High-energy laser-plasma interactions

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Contents

Chapter 1

Some notes on lasers

Consider a wave that depends on time t and single spatial dimension x, which is orthogonal to the direction of propagation.

A wave is spatially coherent at a given time t, position x, and separation distance L if the phase difference between the points x and $x + L$ at time t is the same as that at a later time $t + dt$. You can keep on picking larger and larger values of L until this is not the case, this L would be the spatial coherence length $L_c = L_c(t, x)$.

A wave is temporally coherent at a given time t, position x, and separation time τ if the phase difference between times t and $t+\tau$ at position x is the same as that between times $t+dt$ and $t + dt + \tau$. You can keep on picking larger and larger values of τ until this is no longer the case, this τ would be the temporal coherence length $\tau_c = \tau_c(t, x)$.

A depiction of temporal coherence is provided in fig. 1.1, which shows a light pulse as a function of time. An arbitrary time t_* is chosen along the light wave, and the wave's phase at that time is θ_1. At the time $t_* + \tau$ the wave has a different phase θ_2, and thus the phase difference between times t_* and $t_* + \tau$ is $\theta_2 - \theta_1$. This is depicted by the orange dots. The green dots are used to highlight the phase difference between times $t_* + dt$ and $t_* + dt + \tau$. In this case, the phases are respectively $\theta_1 + d\theta$ and $\theta_2 + d\theta$, and hence the phase difference is still maintained at $\theta_2 - \theta_1$.

Rather than picking the time t_* to be at an arbitrary location along the wave, in fig. 1.2a we choose t_* to be at the beginning of the wave, and also choose τ to be just a bit smaller than the entire duration of the pulse. For this case, the phase difference between t_* and $t_* + \tau$ is $d\theta$, which is equal to the phase difference between $t_* + dt$ and $t_* + dt + \tau$. On the other hand, in fig. 1.2b we have chosen τ to be just a bit larger so as to be equal to the duration of the wave. For this case the phase difference between t_* and $t_* + \tau$ is zero, which is not the same as the phase difference between $t_* + dt$ and $t_* + dt + \tau$, namely $d\theta$. Thus, the maximum value of τ that maintains a constant phase difference has been reached, and therefore by definition this value is referred to as the temporal coherence length τ_c at time t_* (other locations of t_*, such as that in fig. 1.1, have a different τ_c).

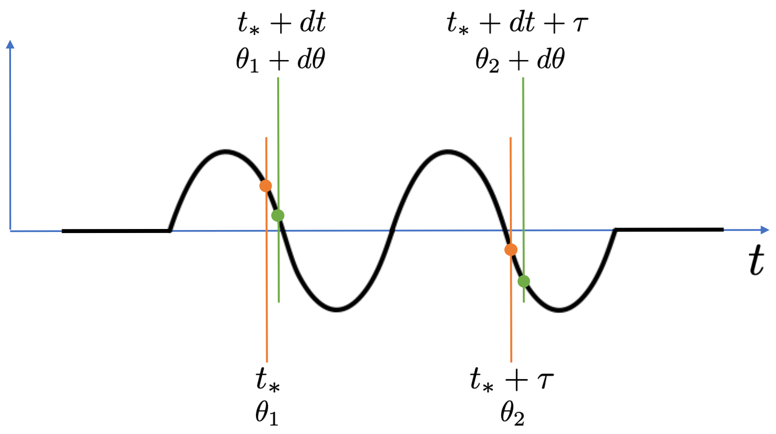
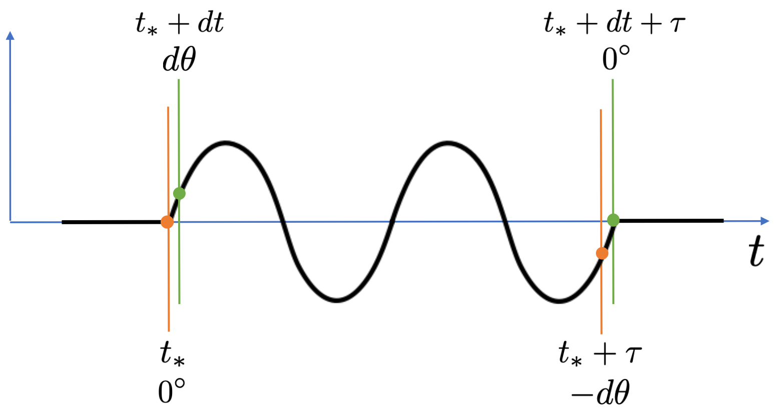
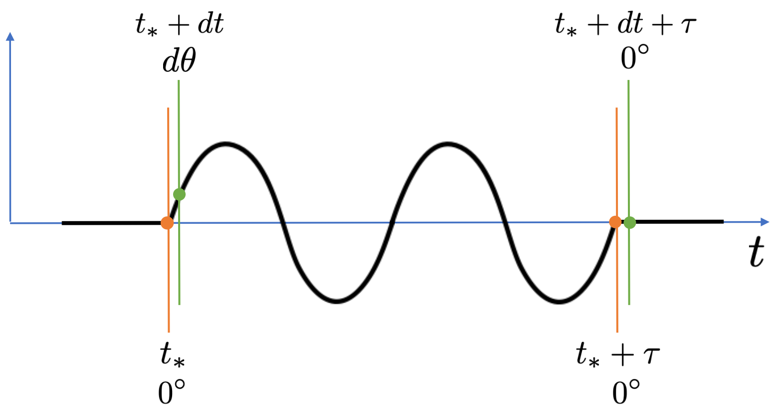


Figure 1.1



(a)



(b)

Figure 1.2: Depiction of temporal coherence for a light pulse.

