\phi_s$, as shown in fig. 3.3. We note that $N_{\theta,\phi} = N_{\theta,\phi}(\theta_s, \phi_s)$. We introduce the differential cross section

$$\frac{d\sigma_{\theta,\phi}}{d\Omega} = \frac{d\sigma_{\theta,\phi}}{d\Omega}(\theta_s, \phi_s), \quad (3.2)$$

which is defined by the following expression in an analogous manner to eq. (3.1),

$$N_{\theta,\phi}d\Omega = \left(\frac{d\sigma_{\theta,\phi}}{d\Omega} \right) In_A. \quad (3.3)$$

It is best to not think of it as a derivative (what does a derivative with respect to solid angle mean?) and instead to simply think of it as a function that depends on θ_s and ϕ_s . Integrating over all θ_s and ϕ_s , i.e.

$$\int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} N_{\theta,\phi}d\Omega = \int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} \frac{d\sigma_{\theta,\phi}}{d\Omega} d\Omega In_A$$

gives eq. (3.1).

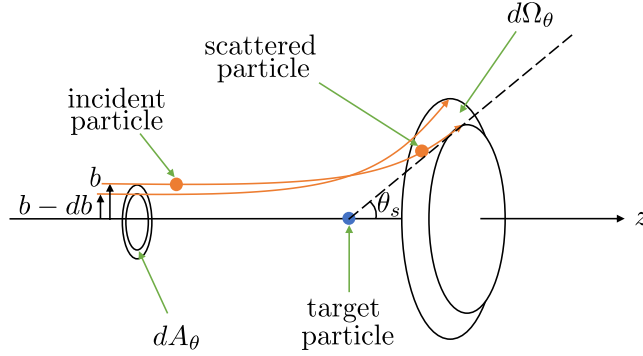


Figure 3.4: Depiction of particle scattering in for axisymmetric interactions.

For various cases the scattering is axisymmetric, that is, it is independent of ϕ_s . Thus

$$N_{\theta,\phi} \rightarrow N_{\theta} \quad \frac{d\sigma_{\theta,\phi}}{d\Omega} \rightarrow \frac{d\sigma_{\theta}}{d\Omega},$$

where

$$N_{\theta} = N_{\theta}(\theta_s),$$

and

$$\frac{d\sigma_{\theta}}{d\Omega} = \frac{d\sigma_{\theta}}{d\Omega}(\theta_s).$$

Integrating eq. (3.3) from $\phi_s = 0$ to $\phi_s = 2\pi$ gives

$$N_{\theta}d\Omega_{\theta} = \frac{d\sigma_{\theta}}{d\Omega}d\Omega_{\theta}In_A, \quad (3.4)$$

where $d\Omega_{\theta} = 2\pi \sin \theta_s d\theta_s$. $N_{\theta}d\Omega_{\theta}$ thus represents the number of particles that are scattered into the infinitesimal band $d\Omega_{\theta}$ on a sphere, where $d\Omega_{\theta}$ is defined by scattering angle θ_s (see fig. 3.4). We will note that there is a one-to-one correspondence between the impact parameter b and the scattering angle θ_s , that is, $b = b(\theta_s)$. In other words, any incident particle scattered out through the infinitesimal band $d\Omega_{\theta}$ would have approached the target-particle through the infinitesimal ring dA_{θ} that corresponds to $d\Omega_{\theta}$. Since there are many target particles, there are many dA_{θ} 's that correspond to the same $d\Omega_{\theta}$. Thus, the total number of particles scattered through $d\Omega_{\theta}$ is given by all the incident particles that cross through the dA_{θ} 's of all the target particles.

The number of incident particles crossing all the infinitesimal rings dA_{θ} is equal to the total number of incident particles I times the probability P that any single incident particle will cross one of those rings. Thus, we can write

$$N_{\theta}d\Omega_{\theta} = IP.$$

The probability that an incident particle will cross one of those rings is simply the ratio of the surface area covered by all the rings in a section of the target material to the total area of that section. The surface area covered by all the rings in a section of area S is given by $(n_AS)dA_{\theta}$. Thus, $P = n_AdA_{\theta}$ and

$$N_{\theta}d\Omega_{\theta} = In_AdA_{\theta}.$$

We now introduce the differential

$$db = \frac{db}{d\theta_s} d\theta_s. \quad (3.5)$$

We note that by definition $d\theta_s$ is positive but db can be either positive or negative depending on the sign of $db/d\theta_s$. The infinitesimal area dA_θ is then given by

$$dA_\theta = 2\pi b|db| = 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s. \quad (3.6)$$

