Plasma turbulence

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1 Hasegawa-Mima

References for this model can be found in Hasegawa and Mima [1977], Horton and Hasegawa [1994].

1.1 Assumptions

1. Singly-charged ions.
2. No shear stresses, collisions, or sources.
3. Cold ion approximation, i.e. $T_e \gg T_i$ and thus $\nabla p_i \approx 0$, Hasegawa and Mima [1977].
4. Isothermal electron fluid, i.e. T_e is constant.
5. Electrostatic field, i.e. $\mathbf{E} = -\nabla\phi$.
6. Magnetic field is constant.
7. Neglect parallel ion velocity, i.e. $u_{i,||} \approx 0$, Hasegawa and Mima [1977].
8. Quasi-neutrality, i.e. $n_i \approx n_e$.
9. Adiabatic electrons, i.e. $n_e = n_0 \exp(e\phi/T_e)$, where $n_0 = n_0(x_1)$.

1.2 Derivation

Using the assumptions in items 1 and 2, the momentum equation for singly-charged ions

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i \left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right) = -\nabla p_i + \nabla \cdot \mathbf{t}_i + \ \mathbf{R}_i + \hat{\mathbf{M}}_i \tag{1}$$

simplifies to

$$m_i n_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i\right) = en_i(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i. \tag{2}$$

Using the assumptions in items 3 and 5, the above becomes

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{e}{m_i}\nabla\phi + \frac{e}{m_i}\mathbf{u}_i \times \mathbf{B}. \tag{3}$$

Introduce the coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and assume $\mathbf{B}$ points in the $\mathbf{e}_3$ direction. Defining the perpendicular velocity as $\mathbf{u}_{i,\perp} = [u_{i,1}, u_{i,2}, 0]^T$ and the perpendicular gradient as $\nabla_\perp = [\partial_1, \partial_2, 0]^T$, we have

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_{i,\perp} = -\frac{e}{m_i} \nabla_\perp \phi + \frac{e}{m_i} \mathbf{u}_i \times \mathbf{B}. \tag{4}$$

Using the assumption in item 7 and noting that $\mathbf{u}_i \times \mathbf{B} = \mathbf{u}_{i,\perp} \times \mathbf{B}$, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i} \nabla_\perp \phi + \frac{e}{m_i} \mathbf{u}_{i,\perp} \times \mathbf{B}. \tag{5}$$

We now introduce the scalings for a characteristic frequency w and length scale r

$$\frac{w}{w_{c,i}} \sim \epsilon \qquad \frac{r_s}{r} \sim \epsilon, \tag{6}$$

where $w_{c,i} = eB/m_i$ is the cyclotron frequency, $r_s = v_s/w_{c,i}$ is a reference length scale, and $v_s = \sqrt{T_e/m_i}$ a reference velocity scale. Given these variables, we assume

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} \sim \mathbf{u}_{i,\perp} w \qquad \nabla_\perp \mathbf{u}_{i,\perp} \sim \frac{\mathbf{u}_{i,\perp}}{r} \qquad \mathbf{E} \sim \mathbf{u}_{i,\perp} B. \tag{7}$$

Finally, we introduce the decomposition $\mathbf{u}_{i,\perp} = \mathbf{u}_{i,\perp}^{(0)} + \mathbf{u}_{i,\perp}^{(1)}$, where $\mathbf{u}_{i,\perp}^{(0)} \sim v_s$ and $\mathbf{u}_{i,\perp}^{(1)} \sim \epsilon v_s$. We use this decomposition in eq. (5) and then divide the PDE by $w_{c,i} v_s$. The order of each element in the resulting equation is as follows

1. $\frac{\partial \mathbf{u}_{i,\perp}^{(0)}}{\partial t} \sim \epsilon$.
2. $\frac{\partial \mathbf{u}_{i,\perp}^{(1)}}{\partial t} \sim \epsilon^2$.
3. $\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon$.
4. $\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^2$.
5. $\mathbf{u}_{i,\perp}^{(1)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon^2$.
6. $\mathbf{u}_{i,\perp}^{(1)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^3$.
7. $-\frac{e}{m_i} \nabla_\perp \phi \sim 1$.
8. $\frac{e}{m_i} \mathbf{u}_{i,\perp}^{(0)} \times \mathbf{B} \sim 1$.
9. $\frac{e}{m_i} \mathbf{u}_{i,\perp}^{(1)} \times \mathbf{B} \sim \epsilon$.

