Physical Kinetics

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Appendix A

Lagrangian and Eulerian PDFs

A.1 Eulerian PDF

Consider an Eulerian velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. The Eulerian PDF $f = f(\mathbf{V}; \mathbf{x}, t)$ gives the probability that the velocity field will have a value of $\mathbf{V}$ at location $\mathbf{x}$ and at time t. We'll also introduce the fine-grained Eulerian PDF $f' = f'(\mathbf{V}; \mathbf{x}, t)$, which is defined as

$$f'(\mathbf{V}; \mathbf{x}, t) = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}). \tag{A.1}$$

Note: a delta function of a 3D argument means the following $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$. The Eulerian PDF can be obtained from the fine-grained Eulerian PDF using

$$f(\mathbf{V}; \mathbf{x}, t) = \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle. \tag{A.2}$$

The proof is as follows,

$$\begin{aligned} \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle &= \langle \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) f(\mathbf{V}'; \mathbf{x}, t) d\mathbf{V}' \\ &= f(\mathbf{V}; \mathbf{x}, t). \end{aligned} \tag{A.3}$$

A.2 Lagrangian PDF

Consider a Lagrangian particle with velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$ and position $\mathbf{x}^+(t, \mathbf{y})$. The Lagrangian PDF $f_L = f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$ gives the probability that the particle that started at location $\mathbf{y}$ at the reference time t_0 will have a velocity $\mathbf{V}$ and position $\mathbf{x}$ at time t. We'll also introduce the fine-grained Eulerian PDF $f'_L = f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$, which is defined as

$$f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}). \tag{A.4}$$

Note: a delta function of a 3D argument means the following $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$. The Lagrangian PDF can be obtained from the fine-grained Lagrangian PDF using

$$f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle. \tag{A.5}$$

The proof is as follows,

$$\begin{aligned} \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle &= \langle \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V})\delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{V}', \mathbf{x}'; t|\mathbf{y}) d\mathbf{V}' d\mathbf{x}' \\ &= f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}). \end{aligned} \tag{A.6}$$

A.3 Relation between Lagrangian and Eulerian PDFs

As a quick side note, we mention that the inverse of $\mathbf{x}^+$ is $\mathbf{y}^+ = \mathbf{y}^+(t, \mathbf{z})$, which gives the initial location of a fluid particle that at time t is located at position $\mathbf{z}$. Thus, $\mathbf{x}^+(t, \mathbf{y}^+(t, \mathbf{z})) = \mathbf{z}$.

We begin as follows

$$\begin{aligned}
\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} &= \int \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})\, d\mathbf{y} \\
&= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})\, d\mathbf{y} \\
&= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V})\delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x})|\det D\mathbf{x}^+|\, d\mathbf{y}, \qquad \text{(A.7)}
\end{aligned}$$

where we have introduced $|\det D\mathbf{x}^+|$, which is the absolute value of the determinant of the Jacobean $\partial\mathbf{x}^+/\partial\mathbf{y}$, and is equal to one for incompressible flows. Using integration by substitution we obtain

$$\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} = \int \delta(\mathbf{u}(\mathbf{z}, t) - \mathbf{V})\delta(\mathbf{z} - \mathbf{x})\, d\mathbf{z} = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \qquad \text{(A.8)}$$

Given the definition of $f'(\mathbf{V}; \mathbf{x}, t)$, we have

$$\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} = f'(\mathbf{V}; \mathbf{x}, t). \qquad \text{(A.9)}$$

Taking the expectation of the above we obtain

$$\int f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})\, d\mathbf{y} = f(\mathbf{V}; \mathbf{x}, t). \qquad \text{(A.10)}$$

A summary of all of the relations derived thus far is given by the following graph

$$\begin{tikzcd}
\text{Eulerian fine-grained PDF} \arrow[d, "\text{eq. (A.2)}"] & \text{Lagrangian fine-grained PDF} \arrow[l, "\text{eq. (A.9)}"'] \arrow[d, "\text{eq. (A.5)}"] \\
\text{Eulerian PDF} & \text{Lagrangian PDF} \arrow[l, "\text{eq. (A.10)}"']
\end{tikzcd}$$

A.4 Evolution equation for fine-grained Eulerian PDF

A.5 Evolution equation for fine-grained Lagrangian PDF