Thus,

$$N_\theta d\Omega_\theta = n_A I 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s.$$

Using eq. (3.4) in the above, we get

$$\frac{d\sigma_\theta}{d\Omega} d\Omega_\theta I n_A = I n_A 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s,$$

or

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin \theta_s} \left| \frac{db}{d\theta_s} \right|. \quad (3.7)$$

3.1.3 Mean free path & collision frequency

The mean free path can be expressed in terms of the cross section as

$$\lambda_m = \frac{1}{n_1 \sigma}. \quad (3.8)$$

Given the particle's speed v , one can also define the collision time as follows

$$\tau_m = \frac{\lambda_m}{v} = \frac{1}{n_1 \sigma v}. \quad (3.9)$$

Finally, the collision frequency is simply the inverse of the collision time, that is

$$\nu_m = \frac{1}{\tau_m} = n_1 \sigma v. \quad (3.10)$$

3.2 Coulomb collisions

3.2.1 Particle equations

Consider two particles, with positions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, velocities $\mathbf{v}_1 = \mathbf{v}_1(t)$ and $\mathbf{v}_2 = \mathbf{v}_2(t)$, charges q_1 and q_2 , and masses m_1 and m_2 , respectively. Their positions and velocities are governed by the following equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1, \quad (3.11)$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2, \quad (3.12)$$

$$m_1 \frac{d\mathbf{v}_1}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \quad (3.13)$$

$$m_2 \frac{d\mathbf{v}_2}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \quad (3.14)$$

We note that the above system consists of twelve equations for twelve unknowns. We now introduce the center-of-mass position $\mathbf{R} = \mathbf{R}(t)$, the center-of-mass velocity $\mathbf{V} = \mathbf{V}(t)$, the shifted position $\mathbf{r} = \mathbf{r}(t)$ and the shifted velocity $\mathbf{v} = \mathbf{v}(t)$ as follows

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$

$$\mathbf{V} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \quad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$$

Thus, in terms of these new four variables, the particle equations can be written as

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}, \quad (3.15)$$

$$\frac{d\mathbf{V}}{dt} = 0, \quad (3.16)$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad (3.17)$$

$$\frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r}}{r^3}, \quad (3.18)$$

where the reduced mass m_r is given by

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}. \quad (3.19)$$

The first two equations above give the trivial solution $\mathbf{V} = \text{constant}$ and $\mathbf{R} = \mathbf{R}(0) + \mathbf{V}t$. Thus, we have reduced the problem from twelve unknowns to six unknowns, namely \mathbf{r} and \mathbf{v} .

3.2.2 Conservation of energy and momentum

Dotting eq. (3.18) by \mathbf{v} gives

$$\begin{aligned} \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \mathbf{v} \cdot \frac{\mathbf{r}}{r^3} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{r}}{r^3} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{1}{2} \frac{dr^2}{dt} \frac{1}{r^3} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{1}{r^2} \frac{dr}{dt} \\ &= -\frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{d}{dt} \left(\frac{1}{r} \right). \end{aligned}$$

For the left hand side above we have

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{1}{2} \frac{dv^2}{dt},$$

and thus we obtain the following expression for conservation of energy

$$\frac{d}{dt} \left(\frac{1}{2} m_r v^2 + \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r} \right) = 0.$$

Crossing eq. (3.18) by \mathbf{r} gives

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0,$$

and thus

$$\frac{d}{dt} [m_r (\mathbf{r} \times \mathbf{v})] = 0.$$

That is, angular momentum is conserved. A consequence of this is that the vector $\mathbf{r} \times \mathbf{v}$ is always pointing in the same direction. Thus, if $\mathbf{r}(0)$ and $\mathbf{v}(0)$ form a plane, then $\mathbf{r}(t)$ and $\mathbf{v}(t)$ need to reside within that same plane for all times t so that $\mathbf{r}(t) \times \mathbf{v}(t)$ points in the same direction as $\mathbf{r}(0) \times \mathbf{v}(0)$. Therefore, the evolution of the position and velocity are confined to a plane and the problem can be reduced from six unknowns to four unknowns. This planar encounter is depicted in fig. 3.5.