Combining the first order terms we obtain

$$0 = -\nabla_\perp \phi + \mathbf{u}_{i,\perp}^{(0)} \times \mathbf{B}, \tag{8}$$

which, upon crossing by $\mathbf{B}$, gives

$$\mathbf{u}_{i,\perp}^{(0)} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B}. \tag{9}$$

Combining the terms of order ϵ we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}^{(0)}}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} = \frac{e}{m_i} \mathbf{u}_{i,\perp}^{(1)} \times \mathbf{B}, \tag{10}$$

which, upon crossing by $\mathbf{B}$, gives

$$\mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i} B} \left[\frac{\partial \nabla_\perp \phi}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\nabla_\perp \phi \right) \right]. \tag{11}$$

The above is the polarization drift. The velocity given by eq. (9) is referred to as the $E \times B$ drift, and the velocity given by eq. (11) as the polarization drift.

Using the assumption in item 2, the continuity equation for ions is

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0. \tag{12}$$

Using the assumption in item 7 the above becomes

$$\frac{\partial n_i}{\partial t} + \nabla_\perp \cdot (n_i \mathbf{u}_{i,\perp}) = 0, \tag{13}$$

or

$$\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp n_i + n_i \nabla_\perp \cdot \mathbf{u}_{i,\perp} = 0, \tag{14}$$

One of the main components of the derivation of the Hasegawa-Mima equation is the assumption that advection is governed by the lowest-order velocity only; that is, by $\mathbf{u}_{i,\perp}^{(0)}$ and not $\mathbf{u}_{i,\perp}^{(1)}$. Thus, the above is written as

$$\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp n_i + n_i \nabla_\perp \cdot \mathbf{u}_{i,\perp} = 0. \tag{15}$$

We note that $\mathbf{u}_{i,\perp}^{(0)}$ is divergence free, and thus we have

$$\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp n_i + n_i \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{16}$$

We divide by n_i to express the density in terms of its logarithm

$$\frac{\partial \ln n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln n_i + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{17}$$

We now use the assumptions in items 8 and 9 to obtain

$$\ln n_i = \ln \left[n_0 \exp \left(\frac{e\phi}{T_e} \right) \right] = \ln n_0 + \frac{e\phi}{T_e}. \tag{18}$$

Taking into account the fact that n_0 is time independent, the continuity equation becomes

$$\frac{\partial}{\partial t} \left(\frac{e\phi}{T_e} \right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left[\ln n_0 + \frac{e\phi}{T_e} \right] + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{19}$$

Since $\mathbf{u}_{i,\perp}^{(0)}$ and $\nabla_\perp \phi$ are orthogonal, the above simplifies to

$$\frac{\partial}{\partial t} \left(\frac{e\phi}{T_e} \right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln n_0 + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0, \tag{20}$$

which we re-write as

$$\frac{\partial}{\partial t}\left(\frac{e\phi}{T_e}\right)+\mathbf{u}_{i,\perp}^{(0)}\cdot\nabla_\perp \ln\left(\frac{n_0}{w_{c,i}}\right)+\nabla_\perp\cdot\mathbf{u}_{i,\perp}^{(1)}=0. \tag{21}$$

Given the definition of the polarization drift, we have

$$\nabla_\perp\cdot\mathbf{u}_{i,\perp}^{(1)}=-\frac{1}{w_{c,i}B}\left\{\frac{\partial\nabla_\perp^2\phi}{\partial t}+\nabla_\perp\cdot\left[\mathbf{u}_{i,\perp}^{(0)}\cdot\nabla_\perp\left(\nabla_\perp\phi\right)\right]\right\}. \tag{22}$$

The second term above is best computed using tensor notation, and we'll use u_j to denote the components of $\mathbf{u}_{i,\perp}^{(0)}$. Thus,

$$\frac{\partial}{\partial x_i}\left[\left(u_j\frac{\partial}{\partial x_j}\right)\frac{\partial\phi}{\partial x_i}\right]=\frac{\partial u_j}{\partial x_i}\frac{\partial^2\phi}{\partial x_j\partial x_i}+u_j\frac{\partial}{\partial x_j}\left(\frac{\partial^2\phi}{\partial x_i\partial x_i}\right). \tag{23}$$