Chapter 2

The baseline plasma model

We rely on the single-material two-fluid model under homentropic assumptions, which is summarized below

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{2.1}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{2.2}$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Zen_i \left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right) = -\nabla p_i, \tag{2.3}$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + en_e \left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) = -\nabla p_e, \tag{2.4}$$

$$p_i = C_i n_i^{\gamma_i}, \tag{2.5}$$

$$p_e = C_e n_e^{\gamma_e}, \tag{2.6}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{2.7}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.9}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.10}$$

$$\mathbf{J} = e(Zn_i \mathbf{u}_i - n_e \mathbf{u}_e), \tag{2.11}$$

$$\rho_q = e(Zn_i - n_e), \tag{2.12}$$

$$p_i = n_i k_B T_i, \tag{2.13}$$

$$p_e = n_e k_B T_e. \tag{2.14}$$

Chapter 3

Longitudinal and transverse waves

3.1 Definitions

The Helmholtz decomposition for a function $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$ is of the following form

$$\mathbf{F} = \mathbf{F}_l + \mathbf{F}_t, \tag{3.1}$$

where $\mathbf{F}_l = \mathbf{F}_l(\mathbf{x}, t)$ is the longitudinal component and $\mathbf{F}_t = \mathbf{F}_t(\mathbf{x}, t)$ the transverse component. These are defined by

$$\nabla \times \mathbf{F}_l = 0, \tag{3.2}$$

$$\nabla \cdot \mathbf{F}_t = 0. \tag{3.3}$$

We'll assume that any vector function $\mathbf{G} = \mathbf{G}(\mathbf{x}, t)$ can be expressed as the real component of

$$\mathbf{G} = \hat{\mathbf{G}} \exp\left[i\left(\int_0^x k_x(x')\,dx' + \int_0^y k_y(y')\,dy' + \int_0^z k_z(z')\,dz' - wt\right)\right]. \tag{3.4}$$

For the above, w a frequency constant in time and space, and $k_x = k_x(x)$, $k_y = k_y(y)$, and $k_z = k_z(z)$ form the wave vector $\mathbf{k} = [k_x, k_y, k_z]$. $\hat{\mathbf{G}} = \hat{\mathbf{G}}(\mathbf{x}, t)$ is a complex vector where the real and complex components point in the same direction. Additionally, we enforce the constraint that if $\nabla \times \mathbf{G} = 0$, then $\nabla \times \hat{\mathbf{G}} = 0$, and similarly, if $\nabla \cdot \mathbf{G} = 0$, then $\nabla \cdot \hat{\mathbf{G}} = 0$. We'll often use a different subscript in the wave vectors and frequencies of different waves. For example, we'll use $\mathbf{k}_e$ for electron-plasma waves, $\mathbf{k}_i$ for ion-acoustic waves, $\mathbf{k}_L$ for laser waves, and $\mathbf{k}_s$ for scattered waves. Similarly for the frequencies w_e, w_i, w_L, w_s.

We note that

$$\begin{aligned}\nabla \exp&\left[i\left(\int_0^x k_x(x')\,dx' + \int_0^y k_y(y')\,dy' + \int_0^z k_z(z')\,dz' - wt\right)\right] \\ &= i\mathbf{k} \exp\left[i\left(\int_0^x k_x(x')\,dx' + \int_0^y k_y(y')\,dy' + \int_0^z k_z(z')\,dz' - wt\right)\right]\end{aligned} \tag{3.5}$$

Given the identity $\nabla \times (\mathbf{A}f) = (\nabla \times \mathbf{A})f - \mathbf{A} \times (\nabla f)$, we can show that

$$\begin{aligned}\nabla \times \mathbf{G} &= \nabla \times \left[\hat{\mathbf{G}} \exp(...)\right] \\ &= \left(\nabla \times \hat{\mathbf{G}}\right) \exp(...) - \hat{\mathbf{G}} \times [\nabla \exp(...)] \\ &= \left(\nabla \times \hat{\mathbf{G}}\right) \exp(...) - \hat{\mathbf{G}} \times [i\mathbf{k} \exp(...)] \\ &= \left(\nabla \times \hat{\mathbf{G}}\right) \exp(...) + i\mathbf{k} \times \mathbf{G}.\end{aligned} \tag{3.6}$$

Given the identity $\nabla \cdot (\mathbf{A}f) = (\nabla \cdot \mathbf{A})f + \mathbf{A} \cdot (\nabla f)$, we can show that

$$\begin{aligned}
\nabla \cdot \mathbf{G} &= \nabla \cdot \left[\hat{\mathbf{G}} \exp(...)\right] \\
&= \left(\nabla \cdot \hat{\mathbf{G}}\right) \exp(...) + \hat{\mathbf{G}} \cdot [\nabla \exp(...)] \\
&= \left(\nabla \cdot \hat{\mathbf{G}}\right) \exp(...) + \hat{\mathbf{G}} \cdot [i\mathbf{k} \exp(...)] \\
&= \left(\nabla \cdot \hat{\mathbf{G}}\right) \exp(...) + i\mathbf{k} \cdot \mathbf{G}.
\end{aligned} \tag{3.7}$$

By definition, $\mathbf{F}_l$ has no curl and $\mathbf{F}_t$ has no divergence. As mentioned earlier, we then require $\hat{\mathbf{F}}_l$ to have no curl and $\hat{\mathbf{F}}_t$ to have no divergence. Using this in eqs. (3.6) and (3.7) allow us to write

$$\mathbf{k} \times \mathbf{F}_l = 0 \tag{3.8}$$

$$\mathbf{k} \cdot \mathbf{F}_t = 0. \tag{3.9}$$

The first expression above says $\mathbf{F}_l$ is parallel to $\mathbf{k}$ and the second says $\mathbf{F}_t$ is orthogonal to $\mathbf{k}$. Thus, $\mathbf{F}_l \cdot \mathbf{F}_t = 0$. We will often have situations where $\nabla \times \mathbf{F} = \nabla \cdot \mathbf{F} = 0$, which by its own does not imply $\mathbf{F} = 0$. However, using eqs. (3.6) and (3.7), this translates to to $\mathbf{k} \times \mathbf{F} = \mathbf{k} \cdot \mathbf{F} = 0$. The latter equality states that $\mathbf{k}$ and $\mathbf{F}$ are orthogonal, that is, the angle between them is $90°$. The former equality leads to $|\mathbf{F}| \sin(90°) = 0$, which in turn means $\mathbf{F} = 0$. To summarize,

$$\nabla \times \mathbf{F} = \nabla \cdot \mathbf{F} = 0 \rightarrow \mathbf{F} = 0. \tag{3.10}$$

For some cases we'll further restrict $\hat{\mathbf{G}}$ in eq. (3.4) such that $\hat{\mathbf{G}} = \hat{\mathbf{G}}(\mathbf{x})$, that is, the time dependence of the wave is fully captured by the $\exp(-iwt)$ term. For this case, we'll often re-write the expression for $\mathbf{G}$ as

$$\mathbf{G} = \tilde{\mathbf{G}} \exp(-iwt), \tag{3.11}$$

where $\tilde{\mathbf{G}} = \tilde{\mathbf{G}}(\mathbf{x})$ is given by

$$\tilde{\mathbf{G}} = \hat{\mathbf{G}} \exp\left[i\left(\int_0^x k_x(x')\,dx' + \int_0^y k_y(y')\,dy' + \int_0^z k_z(z')\,dz'\right)\right]. \tag{3.12}$$

Finally, a further simplification occurs when $\mathbf{k}$ and $\hat{\mathbf{G}}$ are assumed to be constant in space. For this case, the expression for $\mathbf{G}$ becomes

$$\mathbf{G} = \hat{\mathbf{G}} \exp\left[i\left(\mathbf{k} \cdot \mathbf{x} - wt\right)\right]. \tag{3.13}$$

These are the so-called plane waves. We end with the cautionary note that the second gradients of $\mathbf{G}$ in eq. (3.4) are not necessarily the same as those of $\mathbf{G}$ in eq. (3.13).