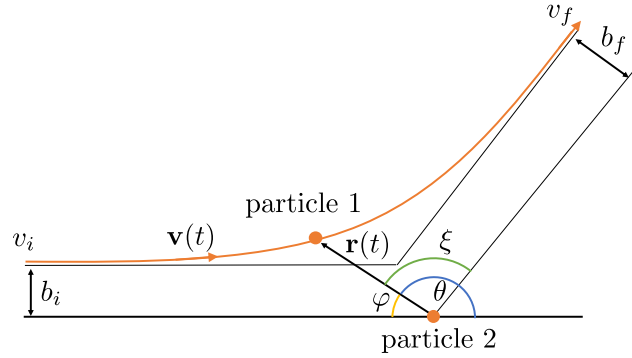


Figure 3.5: Depiction of Coulomb scattering.

If we refer to the plane shown in fig. 3.5 as the $x - y$ plane, then one can tell that the angular-momentum vector points in the negative z direction. We will denote the magnitude of the conserved angular momentum by L , and thus we can write

$$m_r (\mathbf{r} \times \mathbf{v}) = -L \hat{\mathbf{z}}. \quad (3.20)$$

A consequence of both conservation of energy and momentum is as follows. Consider the two limiting states of particle 1—the initial state v_i, b_i and the final state v_f, b_f . Assuming the potential energy is very low at sufficiently early and late times, conservation of energy gives

$$\frac{1}{2} m_r v_i^2 = \frac{1}{2} m_r v_f^2, \quad (3.21)$$

that is, $v_i = v_f$ (note that for other scattering processes, e.g. Compton scattering, this is not necessarily the case). For the angular momentum of the initial state we have

$$\begin{aligned} m_r (\mathbf{r}_i \times \mathbf{v}_i) &= m_r \sin(-\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\pi - \varphi_i) r_i v_i \hat{\mathbf{z}} \\ &= -m_r \sin(\varphi_i) r_i v_i \hat{\mathbf{z}} = -m_r b_i v_i \hat{\mathbf{z}} \end{aligned} \quad (3.22)$$

Similarly, for the angular momentum of the final state we have

$$m_r (\mathbf{r}_f \times \mathbf{v}_f) = m_r \sin(-\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r \sin(\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r b_f v_f \hat{\mathbf{z}}. \quad (3.23)$$

Equating the last two relationships gives $m_r b_i v_i = m_r b_f v_f$. Since $v_i = v_f$, we finally have $b_i = b_f = b$. Using eq. (3.20) in eq. (3.22), we can also write

$$L = m_r b v_i. \quad (3.24)$$

3.2.3 Polar coordinates

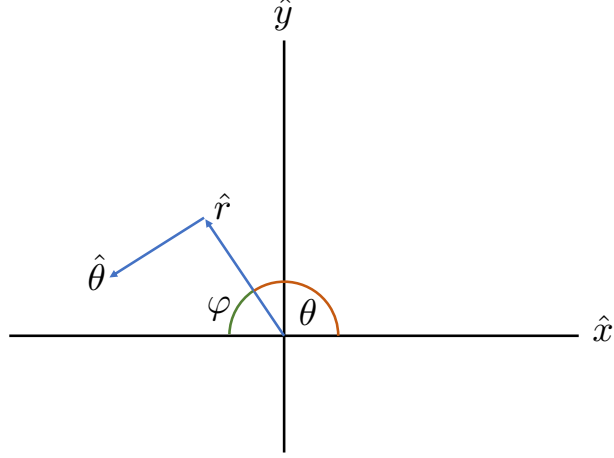


Figure 3.6: Polar coordinates in plane of interaction.

Using polar coordinates, as shown in fig. 3.6, we get

$$r_x = r \cos \theta = r \cos(\pi - \varphi) = -r \cos \varphi,$$

$$r_y = r \sin \theta = r \sin(\pi - \varphi) = r \sin \varphi.$$

Also, since $\mathbf{r} = r\hat{\mathbf{r}}$, we have

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}, \end{aligned}$$

and

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}. \end{aligned}$$

The radial component of eq. (3.18) thus becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0m_r}\frac{1}{r^2}.$$

Since $\theta = \pi - \varphi$, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0m_r}\frac{1}{r^2}. \quad (3.25)$$