Using the definition of $\mathbf{u}_{i,\perp}^{(0)}$, the first term on the right-hand side above can be expressed as

$$\begin{aligned}\frac{\partial u_j}{\partial x_i}\frac{\partial^2\phi}{\partial x_j\partial x_i}&=-\frac{1}{B^2}\epsilon_{jpq}\frac{\partial^2\phi}{\partial x_p\partial x_i}B_q\frac{\partial^2\phi}{\partial x_j\partial x_i}\\&=-\frac{1}{B^2}\epsilon_{qjp}\frac{\partial^2\phi}{\partial x_j\partial x_i}\frac{\partial^2\phi}{\partial x_p\partial x_i}B_q\\&=-\frac{1}{B^2}\epsilon_{qjp}\left(\frac{\partial^2\phi}{\partial x_j\partial x_1}\frac{\partial^2\phi}{\partial x_p\partial x_1}+\frac{\partial^2\phi}{\partial x_j\partial x_2}\frac{\partial^2\phi}{\partial x_p\partial x_2}\right)B_q.\end{aligned} \tag{24}$$

Since $\epsilon_{qjp}\partial_j a\partial_p a\to\nabla a\times\nabla a=0$ for any scalar a, the term above is identically zero. Thus, we have

$$\nabla_\perp\cdot\mathbf{u}_{i,\perp}^{(1)}=-\frac{1}{w_{c,i}B}\left[\frac{\partial\nabla_\perp^2\phi}{\partial t}+\mathbf{u}_{i,\perp}^{(0)}\cdot\nabla_\perp\left(\nabla_\perp^2\phi\right)\right], \tag{25}$$

and eq. (21) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\frac{e\phi}{T_e}\right)+\mathbf{u}_{i,\perp}^{(0)}\cdot\nabla_\perp\left[\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\ln\left(\frac{n_0}{w_{c,i}}\right)\right]=0. \tag{26}$$

Plugging in for $\mathbf{u}_{i,\perp}^{(0)}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\frac{e\phi}{T_e}\right)-\left(\nabla_\perp\phi\times\frac{\mathbf{b}}{B}\right)\cdot\nabla_\perp\left[\frac{1}{w_{c,i}B}\nabla_\perp^2\phi-\ln\left(\frac{n_0}{w_{c,i}}\right)\right]=0. \tag{27}$$

We now introduce the normalizations

$$\phi(t,\mathbf{x})=\frac{T_e}{e}\hat{\phi}(\hat{t},\hat{\mathbf{x}})\qquad n_0(x_1)=\hat{n}_0(\hat{x}_1), \tag{28}$$

where $\hat{t}=tw_{c,i}$ and $\hat{\mathbf{x}}=\mathbf{x}/r_s$. Neglecting the hat notation for the sake of simplicity, eq. (27) finally becomes

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2\phi-\phi\right)-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp\left[\nabla_\perp^2\phi-\ln\left(\frac{n_0}{w_{c,i}}\right)\right]=0. \tag{29}$$

Using the following expansion

$$(\nabla_\perp \phi \times \mathbf{b}) \cdot \nabla_\perp = \frac{\partial \phi}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial \phi}{\partial x_1}\frac{\partial}{\partial x_2}, \tag{30}$$

The Hasegawa-Mima equation can be written as

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2 \phi - \phi\right) - \frac{\partial \phi}{\partial x_2}\frac{\partial \nabla_\perp^2 \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_1}\frac{\partial \nabla_\perp^2 \phi}{\partial x_2} + \beta \frac{\partial \phi}{\partial x_2} = 0, \tag{31}$$

where

$$\beta = \frac{\partial}{\partial x_1} \ln\left(\frac{n_0}{w_{c,i}}\right). \tag{32}$$

1.3 Spectral space

In this section we derive the equation for the Fourier coefficient $\hat{\phi}_{\mathbf{n}} = \hat{\phi}(t)_{\mathbf{n}}$, which relates to the potential through the following

$$\phi(t,x) = \sum_{\mathbf{n}=-\infty}^{\infty} \hat{\phi}_{\mathbf{n}}(t) e^{i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}, \tag{33}$$

$$\hat{\phi}_{\mathbf{n}}(t) = \frac{1}{L^2}\int_{L^2} \phi(t,\mathbf{x}) e^{-i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}\, d\mathbf{x}. \tag{34}$$