3.2 Electron-plasma and ion-acoustic waves

For both electron-plasma and ion-acoustic waves we can assume the magnetic field does not change. Thus, Faraday's law gives

$$\nabla \times \mathbf{E} = \nabla \times \mathbf{E}_t = 0. \tag{3.14}$$

By definition, $\nabla \cdot \mathbf{E}_t = 0$. Thus, using eq. (3.10) we get $\mathbf{E}_t = 0$, that is, $\mathbf{E} = \mathbf{E}_l$.

For electron-plasma waves, we have (see notes on electron-plasma waves)

$$\frac{\partial n_{e0}\mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e}\mathbf{E}_1 = -\frac{\gamma_e k_B T_e}{m_e}\nabla n_{e1}. \tag{3.15}$$

We can use eq. (3.13) to write the above in spectral form and thus obtain

$$-iwn_{e0}\hat{\mathbf{u}}_{e1} + \frac{en_{e0}}{m_e}\hat{\mathbf{E}}_{1,l} = -\mathbf{k}_e\frac{\gamma_e p_{e0}}{n_{e0}m_e}\hat{n}_{e1}. \tag{3.16}$$

Since the second term on the left-hand side and the term on the right-hand side point along $\mathbf{k}_e$, $\hat{\mathbf{u}}_{e1}$ also points along $\mathbf{k}_e$, that is, $\hat{\mathbf{u}}_{e1} = \hat{\mathbf{u}}_{e1,l}$.

For ion-acoustic waves, we have (see notes on ion-acoustic waves)

$$\frac{\partial n_{i0}\mathbf{u}_{i1}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_1 = -\frac{\gamma_i k_B T_i}{m_i}\nabla n_{i1}. \tag{3.17}$$

We can use eq. (3.13) to write the above in spectral form and thus obtain

$$-iwn_{i0}\hat{\mathbf{u}}_{i1} - \frac{Zen_{i0}}{m_i}\hat{\mathbf{E}}_{1,l} = -\mathbf{k}_i\frac{\gamma_i p_{i0}}{n_{i0}m_i}\hat{n}_{i1}. \tag{3.18}$$

Since the second term on the left-hand side and the term on the right-hand side point along $\mathbf{k}_i$, $\hat{\mathbf{u}}_{i1}$ also points along $\mathbf{k}_i$, that is, $\hat{\mathbf{u}}_{i1} = \hat{\mathbf{u}}_{i1,l}$. Finally, we note that the electric field being purely longitudinal is in agreement with the linearized electron momentum equation, namely (see notes on ion acoustic waves)

$$en_{e0}\mathbf{E}_1 = -\gamma_e k_B T_e \nabla n_{e1}. \tag{3.19}$$

Chapter 4

Electromagnetic waves in plasmas

In introductory electrodynamics, one typically studies electromagnetic waves in vacuum, that is, for cases where $\rho_q = \mathbf{J} = 0$. In this section we relax both of these assumptions. Consider the electric and magnetic fields as well as the scalar and vector potentials, which satisfy

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.1}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4.2}$$

For the above, we choose $\nabla \cdot \mathbf{A} = 0$. Using the fact that the magnetic field is solenoidal, we have

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{B}_l + \nabla \cdot \mathbf{B}_t = \nabla \cdot \mathbf{B}_l = 0.$$

However, by definition, $\nabla \times \mathbf{B}_l = 0$ as well. Thus, using eq. (3.10), we have $\mathbf{B}_l = 0$. The same argument applies to the vector potential, and thus $\mathbf{A}_l = 0$. For the electric field, we have

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}_l + \nabla \cdot \mathbf{E}_t = \nabla \cdot \mathbf{E}_l.$$

Taking the divergence of eq. (4.1), we get

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla\phi)\,. \tag{4.3}$$

Combining the last two equations gives

$$\nabla \cdot (\mathbf{E}_l + \nabla\phi) = 0.$$

By definition, we also have

$$\nabla \times (\mathbf{E}_l + \nabla\phi) = 0.$$

Thus, using eq. (3.10), we have $\mathbf{E}_l = -\nabla\phi$. A similar argument can be used to show $\mathbf{E}_t = -\partial\mathbf{A}/\partial t$. Our goal in this section will be to determine equations for $\mathbf{E}_l$, $\mathbf{E}_t$ and $\mathbf{B}$.

We'll begin with the conservation of charge equation

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

which we re-write as

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J}_l = 0,$$

Using Poisson's equation $\nabla^2\phi = -\rho_q/\epsilon_0$ in the above, we get

$$\frac{\partial}{\partial t}\left(-\epsilon_0\nabla^2\phi\right) + \nabla \cdot \mathbf{J}_l = 0,$$

or

$$\nabla \cdot \left(\frac{\partial \nabla \phi}{\partial t} - \frac{1}{\epsilon_0} \mathbf{J}_l \right) = 0.$$

However, by definition, we also have

$$\nabla \times \left(\frac{\partial \nabla \phi}{\partial t} - \frac{1}{\epsilon_0} \mathbf{J}_l \right) = 0.$$

Using eq. (3.10), we conclude

$$\frac{\partial \nabla \phi}{\partial t} = \frac{1}{\epsilon_0} \mathbf{J}_l. \tag{4.4}$$

This gives the equation for $\mathbf{E}_l$, namely,

$$\frac{\partial^2 \mathbf{E}_l}{\partial t^2} + \frac{1}{\epsilon_0} \frac{\partial \mathbf{J}_l}{\partial t} = 0. \tag{4.5}$$

Both $\mathbf{E}_t$ and $\mathbf{B}$ can be extracted from $\mathbf{A}$, so now we proceed to find an equation for the transverse vector potential. Ampere's law with Maxwell's correction gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

