Functional analysis

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1 Linear forms

- A linear form (or linear functional) is a mapping $w : V \to \mathbb{R}$ such that

$$w(\mathbf{v}_1 + \mathbf{v}_2) = w(\mathbf{v}_1) + w(\mathbf{v}_2) \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V, \tag{1}$$

$$w(\alpha \mathbf{v}) = \alpha w(\mathbf{v}) \qquad \forall \alpha \in \mathbb{R}, \mathbf{v} \in V. \tag{2}$$

- A bilinear form (or bilinear functional) is a mapping $w : V \times U \to \mathbb{R}$ such that

$$w(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{u}) = w(\mathbf{v}_1, \mathbf{u}) + w(\mathbf{v}_2, \mathbf{u}) \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V, \mathbf{u} \in U, \tag{3}$$

$$w(\mathbf{v}, \mathbf{u}_1 + \mathbf{u}_2) = w(\mathbf{v}, \mathbf{u}_1) + w(\mathbf{v}, \mathbf{u}_2) \qquad \forall \mathbf{v} \in V, \mathbf{u}_1, \mathbf{u}_2 \in U, \tag{4}$$

$$w(\alpha \mathbf{v}, \mathbf{u}) = \alpha w(\mathbf{v}, \mathbf{u}) \qquad \forall \alpha \in \mathbb{R}, \mathbf{v} \in V, \mathbf{u} \in U, \tag{5}$$

$$w(\mathbf{v}, \alpha \mathbf{u}) = \alpha w(\mathbf{v}, \mathbf{u}) \qquad \forall \alpha \in \mathbb{R}, \mathbf{v} \in V, \mathbf{u} \in U, \tag{6}$$

- A multi-linear form (or multi-linear functional) is a mapping $w : V^{(1)} \times ... V^{(n)} \to \mathbb{R}$ such that

$$w(\mathbf{v}^{(1)}, ..., \mathbf{v}_1^{(i)} + \mathbf{v}_2^{(i)}, ..., \mathbf{v}^{(n)}) = w(\mathbf{v}^{(1)}, ..., \mathbf{v}_1^{(i)}, ..., \mathbf{v}^{(n)}) + w(\mathbf{v}^{(1)}, ..., \mathbf{v}_2^{(i)}, ..., \mathbf{v}^{(n)}), \tag{7}$$

$$w(\mathbf{v}^{(1)}, ..., \alpha \mathbf{v}^{(i)}, ..., \mathbf{v}^{(n)}) = \alpha w(\mathbf{v}^{(1)}, ..., \mathbf{v}^{(i)}, ..., \mathbf{v}^{(n)}), \tag{8}$$

$\forall i$, $\forall \alpha \in \mathbb{R}$, $\forall \mathbf{v}^{(1)} \in V^{(1)}$, ... , $\forall \mathbf{v}^{(n)} \in V^{(n)}$, and $\forall \mathbf{v}_1^{(i)}, \mathbf{v}_2^{(i)} \in V^{(i)}$.

2 Some terminology

- Cauchy sequence: a sequence $v_1,\ v_2,\ v_3,\ \dots$ is a Cauchy sequence if for every possitive real number ϵ, there is a possitive integer N such that for all possitive integers $m, n > N$, $||v_m - v_n|| < \epsilon$.
- Complete inner produce space: an inner product space $\mathcal{V}$ is complete if every Cauchy sequence $\{v_i\}_{i=1}^{\infty}$ in $\mathcal{V}$ has a limit $v = \lim v_i \in \mathcal{V}$.
- Compact set: a set is compact if it is bounded and closed.
- Coercive bilinear form: a bilinear form $a(:,:)$ is coercive in a Hilbert space $\mathcal{V}$ if

$$a(v,v) \geq \alpha ||v||_{\mathcal{V}}^2, \qquad \forall v \in \mathcal{V}, \qquad \text{with } \alpha > 0. \tag{9}$$

- Bounded linear form: a linear form is bounded in the normed vector space $\mathcal{V}$ if there exists an $M > 0$ such that $|L(v)| \leq M||v||$, for every $v \in \mathcal{V}$.
- Bounded bilinear form: a bilinear form is bounded in the normed vector space $\mathcal{V}$ if there exists and $M > 0$ such that $|a(w,v)| \leq M||w||\ ||v||$, for every $w, v \in \mathcal{V}$.

3 Useful equalities and inequalities

$$|ab| = |a||b| \qquad \forall a, b \in \mathbb{C} \tag{10}$$

$$|a+b| \leq |a| + |b| \qquad \forall a, b \in \mathbb{C} \tag{11}$$

$$|(w,v)| \leq ||w||\ ||v|| \qquad \forall v, w \in \text{Inner product space (Cauchy-Schwarz inequality)} \tag{12}$$

$$||w+v|| \leq ||w|| + ||v|| \qquad \forall v, w \in \text{Normed space (triangle inequality)} \tag{13}$$

$$\left| \int_a^b v(x)\, dx \right| \leq \int_a^b |v(x)|\, dx \tag{14}$$

$$\left\| \int_a^b v(x,y)\, dx \right\| \leq \int_a^b ||v(x,y)||\, dx \text{ where the norm is over the y-domain.} \tag{15}$$

4 The weak derivative

If $v \in C^1(\bar{\Omega})$, then through integration by parts

$$\int_\Omega \frac{\partial v}{\partial x_i} \phi\, dV = -\int_\Omega v \frac{\partial \phi}{\partial x_i}\, dV \qquad \forall \phi \in C_0^1(\Omega). \tag{16}$$

However, if $v \in L_2(\Omega)$ but not necessarily in $C^1(\bar{\Omega})$, we cannot write the equation above. Instead, we ask, is there a function w such that the following holds?

$$\int_\Omega w\phi\, dV = -\int_\Omega v \frac{\partial \phi}{\partial x_i}\, dV \qquad \forall \phi \in C_0^1(\Omega). \tag{17}$$

This can be rewritten as $(w, \phi) = L(\phi)$, where $L(\phi) = -\int_\Omega v \frac{\partial \phi}{\partial x_i} dV$. If $L(\phi)$ is bounded in L_2, Riesz' representation theorem then states a unique solution $w \in L_2(\Omega)$ exists. This w is the weak derivative.

