Introduction

Alejandro Campos

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Governing equations

1.1 Particle description

Table 1.1: Various coordinates of classical mechanics.

Classical coordinates	$\mathbf{x}(t)$	$\mathbf{v}(t)$
Generalized coordinates	$\mathbf{q}$	$\dot{\mathbf{q}}$
Canonical coordinates	$\mathbf{q}$	$\mathbf{p}$
Time-dependent canonical coordinates	$\tilde{\mathbf{q}}(t)$	$\tilde{\mathbf{p}}(t)$

1.1.1 Lagrangian mechanics

- Define the position $\mathbf{x} = \mathbf{x}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ of a particle.

- Define the Lagrangian as $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the generalized position and generalized velocity, respectively.

- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}}\right] = \left(\frac{\partial L}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}}. \tag{1.1}$$

- For example, the Lagrangian of a particle in an electromagnetic field where $\phi = \phi(\mathbf{q}, t)$ is the electric potential and $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$ is the magnetic potential, is

$$L = \frac{1}{2}m\dot{q}_i\dot{q}_i + e\dot{q}_i A_i - e\phi. \tag{1.2}$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e\dot{q}_j \frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i} \tag{1.3}$$

$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i + eA_i \tag{1.4}$$

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$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] = \frac{d}{dt}\left[mv_i + eA_i(\mathbf{x},t)\right]$$

$$= m\frac{dv_i}{dt} + ev_j\left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.5}$$

Thus, the Euler-Lagrange equation becomes

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.6}$$

In vector notation, this is written as

$$m\frac{d\mathbf{v}}{dt} = \left(-e\mathbf{v}\cdot\nabla_q\mathbf{A} - e\frac{\partial \mathbf{A}}{\partial t} + e\nabla_q(\mathbf{v}\cdot\mathbf{A}) - e\nabla_q\phi\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.7}$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}, \tag{1.8}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{q},t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{q},t)$.

1.1.2 Hamiltonian mechanics

- Define the Hamiltonian as $H = H(\mathbf{q},\mathbf{p},t)$, where $\mathbf{q}$ and $\mathbf{p}$ are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.

- The Hamiltonian is obtained from the Lagrangian using

$$H = (\dot{\mathbf{q}}\cdot\mathbf{p} - L)_{\dot{\mathbf{q}}=\mathbf{f}(\mathbf{q},\mathbf{p})}, \tag{1.9}$$

where the dependency of $\dot{\mathbf{q}}$ on $\mathbf{q}$ and $\mathbf{p}$ is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \tag{1.10}$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[\dot{q}_i p_i - \left(\frac{1}{2}m\dot{q}_i\dot{q}_i + e\dot{q}_i A_i - e\phi\right)\right]_{\dot{\mathbf{q}}=f(\mathbf{q},\mathbf{p})}. \tag{1.11}$$

Evaluating eq. (1.10) gives $p_i = m\dot{q}_i + eA_i$, which allows us to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$ and $\mathbf{p}$ as $\dot{q}_i = \frac{1}{m}(p_i - eA_i)$. Thus

$$H = \frac{1}{m}(p_i - eA_i)p_i - \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) - e\frac{1}{m}(p_i - eA_i)A_i + e\phi$$

$$= \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) + e\phi. \tag{1.12}$$

- We introduce the variables $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$, which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \tag{1.13}$$

$$\tilde{\mathbf{p}} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}} \tag{1.14}$$

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- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.15}$$

$$\frac{d\tilde{p}_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.16}$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x},t) \tag{1.17}$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m\frac{dv_i}{dt} + ev_j\left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.18}$$

$$\begin{aligned}\frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i}\left[\frac{1}{2m}(p_j - eA_j)(p_j - eA_j) + e\phi\right]\\&= \frac{1}{m}(p_j - eA_j)\frac{\partial}{\partial q_i}(p_j - eA_j) + e\frac{\partial\phi}{\partial q_i}\\&= -\frac{e}{m}(p_j - eA_j)\frac{\partial A_j}{\partial q_i} + e\frac{\partial\phi}{\partial q_i}\end{aligned} \tag{1.19}$$

$$\begin{aligned}\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m}\left[mv_j + eA_j(\mathbf{x},t) - eA_j(\mathbf{x},t)\right]\left(\frac{\partial A_j}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial\phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}\\&= \left(-ev_j\frac{\partial A_j}{\partial q_i} + e\frac{\partial\phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}.\end{aligned} \tag{1.20}$$

Equation (1.16) thus leads to

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial\phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.21}$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.22}$$

1.2 Kinetic description

We denote the distribution function for a species s as $f_s = f_s(\mathbf{r},\mathbf{v},t)$, where $\mathbf{r}$ and $\mathbf{v}$ are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_s\, d\mathbf{r}d\mathbf{v} = N_s, \tag{1.23}$$

where N_s is the total number of particles corresponding to species s.

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution function of N species along with Maxwell's equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = C_s + S_s \tag{1.24}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0} \tag{1.25}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.26}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.27}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.28}$$

$$\mathbf{J} = \sum_s^N Z_s e \int \mathbf{v} f_s \, d\mathbf{v} \tag{1.29}$$

$$\rho_q = \sum_s^N Z_s e \int f_s \, d\mathbf{v}. \tag{1.30}$$

In the above,

- m_s is the species mass

- e is the charge

- Z_s the charge number

- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ the charge current

- $\rho_q = \rho_q(\mathbf{r}, t)$ the charge density

- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the electric field

- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ the magnetic field.

The terms C_s and S_s represent collision and source terms.

If we express the collision term in the usual way, that is $C_s = \sum_r^N C_{sr}$, then we can make the following statements:

1. Conservation of particles:

$$\int C_{ss} \, d\mathbf{v} = 0 \quad \forall s \qquad \int C_{sr} \, d\mathbf{v} = 0 \quad \forall s, r | r \neq s. \tag{1.31}$$

2. Conservation of momentum:

$$\int m_s \mathbf{v} C_{ss} \, d\mathbf{v} = 0 \quad \forall s \qquad \sum_s^N \sum_{r, r \neq s}^N \int m_s \mathbf{v} C_{sr} d\mathbf{v} = 0. \tag{1.32}$$

3. Conservation of energy:

$$\int \frac{1}{2} m_s v^2 C_{ss} \, d\mathbf{v} = 0 \quad \forall s \qquad \sum_s^N \sum_{r, r \neq s}^N \int \frac{1}{2} m_s v^2 C_{sr} \, d\mathbf{v} = 0. \tag{1.33}$$

1.3 Fluid description

We now define the particle density $n_s = n_s(\mathbf{r}, t)$, the fluid velocity $\mathbf{u}_s = \mathbf{u}_s(\mathbf{r}, t)$ and the fluid energy per unit mass $E_s = E_s(\mathbf{r}, t)$ as follows

$$n_s = \int f_s \, d\mathbf{v} \tag{1.34}$$

$$\mathbf{u}_s = \frac{1}{n_s} \int \mathbf{v} f_s \, d\mathbf{v} \tag{1.35}$$

$$E_s = \frac{1}{n_s} \int \frac{1}{2} v^2 f_s \, d\mathbf{v}. \tag{1.36}$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_s}{\partial t} + \nabla \cdot (\mathbf{v} f_s) + \nabla_v \cdot \left[\frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] = C_s + S_s \tag{1.37}$$

1.3.1 Mass

Integrating eq. (1.37) over all $\mathbf{v}$ we obtain

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = \hat{S}_s \tag{1.38}$$

where

$$\hat{S}_s = \int S_s \, d\mathbf{v} \tag{1.39}$$

is an external source of mass.

1.3.2 Momentum

Multiplying eq. (1.37) by $\mathbf{v}$ and then integrating over all $\mathbf{v}$ leads to

$$\frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot \left(\int \mathbf{v} \mathbf{v} f_s \, d\mathbf{v} \right) + \int \nabla_v \cdot \left[\mathbf{v} \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] - \nabla_v \mathbf{v} \cdot \left[\frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] d\mathbf{v} =$$
$$\sum_{r, r \neq s}^{N} \int \mathbf{v} C_{sr} \, d\mathbf{v} + \int \mathbf{v} S_s \, d\mathbf{v}. \tag{1.40}$$

We note that the third term in eq. (1.40) is zero since we are integrating over all space, and that $\nabla_v \mathbf{v}$ is the identity matrix. We thus have

$$\frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot \left(\int \mathbf{v} \mathbf{v} f_s \, d\mathbf{v} \right) - \frac{Z_s e n_s}{m_s} (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) =$$
$$\sum_{r, r \neq s}^{N} \int \mathbf{v} C_{sr} \, d\mathbf{v} + \int \mathbf{v} S_s \, d\mathbf{v}. \tag{1.41}$$

To proceed, we decompose $\mathbf{v}$ into a mean and a fluctuation, that is, $\mathbf{v} = \mathbf{u}_s + \mathbf{w}_s$. Using this decomposition

$$\int \mathbf{v} \mathbf{v} f_s \, d\mathbf{v} = \int (\mathbf{u}_s \mathbf{u}_s + 2 \mathbf{u}_s \mathbf{w}_s + \mathbf{w}_s \mathbf{w}_s) f_s \, d\mathbf{v} = n_s \mathbf{u}_s \mathbf{u}_s + \int \mathbf{w}_s \mathbf{w}_s f_s \, d\mathbf{v}. \tag{1.42}$$

Thus, eq. (1.41) becomes

$$\frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} +$$
$$\sum_{r,r\neq s}^{N} \int \mathbf{v} C_{sr}\, d\mathbf{v} + \int \mathbf{v} S_s\, d\mathbf{v}. \quad (1.43)$$

Conservation of particles is used to modify the collisional term as follows

$$\sum_{r,r\neq s}^{N} \int \mathbf{v} C_{sr}\, d\mathbf{v} = \sum_{r,r\neq s}^{N} \int (\mathbf{u}_s + \mathbf{w}_s) C_{sr}\, d\mathbf{v}$$
$$= \sum_{r,r\neq s}^{N} \int \mathbf{w}_s C_{sr}\, d\mathbf{v} \quad (1.44)$$

Thus, we now have

$$\frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} +$$
$$\sum_{r,r\neq s}^{N} \int \mathbf{w}_s C_{sr}\, d\mathbf{v} + \int \mathbf{v} S_s\, d\mathbf{v}. \quad (1.45)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) - Z_s e n_s(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_s + \mathbf{R}_s + \hat{\mathbf{M}}_s, \quad (1.46)$$

where the stress tensor is

$$\boldsymbol{\sigma}_s = -\int m_s \mathbf{w}_s \mathbf{w}_s f_s\, d\mathbf{v}, \quad (1.47)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_s = \sum_{r,r\neq s}^{N} \int m_s \mathbf{w}_s C_{sr}\, d\mathbf{v}, \quad (1.48)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_s = \int m_s \mathbf{v} S_s\, d\mathbf{v}. \quad (1.49)$$

The stress tensor is typically decomposed into isotropic p_s and anisotropic (shear) $\mathbf{t}_s$ tensors as follows

$$\boldsymbol{\sigma}_s = -p_s \mathbf{I} + \mathbf{t}_s, \quad (1.50)$$

where p_s is given by

$$p_s = \frac{1}{3} \int m_s (\mathbf{w}_s \cdot \mathbf{w}_s) f_s d\mathbf{v}. \quad (1.51)$$

Thus, conservation of momentum becomes

$$\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) - Z_s e n_s(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = -\nabla p_s + \nabla \cdot \mathbf{t}_s + \mathbf{R}_s + \hat{\mathbf{M}}_s. \quad (1.52)$$

1.3.3   Energy

Multiplying eq. (1.37) by $\frac{1}{2}v^2$ and then integrating over all $\mathbf{v}$ leads to

$$\frac{\partial n_s E_s}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})\mathbf{v} f_s \, d\mathbf{v} \right] + \int \nabla_v \cdot \left[\frac{1}{2}(\mathbf{v} \cdot \mathbf{v})\frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right]$$

$$- \nabla_v \left[\frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \right] \cdot \left[\frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] \, d\mathbf{v} = \sum_{r,r \neq s}^{N} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})C_{sr} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})S_s \, d\mathbf{v}. \quad (1.53)$$

We note that the third term above is zero since we are integrating over all space, and that $\nabla_v[1/2(\mathbf{v} \cdot \mathbf{v})] = \mathbf{v}$. Thus, we have

$$\frac{\partial n_s E_s}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})\mathbf{v} f_s \, d\mathbf{v} \right] - \frac{Z_s e n_s}{m_s}\mathbf{E} \cdot \mathbf{u}_s =$$

$$\sum_{r,r \neq s}^{N} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})C_{sr} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})S_s \, d\mathbf{v}. \quad (1.54)$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})\mathbf{v} f_s \, d\mathbf{v} = \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})(\mathbf{u}_s + \mathbf{w}_s) f_s \, d\mathbf{v} = n_s E_s \mathbf{u}_s + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})\mathbf{w}_s f_s \, d\mathbf{v} \quad (1.55)$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})\mathbf{w}_s f_s \, d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_s \cdot \mathbf{u}_s + 2\mathbf{u}_s \cdot \mathbf{w}_s + \mathbf{w}_s \cdot \mathbf{w}_s)\mathbf{w}_s f_s \, d\mathbf{v} \quad (1.56)$$

$$= \mathbf{u}_s \cdot \int \mathbf{w}_s \mathbf{w}_s f_s \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s)\mathbf{w}_s f_s \, d\mathbf{v}. \quad (1.57)$$

Using the expressions above, eq. (1.54) becomes

$$\frac{\partial n_s E_s}{\partial t} + \nabla \cdot (n_s E_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s}\mathbf{E} \cdot \mathbf{u}_s = -\nabla \cdot \left(\mathbf{u}_s \cdot \int \mathbf{w}_s \mathbf{w}_s f_s \, d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s)\mathbf{w}_s f_s \, d\mathbf{v}$$

$$+ \sum_{r,r \neq s}^{N} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})C_{sr} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})S_s \, d\mathbf{v}. \quad (1.58)$$

Conservation of particles is used to modify the collisional term as follows

$$\sum_{r,r \neq s}^{N} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})C_{sr} \, d\mathbf{v} = \sum_{r,r \neq s}^{N} \int \frac{1}{2}[(\mathbf{u}_s + \mathbf{w}_s) \cdot (\mathbf{u}_s + \mathbf{w}_s)]C_{sr} \, d\mathbf{v}$$

$$= \mathbf{u}_s \cdot \sum_{r,r \neq s}^{N} \int \mathbf{w}_s C_{sr} \, d\mathbf{v} + \sum_{r,r \neq s}^{N} \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s)C_{sr} \, d\mathbf{v} \quad (1.59)$$