We introduce the operator $\mathcal{F}\{\}_{\mathbf{n}}$, which is defined by

$$\mathcal{F}\{\phi(t,\mathbf{x})\}_{\mathbf{n}} = \frac{1}{L^2}\int_{L^2} \phi(t,\mathbf{x}) e^{-i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}\, d\mathbf{x}. \tag{35}$$

The equation for $\hat{\phi}_{\mathbf{n}}$ is obtained by applying this operator to eq. (29). Thus, the time derivative term in that equation becomes

$$\mathcal{F}\left\{\frac{\partial}{\partial t}\left(\nabla_\perp^2 - \phi\right)\right\}_{\mathbf{n}} = \frac{\partial}{\partial t}\mathcal{F}\left\{\nabla_\perp^2 \phi - \phi\right\}_{\mathbf{n}} = \frac{\partial}{\partial t}\left(-k_{\mathbf{n}}^2 \hat{\phi}_{\mathbf{n}} - \hat{\phi}_{\mathbf{n}}\right) = -\left(1 + k_{\mathbf{n}}^2\right)\frac{\partial \hat{\phi}_{\mathbf{n}}}{\partial t}. \tag{36}$$

We assume $\nabla \ln(n_o/w_{ci})$ is constant in space. Thus, the term containing the inhomogeneity becomes

$$\begin{aligned}
&\mathcal{F}\left\{(\nabla_\perp \phi \times \mathbf{b}) \cdot \nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right)\right\}_{\mathbf{n}} \\
&\quad = \left(\mathcal{F}\left\{\nabla_\perp \phi\right\}_{\mathbf{n}} \times \mathbf{b}\right) \cdot \nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right) = i\left(\mathbf{k}_{\mathbf{n}} \times \mathbf{b}\right) \cdot \nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right)\hat{\phi}_{\mathbf{n}}.
\end{aligned} \tag{37}$$

The remaining term is computed as follows

$$\begin{aligned}
&\mathcal{F}\left\{-\left(\nabla_\perp \phi \times \mathbf{b}\right) \cdot \nabla_\perp^3 \phi\right\}_{\mathbf{n}} \\
&\quad = \mathcal{F}\left\{-\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left[\hat{\phi}_{\mathbf{n}'} i\left(\mathbf{k}_{\mathbf{n}'} \times \mathbf{b}\right) e^{i\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{x}}\right] \cdot \left[\hat{\phi}_{\mathbf{n}''}\left(-ik_{\mathbf{n}''}^2 \mathbf{k}_{\mathbf{n}''}\right) e^{i\mathbf{k}_{\mathbf{n}''}\cdot\mathbf{x}}\right]\right\}_{\mathbf{n}} \\
&\quad = -\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'} \times \mathbf{b}\right) \cdot \mathbf{k}_{\mathbf{n}''} k_{\mathbf{n}''}^2 \hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\mathcal{F}\left\{e^{i\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{x}} e^{i\mathbf{k}_{\mathbf{n}''}\cdot\mathbf{x}}\right\} \\
&\quad = \sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'} \times \mathbf{k}_{\mathbf{n}''}\right) \cdot \mathbf{b} k_{\mathbf{n}''}^2 \hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''}
\end{aligned} \tag{38}$$

Since $\mathbf{n}'$ and $\mathbf{n}''$ are just symbolic variables for the summation, we can write the above as follows

$$\begin{aligned}
&\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}} \\
&\qquad=\frac{1}{2}\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''} \\
&\qquad\quad+\frac{1}{2}\sum_{\mathbf{n}''=-\infty}^{\infty}\sum_{\mathbf{n}'=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}''}\times\mathbf{k}_{\mathbf{n}'}\right)\cdot\mathbf{b}k_{\mathbf{n}'}^2\hat{\phi}_{\mathbf{n}''}\hat{\phi}_{\mathbf{n}'}\delta_{\mathbf{n},\mathbf{n}''+\mathbf{n}'} \\
&\qquad=\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\frac{1}{2}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''}
\end{aligned}\tag{39}$$

Thus, we finally have

$$\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}}=\sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\tag{40}$$

