Coordinate system for Tokamaks

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1 Basic definitions

1.1 Eulerian coordinates

Consider our traditional Eucledian coordinate system given by coordinates (x^1, x^2, x^3) and unit vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2$. The position vector is $\mathbf{x} = x^1\mathbf{x}_1 + x^2\mathbf{x}_2 + x^3\mathbf{x}_3$.

1.2 Curvilinear Coordinates

We will now define a new coordinate system, a curvilinear coordinate system, in relation to the standard Eucledian coordinates. To do so we first define the transformation

$$\hat{u}^i = \hat{u}^i(x^1, x^2, x^3). \tag{1}$$

and we label its inverse as

$$\hat{x}^i = \hat{x}^i(u^1, u^2, u^3) \tag{2}$$

Thus, we can write

$$\hat{x}^i(\hat{u}^1, \hat{u}^2, \hat{u}^3) = x^i \tag{3}$$

$$\hat{u}^i(\hat{x}^1, \hat{x}^2, \hat{x}^3) = u^i \tag{4}$$

We can now take the derivative of either eq. (3) or eq. (4). For example, the derivative d/du^j of eq. (4) gives

$$\left(\frac{\partial \hat{u}^i}{\partial x^k}\right)_{x^i=\hat{x}^i} \frac{\partial \hat{x}^k}{\partial u^j} = \delta^i_j. \tag{5}$$

We can evaluate the above at $u^i = \hat{u}^i$, so that

$$\frac{\partial \hat{u}^i}{\partial x^k} \left(\frac{\partial \hat{x}^k}{\partial u^j}\right)_{u^i=\hat{u}^i} = \delta^i_j. \tag{6}$$

We now define two basis vectors as follows

$$\mathbf{e}_i = \left(\frac{\partial \hat{x}^1}{\partial u^i}\right)_{u^i=\hat{u}^i} \mathbf{x}_1 + \left(\frac{\partial \hat{x}^2}{\partial u^i}\right)_{u^i=\hat{u}^i} \mathbf{x}_2 + \left(\frac{\partial \hat{x}^3}{\partial u^i}\right)_{u^i=\hat{u}^i} \mathbf{x}_3 \tag{7}$$

$$\mathbf{e}^i = \frac{\partial \hat{u}^i}{\partial x^1}\mathbf{x}^1 + \frac{\partial \hat{u}^i}{\partial x^2}\mathbf{x}^2 + \frac{\partial \hat{u}^i}{\partial x^3}.\mathbf{x}^3 \tag{8}$$

The dot product of these two vectors is given by

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{9}$$

The two coordinate bases are not necessarily constant, orthogonal, of unit length, or dimensionless.

At the end, one way to think about it is that for every point $[x^1, x^2, x^3]$ in the Eucledian coordiante system, there is a coresponding coordinate given by $[\hat{u}^1, \hat{u}^2, \hat{u}^3]$, and that at every point of these new coordinates, there are two coordinate bases, given by eq. (7) and eq. (8). The latter basis is typically expresses as $\nabla\hat{u}^1$,$\nabla\hat{u}^2$, and $\nabla\hat{u}^3$.

1.3 Vectors

Since there are two coordinate bases, one can define two types of vectors in curvilinear coordinate systems. One is a vector in terms of contravariant components v^i

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3, \tag{10}$$

and the other a vector in terms of covariant components v_i

$$\mathbf{v} = v_1\mathbf{e}^1 + v_2\mathbf{e}^2 + v_3\mathbf{e}^3. \tag{11}$$

Note that, due to eq. (9), we have $v^i = \mathbf{v} \cdot \mathbf{e}^i$ and $v_i = \mathbf{v} \cdot \mathbf{e}_i$.

We now define the metric coefficients g_{ij} and g^{ij} as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{12}$$
$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j \tag{13}$$
$$\tag{14}$$

Thus, the dot product of two vectors in the curvilinear reference frame simplifies to the following

$$\mathbf{v} \cdot \mathbf{w} = v^i w_i = v_i w^i = g_{ij} v^i w^j = g^{ij} v_i w_j. \tag{15}$$

The cross product can be computed as

$$\mathbf{v} \times \mathbf{w} = \epsilon_{ijk}\sqrt{g} v^i w^j \mathbf{e}^k \tag{16}$$
$$\mathbf{v} \times \mathbf{w} = \epsilon^{ijk}\frac{1}{\sqrt{g}} v_i w_j \mathbf{e}_k, \tag{17}$$

where $g = \det[g_{ij}]$. One can also define $g^{-1} = \det[g^{ij}]$.