The above is re-written as

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \left(-\frac{\partial \nabla \phi}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right),$$

which gives

$$\frac{\partial^2 \mathbf{A}}{\partial t} - \frac{1}{\mu_0 \epsilon_0} \nabla^2 \mathbf{A} = \frac{1}{\epsilon_0} \mathbf{J} - \frac{\partial \nabla \phi}{\partial t},$$

or

$$\frac{\partial^2 \mathbf{A}}{\partial t} - c_0^2 \nabla^2 \mathbf{A} = \frac{1}{\epsilon_0} \mathbf{J} - \frac{\partial \nabla \phi}{\partial t},$$

where $c_0 = 1/\sqrt{\mu_0 \epsilon_0}$. Expanding the current density as $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$, and using eq. (4.4), we get

$$\frac{\partial^2 \mathbf{A}}{\partial t} - c_0^2 \nabla^2 \mathbf{A} = \frac{1}{\epsilon_0} \mathbf{J}_t. \tag{4.6}$$

Using the functional form in eq. (3.13) for $\mathbf{A}$ and $\mathbf{J}_t$ gives

$$-w^2 \hat{\mathbf{A}} + k^2 c_0^2 \hat{\mathbf{A}} = \frac{1}{\epsilon_0} \hat{\mathbf{J}}_t. \tag{4.7}$$

That is, $\mathbf{A}$ and $\mathbf{J}_t$ point in the same direction.

Taking the time derivative of eq. (4.6) gives the equation for $\mathbf{E}_t$, that is

$$\frac{\partial^2 \mathbf{E}_t}{\partial t^2} - c_0^2 \nabla^2 \mathbf{E}_t + \frac{1}{\epsilon_0} \frac{\partial \mathbf{J}_t}{\partial t} = 0. \tag{4.8}$$

Taking the curl of eq. (4.6) gives the equation for $\mathbf{B}$, that is

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - c_0^2 \nabla^2 \mathbf{B} - \frac{1}{\epsilon_0} \nabla \times \mathbf{J}_t = 0. \tag{4.9}$$

Using the functional form in eq. (3.13) for $\mathbf{E}_t$, $\mathbf{B}$ and $\mathbf{J}_t$ gives

$$-w^2\hat{\mathbf{E}}_t + k^2 c_0^2 \hat{\mathbf{E}}_t - \frac{iw}{\epsilon_0}\hat{\mathbf{J}}_t = 0, \tag{4.10}$$

$$-w^2\mathbf{B} + k^2 c_0^2 \mathbf{B} - \frac{i}{\epsilon_0}\mathbf{k} \times \mathbf{J}_t = 0. \tag{4.11}$$

That is, $\mathbf{E}_t$ points in the same direction as $\mathbf{J}_t$, which as shown before points in the same direction as $\mathbf{A}$. Additionally, $\mathbf{B}$ points in the direction of $\mathbf{k} \times \mathbf{J}_t$, that is, it is orthogonal to $\mathbf{E}_t$.

We briefly note that taking the curl of eq. (2.9) and using eq. (2.10) gives the wave equation for the total electric field $\mathbf{E}$, that is

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c_0^2 \nabla^2 \mathbf{E} + c_0^2 \nabla(\nabla \cdot \mathbf{E}) + \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t} = 0. \tag{4.12}$$

The above can be considered as the sum of the following three equations

$$\begin{aligned}
\frac{\partial^2 \mathbf{E}_l}{\partial t^2} + \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}_l}{\partial t} &= 0,\\
\frac{\partial^2 \mathbf{E}_t}{\partial t^2} - c_0^2 \nabla^2 \mathbf{E}_t + \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}_t}{\partial t} &= 0,\\
-c_0^2 \nabla^2 \mathbf{E}_l + c_0^2 \nabla(\nabla \cdot \mathbf{E}_l) &= 0.
\end{aligned}$$

The first is the equation for the longitudinal electric field, that is eq. (4.5). The second is the equation for the transverse electric field, that is eq. (4.8). The third equation above follows from the vector identity $\nabla \times (\nabla \times \mathbf{F}) = -\nabla^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F})$ and the fact that $\nabla \times \mathbf{E}_l = 0$.

It will often be the case that transverse waves will oscillate at such a fast rate that the ions, which have a large inertia, will be unable to react quickly enough. Thus, we can assume $\mathbf{u}_{i,t} = 0$. Given the definition of the current density in eq. (2.11), the transverse current density is expressed as $\mathbf{J}_t = e\left(Zn_i \mathbf{u}_{i,t} - n_e \mathbf{u}_{e,t}\right)$, which now simplifies to

$$\mathbf{J}_t = -en_e \mathbf{u}_{e,t}. \tag{4.13}$$

Thus, the transverse electron velocity $\mathbf{u}_{e,t}$ points in the same direction as $\mathbf{J}_t$, which is the same direction as $\mathbf{E}_t$ and $\mathbf{A}$. The next section focuses on deriving an expression for $\mathbf{u}_{e,t}$.

We begin with eq. (2.4), the electron momentum equation, which, due to the electron continuity equation, can be written as

$$m_e n_e \frac{\partial \mathbf{u}_e}{\partial t} + m_e n_e \mathbf{u}_e \cdot \nabla \mathbf{u}_e + en_e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) = -\nabla p_e,$$

or

$$\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e + \frac{e}{m_e}\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) = -\frac{1}{n_e m_e}\nabla p_e,$$

Using the scalar and vector potentials we have

$$\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e + \frac{e}{m_e}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_e \times (\nabla \times \mathbf{A})\right] = -\frac{1}{n_e m_e}\nabla p_e.$$

Using the vector identity $\nabla\left(F^2/2\right) = \mathbf{F} \times (\nabla \times \mathbf{F}) + \mathbf{F} \cdot \nabla \mathbf{F}$, we write the above as

$$\frac{\partial \mathbf{u}_e}{\partial t} - \mathbf{u}_e \times (\nabla \times \mathbf{u}_e) + \nabla\left(\frac{u_e^2}{2}\right) + \frac{e}{m_e}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_e \times (\nabla \times \mathbf{A})\right] = -\frac{1}{n_e m_e}\nabla p_e, \tag{4.14}$$

which is equivalent to

$$\frac{\partial \mathbf{u}_e}{\partial t} - \mathbf{u}_e \times (\nabla \times \mathbf{u}_{e,t}) + \nabla\left(\frac{u_e^2}{2}\right) + \frac{e}{m_e}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_e \times (\nabla \times \mathbf{A})\right] = -\frac{1}{n_e m_e}\nabla p_e. \quad (4.15)$$

