

# Physical Kinetics

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December 30, 2023

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# Chapter 1

## Lagrangian and Eulerian PDFs

### 1.1 Eulerian PDF

Consider an Eulerian velocity field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ . The Eulerian PDF  $f = f(\mathbf{V}; \mathbf{x}, t)$  gives the probability that the velocity field will have a value of  $\mathbf{V}$  at location  $\mathbf{x}$  and at time  $t$ . We'll also introduce the fine-grained Eulerian PDF  $f' = f'(\mathbf{V}; \mathbf{x}, t)$ , which is defined as

$$f'(\mathbf{V}; \mathbf{x}, t) = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}). \quad (1.1)$$

Note: a delta function of a 3D argument means the following  $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$ . The Eulerian PDF can be obtained from the fine-grained Eulerian PDF using

$$f(\mathbf{V}; \mathbf{x}, t) = \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle. \quad (1.2)$$

The proof is as follows,

$$\begin{aligned} \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle &= \langle \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) f(\mathbf{V}'; \mathbf{x}, t) d\mathbf{V}' \\ &= f(\mathbf{V}; \mathbf{x}, t). \end{aligned} \quad (1.3)$$

### 1.2 Lagrangian PDF

Consider a Lagrangian particle with velocity  $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$  and position  $\mathbf{x}^+(t, \mathbf{y})$ . The Lagrangian PDF  $f_L = f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$  gives the probability that the particle that started at location  $\mathbf{y}$  at the reference time  $t_0$  will have a velocity  $\mathbf{V}$  and position  $\mathbf{x}$  at time  $t$ . We'll also introduce the fine-grained Eulerian PDF  $f'_L = f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$ , which is defined as

$$f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}). \quad (1.4)$$

Note: a delta function of a 3D argument means the following  $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$ . The Lagrangian PDF can be obtained from the fine-grained Lagrangian PDF using

$$f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle. \quad (1.5)$$

The proof is as follows,

$$\begin{aligned} \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle &= \langle \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{V}', \mathbf{x}'; t|\mathbf{y}) d\mathbf{V}' d\mathbf{x}' \\ &= f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}). \end{aligned} \quad (1.6)$$

### 1.3 Relation between Lagrangian and Eulerian PDFs

As a quick side note, we mention that the inverse of  $\mathbf{x}^+$  is  $\mathbf{y}^+ = \mathbf{y}^+(t, \mathbf{z})$ , which gives the initial location of a fluid particle that at time  $t$  is located at position  $\mathbf{z}$ . Thus,  $\mathbf{x}^+(t, \mathbf{y}^+(t, \mathbf{z})) = \mathbf{z}$ .

We begin as follows

$$\begin{aligned} \int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} &= \int \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) |\det D\mathbf{x}^+| d\mathbf{y}, \end{aligned} \quad (1.7)$$

where we have introduced  $|\det D\mathbf{x}^+|$ , which is the absolute value of the determinant of the Jacobean  $\partial\mathbf{x}^+/\partial\mathbf{y}$ , and is equal to one for incompressible flows. Using integration by substitution we obtain

$$\int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = \int \delta(\mathbf{u}(\mathbf{z}, t) - \mathbf{V}) \delta(\mathbf{z} - \mathbf{x}) d\mathbf{z} = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \quad (1.8)$$

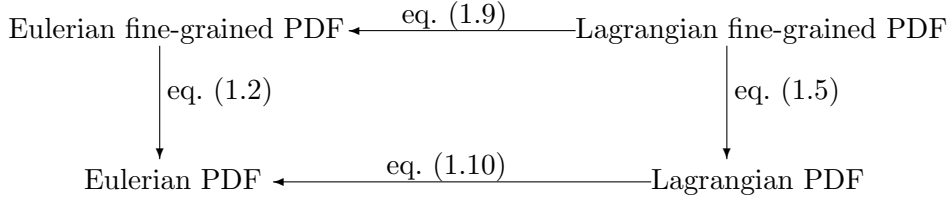
Given the definition of  $f'(\mathbf{V}; \mathbf{x}, t)$ , we have

$$\int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = f'(\mathbf{V}; \mathbf{x}, t). \quad (1.9)$$

Taking the expectation of the above we obtain

$$\int f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = f(\mathbf{V}; \mathbf{x}, t). \quad (1.10)$$

A summary of all of the relations derived thus far is given by the following graph



### 1.4 Evolution equation for fine-grained Eulerian PDF

### 1.5 Evolution equation for fine-grained Lagrangian PDF