2 Calculus

2.1 Integration

- Volume integrals: Define dV_u as an infinitesimal volume in curvilinear coordinates, Ω_u as a finite volume of integration in curvilinear coordinates, and f_u as a function whose input is in curvilinear coordinates, that is, $f_u = f_u(u^1, u^2, u^3)$. Then, the volume integral can be computed using

$$\int_{\Omega_u} f_u \, dV_u = \int_{\Omega_u} f_u \, J du^1 du^2 du^3, \tag{18}$$

where J is the Jacobian. Given that Eulerian coordinates can be thought of as an instance of curvilinear coordinates, we have

$$\int_{\Omega_u} f_u \, dV_u = \int_{\Omega_x} f_x \, dV_x = \int_{\Omega_x} f_x \, dx^1 dx^2 dx^3 \tag{19}$$

One thing to note is that $f_x(x^1, x^2, x^3)$ and $f_u(u^1, u^2, u^3)$ are equal when evaluated at the same point in space. In other words, these functions satisfy

$$f_u(u^1, u^2, u^3) = f_x(\hat{x}^1, \hat{x}^2, \hat{x}^3). \tag{20}$$

Equating eqs. (18) and (19) allows us to write the integration by substitution rule

$$\int_{\Omega_x} f_x(x^1, x^2, x^3) \, dx^1 dx^2 dx^3 = \int_{\Omega_u} f_x(\hat{x}^1, \hat{x}^2, \hat{x}^3) \, J du^1 du^2 du^3 \tag{21}$$

- Surface integrals: Define dS_u as an infinitesimal surface in curvilinear coordinates, Γ_u as a finite surface of integration in curvilinear coordinates that belongs to the $u^1 =$ constant surfaces, and f_u as a function whose input is defined using curvilinear coordinates. Then, a surface integral can be computed using

$$\int_{\Gamma_u} f_u \, dS_u = \int_{\Gamma_u} f_u \, J|\nabla \hat{u}^1| du^2 du^3. \tag{22}$$

Note that now we can write

$$\int_{\Omega_u} f_u \, dV_u = \int_{u_l^1}^{u_u^1} \int_{\Gamma_u} f_u \, J du^1 du^2 du^3 = \int_{u_l^1}^{u_u^1} \int_{\Gamma_u} f_u \, \frac{J|\nabla \hat{u}^1| du^2 du^3}{|\nabla \hat{u}^1|} du^1 = \int_{u_l^1}^{u_u^1} \int_{\Gamma_u} f_u \, \frac{dS_u}{|\nabla \hat{u}^1|} du^1. \tag{23}$$

2.2 Differentiation

2.2.1 The grad operator

Consider the function $f_x = f_x(x^1, x^2, x^3)$ and the grad operator, which is

$$\nabla f_x = \frac{\partial f_x}{\partial x^1}\mathbf{x}_1 + \frac{\partial f_x}{\partial x^2}\mathbf{x}_2 + \frac{\partial f_x}{\partial x^3}\mathbf{x}_3. \tag{24}$$

We now introduce the function $f_u = f_u(u^1, u^2, u^3)$ and note that $f_x = f_u(\hat{u}^1, \hat{u}^2, \hat{u}^3)$. Thus

$$\frac{\partial f_x}{\partial x^1} = \left(\frac{\partial f_u}{\partial u^1}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^1}{\partial x^1} + \left(\frac{\partial f_u}{\partial u^2}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^2}{\partial x^1} + \left(\frac{\partial f_u}{\partial u^3}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^3}{\partial x^1}, \tag{25}$$

$$\frac{\partial f_x}{\partial x^2} = \left(\frac{\partial f_u}{\partial u^1}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^1}{\partial x^2} + \left(\frac{\partial f_u}{\partial u^2}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^2}{\partial x^2} + \left(\frac{\partial f_u}{\partial u^3}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^3}{\partial x^2}, \tag{26}$$

$$\frac{\partial f_x}{\partial x^3} = \left(\frac{\partial f_u}{\partial u^1}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^1}{\partial x^3} + \left(\frac{\partial f_u}{\partial u^2}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^2}{\partial x^3} + \left(\frac{\partial f_u}{\partial u^3}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^3}{\partial x^3}. \tag{27}$$

Using the definition of $\mathbf{e}^i$, the grad operator can be written as

$$\nabla f_x = \left(\frac{\partial f_u}{\partial u^1}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \mathbf{e}^1 + \left(\frac{\partial f_u}{\partial u^2}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \mathbf{e}^2 + \left(\frac{\partial f_u}{\partial u^3}\right)_{\mathbf{u}=\hat{\mathbf{u}}} \mathbf{e}^3. \tag{28}$$

This shows the equivalence between the grad operator in Eulerian coordinates and curvilinear coordinates.

2.2.2 The divergence operator

2.2.3 The curl operator

The curl is given by

$$\nabla \times A = \epsilon^{ijk} \frac{1}{\sqrt{g}} \frac{\partial A_j}{\partial u^i} \mathbf{e}_k \tag{29}$$

3 Flux coordinates

Imagine that Eucledian space is permeated by a set of surfaces, which we call flux surfaces. Each of those surfaces is labeled with a different value of the variable ψ. We also note that the flux surfaces are not stationary, they can move around as time progresses.