We'll now introduce a more specific coordinate system. We'll be dealing with at most three waves at a time: a laser wave, a scattered wave, and a plasma wave (either electron-plasma or ion-acoustic wave). We'll assume all three of these waves lie on a so-called base plane. That is, $\mathbf{k}_e$ (or $\mathbf{k}_i$), $\mathbf{k}_L$, and $\mathbf{k}_s$ all point along this plane. We now choose the main transverse direction, that is, the direction of $\mathbf{u}_{e,t}$, $\mathbf{J}_t$, $\mathbf{E}_t$, and $\mathbf{A}$ to be the direction orthogonal to this plane, so that these vectors are orthogonal to any $\mathbf{k}$. As an aside, we note that the longitudinal and transverse components of the electron velocity can belong to different waves. That is

$$\mathbf{u}_{e,l} = \hat{\mathbf{u}}_{e,l} \exp\left[i(\mathbf{k}_p \cdot \mathbf{x} - w_p t)\right] \quad (4.16)$$
$$\mathbf{u}_{e,t} = \hat{\mathbf{u}}_{e,t} \exp\left[i(\mathbf{k}_q \cdot \mathbf{x} - w_q t)\right]. \quad (4.17)$$

The vectors $\nabla\left(u_e^2/2\right)$, $\nabla\phi$ and ∇p_e are all by definition longitudinal. As eq. (3.8) states, longitudinal vectors point along their wave vectors. Since we chose all wave vectors to be confined to the base plane, $\nabla\left(u_e^2/2\right)$, $\nabla\phi$ and ∇p_e do not have a component along the main transverse direction. As a result, the component of eq. (4.15) along the main transverse direction simplifies to

$$\frac{\partial \mathbf{u}_{e,t}}{\partial t} - \mathbf{u}_{e,l} \times (\nabla \times \mathbf{u}_{e,t}) + \frac{e}{m_e}\left[-\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}_{e,l} \times (\nabla \times \mathbf{A})\right] = 0. \quad (4.18)$$

Using $c = w/k$, we show the following scalings

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial \mathbf{u}_{e,t}}{\partial t} &= -\frac{iw\mathbf{u}_{e,t}}{c^2} \sim i\frac{\mathbf{u}_{e,t}}{c}k, \\
\frac{1}{c^2}\mathbf{u}_{e,l} \times (\nabla \times \mathbf{u}_{e,t}) &= \frac{i\mathbf{u}_{e,l} \times (\mathbf{k} \times \mathbf{u}_{e,t})}{c^2} \sim i\frac{\mathbf{u}_{e,l}}{c}\frac{\mathbf{u}_{e,t}}{c}k, \\
\frac{1}{c^2}\frac{\partial \mathbf{A}}{\partial t} &= -\frac{iw\mathbf{A}}{c^2} \sim i\frac{\mathbf{A}}{c}k, \\
\frac{1}{c^2}\mathbf{u}_{e,l} \times (\nabla \times \mathbf{A}) &= \frac{i\mathbf{u}_{e,l} \times (\mathbf{k} \times \mathbf{A})}{c^2} \sim i\frac{\mathbf{u}_{e,l}}{c}\frac{\mathbf{A}}{c}k.
\end{aligned} \quad (4.19)$$

Thus, assuming $\mathbf{u}_{e,l} \ll c$, the terms involving the double cross product are smaller than those involving the time derivative. As a result, eq. (4.18) becomes

$$\frac{\partial \mathbf{u}_{e,t}}{\partial t} - \frac{e}{m_e}\frac{\partial \mathbf{A}}{\partial t} = 0. \quad (4.20)$$

Using eq. (3.11), the above is equivalent to

$$-iw\mathbf{u}_{e,t} + iw\frac{e\mathbf{A}}{m_e} = 0, \quad (4.21)$$

which upon re-arranging gives

$$\mathbf{u}_{e,t} = \frac{e\mathbf{A}}{m_e}. \quad (4.22)$$

Using both the transverse current given by eq. (4.13) and the transverse velocity given by eq. (4.22), eq. (4.6) can be re-written as

$$\frac{\partial^2 \mathbf{A}}{\partial t} - c_0^2\nabla^2\mathbf{A} = -\frac{en_e}{\epsilon_0}\mathbf{u}_{e,t} = -\frac{e^2 n_e}{\epsilon_0 m_e}\mathbf{A}.$$

We now use the decomposition $n_e = n_{e0} + n_{e1}$, where n_{e0} is time independent. The above becomes

$$\frac{\partial^2 \mathbf{A}}{\partial t} + w_{pe}^2 \mathbf{A} - c_0^2 \nabla^2 \mathbf{A} = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}, \tag{4.23}$$

where $w_{pe}^2 = e^2 n_{e0} / m_e \epsilon_0$.

Chapter 5

Electromagnetic waves in a stable plasma

5.1 The vector potential

We start with eq. (4.23), but focus on the stable-plasma case, that is, $n_{e1} = 0$. Thus, we have

$$\frac{\partial^2 \mathbf{A}}{\partial t} + w_{pe}^2 \mathbf{A} - c_0^2 \nabla^2 \mathbf{A} = 0. \tag{5.1}$$

We note that n_{e0} is only time independent, that is, it is still allowed to vary across space. As a result, w_{pe}^2 is also allowed to vary across space. Using eq. (3.11) for the vector potential, eq. (5.1) becomes

$$-w^2 \mathbf{A} + w_{pe}^2 \mathbf{A} - c_0^2 \nabla^2 \mathbf{A} = 0.$$

We re-write the above as

$$\frac{w^2}{c_0^2} \mathbf{A} - \frac{w^2}{c_0^2} \frac{w_{pe}^2}{w^2} \mathbf{A} + \nabla^2 \mathbf{A} = 0.$$

Defining $\epsilon = 1 - w_{pe}^2/w^2$, we ultimately get

$$\frac{w^2}{c_0^2} \epsilon \mathbf{A} + \nabla^2 \mathbf{A} = 0. \tag{5.2}$$

We now consider the case of a *uniform* stable plasma, that is, a plasma where n_{e0} is uniform across space, and thus w_{pe} and ϵ are also uniform across space. Using the standard plane-wave expression $\mathbf{A} = \hat{\mathbf{A}} \exp[i(\mathbf{k} \cdot \mathbf{x} - wt)]$ in eq. (5.2) gives the following dispersion relation

$$\frac{w^2}{c_0^2} \epsilon = k^2. \tag{5.3}$$

We expand the above to obtain

$$w^2 - w_{pe}^2 = c_0^2 k^2.$$

Taking the derivative $\partial/\partial k$ on both sides we get

$$2w \frac{\partial w}{\partial k} = 2c_0^2 k,$$

which in turn gives the following expressions for the group velocity v_g

$$v_g = \frac{c_0^2 k}{w}. \tag{5.4}$$

Using eq. (5.3), we can also write the above as

$$v_g = \frac{c_0^2 k}{w} = c_0^2 \frac{\sqrt{\epsilon}}{c_0} = c_0 \sqrt{\epsilon}. \tag{5.5}$$