More generally, if $v \in C^k(\bar{\Omega})$, then through integration by parts

$$\int_\Omega D^\alpha v \phi \, dV = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, dV \qquad \forall |\alpha| \leq k, \, \forall \phi \in C_0^{|\alpha|}(\Omega). \tag{18}$$

However, if $v \in L_2(\Omega)$ but not necessarily in $C^k(\bar{\Omega})$, we cannot write the equation above. Instead, we ask, is there a function w such that the following holds?

$$\int_\Omega w \phi \, dV = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, dV \qquad \forall \phi \in C_0^{|\alpha|}(\Omega). \tag{19}$$

As before, if the left-hand-side operator is bounded, then we have a unique solution $w \in L_2(\Omega)$. This w is the weak derivative. Often, weak derivatives are referred to as "$D^\alpha v$ in the weak sense."

5 Function spaces

We label function spaces using \mathcal notation, except for the three main function spaces for continuous, square-integrable, and Sobolev functions.

- C^0: the set of all continuous functions
- C^k the set of all functions whose derivatives up to order k all exist and are continuous. These are called continuously differentiable functions of order k.
- L_2: the set of all functions that are square integrable.
- H^k: the set of all L_2 functions whose weak partial derivatives up to order k also belong to L_2.
- Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth or polygonal boundary. Then part of the Sobolev embedding theorem can be written as

$$H^k(\Omega) \subset C^l(\bar{\Omega}) \text{ if } k > l + d/2. \tag{20}$$

Thus, we have $H^m(\Omega) \subset C^{m-1}(\bar{\Omega})$ for $\Omega \in \mathbb{R}$ and $H^m(\Omega) \subset C^{m-2}(\bar{\Omega})$ for $\Omega \in \mathbb{R}^2$ or $\mathbb{R}^3$.

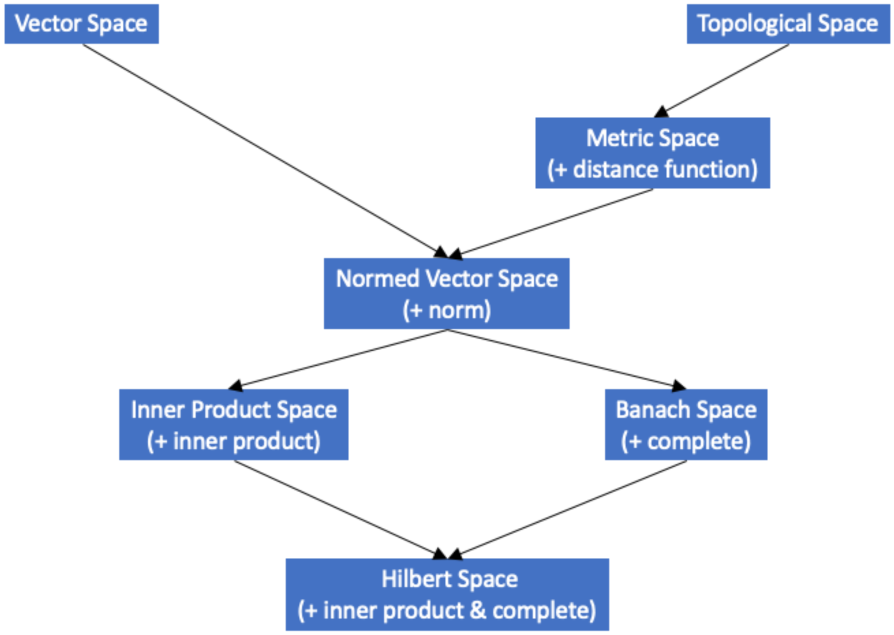

Figure 1: Map of function spaces.

Space	Norm	Inner product
$C(M)$	$\lvert\lvert v\rvert\rvert_C = \sup\limits_{x\in M} \lvert v(x)\rvert$	X
$C^k(M)$	$\lvert\lvert v\rvert\rvert_{C^k} = \max\limits_{\lvert\alpha\rvert\leq k} \lvert\lvert D^\alpha v\rvert\rvert_C$ $\lvert v\rvert_{C^k} = \max\limits_{\lvert\alpha\rvert = k} \lvert\lvert D^\alpha v\rvert\rvert_C$	X
$L_p(\Omega)$	$\lvert\lvert v\rvert\rvert_{L_p} = \left(\int_\Omega \lvert v\rvert^p \, dV\right)^{1/p}$	X
$L_2(\Omega)$	$\lvert\lvert v\rvert\rvert_{L_2} = \left(\int_\Omega \lvert v\rvert^2 \, dV\right)^{1/2}$	$(v, w) = \int_\Omega v w^* \, dV$
$H^k(\Omega)$	$\lvert\lvert v\rvert\rvert_k = \left(\sum\limits_{\lvert\alpha\rvert\leq k} \lvert\lvert D^\alpha v\rvert\rvert^2\right)^{1/2}$ $\lvert v\rvert_k = \left(\sum\limits_{\lvert\alpha\rvert = k} \lvert\lvert D^\alpha v\rvert\rvert^2\right)^{1/2}$	$(v, w)_k = \sum\limits_{\lvert\alpha\leq k\rvert} (D^\alpha v, D^\alpha w)$