Combining the results above, we obtain

$$\frac{\partial\hat{\phi}_{\mathbf{n}}}{\partial t}+iw_{\mathbf{n}}\hat{\phi}_{\mathbf{n}}=\sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''}\Lambda_{\mathbf{n}',\mathbf{n}''}^{\mathbf{n}}\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''},\tag{41}$$

where

$$w_{\mathbf{n}}=-\frac{\left(\mathbf{k}_{\mathbf{n}}\times\mathbf{b}\right)}{1+k_{\mathbf{n}}^2}\cdot\nabla_\perp\ln\left(\frac{n_o}{w_{ci}}\right),\tag{42}$$

and

$$\Lambda_{\mathbf{n}',\mathbf{n}''}^{\mathbf{n}}=\frac{1}{2}\frac{\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)}{1+k_{\mathbf{n}}^2}.\tag{43}$$

Note that $w_{\mathbf{n}}$ can also be written as

$$w_{\mathbf{n}}=-\frac{k_{2,\mathbf{n}}\mathbf{e}_1-k_{1,\mathbf{n}}\mathbf{e}_2}{1+k_{\mathbf{n}}^2}\cdot\beta\mathbf{e}_1=-\frac{k_{2,\mathbf{n}}\beta}{1+k_{\mathbf{n}}^2}.\tag{44}$$

2 Hasegawa-Wakatani

References for this model can be found in Wakatani and Hasegawa [1984], Hasegawa and Wakatani [1987].

2.1 Assumptions

1. Singly-charged ions.
2. No shear stresses in the electron momentum equation, no collisions in the ion momentum equation, no sources.
3. Cold ion approximation, i.e. $T_e \gg T_i$ and thus $\nabla p_i \approx 0$.
4. Isothermal electron fluid, i.e. T_e is constant.

5. Electrostatic field, i.e. $\mathbf{E} = -\nabla\phi$.

6. Magnetic field is constant.

7. Neglect parallel ion velocity, i.e. $u_{i,||} \approx 0$.

8. Quasi-neutrality, i.e. $n_i \approx n_e$.

9. $n_i = n_0 + n'$, where $n_0 = n_0(x_1)$ and n' is smaller than n_0.

10. Perpendicular components of the ion shear-stress term are modeled as $(\nabla \cdot \mathbf{t}_i)_\perp/(m_i n_i) = \nu\nabla_\perp^2 \mathbf{u}_{i,\perp}$.

11. Assume infinitesimally small electron mass, i.e. $m_e \to 0$.

2.2 Derivation

Using the assumptions in items 1 and 2, the momentum equation for singly-charged ions

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i + \nabla \cdot \mathbf{t}_i + \ \mathbf{R}_i + \hat{\mathbf{M}}_i \tag{45}$$

simplifies to

$$m_i n_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right) = en_i(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i + \nabla \cdot \mathbf{t}. \tag{46}$$

Using the assumptions in items 3 and 5, the above becomes

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{e}{m_i}\nabla\phi + \frac{e}{m_i}\mathbf{u}_i \times \mathbf{B} + \frac{\nabla \cdot \mathbf{t}_i}{m_i n_i}. \tag{47}$$

As before, introduce the coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and assume $\mathbf{B}$ points in the $\mathbf{e}_3$ direction. Defining the perpendicular velocity as $\mathbf{u}_{i,\perp} = [u_{i,1}, u_{i,2}, 0]^T$, the perpendicular gradient as $\nabla_\perp = [\partial_1, \partial_2, 0]^T$, and the perpendicular shear stress as $(\nabla \cdot \mathbf{t}_i)_\perp = [(\nabla \cdot \mathbf{t})_1, (\nabla \cdot \mathbf{t})_2, 0]^T$, we have

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_{i,\perp} = -\frac{e}{m_i}\nabla_\perp\phi + \frac{e}{m_i}\mathbf{u}_i \times \mathbf{B} + \frac{(\nabla \cdot \mathbf{t}_i)_\perp}{m_i n_i}. \tag{48}$$

Using the assumption in item 7 and noting that $\mathbf{u}_i \times \mathbf{B} = \mathbf{u}_{i,\perp} \times \mathbf{B}$, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i}\nabla_\perp\phi + \frac{e}{m_i}\mathbf{u}_{i,\perp} \times \mathbf{B} + \frac{(\nabla \cdot \mathbf{t}_i)_\perp}{m_i n_i}. \tag{49}$$

Finally, using the assumption in item 10, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i}\nabla_\perp\phi + \frac{e}{m_i}\mathbf{u}_{i,\perp} \times \mathbf{B} + \nu\nabla_\perp^2 \mathbf{u}_{i,\perp}. \tag{50}$$