5.2 The electric field

Taking the time derivative of eq. (5.1) gives the equation for $\mathbf{E}_t$, that is

$$\frac{\partial^2 \mathbf{E}_t}{\partial t^2} + w_{pe}^2 \mathbf{E}_t - c_0^2 \nabla^2 \mathbf{E}_t = 0. \tag{5.6}$$

Using eq. (3.11) for the electric field, the time derivative in eq. (5.6) evaluates such that

$$-w^2 \mathbf{E}_t + w_{pe}^2 \mathbf{E}_t - c_0^2 \nabla^2 \mathbf{E}_t = 0.$$

We re-write the above as

$$\frac{w^2}{c_0^2} \mathbf{E}_t - \frac{w^2}{c_0^2} \frac{w_{pe}^2}{w^2} \mathbf{E}_t + \nabla^2 \mathbf{E}_t = 0,$$

which becomes

$$\frac{w^2}{c_0^2} \epsilon \mathbf{E}_t + \nabla^2 \mathbf{E}_t = 0. \tag{5.7}$$

5.3 The magnetic field

Taking the curl of eq. (5.1) gives the equation for $\mathbf{B}$, that is

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + w_{pe}^2 \mathbf{B} - c_0^2 \nabla^2 \mathbf{B} + \nabla w_{pe}^2 \times \mathbf{A} = 0. \tag{5.8}$$

Using eq. (3.11) for the magnetic field, the time derivative in eq. (5.8) evaluates such that

$$-w^2 \mathbf{B} + w_{pe}^2 \mathbf{B} - c_0^2 \nabla^2 \mathbf{B} + \nabla w_{pe}^2 \times \mathbf{A} = 0.$$

We re-write the above as

$$\frac{w^2}{c_0^2} \mathbf{B} - \frac{w^2}{c_0^2} \frac{w_{pe}^2}{w^2} \mathbf{B} + \nabla^2 \mathbf{B} - \frac{w^2}{c_0^2} \nabla \left(\frac{w_{pe}^2}{w^2} \right) \times \mathbf{A} = 0,$$

which becomes

$$\frac{w^2}{c_0^2} \epsilon \mathbf{B} + \nabla^2 \mathbf{B} + \frac{w^2}{c_0^2} \nabla \epsilon \times \mathbf{A} = 0.$$

Using eq. (5.2) we get

$$\frac{w^2}{c_0^2} \epsilon \mathbf{B} + \nabla^2 \mathbf{B} - \frac{1}{\epsilon} \nabla \epsilon \times \nabla^2 \mathbf{A} = 0.$$

The vector identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$ gives $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A}$, or

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A}. \tag{5.9}$$

Thus, we finally get

$$\frac{w^2}{c_0^2} \epsilon \mathbf{B} + \nabla^2 \mathbf{B} + \frac{1}{\epsilon} \nabla \epsilon \times (\nabla \times \mathbf{B}) = 0. \tag{5.10}$$

As a side note, we can use the expressions above to write Ampere's law in a new form. We combine the vector identity eq. (5.9) with eq. (5.2) to obtain

$$\nabla \times \mathbf{B} = \frac{w^2}{c_0^2}\epsilon\mathbf{A}.$$

Using eq. (3.11) for the vector potential, the expression $\mathbf{E}_t = -\partial\mathbf{A}/\partial t$ gives

$$\mathbf{E}_t = iw\mathbf{A}.$$

Thus, the curl of $\mathbf{B}$ can be expressed as

$$\nabla \times \mathbf{B} = -i\frac{w}{c_0^2}\epsilon\mathbf{E}_t. \tag{5.11}$$

As mentioned in section 5.1, for a uniform stable plasma we have ϵ equal to a constant. Thus, eq. (5.10) becomes identical to eq. (5.7), that is, the wave forms of $\mathbf{E}_t$ and $\mathbf{B}$ are the same.

Chapter 6

Stimulated Raman and Brillouin instabilities

6.1 Linearization

The following decompositions will be used in the derivation of stimulated Raman and Brillouin instabilities:

$$
\begin{aligned}
n_i &= n_{i0} + n_{i1}, \\
n_e &= n_{e0} + n_{e1}, \\
p_i &= p_{i0} + p_{i1}, \\
p_e &= p_{e0} + p_{e1}, \\
\mathbf{u}_{i,l} &= \mathbf{u}_{i0,l} + \mathbf{u}_{i1,l}, \\
\mathbf{u}_{e,l} &= \mathbf{u}_{e0,l} + \mathbf{u}_{e1,l}, \\
\mathbf{E}_l &= \mathbf{E}_{0,l} + \mathbf{E}_{1,l}, \\
\mathbf{A} &= \mathbf{A}_L + \mathbf{A}_s.
\end{aligned} \tag{6.1}
$$

For these decompositions, we'll assume

1. Terms with a subscript 1 are small and thus products of two small quantities can be neglected.

2. $\mathbf{u}_{i0,l}$, $\mathbf{u}_{e0,l}$, and $\mathbf{E}_{0,l}$,are zero.

3. n_{i0}, n_{e0}, p_{i0}, and p_{e0} are uniform in space and time.

Thus, unlike the previous section, we do not assume the plasma is stable, that is, we assume fluctuations such as n_{e1} are small but non zero. $\mathbf{A}_L$ is the vector potential associated with the laser light, and $\mathbf{A}_s$ is the potential associated with the scattered light. For linearization purposes, we'll assume $\mathbf{A}_s$ is small.

Using the decomposition for $\mathbf{A}$, eq. (4.23) is written as

$$
\frac{\partial^2 \mathbf{A}_L}{\partial t} + \frac{\partial^2 \mathbf{A}_s}{\partial t} + w_{pe}^2 \mathbf{A}_L + w_{pe}^2 \mathbf{A}_s - c_0^2 \nabla^2 \mathbf{A}_L - c_0^2 \nabla^2 \mathbf{A}_s = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_L - \frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_s, \tag{6.2}
$$

Dropping products of small quantities we have

$$
\frac{\partial^2 \mathbf{A}_L}{\partial t} + \frac{\partial^2 \mathbf{A}_s}{\partial t} + w_{pe}^2 \mathbf{A}_L + w_{pe}^2 \mathbf{A}_s - c_0^2 \nabla^2 \mathbf{A}_L - c_0^2 \nabla^2 \mathbf{A}_s = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_L. \tag{6.3}
$$

We'll assume the laser light is stable, that is, it satisfies eq. (5.1), which we re-write below

$$\frac{\partial^2 \mathbf{A}_L}{\partial t} + w_{pe}^2 \mathbf{A}_L - c_0^2 \nabla^2 \mathbf{A}_L = 0. \tag{6.4}$$

Thus, eq. (6.3) becomes

$$\frac{\partial^2 \mathbf{A}_s}{\partial t} + w_{pe}^2 \mathbf{A}_s - c_0^2 \nabla^2 \mathbf{A}_s = -\frac{e^2 n_{e1}}{\epsilon_0 m_e} \mathbf{A}_L. \tag{6.5}$$

The above shows that the fluctuating n_{e1} couples with the laser light to serve as a source for the scattered light.

