Electromagnetism

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This document first focuses on electrostatics and magnetostatics, which can be understood as follows

$$\begin{aligned}
&\text{stationary charges} \rightarrow \text{constant electric fields} = \text{electrostatics} \\
&\text{stationary currents} \rightarrow \text{constant magnetic fields} = \text{magnetostatics.}
\end{aligned} \tag{1}$$

1 Electrostatics

- Coulomb's Law
$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}} \tag{2}$$
- Electric Field $\mathbf{E}$ derived from $\mathbf{F} = Q\mathbf{E}$
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{3}$$
- If there are multiple point charges
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i \tag{4}$$
- **Charge distributions and fields:** if the charges are so small and so numerous that they can be described using a continuous distribution (i.e. $q_i \rightarrow dq = \rho d\tau$, where ρ is a charge density and $d\tau$ and infinitesimal volume)
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{r^2}\hat{\mathbf{r}}d\tau' \tag{5}$$
If the charge distribution is localized to a surface or a line, then the analogous of the above is
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{r}')}{r^2}\hat{\mathbf{r}}da' \qquad \text{or} \qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{r^2}\hat{\mathbf{r}}dl' \tag{6}$$
Taking the divergence and curl of eq. (5):
$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{7}$$
$$\nabla\times\mathbf{E} = 0 \tag{8}$$

- **Fields and potentials**
  Since $\nabla \times \mathbf{E} = 0$ we have
$$\mathbf{E} = -\boldsymbol{\nabla} V. \tag{9}$$
  where V is the electric potential. Fundamental theorem of calculus can be used to express the potential $V(\mathbf{r})$ as
$$V(\mathbf{r}) - V(\mathcal{O}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{10}$$
  where $\mathcal{O}$ is the reference point, at which one usually defines $V(\mathcal{O}) = 0$ (e.g. sea-level as the altitude at which height is equal to zero).

- **Charge distributions and potentials**
  Divergence of eq. (9) gives
$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{11}$$
  whose solution is
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{r} d\tau'. \tag{12}$$

- Define potential energy U as the negative of the work required to move charge Q from $\mathbf{a}$ to $\mathbf{b}$.
$$U = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})] \tag{13}$$
  If the reference point is infinity, then $U(\mathbf{r}) = QV(\mathbf{r})$.

- Potential energy of a set of charges q_i
$$U = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{14}$$
  where $V(\mathbf{r}_i)$ is the potential due to all charges except the one at $\mathbf{r}_i$. The continuous form is
$$U = \frac{1}{2}\int \rho V \, d\tau = \frac{\epsilon_0}{2}\int E^2 \, d\tau \tag{15}$$
  were now V represents the potential due to all charges. Thus, if ρ is such that it defines a set of point charges (e.g. $\delta(\mathbf{r})$), then eq. (15) would be equal to eq. (14) plus the additional terms corresponding to $i = j$. Those additional terms correspond to the energy required to create point charges, which is infinity.

- Electrostatic conductors: materials whose charges are free to move but are in a state of electrostatic equilibrium. $\mathbf{E} = 0$ inside, since if it were not, then charges would move and the material would not be in electrostatic equilibrium. As a consequence, $\rho = 0$ inside, all the charge is on the surface, and $\mathbf{E}$ is perpendicular to the outer surface.

- If there is a cavity within the conductor, and within the cavity a charge q, an amount $-q$ of charge will reside in the inner surface, and an amount q on the outer surface, and that configuration will lead to $\mathbf{E} = 0$ inside the conductor.

- Faraday cage: if there are no charges within such cavity, then $\mathbf{E} = 0$ within the cavity as well, regardless of how many charges are outside the conductor. If $\mathbf{E}$ was not zero inside the cavity, then its field lines would start and end on the cavity walls. Letting the field lines be part of a closed loop, the rest of which is inside the conductor, then the line integral along the closed loop would be positive, in violation of $\nabla \times \mathbf{E} = 0$.
- A capacitor consists of two conductors, one with charge Q and the other with charge $-Q$. The constant of proportionality between Q and the voltage difference between the two conductors is the capacitance $C = Q/V$. The energy stored in a capacitor is $W = \frac{1}{2}CV^2$.