The same scaling analysis performed for the derivation of the Hasegawa Mima equation is now applied. The only new term in eq. (50) is the viscous term. We note that the kinematic viscosity ν scales as

$$\nu \sim r^2 w. \tag{51}$$

Thus, the viscous stress term leads to the following scalings

1. $\nu \nabla_\perp^2 \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon$

2. $\nu \nabla_\perp^2 \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^2$

As before, the first order terms lead to the $E \times B$ drift

$$\mathbf{u}_{i,\perp}^{(0)} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B}. \tag{52}$$

However, the equation for terms of order ϵ now contains the viscous term as shown below

$$\frac{\partial \mathbf{u}_{i,\perp}^{(0)}}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \mathbf{u}_{i,\perp}^{(0)} = \frac{e}{m_i} \mathbf{u}_{i,\perp}^{(1)} \times \mathbf{B} + \nu \nabla_\perp^2 \mathbf{u}_{i,\perp}^{(0)}. \tag{53}$$

Upon crossing by $\mathbf{B}$, the above gives

$$\mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B} \left[\frac{\partial \nabla_\perp \phi}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\nabla_\perp \phi \right) \right] + \frac{\nu}{w_{c,i}B} \nabla_\perp^2 (\nabla_\perp \phi). \tag{54}$$

That is, an additional viscous term appears in the polarization drift.

As shown in the derivation of the Hasegawa-Mima equation, the continuity equation for ions can be expressed in the form of eq. (17), which is repeated below for convenience

$$\frac{\partial \ln n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \ln n_i + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{55}$$

Given the assumption in item 9, the natural logarithm of density is re-written as follows

$$\ln n_i = \ln \left(n_0 + n' \right) = \ln \left[n_0 \left(1 + \frac{n'}{n_0} \right) \right] = \ln n_0 + \ln \left(1 + \frac{n'}{n_0} \right). \tag{56}$$

We now introduce $n = n'/n_0$, which is small due to the assumption in item 9. Thus, a Taylor series expansion would allow us to write

$$\ln n_i = \ln n_0 + n. \tag{57}$$

Since n_0 is time independent (assumption in item 9), the continuity equation becomes

$$\frac{\partial n}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\ln n_0 + n \right) + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{58}$$

Using the same derivation for eq. (25), we now have

$$\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B} \left[\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\nabla_\perp^2 \phi \right) \right] + \frac{\nu}{w_{c,i}B} \nabla_\perp^4 \phi. \tag{59}$$

Plugging this in eq. (58), we obtain

$$\frac{\partial}{\partial t} \left(\frac{1}{w_{c,i}B} \nabla_\perp^2 \phi - n \right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left(\frac{1}{w_{c,i}B} \nabla_\perp^2 \phi - n - \ln n_0 \right) - \frac{\nu}{w_{c,i}B} \nabla_\perp^4 \phi = 0. \tag{60}$$

Plugging in for $\mathbf{u}_{i,\perp}^{(0)}$,

$$\frac{\partial}{\partial t} \left(\frac{1}{w_{c,i}B} \nabla_\perp^2 \phi - n \right) - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B} \right) \cdot \nabla_\perp \left(\frac{1}{w_{c,i}B} \nabla_\perp^2 \phi - n - \ln n_0 \right) - \frac{\nu}{w_{c,i}B} \nabla_\perp^4 \phi = 0. \tag{61}$$

Using the assumptions in items 1 and 2, the momentum equation for electrons

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \ \mathbf{R}_e + \hat{\mathbf{M}}_e \tag{62}$$

simplifies to

$$m_e n_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e \right) = -e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e + \mathbf{R}_e. \tag{63}$$

Using the assumptions in items 4, 5 and 11, the above simplifies to

$$0 = -e n_e (-\nabla \phi + \mathbf{u}_e \times \mathbf{B}) - T_e \nabla n_e + \mathbf{R}_e. \tag{64}$$

Dividing by $-en_e$, we get

$$0 = -\nabla \phi + \mathbf{u}_e \times \mathbf{B} + \frac{T_e}{e} \nabla \ln n_e - \frac{1}{e n_e} \mathbf{R}_e. \tag{65}$$

We now focus on the component of the equation above that is parallel to $\mathbf{B}$, that is

$$0 = -\nabla_{||} \phi + \frac{T_e}{e} \nabla_{||} \ln n_e - \frac{1}{e n_e} R_{e,||}. \tag{66}$$