Thus, we now have

$$\frac{\partial n_s E_s}{\partial t} + \nabla \cdot (n_s E_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s}\mathbf{E} \cdot \mathbf{u}_s = -\nabla \cdot \left(\mathbf{u}_s \cdot \int \mathbf{w}_s \mathbf{w}_s f_s \, d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s)\mathbf{w}_s f_s \, d\mathbf{v}$$

$$+ \mathbf{u}_s \cdot \sum_{r,r \neq s}^{N} \int \mathbf{w}_s C_{sr} \, d\mathbf{v} + \sum_{r,r \neq s}^{N} \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s)C_{sr} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})S_s \, d\mathbf{v}. \quad (1.60)$$

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Multiplying by mass leads to the following equation

$$\frac{\partial m_s n_s E_s}{\partial t} + \nabla \cdot (m_s n_s E_s \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \nabla \cdot \mathbf{q}_s$$
$$+ \mathbf{u}_s \cdot \mathbf{R}_s + Q_s + \hat{Q}_s, \quad (1.61)$$

where heat flux due to random motion is

$$\mathbf{q}_s = \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) \mathbf{w}_s f_s \, d\mathbf{v}, \quad (1.62)$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_s = \sum_{r, r \neq s}^{N} \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) C_{sr} \, d\mathbf{v}, \quad (1.63)$$

and the external source of energy is

$$\hat{Q}_s = \int \frac{1}{2} m_s (\mathbf{v} \cdot \mathbf{v}) S_s \, d\mathbf{v}. \quad (1.64)$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\frac{\partial m_s n_s E_s}{\partial t} + \nabla \cdot (m_s n_s E_s \mathbf{u}_s + p_s \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \nabla \cdot (\mathbf{u}_s \cdot \mathbf{t}_s) - \nabla \cdot \mathbf{q}_s$$
$$+ \mathbf{u}_s \cdot \mathbf{R}_s + Q_s + \hat{Q}_s, \quad (1.65)$$

We also note that the energy $m_s n_s E_s$ can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (1.42) one obtains

$$
\begin{aligned}
m_s n_s E_s &= \int \frac{1}{2} m_s (\mathbf{v} \cdot \mathbf{v}) f_s \, d\mathbf{v} \\
&= \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) f_s \, d\mathbf{v} + \frac{1}{2} m_s n_s (\mathbf{u}_s \cdot \mathbf{u}_s) \\
&= \frac{3}{2} p_s + \frac{1}{2} m_s n_s (\mathbf{u}_s \cdot \mathbf{u}_s) \\
&= \frac{3}{2} p_s + m_s n_s K_s.
\end{aligned}
\quad (1.66)
$$

where $K_s = \frac{1}{2} \mathbf{u}_s \cdot \mathbf{u}_s$ is the kinetic energy of species s.

1.3.4 Kinetic and internal energies

The equation for the kinetic energy is obtained by dotting eq. (1.52) with $\mathbf{u}_s$. For this, we first show that

$$\mathbf{u}_s \cdot \left[\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) \right] \tag{1.67}$$

$$= \mathbf{u}_s \cdot \left\{ \left[\frac{\partial m_s n_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s) \right] \mathbf{u}_s + m_s n_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) \right\} \tag{1.68}$$

$$= \mathbf{u}_s \cdot \left[m_s \hat{S}_s \mathbf{u}_s + m_s n_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) \right] \tag{1.69}$$

$$= 2 m_s \hat{S}_s K_s + m_s n_s \left(\frac{\partial K_s}{\partial t} + \mathbf{u}_s \cdot \nabla K_s \right) \tag{1.70}$$

$$= m_s \hat{S}_s K_s + \left[\frac{\partial m_s n_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s) \right] K_s + m_s n_s \left(\frac{\partial K_s}{\partial t} + \mathbf{u}_s \cdot \nabla K_s \right) \tag{1.71}$$

$$= m_s \hat{S}_s K_s + \frac{\partial m_s n_s K_s}{\partial t} + \nabla \cdot (m_s n_s K \mathbf{u}_s). \tag{1.72}$$

Thus, the equation for the turbulent kinetic energy is

$$\frac{\partial m_s n_s K_s}{\partial t} + \nabla \cdot (m_s n_s K \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s =$$
$$\nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s + \mathbf{u}_s \cdot \mathbf{R}_s + \mathbf{u}_s \cdot \hat{\mathbf{M}}_s - m_s K_s \hat{S}_s. \tag{1.73}$$

Subtracting the above equation from eq. (1.65) leads to

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_s \right) + \nabla \cdot \left(\frac{3}{2} p_s \mathbf{u}_s \right) = \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s - \nabla \cdot \mathbf{q}_s + Q_s + \hat{Q}_s - \mathbf{u}_s \cdot \hat{\mathbf{M}}_s + m_s K_s \hat{S}_s. \tag{1.74}$$

1.3.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = \hat{S}_s, \tag{1.75}$$

- Momentum

$$\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) - Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_s + \mathbf{R}_s + \hat{\mathbf{M}}_s, \tag{1.76}$$

- Total Energy

$$\frac{\partial m_s n_s E_s}{\partial t} + \nabla \cdot (m_s n_s E_s \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \nabla \cdot \mathbf{q}_s$$
$$+ \mathbf{u}_s \cdot \mathbf{R}_s + Q_s + \hat{Q}_s, \tag{1.77}$$

- Kinetic Energy

$$\frac{\partial m_s n_s K_s}{\partial t} + \nabla \cdot (m_s n_s K \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s =$$
$$\nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s + \mathbf{u}_s \cdot \mathbf{R}_s + \mathbf{u}_s \cdot \hat{\mathbf{M}}_s - m_s K_s \hat{S}_s. \tag{1.78}$$

- Internal Energy

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_s \right) + \nabla \cdot \left(\frac{3}{2} p_s \mathbf{u}_s \right) = \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s - \nabla \cdot \mathbf{q}_s + Q_s + \hat{Q}_s - \mathbf{u}_s \cdot \hat{\mathbf{M}}_s + m_s K_s \hat{S}_s. \tag{1.79}$$

Chapter 2

Variants of the fluid model

2.1 The multi-fluid model

The following assumptions are used:

1. No sources ($\hat{S}_s = \hat{\mathbf{M}}_s = \hat{Q}_s = 0$ for all s).

2.1.1 Multi-material multi-fluid model

We define a material k as a collection of species s. Rather than labeling species using the single subscript s, we'll use the subscript k, i, s and k, e, where the former denotes the ion species s that belongs to material k, and the latter denotes the electron species that belongs to material k. We also note that the number of ion species within a material will be denoted as N_k, and the total number of materials as M.

Thus, the governing equations are

$$\frac{\partial n_{k,i,s}}{\partial t} + \nabla \cdot (n_{k,i,s}\mathbf{u}_{k,i,s}) = 0, \tag{2.1}$$

$$\frac{\partial n_{k,e}}{\partial t} + \nabla \cdot (n_{k,e}\mathbf{u}_{k,e}) = 0, \tag{2.2}$$

$$\frac{\partial m_{k,i,s}n_{k,i,s}\mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (m_{k,i,s}n_{k,i,s}\mathbf{u}_{k,i,s}\mathbf{u}_{k,i,s}) - Z_{k,i,s}en_{k,i,s}(\mathbf{E} + \mathbf{u}_{k,i,s} \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_{k,i,s} + \mathbf{R}_{k,i,s}, \tag{2.3}$$

$$\frac{\partial m_{k,e}n_{k,e}\mathbf{u}_{k,e}}{\partial t} + \nabla \cdot (m_{k,e}n_{k,e}\mathbf{u}_{k,e}\mathbf{u}_{k,e}) + en_{k,e}(\mathbf{E} + \mathbf{u}_{k,e} \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_{k,e} + \mathbf{R}_{k,e}, \tag{2.4}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_{k,i,s}\right) + \nabla \cdot \left(\frac{3}{2}p_{k,i,s}\mathbf{u}_{k,i,s}\right) = \boldsymbol{\sigma}_{k,i,s} : \nabla\mathbf{u}_{k,i,s} - \nabla \cdot \mathbf{q}_{k,i,s} + Q_{k,i,s}, \tag{2.5}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_{k,e}\right) + \nabla \cdot \left(\frac{3}{2}p_{k,e}\mathbf{u}_{k,e}\right) = \boldsymbol{\sigma}_{k,e} : \nabla\mathbf{u}_{k,e} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e}, \tag{2.6}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.7}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.9}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}, \tag{2.10}$$

$$\mathbf{J}_k = e\left(\sum_s^{N_k} Z_{k,i,s}n_{k,i,s}\mathbf{u}_{k,i,s} - n_{k,e}\mathbf{u}_{k,e}\right), \tag{2.11}$$

$$\mathbf{J} = \sum_k^M \mathbf{J}_k \tag{2.12}$$

$$\rho_{q,k} = e\left(\sum_s^{N_k} Z_{k,i,s}n_{k,i,s} - n_{e,k}\right), \tag{2.13}$$

$$\rho_q = \sum_k^M \rho_{q,k}, \tag{2.14}$$

$$p_{k,i,s} = n_{k,i,s}k_BT_{k,i,s}, \tag{2.15}$$

$$p_{k,e} = n_{k,e}k_BT_{k,e}. \tag{2.16}$$

2.1.2 Single-material multi-fluid model

The equations are the same as the multi-material formulation, but there is only one material now and hence the k subscript can be dropped. Thus we have

$$\frac{\partial n_{i,s}}{\partial t} + \nabla \cdot (n_{i,s}\mathbf{u}_{i,s}) = 0, \tag{2.17}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\mathbf{u}_e) = 0, \tag{2.18}$$

$$\frac{\partial m_{i,s}n_{i,s}\mathbf{u}_{i,s}}{\partial t} + \nabla \cdot (m_{i,s}n_{i,s}\mathbf{u}_{i,s}\mathbf{u}_{i,s}) - Z_{i,s}en_{i,s}(\mathbf{E} + \mathbf{u}_{i,s} \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_{i,s} + \mathbf{R}_{i,s}, \tag{2.19}$$

$$\frac{\partial m_e n_e\mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e\mathbf{u}_e\mathbf{u}_e) + en_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e, \tag{2.20}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_{i,s}\right) + \nabla \cdot \left(\frac{3}{2}p_{i,s}\mathbf{u}_{i,s}\right) = \boldsymbol{\sigma}_{i,s} : \nabla\mathbf{u}_{i,s} - \nabla \cdot \mathbf{q}_{i,s} + Q_{i,s}, \tag{2.21}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_e\right) + \nabla \cdot \left(\frac{3}{2}p_e\mathbf{u}_e\right) = \boldsymbol{\sigma}_e : \nabla\mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \tag{2.22}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.23}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.24}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.25}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}, \tag{2.26}$$

$$\mathbf{J} = e\left(\sum_s^N Z_{i,s}n_{i,s}\mathbf{u}_{i,s} - n_e\mathbf{u}_e\right), \tag{2.27}$$

$$\rho_q = e\left(\sum_s^N Z_{i,s}n_{i,s} - n_e\right), \tag{2.28}$$

$$p_{i,s} = n_{i,s}k_BT_{i,s}, \tag{2.29}$$

$$p_e = n_ek_BT_e. \tag{2.30}$$

2.1.3 Single-material two-fluid model

If there is only one ion species, then we have the two-fluid model

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{2.31}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{2.32}$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_i + \mathbf{R}_i, \tag{2.33}$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + en_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e, \tag{2.34}$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_i \right) + \nabla \cdot \left(\frac{3}{2} p_i \mathbf{u}_i \right) = \boldsymbol{\sigma}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \tag{2.35}$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_e \right) + \nabla \cdot \left(\frac{3}{2} p_e \mathbf{u}_e \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \tag{2.36}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.37}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.38}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.39}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.40}$$

$$\mathbf{J} = e(Zn_i \mathbf{u}_i - n_e \mathbf{u}_e), \tag{2.41}$$

$$\rho_q = e(Zn_i - n_e). \tag{2.42}$$

$$p_i = n_i k_B T_i, \tag{2.43}$$

$$p_e = n_e k_B T_e. \tag{2.44}$$

These equations correspond to eq. (2.22) in Freidberg's Ideal MHD book, but for ions that are not singly charged.

We can also introduce the following extra assumptions

1. No stresses $\mathbf{t}_s$, heat flux $\mathbf{q}_s$, and collisions $\mathbf{R}_s$, Q_s. This in itself would make the flow isentropic.

2. The flow is not just isentropic but also homentropic. This allows for eqs. (2.49) and (2.50) to hold across all space, not just along streamlines.

With these, the governing equations become

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{2.45}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{2.46}$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i, \tag{2.47}$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + en_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e, \tag{2.48}$$

$$p_i = C_i n_i^{\gamma_i}, \tag{2.49}$$

$$p_e = C_e n_e^{\gamma_e}, \tag{2.50}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.51}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.52}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.53}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.54}$$

$$\mathbf{J} = e(Z n_i \mathbf{u}_i - n_e \mathbf{u}_e), \tag{2.55}$$

$$\rho_q = e(Z n_i - n_e), \tag{2.56}$$

$$p_i = n_i k_B T_i, \tag{2.57}$$

$$p_e = n_e k_B T_e. \tag{2.58}$$

2.2 Single-fluid formulation

A single-fluid formulation is defined as that for which there is a single velocity vector. For the single-fluid formulation, we'll use the following assumptions

1. $m_{k,e} \approx 0$ since $m_{k,e} \ll m_{k,i,s}$.

2. Quasi-neutrality, i.e. $\sum_s^{N_k} Z_{k,i,s} n_{k,i,s} = n_{k,e}$.

2.2.1 Single-fluid variables

To derive the single-fluid equations we first introduce a set of new variables.