We now introduce the function $\hat{\psi} = \hat{\psi}(t, x^1, x^2, x^3)$. This function is defined in such a way that for all values x^1, x^2, x^3 that are part of a given flux surface at a specific time t, then $\hat{\psi}$ will evaluate to the value of ψ corresponding to that flux surface. The velocity of the flux surfaces is given by $\mathbf{V}_\psi = \mathbf{V}_\psi(t, x^1, x^2, x^3)$. Thus, by definition

$$\frac{\partial \hat{\psi}}{\partial t} + \mathbf{V}_\psi \cdot \nabla \hat{\psi} = 0. \tag{30}$$

A flux coordinate is defined as one in which $\hat{u}^1 = \hat{\psi}$.

3.1 Flux-surface averaging

To begin, we define the following. $D(\psi, t)$ is the volume enclosed at time t by the flux surface labelled by ψ. The surface of $D(\psi, t)$ is labelled as $\partial D(\psi, t)$. Additionally, $\Delta(\psi, t) = D(\psi+\Delta\psi, t) - D(\psi, t)$.

The flux surface average of a function is given by

$$\langle f \rangle_\psi = \lim_{\Delta\psi \to 0} \frac{\int_{\Delta(\psi,t)} f\, dV}{\int_{\Delta(\psi,t)} dV}. \tag{31}$$

This can be re-written as shown below

$$\langle f \rangle_\psi = \lim_{\Delta\psi \to 0} \frac{\frac{1}{\Delta\psi} \int_{\Delta(\psi,t)} f\, dV}{\frac{1}{\Delta(\psi)} \int_{\Delta(\psi,t)} dV} = \lim_{\Delta\psi \to 0} \frac{\frac{1}{\Delta\psi} \left(\int_{D(\psi+\Delta\psi,t)} f\, dV - \int_{D(\psi,t)} f\, dV \right)}{\frac{1}{\Delta\psi} \left(\int_{D(\psi+\Delta\psi,t)} dV - \int_{D(\psi,t)} dV \right)} = \frac{\frac{\partial}{\partial\psi} \int_{D(\psi,t)} f\, dV}{\frac{\partial}{\partial\psi} \int_{D(\psi,t)} dV}. \tag{32}$$

Defining $V' = V'(\psi, t)$ as

$$V' = \frac{\partial}{\partial\psi} \int_{D(\psi,t)} dV. \tag{33}$$

the second expression for the flux surface average is written as

$$\langle f \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{D(\psi,t)} f \, dV. \tag{34}$$

A third expression for $\langle g \rangle_\psi$ follows from using eq. (23) for the above. Thus,

$$\langle f \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_0^\psi \int_{\partial D(\psi',t)} f \, \frac{dS}{|\nabla \hat{\psi}|} d\psi' = \frac{1}{V'} \int_{\partial D(\psi,t)} f \, \frac{dS}{|\nabla \hat{\psi}|}. \tag{35}$$

3.1.1 Average of spatial derivatives

We use the second definition of the flux-surface average, given by eq. (34), and then the divergence theorem to obtain

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{D(\psi,t)} \nabla \cdot \mathbf{A} \, dV = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{\partial D(\psi,t)} \mathbf{A} \cdot \frac{\nabla \hat{\psi}}{|\nabla \hat{\psi}|} \, dS. \tag{36}$$

We now use the third definition eq. (35) to obtain

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \mathbf{A} \cdot \nabla \hat{\psi} \rangle_\psi. \tag{37}$$

3.1.2 Average of time derivatives

Using the Reynolds transport theorem we show

$$\begin{aligned} \frac{\partial}{\partial t} \int_{D(\psi,t)} f \, dV &= \int_{D(\psi,t)} \frac{\partial f}{\partial t} \, dV + \int_{\partial D(\psi,t)} f \mathbf{V}_\psi \cdot \frac{\nabla \hat{\psi}}{|\nabla \hat{\psi}|} \, dS \\ &= \int_{D(\psi,t)} \frac{\partial f}{\partial t} \, dV + V' \langle f \mathbf{V}_\psi \cdot \nabla \hat{\psi} \rangle_\psi. \end{aligned} \tag{38}$$

We now take the derivative of both sides by ψ and then divide by V'.

$$\frac{1}{V'} \frac{\partial}{\partial t} V' \langle f \rangle_\psi = \left\langle \frac{\partial f}{\partial t} \right\rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle f \mathbf{V}_\psi \cdot \nabla \hat{\psi} \rangle_\psi. \tag{39}$$

Re-arranging and using eq. (30)

$$\left\langle \frac{\partial f}{\partial t} \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial t} V' \langle f \rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle f \frac{\partial \hat{\psi}}{\partial t} \right\rangle_\psi. \tag{40}$$