Using quasineutrality to replace n_e by n_i, and plugging in eq. (57) for n_i gives

$$0 = -\nabla_{||} \phi + \frac{T_e}{e} \nabla_{||} (\ln n_0 + n) - \frac{1}{e n_e} R_{e,||}. \tag{67}$$

We note that the gradient of $\ln n_0$ above is zero since n_0 does not vary along the direction of the magnetic field. The definition of the electron collision term is $\mathbf{R}_e = (m_e \nu_{ei}/e)\mathbf{J}$. Using the definition of the resistivity $\eta = m_e \nu_{ei}/e^2 n_e$, we get $\mathbf{R}_e = e n_e \eta \mathbf{J}$. Thus, we now have

$$0 = -\nabla_{||} \phi + \frac{T_e}{e} \nabla_{||} n - \eta J_{||}, \tag{68}$$

which, upon re-arranging, gives

$$J_{||} = \frac{T_e}{e\eta} \nabla_{||} \left(n - \frac{e\phi}{T_e} \right). \tag{69}$$

The perpendicular component of eq. (65) is as follows

$$0 = -\nabla_{\perp} \phi + \mathbf{u}_e \times \mathbf{B} + \frac{T_e}{e} \nabla_{\perp} \ln n_e - \frac{1}{e n_e} \mathbf{R}_{e,\perp}. \tag{70}$$

Since $\mathbf{u}_e \times \mathbf{B} = \mathbf{u}_{e,\perp} \times \mathbf{B}$ we have

$$0 = -\nabla_{\perp} \phi + \mathbf{u}_{e,\perp} \times \mathbf{B} + \frac{T_e}{e} \nabla_{\perp} \ln n_e - \frac{1}{e n_e} \mathbf{R}_{e,\perp}. \tag{71}$$

Again, using the definition of the electron collision term, we get

$$0 = -\nabla_{\perp} \phi + \mathbf{u}_{e,\perp} \times \mathbf{B} + \frac{T_e}{e} \nabla_{\perp} \ln n_e - \eta \mathbf{J}_{\perp}. \tag{72}$$

Crossing the above by $\mathbf{B}$ gives

$$\mathbf{u}_{e,\perp} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B} - \frac{T_e}{en_e B}\mathbf{b} \times \nabla_\perp n_e + \frac{\eta}{B}\mathbf{b} \times \mathbf{J}_\perp. \tag{73}$$

Typically the last term on the right-hand side above is significantly smaller, and thus it can be neglected. The electron velocity is thus

$$\mathbf{u}_{e,\perp} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B} - \frac{T_e}{en_e B}\mathbf{b} \times \nabla_\perp n_e. \tag{74}$$

The continuity equation for electrons is as follows

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0. \tag{75}$$

We split the convection term in the above into the perpendicular and parallel components

$$\frac{\partial n_e}{\partial t} + \nabla_\perp \cdot (n_e \mathbf{u}_{e,\perp}) + \nabla_{||} \cdot (n_e u_{e,||}) = 0. \tag{76}$$

Given the assumption in item 7, we have $J_{||} = en_e(u_{i,||} - u_{e,||}) = -en_e u_{e,||}$. Thus, the above becomes

$$\frac{\partial n_e}{\partial t} + \nabla_\perp \cdot (n_e \mathbf{u}_{e,\perp}) = \frac{1}{e}\nabla_{||} J_{||}. \tag{77}$$

Using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$ we show that

$$\nabla_\perp \cdot (\mathbf{b} \times \nabla_\perp n_e) = \nabla_\perp n_e \cdot (\nabla_\perp \times \mathbf{b}) - \mathbf{b} \cdot (\nabla_\perp \times \nabla_\perp n_e) = 0. \tag{78}$$

As a result, the second term on the right-hand side of eq. (74) does not contribute to $\nabla_\perp \cdot (n_e \mathbf{u}_{e,\perp})$. The electron continuity equation then becomes

$$\frac{\partial n_e}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B}\right) \cdot \nabla_\perp n_e = \frac{1}{e}\nabla_{||} J_{||}. \tag{79}$$

Dividing by n_e and using quasi-neutrality

$$\frac{\partial \ln n_i}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B}\right) \cdot \nabla_\perp \ln n_i = \frac{1}{en_i}\nabla_{||} J_{||}. \tag{80}$$