The total density ρ is defined as

$$\rho = m/V. \tag{2.59}$$

The volume fraction η_k and mass fraction Y_k are defined as

$$\eta_k = \frac{V_k}{V}, \tag{2.60}$$

$$Y_k = \frac{m_k}{m}. \tag{2.61}$$

Note that both η_k and Y_k sum up to one over all k. A new density is also introduced, namely ρ_k which is defined as

$$\rho_k = \frac{m_k}{V_k}. \tag{2.62}$$

This can be expressed in terms of the other variables defined above by

$$\rho_k = \rho \frac{Y_k}{\eta_k}. \tag{2.63}$$

The volume fractions $\eta_{k,i,s}$ and $\eta_{k,e}$ are defined as

$$\eta_{k,i,s} = \frac{V_{k,i,s}}{V_k} \qquad \eta_{k,e} = \frac{V_{k,e}}{V_k} \tag{2.64}$$

The mass fractions $Y_{k,i,s}$ and $Y_{k,e}$ are defined as

$$Y_{k,i,s} = \frac{m_{k,i,s}}{m_k} \qquad Y_{k,e} = \frac{m_{k,e}}{m_k}. \tag{2.65}$$

Note that

$$1 = \sum_s^{N_k} \eta_{k,i,s} + \eta_{k,e} \approx \sum_s^{N_k} \eta_{k,i,s} = \eta_{k,i}, \tag{2.66}$$

and

$$1 = \sum_s^{N_k} Y_{k,i,s} + Y_{k,e} \approx \sum_s^{N_k} Y_{k,i,s} = Y_{k,i}. \tag{2.67}$$

New densities are also introduced, namely $\rho_{k,i,s}$ and $\rho_{k,e}$, which are defined as

$$\rho_{k,i,s} = \frac{m_{k,i,s}}{V_{k,i,s}} \qquad \rho_{k,e} = \frac{m_{k,e}}{V_{k,e}}. \tag{2.68}$$

These can be expressed in terms of the other variables defined above as follows

$$\rho_{k,i,s} = \rho_k \frac{Y_{k,i,s}}{\eta_{k,i,s}} \qquad \rho_{k,e} = \rho_k \frac{Y_{k,e}}{\eta_{k,e}}. \tag{2.69}$$

In terms of the number densities $n_{k,i,s}$ and $n_{k,e}$, we have

$$\rho Y_k Y_{k,i,s} = m_{k,i,s} n_{k,i,s} \qquad \rho Y_k Y_{k,e} = m_e n_{k,e}. \tag{2.70}$$

where $m_{k,i,s}$ is the atomic weight of species s and m_e is the weight of an electron. The ion velocity of a material is given by $\mathbf{u}_{k,i}$ and the velocity of all materials by $\mathbf{u}$. These are defined below,

$$\mathbf{u}_{k,i} = \sum_s^{N_k} Y_{k,i,s} \mathbf{u}_{k,i,s} \tag{2.71}$$

$$\mathbf{u}_k = \mathbf{u}_{k,i} + Y_{k,e} \mathbf{u}_{k,e} = \mathbf{u}_{k,i} \tag{2.72}$$

$$\mathbf{u} = \sum_k^M Y_k \mathbf{u}_k. \tag{2.73}$$

If there is only one material, the relations above simplify as follows

$$\eta_{i,s} = \frac{V_{i,s}}{V} \qquad \eta_e = \frac{V_e}{V}, \tag{2.74}$$

$$Y_{i,s} = \frac{m_{i,s}}{m} \qquad Y_e = \frac{m_e}{m}, \tag{2.75}$$

$$1 = \sum_s^N \eta_{i,s} + \eta_e \approx \sum_s^N \eta_{i,s} = \eta_i, \tag{2.76}$$

$$1 = \sum_s^N Y_{i,s} + Y_e \approx \sum_s^N Y_{i,s} = Y_i, \tag{2.77}$$

$$\rho_{i,s} = \frac{m_{i,s}}{V_{i,s}} \qquad \rho_s = \frac{m_e}{V_e}, \tag{2.78}$$

$$\rho_{i,s} = \rho \frac{Y_{i,s}}{\eta_{i,s}} \qquad \rho_e = \rho \frac{Y_e}{\eta_e}, \tag{2.79}$$

$$\rho Y_{i,s} = m_{i,s} n_{i,s} \qquad \rho Y_e = m_e n_e, \tag{2.80}$$

$$\mathbf{u}_i = \sum_s^N Y_{i,s} \mathbf{u}_{i,s}. \tag{2.81}$$

$$\mathbf{u} = \mathbf{u}_i + Y_e \mathbf{u}_e = \mathbf{u}_i. \tag{2.82}$$

2.2.2 Total sum of collisional terms

In this section we derive the total sum across all species (ions and electrons) of the collisional terms $\mathbf{R}_s$ and Q_s. Use eq. (1.44) to re-write $\mathbf{R}_s$ as

$$\mathbf{R}_s = \sum_{r,r \neq s}^N \int m_s \mathbf{w}_s C_{sr} \, d\mathbf{v} = \sum_{r,r \neq s}^N \int m_s \mathbf{v} C_{sr} \, d\mathbf{v},$$

where N is the total number of species. Using eq. (1.32) this leads to

$$\sum_s^N \mathbf{R}_s = 0. \tag{2.83}$$

Similarly, we use eq. (1.59) to re-write Q_s as

$$Q_s = \sum_{r,r \neq s}^N \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) C_{sr} \, d\mathbf{v}$$

$$= \sum_{r,r \neq s}^N \int \frac{1}{2} m_s (\mathbf{v} \cdot \mathbf{v}) C_{sr} \, d\mathbf{v} - \mathbf{u}_s \cdot \sum_{r,r \neq s}^N \int m_s \mathbf{w}_s C_{sr} \, d\mathbf{v}$$

Summing over all species and using eq. (1.33) leads to

$$\sum_s^N Q_s = -\sum_s^N \mathbf{u}_s \cdot \mathbf{R}_s. \tag{2.84}$$

However, for the multi-material formulation we'll enforce slightly different requirements. We first note that ion species and electrons of one material do not collided with those of a different material, that is, materials are fully segregated. Additionally, we introduce the following notation

$$\mathbf{R}_{k,i,s} = \sum_{r,r \neq s}^{N_k + e} \int m_{k,i,s} \mathbf{v} C_{s,r} \, d\mathbf{v},$$

for ion species of material k colliding with other species or electrons of material k, and

$$\mathbf{R}_{k,e} = \sum_r^{N_k} \int m_{k,e} \mathbf{v} C_{e,r} \, d\mathbf{v},$$

for electrons of material k colliding with other species of material k. We then require

$$\sum_s^{N_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e} = 0. \tag{2.85}$$

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Similarly for collisional energy, we introduce the terms

$$Q_{k,i,s} = \sum_{r,r \neq s}^{N_k+e} \int \frac{1}{2} m_{k,i,s} \left(\mathbf{v} \cdot \mathbf{v} \right) C_{sr} \, d\mathbf{v} - \mathbf{u}_{k,i,s} \cdot \sum_{r,r \neq s}^{N_k+e} \int m_{k,i,s} \mathbf{w}_{k,i,s} C_{sr} \, d\mathbf{v},$$

$$Q_{k,e} = \sum_{r}^{N_k} \int \frac{1}{2} m_{k,e} \left(\mathbf{v} \cdot \mathbf{v} \right) C_{er} \, d\mathbf{v} - \mathbf{u}_{k,e} \cdot \sum_{r}^{N_k} \int m_{k,e} \mathbf{w}_{k,e} C_{er} \, d\mathbf{v},$$

We then require

$$\sum_{s}^{N_k} Q_{k,i,s} + Q_{k,e} = \sum_{s}^{N_k} \mathbf{u}_{k,i,s} \cdot \mathbf{R}_{k,i,s} - \mathbf{u}_{k,e} \cdot \mathbf{R}_{k,e}. \tag{2.86}$$

2.2.3 The general model

Mass

Multiplying eq. (2.1) by $m_{i,s}$ and using eq. (2.70) gives

$$\frac{\partial \rho Y_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s}) = 0.$$

We introduce the species diffusion vector

$$\mathbf{j}_{k,i,s} = \rho Y_k Y_{k,i,s} \mathbf{w}_{k,i,s}, \tag{2.87}$$

where

$$\mathbf{w}_{k,i,s} = \mathbf{u}_{k,i,s} - \mathbf{u}. \tag{2.88}$$

The mass-conservation equation is then written as

$$\frac{\partial \rho Y_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\rho Y_k Y_{k,i,s} \mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i,s}.$$

Note that adding over all s of a material gives

$$\frac{\partial \rho Y_k}{\partial t} + \nabla \cdot (\rho Y_k \mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i},$$

where

$$\mathbf{j}_{k,i} = \rho Y_k \mathbf{w}_{k,i}, \tag{2.89}$$

and

$$\mathbf{w}_{k,i} = \mathbf{u}_{k,i} - \mathbf{u}. \tag{2.90}$$

Multiplying eq. (2.1) by $Z_{k,i,s}$ and summing over all s of a material gives

$$\frac{\partial}{\partial t} \sum_{s}^{N_k} Z_{k,i,s} n_{i,s} + \nabla \cdot \left(\sum_{s}^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \right) = 0.$$

Subtracting eq. (2.2) from the above and using the assumption of quasi-neutrality in item 2 gives

$$\nabla \cdot \left(\sum_{s}^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e} \right) = 0,$$

which is equivalent to

$$\nabla \cdot \mathbf{J}_k = 0.$$

Summing over all k gives

$$\nabla \cdot \mathbf{J} = 0.$$

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Momentum

We now add eqs. (2.3) and (2.4) and use item 1 to obtain

$$\frac{\partial m_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (m_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s}) - e\left(Z_{k,i,s} n_{k,i,s} - n_{k,e}\right) \mathbf{E}$$
$$- e\left(Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e}\right) \times \mathbf{B} = \nabla \cdot (\boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e}) + \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}.$$

Using eq. (2.70) the above becomes

$$\frac{\partial \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (\rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s}) - e\left(Z_{k,i,s} n_{k,i,s} - n_{k,e}\right) \mathbf{E}$$
$$- e\left(Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e}\right) \times \mathbf{B} = \nabla \cdot (\boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e}) + \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}.$$

Summing over all s of a material, using the assumption of item 2 and the definition of the current in eq. (2.11) the above simplifies to

$$\frac{\partial}{\partial t} \sum_s^{N_k} \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_s^{N_k} \rho Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s} - \mathbf{J}_k \times \mathbf{B} =$$
$$\nabla \cdot \left(\sum_s^{N_k} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) + \sum_s^{N_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}.$$

Using eq. (2.85) the last two terms above vanish. We now write the momentum equation as

$$\frac{\partial}{\partial t} \sum_s^{N_k} \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_s^{N_k} \left[\rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u} + \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} \left(\mathbf{u}_{k,i,s} - \mathbf{u} \right) \right] - \mathbf{J}_k \times \mathbf{B} =$$
$$\nabla \cdot \left(\sum_s^{N_k} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right).$$

Using the definition of $\mathbf{j}_{k,i,s}$ the above becomes

$$\frac{\partial}{\partial t} \sum_s^{N_k} \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_s^{N_k} \left(\rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u} \right) - \mathbf{J}_k \times \mathbf{B} =$$
$$\nabla \cdot \left(\sum_s^{N_k} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) - \nabla \cdot \sum_s^{N_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s}.$$

We now sum over all materials to obtain

$$\frac{\partial}{\partial t} \sum_k^{M} \sum_s^{N_k} \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_k^{M} \sum_s^{N_k} \left(\rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u} \right) - \mathbf{J} \times \mathbf{B} =$$
$$\nabla \cdot \sum_k^{M} \left(\sum_s^{N_k} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) - \nabla \cdot \sum_k^{M} \sum_s^{N_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s}.$$

Using the definition of $\mathbf{u}$ we get

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k^{M} \left(\sum_s^{N_k} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) - \nabla \cdot \sum_k^{M} \sum_s^{N_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s}.$$

Given the assumption in item 1, eq. (2.4) for electron momentum becomes

$$en_{k,e}(\mathbf{E} + \mathbf{u}_{k,e} \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_{k,e} + \mathbf{R}_{k,e},$$

Using the definition of the current in eq. (2.11), the left-hand side can be re-written as

$$en_{k,e}(\mathbf{E} + \mathbf{u}_{k,e} \times \mathbf{B}) = en_{k,e}\mathbf{E} + \left(e \sum_s^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - \mathbf{J}_k \right) \times \mathbf{B},$$

and thus the electron momentum equation becomes

$$en_{k,e}\mathbf{E} + e \sum_s^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \times \mathbf{B} = \mathbf{J}_k \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_{k,e} + \mathbf{R}_{k,e},$$

Using the definition of $\mathbf{j}_{k,i,s}$, we can write

$$\sum_s^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \times \mathbf{B} = \sum_s^{N_k} [Z_{k,i,s} n_{k,i,s} \mathbf{u} + Z_{k,i,s} n_{k,i,s} (\mathbf{u}_{k,i,s} - \mathbf{u})] \times \mathbf{B}$$

$$= \sum_s^{N_k} \left(Z_{k,i,s} n_{k,i,s} \mathbf{u} + Z_{k,i,s} n_{k,i,s} \frac{\mathbf{j}_{k,i,s}}{\rho Y_k Y_{k,i,s}} \right) \times \mathbf{B}.$$

Using eq. (2.70) we get

$$\sum_s^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \times \mathbf{B} = \sum_s^{N_k} \left(Z_{k,i,s} n_{k,i,s} \mathbf{u} + \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \right) \times \mathbf{B}$$

$$= \sum_s^{N_k} Z_{k,i,s} n_{k,i,s} \mathbf{u} \times \mathbf{B} + \sum_s^{N_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \times \mathbf{B}$$

$$= n_{k,e} \mathbf{u} \times \mathbf{B} + \sum_s^{N_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \times \mathbf{B}.$$

Thus, the electron momentum equation becomes

$$en_{k,e}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{J}_k \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_{k,e} + \mathbf{R}_{k,e} - e \sum_s^{N_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \times \mathbf{B}.$$

We define the total electron density as

$$n_e = \sum_k^M n_{k,e}.$$

Thus, summing over all k gives

$$en_e(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k^M \boldsymbol{\sigma}_{k,e} + \sum_k^M \mathbf{R}_{k,e} - e \sum_k^M \sum_s^{N_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \times \mathbf{B}.$$

or

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} \left(\mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k^M \boldsymbol{\sigma}_{k,e} + \sum_k^M \mathbf{R}_{k,e} \right) - \sum_k^M \sum_s^{N_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{n_e m_{k,i,s}} \times \mathbf{B}. \qquad (2.91)$$

Internal energy

Use the definition of $\mathbf{w}_{k,i,s}$ to re-write eq. (2.5) and eq. (2.6) as follows

