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1 Governing equations

Define $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $e = e(\mathbf{x}, t)$, and $p = p(\mathbf{x}, t)$.

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \tag{2}$$

$$\rho \left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e \right) = -p \nabla \cdot \mathbf{u} \tag{3}$$

2 Finite element expansion

Define the coefficients $R_i = R_i(t)$, $\mathbf{U}_i = \mathbf{U}_i(t)$, $e_i = e_i(t)$, $p_i = p_i(t)$, as well as the basis functions $\phi_i = \phi_i(\mathbf{x}, t)$, and $w_i = w_i(\mathbf{x}, t)$.

$$\rho = \sum_i^{N_\rho} R_i \phi_i, \tag{4}$$

$$\mathbf{u} = \sum_i^{N_u} \mathbf{U}_i w_i, \tag{5}$$

$$e = \sum_i^{N_e} E_i \phi_i, \tag{6}$$

$$p = \sum_i^{N_p} P_i \phi_i. \tag{7}$$

The basis functions satisfy

$$\frac{\partial \phi_i}{\partial t} + \mathbf{u} \cdot \nabla \phi_i = 0, \tag{8}$$

$$\frac{\partial w_i}{\partial t} + \mathbf{u} \cdot \nabla w_i = 0. \tag{9}$$

3 Semi-discrete momentum conservation

We begin by showing that

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= \left(\sum_i^{N_u} \frac{d\mathbf{U}_i}{dt} w_i + \mathbf{U}_i \frac{\partial w_i}{\partial t}\right) + \mathbf{u}\cdot\left(\sum_i^{N_u} \mathbf{U}_i \nabla w_i\right) \\ &= \sum_i^{N_u} \frac{d\mathbf{U}_i}{dt} w_i + \mathbf{U}_i \left(\frac{\partial w_i}{\partial t} + \mathbf{u}\cdot\nabla w_i\right) \\ &= \sum_i^{N_u} \frac{d\mathbf{U}_i}{dt} w_i. \qquad (10)\end{aligned}$$

Define $\Omega = \Omega(t)$. The finite element formulation of the momentum equation is thus

$$\int_\Omega \rho \sum_i^{N_u} \frac{d\mathbf{U}_i}{dt} w_i w_j \, dV = \int_\Omega \sum_i^{N_p} P_i \phi_i \nabla w_j \, dV \qquad \text{for } j = 1, ..., N_u. \qquad (11)$$

The above is re-written as

$$\sum_i^{N_u} \frac{d\mathbf{U}_i}{dt} m_{ij} = \sum_i^{N_p} P_i \mathbf{d}_{ij} \qquad \text{for } j = 1, ..., N_u. \qquad (12)$$

where the mass bilinear form m_{ij} is given by

$$m_{ij} = \int_\Omega \rho w_i w_j \, dV, \qquad (13)$$

and the derivative bilinear form $\mathbf{d}_{ij}$ by

$$\mathbf{d}_{ij} = \int_\Omega \phi_i \nabla w_j \, dV. \qquad (14)$$

In terms of its components, eq. (12)

$$\begin{aligned}\sum_i^{N_u} \frac{dU_{x,i}}{dt} m_{ij} &= \sum_i^{N_p} P_i d_{x,ij} \qquad \text{for } j = 1, ..., N_u, \\ \sum_i^{N_u} \frac{dU_{y,i}}{dt} m_{ij} &= \sum_i^{N_p} P_i d_{y,ij} \qquad \text{for } j = 1, ..., N_u, \\ \sum_i^{N_u} \frac{dU_{z,i}}{dt} m_{ij} &= \sum_i^{N_p} P_i d_{z,ij} \qquad \text{for } j = 1, ..., N_u. \qquad (15)\end{aligned}$$