2 Magnetostatiscs

- Lorentz force law: $\mathbf{F} = Q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$
- Given the charge densities λ, σ, and ρ
  - Current [Amperes]: the amount of charge that passes a point in a small amount of time.
  $$\mathbf{I} = \lambda \mathbf{v} \tag{16}$$
  - Surface current density: the amount of charge that passes a line in a small amount of time.
  $$\mathbf{K} = \sigma \mathbf{v} \tag{17}$$
  - Volume current density: the amount of charge that passes an area in a small amount of time.
  $$\mathbf{J} = \rho \mathbf{v} \tag{18}$$
- Magnetic component of Lorentz force
$$\mathbf{F}_{\text{mag}} = \int \mathbf{I} \times \mathbf{B}\, dl = \int \mathbf{K} \times \mathbf{B}\, da = \int \mathbf{J} \times \mathbf{B}\, d\tau \tag{19}$$
- Conservation of current
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{20}$$
- **Charge currents and fields**
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, dl' \tag{21}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, da' \tag{22}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, d\tau' \tag{23}$$

  Taking the divergence and curl of eq. (23):
$$\nabla \cdot \mathbf{B} = 0 \tag{24}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{25}$$

- A steady straight-line current leads to a circular magnetic field around it. A steady circular current leads to a straight magnetic field line along the axis of the circle.

- **Fields and potentials**
  Since $\nabla \cdot \mathbf{B} = 0$ we have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{26}$$

  where $\mathbf{A}$ is the magnetic vector potential.

- **Charge currents and potentials**
  The magnetic field is not altered if a function whose curl vanishes (that is $\nabla\lambda$) is added to $\mathbf{A}$. Thus, λ can be picked to make $\mathbf{A}$ divergence-less. Taking the curl of $\mathbf{B}$ then leads to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \tag{27}$$

  whose solution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau'. \tag{28}$$

3 Electric Fields in Matter

4 Magnetic Fields in Matter

5 Electrodynamics

5.1 Ohm's Law

- Ohm's law refers to the proportionality between the force per unit charge applied to charged elements and the resulting volume current that occurs. That is,

$$\mathbf{J} = \sigma \mathbf{f}, \tag{29}$$

  where $\mathbf{f}$ is the force per unit charge, and the proportionality σ is the conductivity. If one neglects the magnetic contribution to $\mathbf{f}$, which is typically done for non-plasmas, then

$$\mathbf{J} = \sigma \mathbf{E}. \tag{30}$$

  For steady currents $(\partial\rho/\partial t = 0)$ and uniform conductivity

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = 0 \tag{31}$$

  and thus, the charge density is zero. This is similar to a conductor, but now we have charges moving.

- Similarly, given an applied voltage, a current will result. The constant of proportionality R, known as the resistance, is given by

$$V = IR. \tag{32}$$

5.2 Electromagnetic induction

- Defined the electromotive force (emf) as

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \tag{33}$$

- The universal flux rule states: whenever the magnetic flux through a loop

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \tag{34}$$

  changes, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{35}$$

  will appear in the loop. This can occur in two ways:

  1. Magnetic field doesn't change, loop changes:
     For example, a loop of wire is pulled to the right through a constant magnetic field. In this case the emf is magnetic.
  2. Magnetic field changes, loop doesn't change:
     There is a stationary loop (any loop, not necessarily a physical loop of wire), and the magnetic field through it changes. In this case, the **changing magnetic field induces an electric field** and thus the emf is electric. Using eq. (35) we get **Faraday's law**

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}, \tag{36}$$

     which, in differential form is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{37}$$

- Lenz's law: Nature abhors a change in flux. Thus, as the magnetic flux changes and it induces an electric field over a loop, the resulting current goes in a direction such that it would create an opposing flux that tries to cancel the original change in magnetic flux.

- Mutual inductance:
  If there is a steady current going through a wire loop, this will create a magnetic field and thus a magnetic flux through another wire loop close by. The constant of proportionality between the flux through the second loop and the current in the first is the mutual inductance M. That is

$$\Phi_2 = M_{21} I \tag{38}$$

  Note: If I ran the same current on loop two, then the flux in loop one would be $\Phi_1 = M_{12} I$. However, it can be shown that $M_{21} = M_{12}$ and thus $\Phi_1 = \Phi_2$.