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_{k,i,s}\right) + \nabla \cdot \left(\frac{3}{2}p_{k,i,s}\mathbf{u}\right) =$$

$$\boldsymbol{\sigma}_{k,i,s} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,i,s} + Q_{k,i,s} - \nabla \cdot \left(\frac{3}{2}p_{k,i,s}\mathbf{w}_{k,i,s}\right) + \boldsymbol{\sigma}_{k,i,s} : \nabla\mathbf{w}_{k,i,s}.$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_{k,e}\right) + \nabla \cdot \left(\frac{3}{2}p_{k,e}\mathbf{u}\right) = \boldsymbol{\sigma}_{k,e} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} - \nabla \cdot \left(\frac{3}{2}p_{k,e}\mathbf{w}_{k,e}\right) + \boldsymbol{\sigma}_{k,e} : \nabla\mathbf{w}_{k,e}.$$

2.2.4 Multi-material multi-species model

We add two extra assumptions

1. All ions of a material move with a single ion velocity $\mathbf{u}_{k,i}$.

2. All ions of a material have the same temperature $T_{k,i}$

If all ions move with a single ion velocity, then the expressions for common velocities given in eqs. (2.71) and (2.72) give

$$\mathbf{u}_{k,i,s} = \mathbf{u}_{k,i} = \mathbf{u}_k. \tag{2.92}$$

Additionally,

$$\mathbf{j}_{k,i,s} = \rho Y_k Y_{k,i,s}\mathbf{w}_{k,i,s} = \rho Y_k Y_{k,i,s}\mathbf{w}_{k,i} = \rho Y_k Y_{k,i,s}\mathbf{w}_k. \tag{2.93}$$

We define a material's ion pressure $p_{k,i}$, ion stress tensor $\boldsymbol{\sigma}_{k,i}$, ion heat flux $q_{k,i}$, and ion collision term $Q_{k,i}$ as

$$p_{k,i} = \sum_s^{N_k} p_{k,i,s}, \tag{2.94}$$

$$\boldsymbol{\sigma}_{k,i} = \sum_s^{N_k} \boldsymbol{\sigma}_{k,i,s}, \tag{2.95}$$

$$\mathbf{q}_{k,i} = \sum_s^{N_k} \mathbf{q}_{k,i,s}, \tag{2.96}$$

$$Q_{k,i} = \sum_s^{N_k} Q_{k,i,s}. \tag{2.97}$$

Adding over all ion species of a material, the ion pressure equation becomes

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_{k,i}\right) + \nabla \cdot \left(\frac{3}{2}p_{k,i}\mathbf{u}\right) = \boldsymbol{\sigma}_{k,i} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} - \nabla \cdot \left(\frac{3}{2}p_{k,i}\mathbf{w}_{k,i}\right) + \boldsymbol{\sigma}_{k,i} : \nabla\mathbf{w}_{k,i}.$$

Similarly, summing over all ion species, the ion equation of state becomes

$$p_{k,i} = \sum_s^{N_k} n_{k,i,s} k_B T_{k,i}.$$

To summarize, we have

$$\frac{\partial \rho Y_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\rho Y_k Y_{k,i,s}\mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i,s}, \tag{2.98}$$

$$\nabla \cdot \mathbf{J} = 0, \tag{2.99}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{uu}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k^M (\boldsymbol{\sigma}_{k,i} + \boldsymbol{\sigma}_{k,e}) - \nabla \cdot \sum_k^M \mathbf{j}_{k,i} \mathbf{u}_k, \tag{2.100}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} \left(\mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k^M \boldsymbol{\sigma}_{k,e} + \sum_k^M \mathbf{R}_{k,e} \right) - \sum_s^{N_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{n_e m_{k,i,s}} \times \mathbf{B}, \tag{2.101}$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_{k,i} \right) + \nabla \cdot \left(\frac{3}{2} p_{k,i} \mathbf{u} \right) = \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} - \nabla \cdot \left(\frac{3}{2} p_{k,i} \mathbf{w}_{k,i} \right) + \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{w}_{k,i}, \tag{2.102}$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_{k,e} \right) + \nabla \cdot \left(\frac{3}{2} p_{k,e} \mathbf{u} \right) = \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} - \nabla \cdot \left(\frac{3}{2} p_{k,e} \mathbf{w}_{k,e} \right) + \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{w}_{k,e}, \tag{2.103}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.104}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.105}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.106}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.107}$$

$$\mathbf{J}_k = en_{k,e} (\mathbf{u}_k - \mathbf{u}_{k,e}), \tag{2.108}$$

$$\mathbf{J} = \sum_k^M \mathbf{J}_k, \tag{2.109}$$

$$\rho_{q,k} = 0, \tag{2.110}$$

$$\rho_q = \sum_k^M \rho_{q,k}, \tag{2.111}$$

$$p_{k,i} = \sum_s^{N_k} n_{k,i,s} k_B T_{k,i}. \tag{2.112}$$

$$p_{k,e} = n_{k,e} k_B T_{k,e}. \tag{2.113}$$

Note that eq. (2.98) gives $\rho Y_k Y_{k,i,s}$. Summing over all s gives ρY_k. Summing over all k gives ρ. Dividing $\rho Y_k Y_{k,i,s}$ by ρY_k gives $Y_{k,i,s}$. Dividing ρY_k by ρ gives Y_k.

2.2.5   Single-material multi-species model

Assuming there is only one material, then $\mathbf{w}_{k,i} = \mathbf{u}_{k,i} - \mathbf{u} = 0$ and thus eq. (2.93) gives $\mathbf{j}_{k,i,s} = 0$. Summing over all s would then give $\mathbf{j}_{k,i} = 0$. Thus, the model in section 2.2.4 becomes

$$\frac{\partial \rho Y_{i,s}}{\partial t} + \nabla \cdot (\rho Y_{i,s} \mathbf{u}) = 0, \tag{2.114}$$

$$\nabla \cdot \mathbf{J} = 0, \tag{2.115}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{uu}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e), \tag{2.116}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} (\mathbf{J} \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e), \tag{2.117}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_i\right) + \nabla \cdot \left(\frac{3}{2}p_i\mathbf{u}\right) = \boldsymbol{\sigma}_i : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i, \tag{2.118}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_e\right) + \nabla \cdot \left(\frac{3}{2}p_e\mathbf{u}\right) = \boldsymbol{\sigma}_e : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot \left(\frac{3}{2}p_e\mathbf{w}_e\right) + \boldsymbol{\sigma}_e : \nabla\mathbf{w}_e, \tag{2.119}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.120}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.121}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.122}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.123}$$

$$\mathbf{J} = en_e\left(\mathbf{u} - \mathbf{u}_e\right), \tag{2.124}$$

$$\rho_q = 0, \tag{2.125}$$

$$p_i = \sum_s^N n_{i,s}k_B T_i, \tag{2.126}$$

$$p_e = n_e k_B T_e. \tag{2.127}$$

2.2.6 Single-material two-species model

If there is only one ion species, along with the electron species, then the model in section 2.2.5 simplifies to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \tag{2.128}$$

$$\nabla \cdot \mathbf{J} = 0, \tag{2.129}$$

$$\frac{\partial \rho\mathbf{u}}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \left(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e\right), \tag{2.130}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e}\left(\mathbf{J} \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e\right), \tag{2.131}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_i\right) + \nabla \cdot \left(\frac{3}{2}p_i\mathbf{u}\right) = \boldsymbol{\sigma}_i : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i, \tag{2.132}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_e\right) + \nabla \cdot \left(\frac{3}{2}p_e\mathbf{u}\right) = \boldsymbol{\sigma}_e : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot \left(\frac{3}{2}p_e\mathbf{w}_e\right) + \boldsymbol{\sigma}_e : \nabla\mathbf{w}_e, \tag{2.133}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.134}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.135}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.136}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.137}$$

$$\mathbf{J} = en_e\left(\mathbf{u} - \mathbf{u}_e\right), \tag{2.138}$$

$$\rho_q = 0, \tag{2.139}$$

$$p_i = n_i k_B T_i, \tag{2.140}$$

$$p_e = n_e k_B T_e. \tag{2.141}$$

2.2.7 Alternate forms of the energy equations

The specific energy of a species s is denoted as e_s. Using our current notation, this becomes $e_{k,i,s}$ for the specific energy of ion species s of material k and $e_{k,e}$ for the specific energy of electrons of material k. The total ion specific energy of a material k is given by

$$e_{k,i} = \sum_s^{N_k} Y_{k,i,s} e_{k,i,s}. \tag{2.142}$$

The total specific energy of a material k is given by

$$e_k = e_{k,i} + Y_{k,e} e_{k,e}. \tag{2.143}$$

The total specific energy is given by

$$e = \sum_k^{M} Y_k e_k. \tag{2.144}$$

The enthalpy of any species s is given by

$$h_s = e_s + \frac{\hat{p}_s}{\rho_s}. \tag{2.145}$$

Using our current notation, this becomes

$$h_{k,i,s} = e_{k,i,s} + \frac{\hat{p}_{k,i,s}}{\rho_{k,i,s}} \qquad h_{k,e} = e_{k,e} + \frac{\hat{p}_{k,e}}{\rho_{k,e}}. \tag{2.146}$$

The total ion enthalpy of a material k is given by

$$h_{k,i} = \sum_s^{N_k} Y_{k,i,s} h_{k,i,s}. \tag{2.147}$$

The total enthalpy of a material k is given by

$$h_k = h_{k,i} + Y_{k,e} h_{k,e}. \tag{2.148}$$

The total enthalpy is given by

$$h = \sum_k^{M} Y_k h_k. \tag{2.149}$$

We note that $\hat{p}_{k,i,s}$ and $\hat{p}_{k,e}$ satisfy

$$p_{k,i,s} = \eta_k \eta_{k,i,s} \hat{p}_{k,i,s}, \tag{2.150}$$

$$p_{k,e} = \eta_k \eta_{k,e} \hat{p}_{k,e}. \tag{2.151}$$

Summing eq. (2.150) over all species and using eq. (2.94) gives

$$p_{k,i} = \eta_k \sum_s^{N_k} \eta_{k,i,s} \hat{p}_{k,i,s},$$

If we define $\hat{p}_{k,i}$ so that it satisfies

$$p_{k,i} = \eta_k \hat{p}_{k,i}, \tag{2.152}$$

then we have

$$\hat{p}_{k,i} = \sum_{s}^{N_k} \eta_{k,i,s}\hat{p}_{k,i,s}, \tag{2.153}$$

Thus, we can now show that the total ion enthalpy can be written as

$$h_{k,i} = e_{k,i} + \sum_{s}^{N_k} \frac{Y_{k,i,s}}{\rho_{k,i,s}}\hat{p}_{k,i,s} = e_{k,i} + \sum_{s}^{N_k} \frac{\eta_{k,i,s}}{\rho_k}\hat{p}_{k,i,s} = e_{k,i} + \frac{\hat{p}_{k,i}}{\rho_k}. \tag{2.154}$$

Using eq. (2.152) we can show that

$$h_{k,i} = e_{k,i} + \frac{\eta_k\hat{p}_{k,i}}{\eta_k\rho_k} = e_{k,i} + \frac{p_{k,i}}{\eta_k\rho_k} = e_{k,i} + \frac{p_{k,i}}{\rho Y_k}. \tag{2.155}$$

Similarly, using eq. (2.151), we can show that

$$h_{k,e} = e_{k,e} + \frac{\eta_k\eta_{k,e}\hat{p}_{k,e}}{\eta_k\eta_{k,e}\rho_{k,e}} = e_{k,e} + \frac{p_{k,e}}{\eta_k\eta_{k,e}\rho_{k,e}} = e_{k,e} + \frac{p_{k,e}}{\eta_k\rho_k Y_{k,e}} = e_{k,e} + \frac{p_{k,e}}{\rho Y_k Y_{k,e}}. \tag{2.156}$$

Thus far we've been working with plasmas that behave as ideal gasses with three degrees of freedom (the three translational degrees of freedom). For ideal gasses we also have

$$p_{k,i,s} = (\gamma - 1)\rho Y_k Y_{k,i,s}e_{k,i,s},$$

$$p_{k,e} = (\gamma - 1)\rho Y_k Y_{k,e}e_{k,e}.$$

Summing $p_{k,i,s}$ over all s gives

$$p_{k,i} = (\gamma - 1)\rho Y_k e_{k,i}. \tag{2.157}$$

Since $\gamma = 5/3$ for the plasmas under consideration, the internal energy equation for ions given by eq. (2.102) can now be written as

$$\frac{\partial \rho Y_k e_{k,i}}{\partial t} + \nabla \cdot (\rho Y_k e_{k,i}\mathbf{u}) = \boldsymbol{\sigma}_{k,i} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} - \nabla \cdot (\rho Y_k e_{k,i}\mathbf{w}_{k,i}) + \boldsymbol{\sigma}_{k,i} : \nabla\mathbf{w}_{k,i}.$$

Similarly, the internal energy equation for electrons given by eq. (2.103) becomes

$$\frac{\partial \rho Y_k Y_{k,e} e_{k,e}}{\partial t} + \nabla \cdot (\rho Y_k Y_{k,e} e_{k,e}\mathbf{u}) = \boldsymbol{\sigma}_{k,e} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} - \nabla \cdot (\rho Y_k Y_{k,e} e_{k,e}\mathbf{w}_{k,e}) + \boldsymbol{\sigma}_{k,e} : \nabla\mathbf{w}_{k,e}.$$