For the angular momentum we have

$$m_r \mathbf{r} \times \mathbf{v} = m_r r \hat{\mathbf{r}} \times \left(\frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\boldsymbol{\theta}} \right) = m_r r^2 \frac{d\theta}{dt} \hat{\mathbf{z}}$$

Using eq. (3.20), we can write the above as

$$m_r r^2 \frac{d\varphi}{dt} = L. \quad (3.26)$$

3.2.4 Particle trajectory

The goal is to find the radial position of the particle as a function of its angular orientation. That is, we want to find $\tilde{r} = \tilde{r}(\tilde{\varphi})$ such that

$$r(t) = \tilde{r}(\varphi(t)). \quad (3.27)$$

To simplify the math, we introduce $\tilde{u} = \tilde{u}(\tilde{\varphi})$ such that $\tilde{u} = 1/\tilde{r}$. Thus

$$\frac{d\tilde{u}}{d\tilde{\varphi}} = -\frac{1}{\tilde{r}^2} \frac{d\tilde{r}}{d\tilde{\varphi}},$$

or, after re-arranging

$$\frac{d\tilde{r}}{d\tilde{\varphi}} = -\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}}. \quad (3.28)$$

We now proceed as follows. Taking the derivative of r , we get

$$\begin{aligned} \frac{dr}{dt} &= \left(\frac{d\tilde{r}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} && [eq. (3.27)] \\ &= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} && [eq. (3.28)] \\ &= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} && [eq. (3.26)] \\ &= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} && [eq. (3.27)] \\ &= \left(-\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} && (3.29) \end{aligned}$$

Taking the derivative of the above, we get

$$\begin{aligned} \frac{d}{dt} \frac{dr}{dt} &= \left[\frac{d}{d\tilde{\varphi}} \left(-\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right) \right]_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \\ &= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} && [eq. (3.26)] \\ &= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} && [eq. (3.27)] \\ &= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2 \tilde{u}^2}{m_r^2} \right)_{\tilde{\varphi}=\varphi(t)} && (3.30) \end{aligned}$$

Plugging the last relation into eq. (3.25) gives

$$\left[-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2\tilde{u}^2}{m_r^2} - \frac{1}{\tilde{u}} \left(\frac{L\tilde{u}^2}{m_r} \right)^2 \right]_{\tilde{\varphi}=\varphi(t)} = \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r} \tilde{u}^2 \right)_{\tilde{\varphi}=\varphi(t)},$$

which, upon re-arranging and dropping the $\varphi(t)$ dependance, becomes

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2 m_r}{4\pi\epsilon_0 L^2} \quad (3.31)$$

Using eq. (3.24) we write the evolution equation for \tilde{u} as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2}{4\pi\epsilon_0 m_r b^2 v_i^2}. \quad (3.32)$$

Introducing the notation

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r v_i^2}, \quad (3.33)$$

the evolution equation for \tilde{u} can be simply expressed as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{b_{90}}{b^2}. \quad (3.34)$$

The boundary conditions for eq. (3.34) are as follows

$$\text{as } \varphi(t) \rightarrow 0, \quad r(t) \rightarrow \infty \quad (3.35)$$

$$\text{as } \varphi(t) \rightarrow 0, \quad \frac{dr(t)}{dt} \rightarrow -v_i \quad (3.36)$$

Given eq. (3.27), eq. (3.35) can only be satisfied if as $\tilde{\varphi} \rightarrow 0$, $\tilde{r} \rightarrow \infty$. Thus, we also have, as $\tilde{\varphi} \rightarrow 0$, $\tilde{u} \rightarrow 0$. Similarly, given eq. (3.29), eq. (3.36) can only be satisfied if as $\tilde{\varphi} \rightarrow 0$

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \rightarrow v_i.$$

Using eq. (3.24) we rewrite the above as

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \rightarrow \frac{1}{b}.$$

The general solution to eq. (3.34) is

$$\tilde{u} = A \cos \tilde{\varphi} + B \sin \tilde{\varphi} - \frac{b_{90}}{b^2}.$$

Applying the boundary conditions, we get

$$\tilde{u} = \frac{b_{90}}{b^2} \cos \tilde{\varphi} + \frac{1}{b} \sin \tilde{\varphi} - \frac{b_{90}}{b^2},$$

which we finally re-write as

$$\frac{1}{\tilde{r}} = \frac{1}{b} \sin \tilde{\varphi} + \frac{b_{90}}{b^2} (\cos \tilde{\varphi} - 1). \quad (3.37)$$