The electron density equation is now written as

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \nabla \cdot \left[\left(n_{e0} + n_{e1} \right) \left(\mathbf{u}_{e,t} + \mathbf{u}_{e0,l} + \mathbf{u}_{e1,l} \right) \right] = 0,$$

which, since $\mathbf{u}_{e,t}$ is transverse, can be written as

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \mathbf{u}_{e,t} \cdot \nabla \left(n_{e0} + n_{e1} \right) + \nabla \cdot \left[\left(n_{e0} + n_{e1} \right) \left(\mathbf{u}_{e0,l} + \mathbf{u}_{e1,l} \right) \right] = 0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e1}}{\partial t} + \mathbf{u}_{e,t} \cdot \nabla n_{e1} + \nabla \cdot \left(n_{e0} \mathbf{u}_{e1,l} \right) = 0.$$

Since $\mathbf{u}_{e,t}$ and ∇n_{e1} are orthogonal, we finally have

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot \left(n_{e0} \mathbf{u}_{e1,l} \right) = 0. \tag{6.6}$$

The ion density equation is now written as

$$\frac{\partial n_{i0} + n_{i1}}{\partial t} + \nabla \cdot \left[\left(n_{i0} + n_{i1} \right) \left(\mathbf{u}_{i,t} + \mathbf{u}_{i0,l} + \mathbf{u}_{i1,l} \right) \right] = 0,$$

As stated in chapter 4, it is often the case that transverse waves oscillate at such a fast rate that the ions, which have large inertia, are unable to react on comparable time scales. Thus, we can assume $\mathbf{u}_{i,t} = 0$,

$$\frac{\partial n_{i0} + n_{i1}}{\partial t} + \nabla \cdot \left[\left(n_{i0} + n_{i1} \right) \left(\mathbf{u}_{i0,l} + \mathbf{u}_{i1,l} \right) \right] = 0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i1}}{\partial t} + \nabla \cdot \left(n_{i0} \mathbf{u}_{i1,l} \right) = 0. \tag{6.7}$$

Consider the electron momentum equation. Subtracting eq. (4.18) from eq. (4.15) gives

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t} - \mathbf{u}_{e,t} \times \left(\nabla \times \mathbf{u}_{e,t} \right) + \nabla \left(\frac{u_e^2}{2} \right) + \frac{e}{m_e} \left[-\nabla \phi + \mathbf{u}_{e,t} \times \left(\nabla \times \mathbf{A} \right) \right] = -\frac{1}{n_e m_e} \nabla p_e. \tag{6.8}$$

Since $\mathbf{u}_{e,t} = e\mathbf{A}/m_e$, the above simplifies to

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t} + \nabla \left(\frac{u_e^2}{2} \right) - \frac{e}{m_e} \nabla \phi = -\frac{1}{n_e m_e} \nabla p_e,$$

or

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t} + \nabla\left(\frac{u_e^2}{2}\right) + \frac{e}{m_e}\mathbf{E}_l = -\frac{1}{n_e m_e}\nabla p_e.$$

Since $\mathbf{u}_{e,l}$ and $\mathbf{u}_{e,t}$ are orthogonal $u_e^2 = \mathbf{u}_e \cdot \mathbf{u}_e = u_{e,l}^2 + u_{e,t}^2$. The electron momentum equation is then

$$\frac{\partial \mathbf{u}_{e,l}}{\partial t} + \nabla\left(\frac{u_{e,l}^2 + u_{e,t}^2}{2}\right) + \frac{e}{m_e}\mathbf{E}_l = -\frac{1}{n_e m_e}\nabla p_e.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e0}\mathbf{u}_{e1,l}}{\partial t} + n_{e0}\nabla\left(\frac{u_{e,t}^2}{2}\right) + \frac{e n_{e0}}{m_e}\mathbf{E}_{1,l} = -\frac{1}{m_e}\nabla p_{e1}.$$

For the transverse electron velocity we have

$$u_{e,t}^2 = \left(\frac{e\mathbf{A}}{m_e}\right)\cdot\left(\frac{e\mathbf{A}}{m_e}\right) = \frac{e^2}{m_e^2}\left(\mathbf{A}_L\cdot\mathbf{A}_L + 2\mathbf{A}_L\cdot\mathbf{A}_s + \mathbf{A}_s\cdot\mathbf{A}_s\right).$$

Since the product of small quantities can be neglected, the $\mathbf{A}_s \cdot \mathbf{A}_s$ term is dropped. We'll also ignore the $\mathbf{A}_L \cdot \mathbf{A}_L$ term, given that $\mathbf{A}_L$ is stable and thus its magnitude does not play a critical role in the growth of the instabilities. Thus, the electron momentum equation becomes

$$\frac{\partial n_{e0}\mathbf{u}_{e1,l}}{\partial t} + \frac{e^2 n_{e0}}{m_e^2}\nabla\left(\mathbf{A}_L\cdot\mathbf{A}_s\right) + \frac{e n_{e0}}{m_e}\mathbf{E}_{1,l} = -\frac{1}{m_e}\nabla p_{e1}. \tag{6.9}$$

Consider now the ion momentum equation, given by eq. (2.3), which we re-write below as

$$\frac{\partial n_i\mathbf{u}_i}{\partial t} + \nabla\cdot\left(n_i\mathbf{u}_i\mathbf{u}_i\right) - \frac{Zen_i}{m_i}\left(\mathbf{E} + \mathbf{u}_i\times\mathbf{B}\right) = -\frac{1}{m_i}\nabla p_i,$$