We re-write the last term in the two equations above as

$$\boldsymbol{\sigma}_{k,s} : \nabla\mathbf{w}_{k,s} = \nabla \cdot (\mathbf{w}_{k,s} \cdot \boldsymbol{\sigma}_{k,s}) - \mathbf{w}_{k,s} \cdot \nabla\boldsymbol{\sigma}_{k,s}$$
$$= -\nabla \cdot (p_{k,s}\mathbf{w}_{k,s}) + \nabla \cdot (\mathbf{w}_{k,s} \cdot \mathbf{t}_{k,s}) - \mathbf{w}_{k,s} \cdot \nabla\boldsymbol{\sigma}_{k,s},$$

where s is equal to i or e. Thus, the ion and electron internal energy equations become

$$\frac{\partial \rho Y_k e_{k,i}}{\partial t} + \nabla \cdot (\rho Y_k e_{k,i}\mathbf{u}) =$$
$$\boldsymbol{\sigma}_{k,i} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} - \nabla \cdot \left[\left(e_{k,i} + \frac{p_{k,i}}{\rho Y_k}\right)\rho Y_k\mathbf{w}_{k,i}\right] + \nabla \cdot (\mathbf{w}_{k,i} \cdot \mathbf{t}_{k,i}) - \mathbf{w}_{k,i} \cdot \nabla\boldsymbol{\sigma}_{k,i}.$$

$$\frac{\partial \rho Y_k Y_{k,e} e_{e,k}}{\partial t} + \nabla \cdot (\rho Y_k Y_{k,e} e_{k,e}\mathbf{u}) =$$
$$\boldsymbol{\sigma}_{k,e} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} - \nabla \cdot \left[\left(e_{k,e} + \frac{p_{k,e}}{\rho Y_k Y_{k,e}}\right)\rho Y_k Y_{k,e}\mathbf{w}_{k,e}\right] + \nabla \cdot (\mathbf{w}_{k,e} \cdot \mathbf{t}_{k,e}) - \mathbf{w}_{k,e} \cdot \nabla\boldsymbol{\sigma}_{k,e}.$$

$$\frac{\partial \rho Y_e e_e}{\partial t} + \nabla \cdot (\rho Y_e e_e \mathbf{u}) = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot \left[\left(e_e + \frac{p_e}{\rho Y_e} \right) \mathbf{j}_e \right] + \nabla \cdot (\mathbf{w}_e \cdot \mathbf{t}_e) - \mathbf{w}_e \cdot \nabla \boldsymbol{\sigma}_e.$$

Since

$$\frac{p_e}{\rho Y_e} = \frac{\eta_e \hat{p}_e}{\rho Y_e} = \frac{\hat{p}_e}{\rho_e},$$

and $h_e = e_e + \hat{p}_e / \rho_e$ we have

$$\frac{\partial \rho Y_e e_e}{\partial t} + \nabla \cdot (\rho Y_e e_e \mathbf{u}) = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot (h_e \mathbf{j}_e) + \nabla \cdot (\mathbf{w}_e \cdot \mathbf{t}_e) - \mathbf{w}_e \cdot \nabla \boldsymbol{\sigma}_e.$$

Finally, we define the total internal energy as

$$e = e_i + Y_e e_e, \tag{2.158}$$

so that adding up the ion and electron internal energy equations gives

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) = (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e) : \nabla \mathbf{u} - \nabla \cdot (\mathbf{q}_i + \mathbf{q}_e) + Q_i + Q_e$$
$$- \nabla \cdot (h_e \mathbf{j}_e) + \nabla \cdot (\mathbf{w}_e \cdot \mathbf{t}_e) - \mathbf{w}_e \cdot \nabla \boldsymbol{\sigma}_e.$$

Using eq. (2.84) we have

$$Q_i + Q_e = \sum_s^N Q_{i,s} + Q_e = - \left(\sum_s^N \mathbf{u}_{i,s} \cdot \mathbf{R}_{i,s} + \mathbf{u}_e \cdot \mathbf{R}_e \right) =$$
$$- \left(\mathbf{u} \cdot \sum_s^N \mathbf{R}_{i,s} + \mathbf{u}_e \cdot \mathbf{R}_e \right) = - \left(\mathbf{u} \cdot \mathbf{R}_i + \mathbf{u}_e \cdot \mathbf{R}_e \right),$$

where

$$\mathbf{R}_i = \sum_s^N \mathbf{R}_{i,s} \tag{2.159}$$

Thus, the total internal energy equation becomes

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) = (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e) : \nabla \mathbf{u} - \nabla \cdot (\mathbf{q}_i + \mathbf{q}_e) - (\mathbf{u} \cdot \mathbf{R}_i + \mathbf{u}_e \cdot \mathbf{R}_e)$$
$$- \nabla \cdot (h_e \mathbf{j}_e) + \nabla \cdot (\mathbf{w}_e \cdot \mathbf{t}_e) - \mathbf{w}_e \cdot \nabla \boldsymbol{\sigma}_e.$$

Chapter 3

Basic waves and parameters

3.1 Electron-plasma and ion-acoustic waves

3.1.1 Linearization

The following decompositions will be used in the derivation of electron-plasma and ion-acoustic waves:

$$n_i = n_{i0} + n_{i1},$$
$$n_e = n_{e0} + n_{e1},$$
$$p_i = p_{i0} + p_{i1},$$
$$p_e = p_{e0} + p_{e1},$$
$$\mathbf{u}_i = \mathbf{u}_{i0} + \mathbf{u}_{i1},$$
$$\mathbf{u}_e = \mathbf{u}_{e0} + \mathbf{u}_{e1},$$
$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1,$$
$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1. \tag{3.1}$$

For these decompositions, we'll assume

1. Terms with a subscript 1 are small and thus products of two small quantities can be neglected.

2. $\mathbf{u}_{i0}$, $\mathbf{u}_{e0}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ are zero.

3. n_{i0}, n_{e0}, p_{i0}, and p_{e0} are uniform in space and time.

Using the variable decompositions in the evolution equation for electron density eq. (2.46), we have

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \nabla \cdot [(n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] = 0.$$

Using assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot (n_{e0}\mathbf{u}_{e1}) = 0. \tag{3.2}$$

Using the variable decompositions in the evolution equation for ion density eq. (2.45), we have

$$\frac{\partial n_{i0} + n_{i1}}{\partial t} + \nabla \cdot [(n_{i0} + n_{i1})(\mathbf{u}_{i0} + \mathbf{u}_{i1})] = 0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i1}}{\partial t} + \nabla \cdot (n_{i0}\mathbf{u}_{i1}) = 0. \tag{3.3}$$

Using the variable decompositions in the momentum equation for electrons eq. (2.48), we have

$$\frac{\partial}{\partial t} \left[m_e \left(n_{e0} + n_{e1} \right) \left(\mathbf{u}_{e0} + \mathbf{u}_{e1} \right) \right] + \nabla \cdot \left[m_e \left(n_{e0} + n_{e1} \right) \left(\mathbf{u}_{e0} + \mathbf{u}_{e1} \right) \left(\mathbf{u}_{e0} + \mathbf{u}_{e1} \right) \right]$$
$$+ e \left(n_{e0} + n_{e1} \right) \left[\left(\mathbf{E}_0 + \mathbf{E}_1 \right) + \left(\mathbf{u}_{e0} + \mathbf{u}_{e1} \right) \times \left(\mathbf{B}_0 + \mathbf{B}_1 \right) \right] = -\nabla \left(p_{e0} + p_{e1} \right).$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e0}\mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e}\mathbf{E}_1 = -\frac{1}{m_e}\nabla p_{e1}. \tag{3.4}$$

Using the variable decompositions in the momentum equation for ions eq. (2.47), we have

$$\frac{\partial}{\partial t} \left[m_i \left(n_{i0} + n_{i1} \right) \left(\mathbf{u}_{i0} + \mathbf{u}_{i1} \right) \right] + \nabla \cdot \left[m_i \left(n_{i0} + n_{i1} \right) \left(\mathbf{u}_{i0} + \mathbf{u}_{i1} \right) \left(\mathbf{u}_{i0} + \mathbf{u}_{i1} \right) \right]$$
$$- Ze \left(n_{i0} + n_{i1} \right) \left[\left(\mathbf{E}_0 + \mathbf{E}_1 \right) + \left(\mathbf{u}_{i0} + \mathbf{u}_{i1} \right) \times \left(\mathbf{B}_0 + \mathbf{B}_1 \right) \right] = -\nabla \left(p_{i0} + p_{i1} \right).$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i0}\mathbf{u}_{i1}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_1 = -\frac{1}{m_i}\nabla p_{i1}. \tag{3.5}$$

We'll often need to take the gradient of the ion and electron pressure. Given $p_s = C_s n_s^{\gamma_s}$, where $s = i, e$, we have

$$\nabla p_s = C_s \gamma_s n_s^{\gamma_s - 1} \nabla n_s = C_s \gamma_s \frac{n_s^{\gamma_s}}{n_s} \nabla n_s = \gamma_s \frac{p_s}{n_s} \nabla n_s,$$

Using the equation of state $p_s = n_s k_B T_s$, the above becomes

$$\nabla p_s = \gamma_s k_B T_s \nabla n_s. \tag{3.6}$$

Given the assumption in item 3, the above simplifies to

$$\nabla p_{s1} = \gamma_s k_B T_s \nabla n_{s1}. \tag{3.7}$$

3.1.2 Electron Plasma Waves

On top of the assumptions in section 3.1.1, we'll assume

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.

2. Uniform ion density, $n_{i1} = 0$.

Combining Equation (3.4) with eq. (3.7) gives

$$\frac{\partial n_{e0}\mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e}\mathbf{E}_1 = -\frac{\gamma_e k_B T_e}{m_e}\nabla n_{e1}. \tag{3.8}$$

Taking the time derivative of eq. (3.2) and using eq. (3.8) leads to the the wave equation for electron density

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{e n_{e0}}{m_e} \nabla \cdot \mathbf{E}_1 = \frac{\gamma_e k_B T_e}{m_e} \nabla^2 n_{e1}. \tag{3.9}$$

For electron plasma waves, we'll assume that n_i varies in space and time so slowly that it can be assumed to be constant. That is, we assume $n_{i1} = 0$. Thus, Gauss's law now takes the form

$$\nabla \cdot \mathbf{E}_1 = \frac{e}{\epsilon_0} \left(Z n_{i0} - n_{e0} - n_{e1} \right).$$

Using the quasi-neutrality assumption ($Z n_{i0} = n_{e0}$)

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_{e1}. \tag{3.10}$$

Plugging the above in the electron wave equation we obtain

$$\frac{\partial^2 n_{e1}}{\partial t^2} + \frac{e^2 n_{e0}}{m_e \epsilon_0} n_{e1} = \frac{\gamma_e k_B T_e}{m_e} \cdot \nabla^2 n_{e1},$$

We now introduce the electron plasma frequency

$$w_{pe} = \left(\frac{n_{e0} e^2}{m_e \epsilon_0} \right)^{1/2}, \tag{3.11}$$

and the thermal velocity

$$v_{Ts} = \sqrt{\frac{k_B T_s}{m_s}}. \tag{3.12}$$

These two allow us to write the equation for n_{e1} as

$$\frac{\partial^2 n_{e1}}{\partial t^2} + w_{pe}^2 n_{e1} - \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0. \tag{3.13}$$

Assuming a mode of the form $n_{e1} = \tilde{n}_{e1} \exp(-iwt)$, where $\tilde{n}_{e1} = \tilde{n}_{e1}(\mathbf{x})$, gives the following

$$\left(w^2 - w_{pe}^2 \right) n_{e1} + \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0. \tag{3.14}$$

If we assume $w_{pe} < w$, then $\tilde{n}_{e1} = \hat{n}_{e1} \exp(i\mathbf{k}_e \cdot \mathbf{x})$ is a solution to eq. (3.14). Plugging in this expression for $\tilde{n}_{e1}$ in eq. (3.14) finally gives

$$w^2 - w_{pe}^2 - \gamma_e v_{Te}^2 k_e^2 = 0. \tag{3.15}$$

The above is the dispersion relation for electron plasma waves.

3.1.3 Ion Acoustic Waves

On top of the assumptions in section 3.1.1, we'll assume

1. Quasi-neutrality for the base flow, $Z n_{i0} = n_{e0}$.

2. Approximate quasi-neutrality for the fluctuations, $Z n_{i1} \approx n_{e1}$.

3. Negligible electron mass, $m_e \to 0$.

Combining eq. (3.5) with eq. (3.7) gives

$$\frac{\partial n_{i0}\mathbf{u}_{i1}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_1 = -\frac{\gamma_i k_B T_i}{m_i}\nabla n_{i1}. \tag{3.16}$$

Taking the time derivative of eq. (3.3) and using eq. (3.16) leads to the the wave equation for ion density

$$\frac{\partial^2 n_{i1}}{\partial t^2} + \frac{Zen_{i0}}{m_i}\nabla \cdot \mathbf{E}_1 = \frac{\gamma_i k_B T_i}{m_i}\nabla^2 n_{i1}. \tag{3.17}$$

For this case, we assume that the mass of the electron, which is significantly smaller than that of the ions, is negligible. Thus, eq. (3.4) simplifies to

$$en_{e0}\mathbf{E}_1 = -\gamma_e k_B T_e \nabla n_{e1}. \tag{3.18}$$

Using the above in the ion wave equation we obtain

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{Zn_{i0}}{n_{e0}}\frac{\gamma_e k_B T_e}{m_i}\nabla^2 n_{e1} + \frac{\gamma_i k_B T_i}{m_i}\nabla^2 n_{i1}.$$

Due to quasi-neutrality, we have $Zn_{i0} = n_{e0}$ and $Zn_{i1} \approx n_{e1}$, which gives

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \left(\frac{Z\gamma_e k_B T_e}{m_i} + \frac{\gamma_i k_B T_i}{m_i}\right)\nabla^2 n_{i1}.$$

We now introduce the following velocity

$$v_s = \sqrt{\frac{Z\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}}, \tag{3.19}$$

which allows us to write the equation for n_{i1} as

$$\frac{\partial^2 n_{i1}}{\partial t^2} - v_s^2 \nabla^2 n_{i1} = 0. \tag{3.20}$$

Assuming a mode of the form $n_{i1} = \tilde{n}_{i1}\exp(-iwt)$, where $\tilde{n}_{i1} = \tilde{n}_{i1}(\mathbf{x})$, gives the following

$$w^2 n_{i1} + v_s^2 \nabla^2 n_{i1} = 0. \tag{3.21}$$

Since w^2 is always positive, $\tilde{n}_{i1} = \hat{n}_{i1}\exp(i\mathbf{k}_i \cdot \mathbf{x})$ is a solution to eq. (3.21). Plugging in this expression for $\tilde{n}_{i1}$ in eq. (3.21) finally gives

$$w^2 - k_i^2 v_s^2 = 0. \tag{3.22}$$

The above is the dispersion relation for ion-acoustic waves.