  Now, imagine the current in loop one changes in time. The magnetic field associated with that current changes in time, and thus the magnetic flux through loop two changes as well. That is,

$$\Phi_2(t) = M I_1(t). \tag{39}$$

Due to Faraday's law an induced emf would be created in the second loop,

$$\mathcal{E}_2(t) = -M\frac{dI(t)}{dt}. \tag{40}$$

This emf creates a current $I_2(t)$ in the second loop.

- Self inductance:
  The changing magnetic field associated with the changing current in loop one also creates a changing flux within this loop. This is given by

  $$\Phi_1(t) = LI_1(t), \tag{41}$$

  where L is the self-inductance. Again, the changing flux leads to an emf within loop one, called the back emf

  $$\mathcal{E}(t) = -L\frac{dI(t)}{dt}. \tag{42}$$

  This emf drives a new current in loop one that opposes the original current change.

- The energy stored in magnetic fields is given by

  $$W = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J}\,d\tau = \frac{1}{2\mu_0}\int B^2\,d\tau. \tag{43}$$

- Ampere's law eq. (25) was derived using assumptions of magnetostatics. Maxwell extended Ampere's law to work for magnetodynamics, so that the divergence of eq. (25) would actually give zero on both sides. Thus, Maxwell's equations are

  $$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{44}$$

  $$\nabla\cdot\mathbf{B} = 0 \tag{45}$$

  $$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{46}$$

  $$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{47}$$

- As shown earlier, Faraday's law indicates that a changing magnetic field induces an electric field. Maxwell's correction to Ampere's law then indicates that a changing electric field induces a magnetic field.

6 Conservation Laws

6.1 Conservation of energy

- Suppose you assemble a distribution of charges and currents, which at time t produce fields $\mathbf{E}$ and $\mathbf{B}$.

- The potential energy of the system, as shown in previous sections, would be

  $$U_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right). \tag{48}$$

- The question then arises, how much energy would be transferred to the charges as these charges are allowed to move?
- Label U_{mech} as the energy density gained by these charges as they are allowed to move.
- The evolution of U_{mech} is then

$$\frac{\partial U_{mech}}{\partial t} = -\frac{\partial U_{em}}{\partial t} - \nabla \cdot \mathbf{S}. \tag{49}$$

In the above $\mathbf{S}$ is the pointing vector and is defined as

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}). \tag{50}$$

It represents the flux of energy across space.

- As the evolution equation above shows, a decreasing potential energy in the electromagnetic field constitutes a transfer of this lost energy to that of the charges.
- In integral form, we write the above as

$$\frac{d}{dt}\int_V U_{mech}\, d\tau = -\frac{d}{dt}\int_V U_{em}\, d\tau - \oint \mathbf{S} \cdot d\mathbf{a}. \tag{51}$$

6.2 Conservation of momentum

- We now ask, how much momentum would be transferred to the charges as these are allowed to move?
- Label P_{mech} as the momentum density gained by the charges as they move around.
- Label P_{em} as the momentum density stored in the electromagnetic fields themselves. This is defined as

$$P_{em} = \mu_0 \epsilon_0 \mathbf{S}. \tag{52}$$

- The evolution of P_{mech} is then

$$\frac{\partial P_{mech}}{\partial t} = -\frac{\partial P_{em}}{\partial t} + \nabla \cdot \mathbf{T}. \tag{53}$$

In the above $\mathbf{T}$ is the Maxwell stress tensor and is defined as

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right). \tag{54}$$

It represents the flux of momentum across space.