3.2.5 The scattering angle

We now drop the tilde notation for the sake of simplicity. That is, for the radial location of an incident particle, we have

$$\frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{b_{90}}{b^2} (\cos \varphi - 1), \quad (3.38)$$

where φ is the independent variable and $r = r(\varphi)$. We want to know the value of φ as r goes to infinity. Using eq. (3.38), and labeling this angle as φ_s , we have

$$0 = \sin \varphi_s + \frac{b_{90}}{b} (\cos \varphi_s - 1).$$

We express the above in terms of the scattering angle $\theta_s = \pi - \varphi_s$,

$$0 = \sin(\pi - \theta_s) + \frac{b_{90}}{b} [\cos(\pi - \theta_s) - 1].$$

or

$$0 = \sin \theta_s + \frac{b_{90}}{b} (-\cos \theta_s - 1).$$

Re-writing the above as

$$\frac{\cos \theta_s + 1}{\sin \theta_s} = \frac{b}{b_{90}},$$

and using the trig identity $\cot(\theta/2) = (\cos \theta + 1)/\sin \theta$, we get

$$\cot\left(\frac{\theta_s}{2}\right) = \frac{b}{b_{90}}. \quad (3.39)$$

3.2.6 The differential cross section

The differential cross section for Coulomb scattering can be computed by making use of eq. (3.7), which is repeated below

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin \theta_s} \left| \frac{db}{d\theta_s} \right|. \quad (3.7)$$

From eq. (3.39) we get,

$$\frac{db}{d\theta} = -\frac{b_{90}}{2} \frac{1}{\sin^2(\theta_s/2)}, \quad (3.40)$$

which, plugging in eq. (3.7), gives

$$\frac{d\sigma_\theta}{d\Omega} = \left[b_{90} \frac{\cot(\theta_s/2)}{\sin \theta_s} \right] \left[\frac{b_{90}}{2} \frac{1}{\sin^2(\theta_s/2)} \right].$$

Using the trig identities $\cot(\theta) = \cos(\theta)/\sin(\theta)$ and $\sin(\theta) = 2 \sin(\theta/2) \cos(\theta/2)$ we get

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b_{90}^2}{4} \frac{1}{\sin^4(\theta_s/2)}. \quad (3.41)$$

3.2.7 Collision integral

$$\Omega_{\alpha\beta}^{(lk)} = \sqrt{\frac{k_B T}{2\pi M_{\alpha\beta}}} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} dg. \quad (3.42)$$

In the above $M_{\alpha\beta}$ is the reduced mass, given by

$$M_{\alpha\beta} = \frac{M_\alpha M_\beta}{M_\alpha + M_\beta}, \quad (3.43)$$

and $\phi_{\alpha\beta}^{(l)}$ is the collision cross section for a given velocity, and is computed as

$$\phi_{\alpha\beta}^{(l)} = 2\pi \int_0^\infty \left(1 - \cos^l \chi_{\alpha\beta}\right) b db. \quad (3.44)$$

The scattering angle $\chi_{\alpha\beta}$ is given by

$$\chi_{\alpha\beta} = \pi - 2 \int_{r_{\alpha\beta}^{\min}}^\infty \frac{b}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{\alpha\beta}(r)}{g^2 k_B T}\right]^{1/2}} dr. \quad (3.45)$$

For a Coulombic interaction between ions, we can define the natural scale for the cross-sectional area as

$$\phi_{\alpha\beta}^{(0)} = \frac{\pi (Z_\alpha Z_\beta e^2)^2}{(2k_B T)^2}. \quad (3.46)$$

Given this definition, we express the collision integral as

$$\Omega_{\alpha\beta} = \sqrt{\frac{\pi}{M_{\alpha\beta}}} \frac{(Z_\alpha Z_\beta e^2)^2}{(2k_B T)^{3/2}} \mathcal{F}_{\alpha\beta}^{lk}, \quad (3.47)$$

where

$$\mathcal{F}_{\alpha\beta}^{(lk)} = \frac{1}{2\phi_0} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} dg \quad (3.48)$$

We note that $\mathcal{F}_{\alpha\beta}^{(lk)} = 4\mathcal{K}_{lk}(g_{\alpha\beta})$, where $\mathcal{K}_{lk}(g_{\alpha\beta})$ is the notation from the Stanton-Murillo paper.