3.2   Plasma parameters

3.2.1   Debye length

To describe the Debye length we'll use the governing equations from section 2.1.3 (without assuming isentropic/homentropic flow) as well as the following assumptions

1. Stationary plasma $\mathbf{u}_i = \mathbf{u}_e = 0$.

2. No shear stresses $\mathbf{t}_s$, heat flux $\mathbf{q}_s$, and collisions $\mathbf{R}_s$, Q_s.

3. Effects of magnetic fields can be neglected, i.e. $\mathbf{B} = 0$.

4. Constant temperatures T_i, T_e.

Thus we have

$$-Zen_i\mathbf{E} = -\nabla p_i,$$

$$en_e\mathbf{E} = -\nabla p_e,$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0},$$

$$\nabla \times \mathbf{E} = 0,$$

$$\rho_q = e(Zn_i - n_e),$$

$$p_i = n_i k_B T_i,$$

$$p_e = n_e k_B T_e.$$

We can simplify the above to obtain

$$Zen_i\nabla\phi = -k_B T_i \nabla n_i,$$

$$-en_e\nabla\phi = -k_B T_e \nabla n_e,$$

$$-\nabla^2\phi = \frac{e}{\epsilon_0}(Zn_i - n_e).$$

The physical domain will be one dimensional with length L, namely, $[-L/2, L/2]$. Thus, we have

$$Zen_i\frac{d\phi}{dx} = -k_B T_i \frac{dn_i}{dx}, \tag{3.23}$$

$$-en_e\frac{d\phi}{dx} = -k_B T_e \frac{dn_e}{dx}, \tag{3.24}$$

$$-\frac{d^2\phi}{dx} = \frac{e}{\epsilon_0}(Zn_i - n_e). \tag{3.25}$$

A voltage V is applied across this domain and thus the boundary condition for the electric potential becomes $\phi(-L/2) - \phi(L/2) = V$. We'll also enforce the condition $\phi(0) = 0$ to fix the arbitrary constant of the potential.

The simplest case to consider is the one for which the one-dimensional domain is empty. Gauss's law then gives

$$\frac{d^2\phi}{dx^2} = 0,$$

which, upon applying the boundary conditions gives the trivial solution $\phi = -(V/L)x$ and a constant electric field $E = V/L$ pointing from left to right.

Now let's fill up the physical space with a plasma consisting of electrons and ions. Integrating eqs. (3.23) and (3.24) we obtain

$$\ln n_i = -\frac{Ze}{k_B T_i}\phi + C_1,$$

$$\ln n_e = \frac{e}{k_B T_e}\phi + C_2,$$

which we re-write as

$$n_i = D_1 \exp\left(-\frac{Ze\phi}{k_B T_i}\right),$$

$$n_e = D_2 \exp\left(\frac{e\phi}{k_B T_e}\right).$$

We introduce n_{i0} and n_{e0} as the densities at $x = 0$. Since $\phi = 0$ at $x = 0$, we have

$$n_i = n_{i0} \exp\left(-\frac{Ze\phi}{k_B T_i}\right),$$

$$n_e = n_{e0} \exp\left(\frac{e\phi}{k_B T_e}\right).$$

A condition for the densities at the center is that they satisfy quasi-neutrality, that is, $Zn_{i0} = n_{e0}$. Note that this does not mean $Zn_i = n_e$. We'll assume the terms inside the exponential are small, so that we can use $\exp\alpha = 1 + \alpha$. Thus we have

$$n_i = n_{i0} - \frac{n_{i0}Ze\phi}{k_B T_i},$$

$$n_e = n_{e0} + \frac{n_{e0}e\phi}{k_B T_e}.$$

Plugin this into eq. (3.25) gives

$$\frac{d^2\phi}{dx^2} - \left(\frac{n_{i0}Z^2 e^2}{\epsilon_0 k_B T_i} + \frac{n_{e0}e^2}{\epsilon_0 k_B T_e}\right)\phi = 0. \tag{3.26}$$

We now introduce the Debye length for a particular species s as

$$\lambda_{Ds} = \left(\frac{\epsilon_0 k_B T_s}{n_{\alpha 0} Z^2 e^2}\right)^{1/2}, \tag{3.27}$$

and the total Debye length λ_D as

$$\frac{1}{\lambda_D} = \sum_s \frac{1}{\lambda_{Ds}}. \tag{3.28}$$

Using this, we can re-write eq. (3.26) simply as

$$\frac{d^2\phi}{dx^2} - \frac{1}{\lambda_D^2}\phi = 0. \tag{3.29}$$

The solution to the above that also conforms to the boundary conditions is

$$\phi = -\frac{V}{2}\frac{\sinh\left(\frac{x}{\lambda_D}\right)}{\sinh\left(\frac{L}{2\lambda_D}\right)}. \tag{3.30}$$

A plot of ϕ and the corresponding E is shown in fig. 3.1 for various values of λ_D/L. As the figure shows, the presence of a plasma shields out the electric field, in contrast to the vacuum case where a constant electric field is present across the domain. Given a fixed domain size L, varying the parameter λ_D determines how well the plasma shields away the electric field.

We note that λ_{De}, w_{pe}, and v_{Te} are all related to each other as shown below

$$\lambda_{De}w_{pe} = \left(\frac{\epsilon_0 k_B T_e}{n_{e0}e^2}\right)^{1/2}\left(\frac{n_{e0}e^2}{m_e\epsilon_0}\right)^{1/2} = \left(\frac{k_B T_e}{m_e}\right)^{1/2} = v_{Te}. \tag{3.31}$$
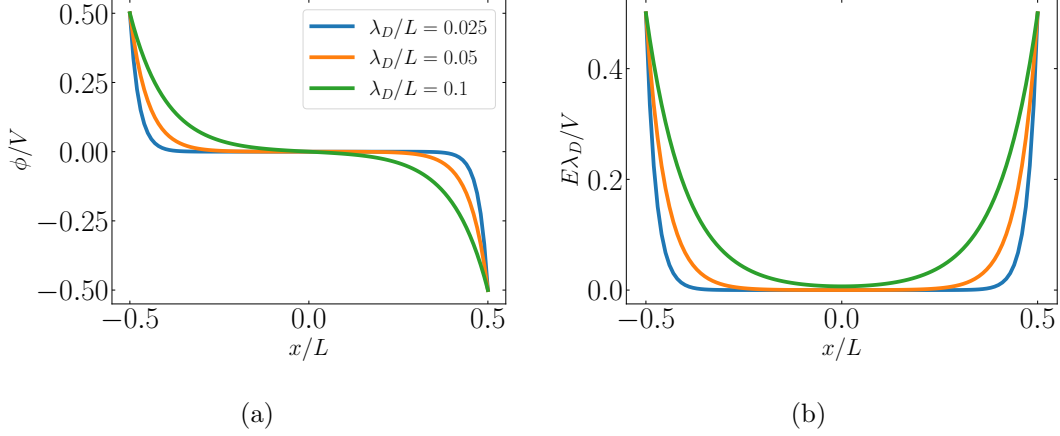
Figure 3.1: Electric potential and field for various values of λ_D/L.

3.2.2 Plasma frequency

We'll use the same setting as that for the derivation of the Debye length. The only difference is that the potential drop across the domain now oscillates with a frequency of w, that is $\phi_1(-L/2) - \phi_1(L/2) = V \exp(-iwt)$.

The starting point is eq. (3.14), that is, the equation for electron-plasma waves after assuming $n_{e1} = \tilde{n}_{e1} \exp(-iwt)$, where $\tilde{n}_{e1} = \tilde{n}_{e1}(\mathbf{x})$. In one dimension this equation takes the form

$$\left(w^2 - w_{pe}^2\right) n_{e1} + \gamma_e v_{Te}^2 \frac{d^2 n_{e1}}{dx^2} = 0.$$

Plugging in eq. (3.10) in the above and using the definition of the electric potential, we get

$$(w^2 - w_{pe}^2) \frac{d^2\phi_1}{dx^2} + \gamma_e v_{Te}^2 \frac{d^4\phi_1}{dx^4} = 0,$$

which we re-write as

$$\frac{d^4\phi_1}{dx^4} + \frac{(w^2 - w_{pe}^2)}{\gamma_e v_{Te}^2} \frac{d^2\phi_1}{dx^2} = 0.$$

Using eq. (3.31) the above becomes

$$\frac{d^4\phi_1}{dx^4} + \frac{(w^2 - w_{pe}^2)}{\gamma_e(\lambda_{De}^2 w_{pe}^2)} \frac{d^2\phi_1}{dx^2} = 0,$$

or

$$\frac{d^4\phi_1}{dx^4} + \frac{1}{\gamma_e \lambda_{De}^2} \left(\frac{w^2}{w_{pe}^2} - 1\right) \frac{d^2\phi_1}{dx^2} = 0.$$

Since we assumed electron number densities of the form $n_{e1} = \tilde{n}_{e1} \exp(-iwt)$, we choose $\phi_1 = \tilde{\phi}_1 \exp(-iwt)$, where $\tilde{\phi}_1 = \tilde{\phi}_1(x)$, so as to satisfy eq. (3.10). Thus, we get

$$\frac{d^4\tilde{\phi}_1}{dx^4} + \frac{1}{\gamma_e \lambda_{De}^2} \left(\frac{w^2}{w_{pe}^2} - 1\right) \frac{d^2\tilde{\phi}_1}{dx^2} = 0.$$

Integrating twice we get

$$\frac{d^2\tilde{\phi}_1}{dx^2} + \frac{1}{\gamma_e \lambda_{De}^2} \left(\frac{w^2}{w_{pe}^2} - 1\right) \tilde{\phi}_1 + C_1 x + C_2 = 0.$$

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We only have one boundary condition, namely $\phi_1(-L/2) - \phi_1(L/2) = V\exp(-iwt)$, yet the equation for ϕ is originally of fourth order. Thus, we prescribe additional boundary conditions such that $C_1 = C_2 = 0$. The equation for $\tilde{\phi}$ then is simply

$$\frac{d^2\tilde{\phi}_1}{dx^2} + \frac{1}{\gamma_e\lambda_{De}^2}\left(\frac{w^2}{w_{pe}^2} - 1\right)\tilde{\phi}_1 = 0. \tag{3.32}$$

Case 1 $w_{pe} \le w$

We introduce the variable α so that it satisfies

$$\frac{1}{\alpha^2} = \frac{1}{\gamma_e\lambda_{De}^2}\left(\frac{w^2}{w_{pe}^2} - 1\right). \tag{3.33}$$

Note that α is always positive. Equation (3.32) becomes

$$\frac{d^2\tilde{\phi}_1}{dx^2} + \frac{1}{\alpha^2}\tilde{\phi}_1 = 0, \tag{3.34}$$

The general solution to the above is

$$\tilde{\phi}_1 = c_1\exp(ix/\alpha) + c_2\exp(-ix/\alpha).$$

We introduce a new boundary condition, namely $\tilde{\phi}_1(0) = 0$, which gives $c_2 = -c_1$. Thus, the solution takes the form

$$\tilde{\phi}_1 = c_1\left[\exp(ix/\alpha) - \exp(-ix/\alpha)\right] = c_1 2i\sin(x/\alpha).$$

Applying the other boundary condition for $\tilde{\phi}$, namely, $\tilde{\phi}_1(-L/2) - \tilde{\phi}_1(L/2) = V$, we get

$$c_1 2i\left[-\sin\left(L/2\alpha\right) - \sin\left(L/2\alpha\right)\right] = V,$$

or

$$c_1 = -\frac{V}{4i\sin\left(L/2\alpha\right)}.$$

Thus, the final solution for ϕ_1 is

$$\phi_1 = -\frac{V}{2}\frac{\sin(x/\alpha)}{\sin(L/2\alpha)}\exp(-iwt). \tag{3.35}$$

Case 2 $w_{pe} \ge w$

We introduce the variable β so that it satisfies

$$\frac{1}{\beta^2} = \frac{1}{\gamma_e\lambda_{De}^2}\left(1 - \frac{w^2}{w_{pe}^2}\right). \tag{3.36}$$

Note that β is always positive. Equation (3.32) becomes

$$\frac{d^2\tilde{\phi}_1}{dx^2} - \frac{1}{\beta^2}\tilde{\phi}_1 = 0, \tag{3.37}$$

The general solution to the above is

$$\tilde{\phi}_1 = c_1\exp(x/\beta) + c_2\exp(-x/\beta).$$

As before, we use the boundary condition $\tilde{\phi}_1(0) = 0$, which gives $c_2 = -c_1$. Thus, the solution takes the form

$$\tilde{\phi}_1 = c_1 \left[\exp(x/\beta) - \exp(-x/\beta) \right] = c_1 2 \sinh(x/\beta).$$

Applying the other boundary condition for $\tilde{\phi}$, namely, $\tilde{\phi}_1(-L/2) - \tilde{\phi}_1(L/2) = V$, we get

$$c_1 2 \left[-\sinh\left(L/2\beta\right) - \sinh\left(L/2\beta\right) \right] = V,$$

or

$$c_1 = -\frac{V}{4 \sin\left(L/2\beta\right)}.$$

Thus, the final solution for ϕ_1 is

$$\phi_1 = -\frac{V}{2} \frac{\sinh(x/\beta)}{\sinh(L/2\beta)} \exp(-iwt). \tag{3.38}$$

Note that the spatial functional form of the above is equivalent to eq. (3.30) if β were to be equal to λ_D.

Add some plots to visualize cases 1 and 2.

It is often important to know the plasma density at which the electron plasma frequency w_{pe} equals the external frequency w. This density is referred to as the critical density n_{cr}. Equating the electron plasma frequency with the external frequency we get

$$\frac{n_{cr}e^2}{m_e\epsilon_0} = w^2,$$

which leads to

$$n_{cr} = \frac{m_e\epsilon_0 w^2}{e^2} \tag{3.39}$$

The above can be re-written as

$$n_{cr} = \frac{m_e\epsilon_0(2\pi\nu)^2}{e^2} = \frac{4\pi^2 m_e\epsilon_0 c^2}{e^2}\frac{1}{\lambda^2} = 1.115 \times 10^{27}\frac{1}{\lambda_{\mu m}^2},$$

where $\lambda_{\mu m}$ is in units of microns and n_{cr} in units of $\#/\text{m}^3$.

3.2.3 The coupling parameter

Coulomb interactions are those which occur when two charge particles head towards each other. We can define two types of Coulomb interactions: strong and weak. Strong Coulomb interactions are those for which the particle's Coulomb potential energy is larger than its kinetic energy, and vice versa for weak Coulomb interactions. Thus, we can also define two types of plasma regimes:

- Strongly-coupled plasmas: plasmas where the Coulomb interactions are mostly strong and thus drive the dynamics of its evolution. Coulomb interactions tend to be strong when the inter-particle distances are small, and thus this regime would be dominated by *short-range* interactions. These plasmas are also described as exhibiting *collisional* behavior, since a strong Coulomb interaction essentially means a collision has occurred.