The longitudinal component of the above is

$$\frac{\partial n_i\mathbf{u}_{i,l}}{\partial t} + \left[\nabla\cdot\left(n_i\mathbf{u}_i\mathbf{u}_i\right)\right]_l - \frac{Zen_i}{m_i}\left(\mathbf{E}_l + \mathbf{u}_{i,t}\times\mathbf{B}\right) = -\frac{1}{m_i}\nabla p_i,$$

where $[\cdot]_l$ denotes longitudinal component. Since $\mathbf{u}_{i,t} = 0$, we have

$$\frac{\partial n_i\mathbf{u}_{i,l}}{\partial t} + \left[\nabla\cdot\left(n_i\mathbf{u}_i\mathbf{u}_i\right)\right]_l - \frac{Zen_i}{m_i}\mathbf{E}_l = -\frac{1}{m_i}\nabla p_i.$$

Using the variable decompositions, we have

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\left(n_{i0}+n_{i1}\right)\left(\mathbf{u}_{i0,l}+\mathbf{u}_{i1,l}\right)\right]\\ &+\left\{\nabla\cdot\left[\left(n_{i0}+n_{i1}\right)\left(\mathbf{u}_{i,t}+\mathbf{u}_{i0,l}+\mathbf{u}_{i1,l}\right)\left(\mathbf{u}_{i,t}+\mathbf{u}_{i0,l}+\mathbf{u}_{i1,l}\right)\right]\right\}_l\\ &\qquad-\frac{Ze}{m_i}\left(n_{i0}+n_{i1}\right)\left(\mathbf{E}_{0,l}+\mathbf{E}_{1,l}\right) = -\frac{1}{m_i}\nabla\left(p_{i0}+p_{i1}\right).\end{aligned}$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i0}\mathbf{u}_{i1,l}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_{1,l} = -\frac{1}{m_i}\nabla p_{i1}. \tag{6.10}$$

Finally, we note that for electron-plasma and ion-acoustic waves (see notes on electron-plasma and ion-acoustic waves) we have

$$\nabla p_{s1} = \gamma_s k_B T_s \nabla n_{s1}. \tag{6.11}$$

6.2 Stimulated Raman Scattering

We employ the same assumptions as for the electron-plasma waves, that is

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.
2. Uniform ion density, $n_{i1} = 0$.

Combining eq. (6.9) with eq. (6.11) gives

$$\frac{\partial n_{e0}\mathbf{u}_{e1,l}}{\partial t} + \frac{e^2 n_{e0}}{m_e^2}\nabla\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) + \frac{en_{e0}}{m_e}\mathbf{E}_{1,l} = -\frac{\gamma_e k_B T_e}{m_e}\nabla n_{e1}. \tag{6.12}$$

Taking the time derivative of eq. (6.6) and using eq. (6.12) leads to the wave equation for electron density

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{e^2 n_{e0}}{m_e^2}\nabla^2\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) - \frac{en_{e0}}{m_e}\nabla\cdot\mathbf{E}_{1,l} = \frac{\gamma_e k_B T_e}{m_e}\nabla^2 n_{e1}.$$

Using

$$\nabla\cdot\mathbf{E}_1 = -\frac{e}{\epsilon_0}n_{e1}. \tag{6.13}$$

we obtain

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{e^2 n_{e0}}{m_e^2}\nabla^2\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) + \frac{e^2 n_{e0}}{m_e\epsilon_0}n_{e1} = \frac{\gamma_e k_B T_e}{m_e}\nabla^2 n_{e1}.$$

or

$$\frac{\partial^2 n_{e1}}{\partial t^2} + w_{pe}^2 n_{e1} - \frac{\gamma_e k_B T_e}{m_e}\nabla^2 n_{e1} = \frac{e^2 n_{e0}}{m_e^2}\nabla^2\left(\mathbf{A}_L \cdot \mathbf{A}_s\right). \tag{6.14}$$

Thus, the scattered laser light $\mathbf{A}_s$ couples with the laser light to serve as a source for the electron-plasma wave.

6.3 Stimulated Brillouin Scattering

We employ the same assumptions as for the ion-acoustic waves, that is

1. Quasi-neutrality for the base flow, $Zn_{i0} = n_{e0}$.
2. Approximate quasi-neutrality for the fluctuations, $Zn_{i1} \approx n_{e1}$.
3. Negligible electron mass, $m_e \to 0$.

Combining eq. (6.10) with eq. (6.11) gives

$$\frac{\partial n_{i0}\mathbf{u}_{i1,l}}{\partial t} - \frac{Zen_{i0}}{m_i}\mathbf{E}_{1,l} = -\frac{\gamma_i k_B T_i}{m_i}\nabla n_{i1}. \tag{6.15}$$

Taking the time derivative of eq. (6.7) and using eq. (6.15) leads to the wave equation for ion density

$$\frac{\partial^2 n_{i1}}{\partial t^2} + \frac{Zen_{i0}}{m_i}\nabla\cdot\mathbf{E}_{1,l} = \frac{\gamma_i k_B T_i}{m_i}\nabla^2 n_{i1}. \tag{6.16}$$

For this case, we assume that the mass of the electron, which is significantly smaller than that of the ions, is negligible. Thus, eq. (6.9) simplifies to

$$\frac{e^2 n_{e0}}{m_e}\nabla\left(\mathbf{A}_L \cdot \mathbf{A}_s\right) + en_{e0}\mathbf{E}_{1,l} = -\gamma_e k_B T_e \nabla n_{e1}. \tag{6.17}$$

Using the above in the ion wave equation we obtain

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{Zn_{i0}}{n_{e0}} \frac{\gamma_e k_B T_e}{m_i} \nabla^2 n_{e1} + \frac{\gamma_i k_B T_i}{m_i} \nabla^2 n_{i1} + \frac{Ze^2 n_{i0}}{m_i m_e} \nabla^2 \left(\mathbf{A}_L \cdot \mathbf{A}_s \right). \tag{6.18}$$

Due to quasi-neutrality, we have $Zn_{i0} = n_{e0}$ and $Zn_{i1} \approx n_{e1}$, which gives

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \left(\frac{Z\gamma_e k_B T_e}{m_i} + \frac{\gamma_i k_B T_i}{m_i} \right) \nabla^2 n_{i1} + \frac{Ze^2 n_{i0}}{m_i m_e} \nabla^2 \left(\mathbf{A}_L \cdot \mathbf{A}_s \right). \tag{6.19}$$

We now use the ion acoustic velocity

$$v_s = \sqrt{\frac{Z\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}}, \tag{6.20}$$

to write the equation for n_{i1} as

$$\frac{\partial^2 n_{i1}}{\partial t^2} - v_s^2 \nabla^2 n_{i1} = \frac{Ze^2 n_{i0}}{m_i m_e} \nabla^2 \left(\mathbf{A}_L \cdot \mathbf{A}_s \right). \tag{6.21}$$

Thus, the scattered laser light $\mathbf{A}_s$ couples with the laser light to serve as a source for the ion-acoustic wave.