Using the expression for $\ln n_i$ in eq. (57), we get

$$\frac{\partial n}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B}\right) \cdot \nabla_\perp (\ln n_0 + n) = \frac{1}{en_i}\nabla_{||} J_{||}. \tag{81}$$

Finally, using the assumption in item 9, we neglect n' in the denominator of the right-hand side, and obtain

$$\frac{\partial n}{\partial t} - \left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B}\right) \cdot \nabla_\perp \left(n + \ln n_0\right) = \frac{1}{en_0}\nabla_{||} J_{||}. \tag{82}$$

Combining eqs. (61), (69) and (82) leads to the dimensional form of the Hasegawa-Wakatani model

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2\phi\right)-\left(\nabla_\perp\phi\times\frac{\mathbf{b}}{B}\right)\cdot\nabla_\perp\left(\frac{1}{w_{c,i}B}\nabla_\perp^2\phi\right)$$
$$=\frac{T_e}{e^2n_0\eta}\nabla_{||}^2\left(n-\frac{e\phi}{T_e}\right)+\frac{\nu}{w_{c,i}B}\nabla_\perp^4\phi. \quad (83)$$

$$\frac{\partial n}{\partial t}-\left(\nabla_\perp\phi\times\frac{\mathbf{b}}{B}\right)\cdot\nabla_\perp\left(n+\ln n_0\right)=\frac{T_e}{e^2n_0\eta}\nabla_{||}^2\left(n-\frac{e\phi}{T_e}\right). \quad (84)$$

It is quite common to replace the parallel-gradient operator $\nabla_{||}$ by a coefficient, say $1/l^2$.

We now introduce the following non-dimensionalization

$$\phi(t,x_1,x_2,x_3)=\frac{T_e}{e}\hat{\phi}(\hat{t},\hat{x}_1,\hat{x}_2,x_3) \quad (85)$$
$$n_0(x_1)=\hat{n}_0(\hat{x}_1) \quad (86)$$
$$n(t,x_1,x_2,x_3)=\hat{n}(\hat{t},\hat{x}_1,\hat{x}_2,x_3), \quad (87)$$

where $\hat{t}=tw_{c,i}$, $\hat{x}_1=x_1/r_s$, and $\hat{x}_2=x_2/r_s$. Neglecting the hat notation for the sake of simplicity, the Hasegawa-Wakatani model in non-dimensional form is written as

$$\frac{\partial\nabla_\perp^2\phi}{\partial t}-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp\left(\nabla_\perp^2\phi\right)=c_1(\phi-n)+c_2\nabla_\perp^4\phi, \quad (88)$$
$$\frac{\partial n}{\partial t}-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp\left(n+\ln n_0\right)=c_1(\phi-n), \quad (89)$$

where

$$c_1=-\frac{T_e}{e^2n_0\eta w_{c,i}}\nabla_{||}^2 \qquad c_2=\frac{\nu}{w_{c,i}r_s^2}. \quad (90)$$

If $\nabla_{||}$ is replaced by $1/l^2$, then the c_1 operator is simply a coefficient.

2.3 Relationship to other models

The Hasegawa-Wakatani eqs. (88) and (89) contain two limits. Assuming c_1 is a coefficient rather than an operator, one of the limits is obtained by letting $c_1=0$. Then, the ϕ and n equations are decoupled, and the ϕ equation corresponds to the third (and only non-zero) component of the 2D Navier-Stokes equations for vorticity

$$\frac{\partial\mathbf{w}}{\partial t}+\mathbf{u}\cdot\nabla_\perp\mathbf{w}=\nu\nabla_\perp^2\mathbf{w}, \quad (91)$$

where

$$\mathbf{u}=\nabla_\perp\times(-\phi\mathbf{b})=-\nabla_\perp\phi\times\mathbf{b}, \quad (92)$$

and

$$\mathbf{w}=\nabla_\perp\times\mathbf{u}=\nabla_\perp^2\phi\mathbf{b}. \quad (93)$$

If, on the other hand, $c_1\to\infty$, then dividing eq. (89) by c_1 shows that $n=\phi$. Subtracting eq. (89) from eq. (88) and assuming $c_2=0$ one obtains

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2\phi-\phi\right)-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp\left(\nabla_\perp^2\phi-\ln n_0\right)=0. \quad (94)$$

The above is the Hasegawa-Mima equation.

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