- As the evolution equation above shows, a decreasing momentum in the electromagnetic field constitutes a transfer of this lost momentum to that of the charges.
- In integral form, we write the above as

$$\frac{d}{dt}\int_V P_{mech}\, d\tau = -\frac{d}{dt}\int_V P_{em}\, d\tau + \oint \mathbf{T} \cdot d\mathbf{a}. \tag{55}$$

7 Electromagnetic waves

7.1 Simple waves

- The simplest kind of waves can be written as

$$u(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi\right) \tag{56}$$

where

$$\begin{aligned} A &: \text{magnitude} \\ \lambda &: \text{wavelength} \\ T &: \text{period} \\ \phi &: \text{phase constant} \end{aligned}$$

Thus, as x goes from zero to λ, for example, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in space. Similarly, as t goes from zero to T, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in time.

- Defining the wavevector and angular frequency as

$$k = \frac{2\pi}{\lambda} \qquad w = \frac{2\pi}{T}, \tag{57}$$

then

$$u(x,t) = A\sin(kx - wt + \phi), \tag{58}$$

- The frequency ν is the inverse of the period, $\nu = 1/T$.
- By inspecting the form of the simple sinusoidal wave above, it is clear that the velocity of the wave is

$$v = \frac{w}{k} = \frac{\lambda}{T} = \lambda\nu. \tag{59}$$

- A general wave can be Fourier decomposed as follows

$$u(x,t) = \sum_n \hat{u}_n e^{i(k_n z - wt + \phi)}, \tag{60}$$

where $k_n = 2\pi n/L$. The above is often re-written as

$$u(x,t) = \sum_n \tilde{u}_n e^{i(k_n z - wt)}, \tag{61}$$

where $\tilde{u}_n = \hat{u}_n e^{i\phi}$.

- For the more general three-dimensional case, a wave is decomposed as follows

$$\mathbf{u}(x,t) = \sum_{\mathbf{n}} \tilde{\mathbf{u}}_{\mathbf{n}} e^{i(\mathbf{k_n}\cdot\mathbf{x} - wt)}, \tag{62}$$

where $\mathbf{k_n} = 2\pi\mathbf{n}/L$ and $\mathbf{n} = [n_1, n_2, n_3]$.

- A plane wave is one for which the only existing $\mathbf{k_n}$'s point along a single direction. Without loss of generality, we can assume this direction is the z direction and thus write

$$\mathbf{u}(x,t) = \sum_{n_3} \tilde{\mathbf{u}}_{n_3} e^{i\left(k_{n_3}\cdot z - wt\right)}, \tag{63}$$

7.2 Electromagnetic waves in vacuum

- The application of eq. (62) to electric and magnetic fields gives

$$\mathbf{E} = \sum_{\mathbf{n}} \tilde{\mathbf{E}}_{\mathbf{n}} e^{i(\mathbf{k_n}\cdot\mathbf{x} - wt)}, \tag{64}$$

  and

$$\mathbf{B} = \sum_{\mathbf{n}} \tilde{\mathbf{B}}_{\mathbf{n}} e^{i(\mathbf{k_n}\cdot\mathbf{x} - wt)}. \tag{65}$$

- For $\rho = \mathbf{J} = 0$, Maxwell's equations can be combined to give the wave equations for $\mathbf{E}$ and $\mathbf{B}$, that is,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\epsilon_0\mu_0}\nabla^2\mathbf{E} = 0, \tag{66}$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{1}{\epsilon_0\mu_0}\nabla^2\mathbf{B} = 0. \tag{67}$$

  The speed of electromagnetic waves is thus $c = 1/\sqrt{\epsilon_0\mu_0}$.

- Using eq. (64) in $\nabla\cdot\mathbf{E} = 0$ gives $\mathbf{k}_n\cdot\tilde{\mathbf{E}}_{\mathbf{n}} = 0$. That is, the $\mathbf{E}$ field is orthogonal to the direction of propagation of the mode.

- Using eq. (65) in $\nabla\cdot\mathbf{B} = 0$ gives $\mathbf{k}_n\cdot\tilde{\mathbf{B}}_{\mathbf{n}} = 0$. That is, the $\mathbf{B}$ field is orthogonal to the direction of propagation of the mode.

- Using eqs. (64) and (65) in $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ gives $\mathbf{k_n}\times\tilde{\mathbf{E}}_{\mathbf{n}} = w\tilde{\mathbf{B}}_{\mathbf{n}}$. That is, the $\mathbf{B}$ field is orthogonal to the $\mathbf{E}$ field.