- Weakly-coupled plasmas: plasmas where the Coulomb interactions are mostly weak, and as a result do not drive the dynamics of its evolution. The plasma dynamics are instead driven by *long-range* effects caused by smooth electromagnetic fields that result from integrating a large number of particles. These plasmas are also described as exhibiting *collective* behavior, since the long-range electromagnetic fields follow from the collective integration of many particles.

We describe an approximate Coulomb potential energy for particles in a plasma as

$$U = \frac{q_s^2}{4\pi\epsilon_0 a_s}. \tag{3.40}$$

The impact parameter that has been used above is a_s, the sphere radius. This provides a decent measure on the average spacing between particles in a plasma. Since the volume of a single particle is $1/n_s$, and if we assume that this volume is given by $4/3\pi a_s^3$, then equating these two gives the expression for the sphere radius

$$a_s = \left(\frac{3}{4\pi n_s}\right)^{1/3}. \tag{3.41}$$

The thermal velocity of a particle is given by

$$v_{T_s} = \sqrt{\frac{k_B T_s}{m_s}} \tag{3.42}$$

A measure of the kinetic energy of a particle is given in terms of the thermal velocity as shown below

$$K = m_s v_{T_s}^2 = k_B T_s \tag{3.43}$$

The ratio of the particle's Coulomb potential energy and its kinetic energy is referred to as the coupling parameter Γ_s. That is

$$\Gamma_s = \frac{q_s^2}{4\pi\epsilon_0 a_s k_B T_s}. \tag{3.44}$$

$\Gamma_s > 1$ denotes a strongly coupled plasma, and $\Gamma_s < 1$ denotes a weakly coupled plasma.

3.2.4 The plasma parameter

The standard plasma parameter Λ_s is defined as

$$\Lambda_s = \frac{4}{3}\pi\lambda_{Ds}^3 n_s. \tag{3.45}$$

There is a one to one relationship between the coupling parameter and the standard plasma parameter. Simple algebra shows that

$$\Gamma_s = (1/3)\Lambda_s^{-2/3}. \tag{3.46}$$

Thus, the coupling and plasma parameters are inversely proportional to each other. $\Lambda_s < 1$ implies strongly-coupled plasmas, and $\Lambda_s > 1$ weakly-coupled plasmas. Since Λ_s represents the number of particles per Debye sphere, it is interesting to see that a large number of particles within such a sphere is needed to be in the weakly-coupled-plasma regime. However, this does not correspond to a plasma with large density, in fact, it corresponds to the opposite. The explicit n_s term in the definition $\Lambda_s = (4/3)\pi\lambda_{Ds}^3 n_s$ is dominated by the n_s in the denominator of λ_{Ds}. In other words, low plasma densities lead to large Debye spheres, which in turn leads to many particles per Debye sphere, and hence a weakly-coupled plasma.

3.2.5   Electron degeneracy

- DeBroglie wavelength

$$\lambda_{Bs} = \frac{h}{\sqrt{2\pi}m_s v_{Ts}} \tag{3.47}$$

- Quantum plasma parameter

$$\chi_s = \frac{4}{3}\pi \lambda_{Bs}^3 n_s \tag{3.48}$$

- Fermi energy:

$$E_{fs} = \frac{\hbar^2}{2m_s} \left(3\pi^2 n_s\right)^{2/3} \tag{3.49}$$

- The Fermi energy can be used to define the Fermi temperature T_{fs}, Fermi velocity v_{fs}, Fermi momentum p_{fs}, and Fermi wave vector k_{fs}

$$E_{fs} = k_B T_{fs} = \frac{1}{2}m_s v_{fs}^2 = \frac{p_{fs}^2}{2m_s} = \frac{(\hbar k_{fs})^2}{2m_s} \tag{3.50}$$

- Degeneracy parameter:

$$\Theta_s = \frac{k_B T_s}{E_{fs}} = \left(\frac{2^{10}\pi}{3^4}\right)^{1/3} \chi_s^{-2/3}. \tag{3.51}$$

Chapter 4

Collisions

4.1 Cross section

4.1.1 General definition

Two particles traveling towards each other can undergo an interaction. Types of interactions include Coulomb collisions between two charged particles, fusion reactions between ions, and photon-matter phenomena such as Compton scattering, the photoelectric effect, and pair production.
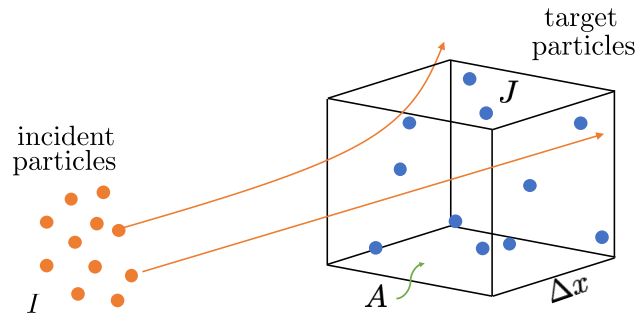

Figure 4.1: Interaction of incident and target particles.

To define the cross section, we'll consider I incident particles heading towards J stationary target particles (see fig. 4.1). Not all of the incident particles will interact with the target particles, some will instead continue to travel in a uniform trajectory. The number of incident particles that do interact with the target particles is labeled as N. The cross section σ is then a constant of proportionality defined by the following equation

$$N = \sigma I n_A. \tag{4.1}$$

In the above, n_A is the areal number density, that is, $n_A = J/A = n\Delta x$, where n is the volume number density.

4.1.2 Differential cross section

Consider the scattering of two particles: an incident and a target particle. If we fix the reference frame to follow the target particle, then the scattering can be depicted as shown in fig. 4.2. The
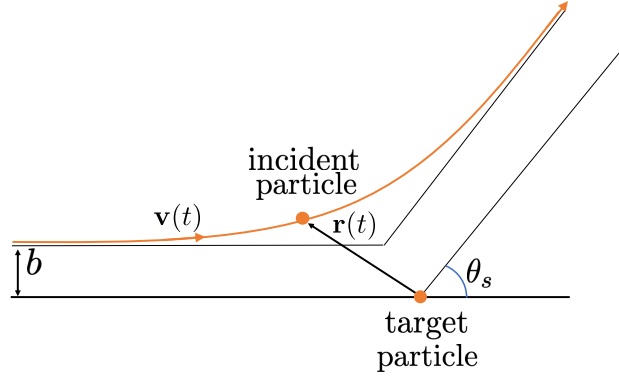
Figure 4.2: Depiction of particle scattering.

displacement parameter is labelled as b, and the scattering angle as θ_s. For three dimensional scattering, the encounter is as shown in fig. 4.3. Not that in that figure the incident particle starts within the $x - z$ plane, and after scattering the particle is confined to a plane that is tilted an angle ϕ_s from the $x - z$ plane.
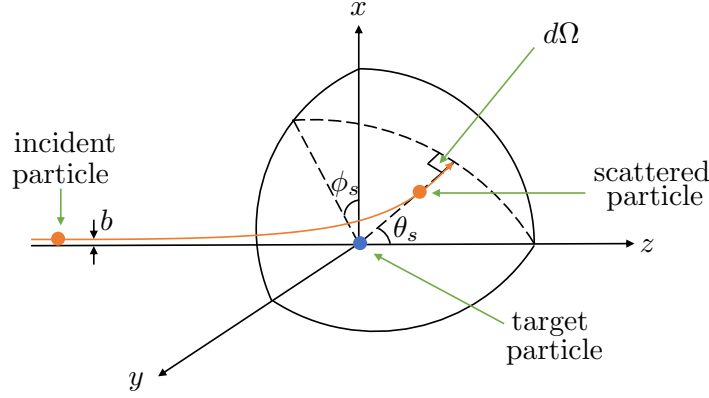

Figure 4.3: Depiction of particle scattering in 3D.

From the entire set of incident particles N that interact with the target particles, one can define an infinitesimal subset $N_{\theta,\phi}d\Omega$ as the number of particles scattered into an infinitesimal solid angle $d\Omega = \sin\theta_s d\theta_s d\phi_s$, as shown in fig. 4.3. We note that $N_{\theta,\phi} = N_{\theta,\phi}(\theta_s, \phi_s)$. We introduce the differential cross section

$$\frac{d\sigma_{\theta,\phi}}{d\Omega} = \frac{d\sigma_{\theta,\phi}}{d\Omega}(\theta_s, \phi_s), \tag{4.2}$$

which is defined by the following expression in an analogous manner to eq. (4.1),

$$N_{\theta,\phi}d\Omega = \left(\frac{d\sigma_{\theta,\phi}}{d\Omega}d\Omega\right) I n_A. \tag{4.3}$$

It is best to not think of it as a derivative (what does a derivative with respect to solid angle mean?) and instead to simply think of it as a function that depends on θ_s and ϕ_s. Integrating over all θ_s and ϕ_s, i.e.

$$\int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} N_{\theta,\phi}d\Omega = \int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} \frac{d\sigma_{\theta,\phi}}{d\Omega}d\Omega I n_A$$
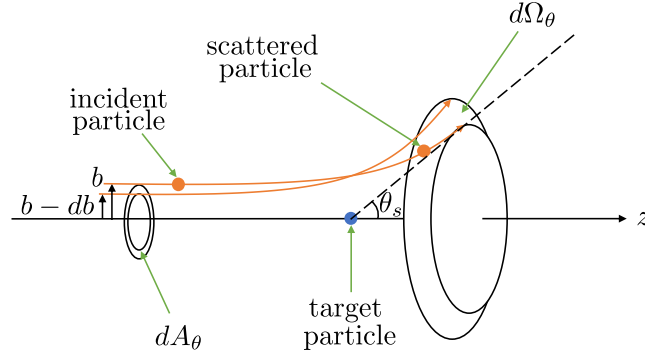
gives eq. (4.1).

Figure 4.4: Depiction of particle scattering in for axi-symmetric interactions.

For various cases the scattering is axi-symmetric, that is, it is independent of ϕ_s. Thus

$$N_{\theta,\phi} \rightarrow N_\theta \qquad \frac{d\sigma_{\theta,\phi}}{d\Omega} \rightarrow \frac{d\sigma_\theta}{d\Omega},$$

where

$$N_\theta = N_\theta(\theta_s),$$

and

$$\frac{d\sigma_\theta}{d\Omega} = \frac{d\sigma_\theta}{d\Omega}(\theta_s).$$

Integrating eq. (4.3) from $\phi_s = 0$ to $\phi_s = 2\pi$ gives

$$N_\theta d\Omega_\theta = \frac{d\sigma_\theta}{d\Omega} d\Omega_\theta I n_A, \tag{4.4}$$

where $d\Omega_\theta = 2\pi \sin\theta_s d\theta_s$. $N_\theta d\Omega_\theta$ thus represents the number of particles that are scattered into the infinitesimal band $d\Omega_\theta$ on a sphere, where $d\Omega_\theta$ is defined by scattering angle θ_s (see fig. 4.4). We will note that there is a one-to-one correspondence between the impact parameter b and the scattering angle θ_s, that is, $b = b(\theta_s)$. In other words, any incident particle scattered out through the infinitesimal band $d\Omega_\theta$ would have approached the target-particle through the infinitesimal ring dA_θ that corresponds to $d\Omega_\theta$. Since there are many target particles, there are many dA_θ's that correspond to the same $d\Omega_\theta$. Thus, the total number of particles scattered through $d\Omega_\theta$ is given by all the incident particles that cross through the dA_θ's of all the target particles.

The number of incident particles crossing all the infinitesimal rings dA_θ is equal to the total number of incident particles I times the probability P that any single incident particle will cross one of those rings. Thus, we can write

$$N_\theta d\Omega_\theta = IP.$$

The probability that an incident particle will cross one of those rings is simply the ratio of the surface area covered by all the rings in a section of the target material to the total area of that section. The surface area covered by all the rings in a section of area S is given by $(n_A S)dA_\theta$. Thus, $P = n_A dA_\theta$ and

$$N_\theta d\Omega_\theta = I n_A dA_\theta.$$

We now introduce the differential

$$db = \frac{db}{d\theta_s} d\theta_s. \tag{4.5}$$

We note that by definition $d\theta_s$ is positive but db can be either positive or negative depending on the sign of $db/d\theta_s$. The infinitesimal area dA_θ is then given by

$$dA_\theta = 2\pi b |db| = 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s. \tag{4.6}$$

Thus,

$$N_\theta d\Omega_\theta = n_A I 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s.$$

Using eq. (4.4) in the above, we get

$$\frac{d\sigma_\theta}{d\Omega} d\Omega_\theta I n_A = I n_A 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s,$$

or

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin\theta_s} \left| \frac{db}{d\theta_s} \right|. \tag{4.7}$$

4.1.3 Mean free path & collision frequency

The mean free path can be expressed in terms of the cross section as

$$\lambda_m = \frac{1}{n_1 \sigma}. \tag{4.8}$$

Given the particle's speed v, on can also define the collision time as follows

$$\tau_m = \frac{\lambda_m}{v} = \frac{1}{n_1 \sigma v}. \tag{4.9}$$

Finally, the collision frequency is simply the inverse of the collision time, that is

$$\nu_m = \frac{1}{\tau_m} = n_1 \sigma v. \tag{4.10}$$

4.2 Coulomb collisions

4.2.1 Particle equations

Consider two particles, with positions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, velocities $\mathbf{v}_1 = \mathbf{v}_1(t)$ and $\mathbf{v}_2 = \mathbf{v}_2(t)$, charges q_1 and q_2, and masses m_1 and m_2, respectively. Their positions and velocities are governed by the following equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1, \tag{4.11}$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2, \tag{4.12}$$

$$m_1 \frac{d\mathbf{v}_1}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \tag{4.13}$$

$$m_2 \frac{d\mathbf{v}_2}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{4.14}$$

We note that the above system consists of twelve equations for twelve unknowns. We now introduce the center-of-mass position $\mathbf{R} = \mathbf{R}(t)$, the center-of-mass velocity $\mathbf{V} = \mathbf{V}(t)$, the shifted position $\mathbf{r} = \mathbf{r}(t)$ and the shifted velocity $\mathbf{v} = \mathbf{v}(t)$ as follows

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$$

Thus, in terms of these new four variables, the particle equations can be written as

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}, \tag{4.15}$$

$$\frac{d\mathbf{V}}{dt} = 0, \tag{4.16}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \tag{4.17}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{\mathbf{r}}{r^3}, \tag{4.18}$$

where the reduced mass m_r is given by

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4.19}$$

The first two equations above give the trivial solution $\mathbf{V} = $ constant and $\mathbf{R} = \mathbf{R}(0) + \mathbf{V}t$. Thus, we have reduced the problem from twelve unknowns to six unknowns, namely $\mathbf{r}$ and $\mathbf{v}$.

