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1 Governing equations

We introduce the flow variables $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $e = e(\mathbf{x}, t)$, and $p = p(\mathbf{x}, t)$. The governing equations that dictate their evolution are

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p, \tag{2}$$

$$\rho \left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e \right) = -p \nabla \cdot \mathbf{u}. \tag{3}$$

2 Finite element expansion

We introduce the coefficients $R_i = R_i(t)$, $\mathbf{U}_i = \mathbf{U}_i(t)$, $E_i = E_i(t)$, $P_i = P_i(t)$, as well as the basis functions $\phi_i = \phi_i(\mathbf{x}, t)$, and $w_i = w_i(\mathbf{x}, t)$. We note that $\mathbf{U}_i$ is a vector whose components are $U_{i,\alpha} = U_{i,\alpha}(t)$ for $\alpha = x, y, z$. These coefficients are used in the following expansions

$$\rho = \sum_j^{N_\rho} R_j \phi_j, \tag{4}$$

$$\mathbf{u} = \sum_j^{N_u} \mathbf{U}_j w_j, \tag{5}$$

$$e = \sum_j^{N_e} E_j \phi_j, \tag{6}$$

$$p = \sum_j^{N_p} P_j \phi_j. \tag{7}$$

The basis functions are defined so that they satisfy

$$\frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j = 0, \tag{8}$$

$$\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j = 0. \tag{9}$$

3 Semi-discrete momentum conservation

We begin by showing that

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= \sum_{j}^{N_u}\left(\frac{d\mathbf{U}_j}{dt}w_j + \mathbf{U}_j\frac{\partial w_j}{\partial t}\right) + \mathbf{u}\cdot\left(\sum_{j}^{N_u}\mathbf{U}_j\nabla w_j\right) \\
&= \sum_{j}^{N_u}\left[\frac{d\mathbf{U}_j}{dt}w_j + \mathbf{U}_j\left(\frac{\partial w_j}{\partial t} + \mathbf{u}\cdot\nabla w_j\right)\right] \\
&= \sum_{j}^{N_u}\frac{d\mathbf{U}_j}{dt}w_j. \qquad (10)
\end{aligned}$$

Define the domain of the problem under consideration as $\Omega = \Omega(t)$. The finite element formulation of the momentum equation is thus

$$\int_\Omega \rho \sum_{j}^{N_u}\frac{d\mathbf{U}_j}{dt}w_j w_i\, dV = \int_\Omega \sum_{j}^{N_p} P_j\phi_j\nabla w_i\, dV \qquad \text{for } i = 1,...,N_u. \qquad (11)$$

The above is re-written as

$$\sum_{j}^{N_u}\frac{d\mathbf{U}_j}{dt}m_{ij} = \sum_{j}^{N_p}P_j\mathbf{d}_{ji} \qquad \text{for } i = 1,...,N_u. \qquad (12)$$

where the mass bilinear form m_{ij} is given by

$$m_{ij} = \int_\Omega \rho w_i w_j\, dV, \qquad (13)$$

and the derivative bilinear form $\mathbf{d}_{ij}$ by

$$\mathbf{d}_{ij} = \int_\Omega \phi_i \nabla w_j\, dV. \qquad (14)$$

Note that $\mathbf{d}_{ij}$ is a vector whose components are $d_{ij,\alpha}$, for $\alpha = x, y, z$, where α determines which component of the ∇ operator is being used. Equation (12) can thus be expanded as

$$\begin{aligned}
\sum_{j}^{N_u}\frac{dU_{j,x}}{dt}m_{ij} &= \sum_{i}^{N_p}P_j d_{ji,x} \qquad \text{for } i = 1,...,N_u, \\
\sum_{j}^{N_u}\frac{dU_{j,y}}{dt}m_{ij} &= \sum_{i}^{N_p}P_j d_{ji,y} \qquad \text{for } i = 1,...,N_u, \\
\sum_{j}^{N_u}\frac{dU_{j,z}}{dt}m_{ij} &= \sum_{i}^{N_p}P_j d_{ji,z} \qquad \text{for } i = 1,...,N_u. \qquad (15)
\end{aligned}$$

We'll now write the above in matrix notation. Introduce the vector $\mathbf{U}_x = \mathbf{U}_x(t)$ whose components are $U_{i,x}$ for $i = 1,...,N_v$. The analogous holds for $\mathbf{U}_y$ and $\mathbf{U}_z$. Similarly, we introduce the matrix

$\mathbf{D}_x = \mathbf{D}_x(t)$ whose components are $d_{ij,x}$. The analogous holds for D_y and D_z. Finally, the matrix $\mathbf{M} = \mathbf{M}(t)$ is that with components m_{ij} and the vector $\mathbf{P} = \mathbf{P}(t)$ is that with components P_j. Equation (15) can now be written as

$$
\begin{aligned}
M\frac{d\mathbf{U}_x}{dt} &= \mathbf{D}_x^T\mathbf{P}, \\
M\frac{d\mathbf{U}_y}{dt} &= \mathbf{D}_y^T\mathbf{P}, \\
M\frac{d\mathbf{U}_z}{dt} &= \mathbf{D}_z^T\mathbf{P}. \qquad (16)
\end{aligned}
$$