4.2.2 Conservation of energy and momentum

Dotting eq. (4.18) by $\mathbf{v}$ gives

$$\begin{aligned}
\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\mathbf{v} \cdot \frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{2}\frac{dr^2}{dt}\frac{1}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}\frac{dr}{dt} \\
&= -\frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{d}{dt}\left(\frac{1}{r}\right).
\end{aligned}$$

For the left hand side above we have

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{1}{2}\frac{dv^2}{dt},$$

and thus we obtain the following expression for conservation of energy

$$\frac{d}{dt}\left(\frac{1}{2}m_r v^2 + \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r}\right) = 0.$$

Crossing eq. (4.18) by $\mathbf{r}$ gives

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0,$$

and thus

$$\frac{d}{dt}\left[m_r \left(\mathbf{r} \times \mathbf{v}\right)\right] = 0.$$

That is, angular momentum is conserved. A consequence of this is that the vector $\mathbf{r} \times \mathbf{v}$ is always pointing in the same direction. Thus, if $\mathbf{r}(0)$ and $\mathbf{v}(0)$ form a plane, then $\mathbf{r}(t)$ and $\mathbf{v}(t)$ need to reside within that same plane for all times t so that $\mathbf{r}(t) \times \mathbf{v}(t)$ points in the same direction as $\mathbf{r}(0) \times \mathbf{v}(0)$. Therefore, the evolution of the position and velocity are confined to a plane and the problem can be reduced from six unknowns to four unknowns. This planar encounter is depicted in fig. 4.5.
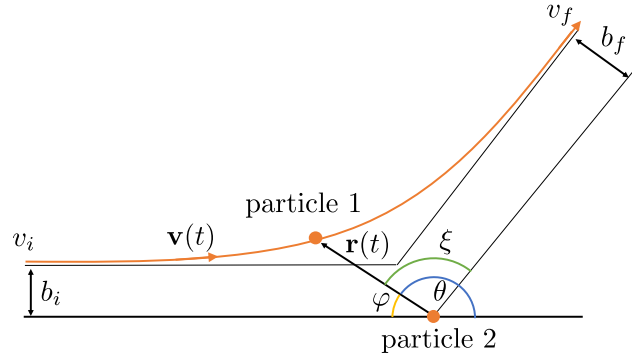

Figure 4.5: Depiction of Coulomb scattering.

If we refer to the plane shown in fig. 4.5 as the $x - y$ plane, then one can tell that the angular-momentum vector points in the negative z direction. We will denote the magnitude of the conserved angular momentum by L, and thus we can write

$$m_r \left(\mathbf{r} \times \mathbf{v}\right) = -L\hat{\mathbf{z}}. \tag{4.20}$$

A consequence of both conservation of energy and momentum is as follows. Consider the two limiting states of particle 1—the initial state v_i, b_i and the final state v_f, b_f. Assuming the potential energy is very low at sufficiently early and late times, conservation of energy gives

$$\frac{1}{2}m_r v_i^2 = \frac{1}{2}m_r v_f^2, \tag{4.21}$$

that is, $v_i = v_f$ (note that for other scattering processes, e.g. Compton scattering, this is not necessarily the case). For the angular momentum of the initial state we have

$$m_r \left(\mathbf{r}_i \times \mathbf{v}_i\right) = m_r \sin(-\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\pi - \varphi_i) r_i v_i \hat{\mathbf{z}}$$
$$= -m_r \sin(\varphi_i) r_i v_i \hat{\mathbf{z}} = -m_r b_i v_i \hat{\mathbf{z}} \quad (4.22)$$

Similarly, for the angular momentum of the final state we have

$$m_r \left(\mathbf{r}_f \times \mathbf{v}_f\right) = m_r \sin(-\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r \sin(\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r b_f v_f \hat{\mathbf{z}}. \tag{4.23}$$

Equating the last two relationships gives $m_r b_i v_i = m_r b_f v_f$. Since $v_i = v_f$, we finally have $b_i = b_f = b$. Using eq. (4.20) in eq. (4.22), we can also write

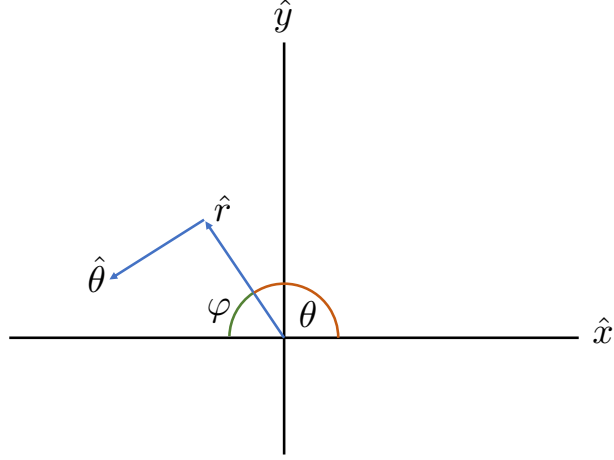$$L = m_r b v_i. \tag{4.24}$$

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4.2.3 Polar coordinates

Figure 4.6: Polar coordinates in plane of interaction.

Using polar coordinates, as shown in fig. 4.6, we get

$$r_x = r\cos\theta = r\cos(\pi - \varphi) = -r\cos\varphi,$$

$$r_y = r\sin\theta = r\sin(\pi - \varphi) = r\sin\varphi.$$

Also, since $\mathbf{r} = r\hat{\mathbf{r}}$, we have

$$
\begin{aligned}
\mathbf{v} = \frac{d\mathbf{r}}{dt} &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\
&= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} \\
&= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}.
\end{aligned}
$$

The radial component of eq. (4.18) thus becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}.$$

Since $\theta = \pi - \varphi$, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}. \tag{4.25}$$

For the angular momentum we have

$$m_r \mathbf{r} \times \mathbf{v} = m_r r \hat{\mathbf{r}} \times \left(\frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\boldsymbol{\theta}} \right) = m_r r^2 \frac{d\theta}{dt} \hat{\mathbf{z}}$$

Using eq. (4.20), we can write the above as

$$m_r r^2 \frac{d\varphi}{dt} = L. \tag{4.26}$$

4.2.4 Particle trajectory

The goal is to find the radial position of the particle as a function of its angular orientation. That is, we want to find $\tilde{r} = \tilde{r}(\tilde{\varphi})$ such that

$$r(t) = \tilde{r}(\varphi(t)). \tag{4.27}$$

To simplify the math, we introduce $\tilde{u} = \tilde{u}(\tilde{\varphi})$ such that $\tilde{u} = 1/\tilde{r}$. Thus

$$\frac{d\tilde{u}}{d\tilde{\varphi}} = -\frac{1}{\tilde{r}^2} \frac{d\tilde{r}}{d\tilde{\varphi}},$$

or, after re-arranging

$$\frac{d\tilde{r}}{d\tilde{\varphi}} = -\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}}. \tag{4.28}$$

We now proceed as follows. Taking the derivative of r, we get

$$\frac{dr}{dt} = \left(\frac{d\tilde{r}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \qquad [eq.\ (4.27)]$$

$$= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \qquad [eq.\ (4.28)]$$

$$= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} \qquad [eq.\ (4.26)]$$

$$= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} \qquad [eq.\ (4.27)]$$

$$= \left(-\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} \tag{4.29}$$

Taking the derivative of the above, we get

$$\frac{d}{dt} \frac{dr}{dt} = \left[\frac{d}{d\tilde{\varphi}} \left(-\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right) \right]_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt}$$

$$= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} \qquad [eq.\ (4.26)]$$

$$= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} \qquad [eq.\ (4.27)]$$

$$= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2 \tilde{u}^2}{m_r^2} \right)_{\tilde{\varphi}=\varphi(t)} \tag{4.30}$$

Plugging the last relation into eq. (4.25) gives

$$\left[-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L^2\tilde{u}^2}{m_r^2} - \frac{1}{\tilde{u}}\left(\frac{L\tilde{u}^2}{m_r}\right)^2 \right]_{\tilde{\varphi}=\varphi(t)} = \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r}\tilde{u}^2\right)_{\tilde{\varphi}=\varphi(t)},$$

which, upon re-arranging and dropping the $\varphi(t)$ dependance, becomes

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2 m_r}{4\pi\epsilon_0 L^2} \tag{4.31}$$

Using eq. (4.24) we write the evolution equation for $\tilde{u}$ as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2}{4\pi\epsilon_0 m_r b^2 v_i^2}. \tag{4.32}$$

Introducing the notation

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r v_i^2}, \tag{4.33}$$

the evolution equation for $\tilde{u}$ can be simply expressed as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{b_{90}}{b^2}. \tag{4.34}$$

The boundary conditions for eq. (4.34) are as follows

$$\text{as } \varphi(t) \to 0, \quad r(t) \to \infty \tag{4.35}$$

$$\text{as } \varphi(t) \to 0, \quad \frac{dr(t)}{dt} \to -v_i \tag{4.36}$$

Given eq. (4.27), eq. (4.35) can only be satisfied if as $\tilde{\varphi} \to 0$, $\tilde{r} \to \infty$. Thus, we also have, as $\tilde{\varphi} \to 0$, $\tilde{u} \to 0$. Similarly, given eq. (4.29), eq. (4.36) can only be satisfied if as $\tilde{\varphi} \to 0$

$$\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r} \to v_i.$$

Using eq. (4.24) we rewrite the above as

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \to \frac{1}{b}.$$

The general solution to eq. (4.34) is

$$\tilde{u} = A\cos\tilde{\varphi} + B\sin\tilde{\varphi} - \frac{b_{90}}{b^2}.$$

Applying the boundary conditions, we get

$$\tilde{u} = \frac{b_{90}}{b^2}\cos\tilde{\varphi} + \frac{1}{b}\sin\tilde{\varphi} - \frac{b_{90}}{b^2},$$

which we finally re-write as

$$\frac{1}{\tilde{r}} = \frac{1}{b}\sin\tilde{\varphi} + \frac{b_{90}}{b^2}\left(\cos\tilde{\varphi} - 1\right). \tag{4.37}$$

4.2.5 The scattering angle

We now drop the tilde notation for the sake of simplicity. That is, for the radial location of an incident particle, we have

$$\frac{1}{r} = \frac{1}{b}\sin\varphi + \frac{b_{90}}{b^2}\left(\cos\varphi - 1\right),\tag{4.38}$$

where φ is the independent variable and $r = r(\varphi)$. We want to know the value of φ as r goes to infinity. Using eq. (4.38), and labeling this angle as φ_s, we have

$$0 = \sin\varphi_s + \frac{b_{90}}{b}\left(\cos\varphi_s - 1\right).$$

We express the above in terms of the scattering angle $\theta_s = \pi - \varphi_s$,

$$0 = \sin(\pi - \theta_s) + \frac{b_{90}}{b}\left[\cos\left(\pi - \theta_s\right) - 1\right].$$

or

$$0 = \sin\theta_s + \frac{b_{90}}{b}\left(-\cos\theta_s - 1\right).$$

Re-writing the above as

$$\frac{\cos\theta_s + 1}{\sin\theta_s} = \frac{b}{b_{90}},$$

and using the trig identity $\cot(\theta/2) = (\cos\theta + 1)/\sin\theta$, we get

$$\cot\left(\frac{\theta_s}{2}\right) = \frac{b}{b_{90}}.\tag{4.39}$$

4.2.6 The differential cross section

The differential cross section for Coulomb scattering can be computed by making use of eq. (4.7), which is repeated below

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|.\tag{4.7}$$

From eq. (4.39) we get,

$$\frac{db}{d\theta} = -\frac{b_{90}}{2}\frac{1}{\sin^2(\theta_s/2)},\tag{4.40}$$

which, plugging in eq. (4.7), gives

$$\frac{d\sigma_\theta}{d\Omega} = \left[b_{90}\frac{\cot(\theta_s/2)}{\sin\theta_s}\right]\left[\frac{b_{90}}{2}\frac{1}{\sin^2(\theta_s/2)}\right].$$

Using the trig identities $\cot(\theta) = \cos(\theta)/\sin(\theta)$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$ we get

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b_{90}^2}{4}\frac{1}{\sin^4(\theta_s/2)}.\tag{4.41}$$

4.2.7 Collision integral

$$\Omega_{\alpha\beta}^{(lk)} = \sqrt{\frac{k_B T}{2\pi M_{\alpha\beta}}} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} \, dg. \tag{4.42}$$

In the above $M_{\alpha\beta}$ is the reduced mass, given by

$$M_{\alpha\beta} = \frac{M_\alpha M_\beta}{M_\alpha + M_\beta}, \tag{4.43}$$

and $\phi_{\alpha\beta}^{(l)}$ is the collision cross section for a given velocity, and is computed as

$$\phi_{\alpha\beta}^{(l)} = 2\pi \int_0^\infty \left(1 - \cos^l \chi_{\alpha\beta}\right) b \, db. \tag{4.44}$$

The scattering angle $\chi_{\alpha\beta}$ is given by

$$\chi_{\alpha\beta} = \pi - 2 \int_{r_{\alpha\beta}^{\min}}^\infty \frac{b}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{\alpha\beta(r)}}{g^2 k_B T}\right]^{1/2}} \, dr. \tag{4.45}$$

For a Coulombic interaction between ions, we can define the natural scale fore the cross-sectional area as

$$\phi_{\alpha\beta}^{(0)} = \frac{\pi \left(Z_\alpha Z_\beta e^2\right)^2}{(2k_B T)^2}. \tag{4.46}$$

Given this definition, we express the collision integral as

$$\Omega_{\alpha\beta} = \sqrt{\frac{\pi}{M_{\alpha\beta}}} \frac{(Z_\alpha Z_\beta e^2)^2}{(2k_B T)^{3/2}} \mathcal{F}_{\alpha\beta}^{lk}, \tag{4.47}$$

where

$$\mathcal{F}_{\alpha\beta}^{(lk)} = \frac{1}{2\phi_0} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} \, dg \tag{4.48}$$

We note that $\mathcal{F}_{\alpha\beta}^{(lk)} = 4\mathcal{K}_{lk}(g_{\alpha\beta})$, where $\mathcal{K}_{lk}(g_{\alpha\beta})$ is the notation from the Stanton-Murillo paper.