ALE finite-element hydrodynamics

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1 Lagrangian governing equations

We consider Lagrangian fluid particles, for which we define the position $\mathbf{x}^+ = \mathbf{x}^+(t, \mathbf{y})$, the density $\rho^+ = \rho^+(t, \mathbf{y})$, the velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$, and the internal energy $e^+ = e^+(t, \mathbf{y})$. The Eulerian counterparts for the density, velocity, and internal energy are, respectively, $\rho = \rho(t, \mathbf{x})$, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, and $e = e(t, \mathbf{x})$. Also consider the volume Ω_0 as the set of all $\mathbf{y}$ vectors that make up the initial domain. The control volume $\Omega^+ = \Omega^+(t, \Omega_0)$ is then defined by

$$\Omega^+ = \{\mathbf{x}^+ : \mathbf{y} \in \Omega_0\}. \tag{1}$$

Note that $\Omega^+(0, \Omega_0) = \Omega_0$.

The governing equations for the Lagrangian fluid particles are derived in my fluid-mechanics notes (see section on kinematics, Lagrangian governing equations, etc.). These are shown below

$$\frac{\partial \mathbf{x}^+}{\partial t} = \mathbf{u}^+, \tag{2}$$

$$\frac{\partial J^+ \rho^+}{\partial t} = 0, \tag{3}$$

$$\rho^+ \frac{\partial \mathbf{u}^+}{\partial t} = (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+}, \tag{4}$$

$$\rho^+ \frac{\partial e^+}{\partial t} = (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+}. \tag{5}$$

In the above, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t, \mathbf{x})$ is the stress tensor, and $J^+ = J^+(t, \mathbf{y})$ is the determinant of the Jacobian matrix $\mathbf{J}^+ = \mathbf{J}^+(t, \mathbf{y})$, which is given by $\mathbf{J}^+ = \partial \mathbf{x}^+ / \partial \mathbf{y}$.

A note on notation. The products that involve a tensor $\boldsymbol{\tau}$ can be expressed in Einstein notation as

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}, \tag{6}$$

$$\boldsymbol{\tau} \cdot \nabla \alpha = \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \tag{7}$$

$$\mathbf{f} \cdot \boldsymbol{\tau} \cdot \nabla \alpha = f_i \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \tag{8}$$

$$\boldsymbol{\tau} : \nabla \mathbf{f} = \tau_{ij} \frac{\partial f_i}{\partial x_j}. \tag{9}$$

where α is a scalar and $\mathbf{f}$ a vector. In these notes we'll mostly be using indices i and j for FE expansions, rather than for Einstein notation.

2 Lagrangian finite elements

We introduce a Lagrangian basis function $\Phi_i^+ = \Phi_i^+(t, \mathbf{y})$ and an Eulerian basis function $\Phi_i = \Phi_i(t, \mathbf{x})$, such that they are related to each other as follows

$$\Phi_i^+(t, \mathbf{y}) = \Phi_i(t, \mathbf{x}^+(t, \mathbf{y})). \tag{10}$$

In the above, $\mathbf{x}^+ = \mathbf{x}^+(t, \mathbf{y})$ is the position of a Lagrangian fluid particle, whose velocity is $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$. We now introduce the Lagrangian variable $f^+ = f^+(t, \mathbf{y})$ and the Eulerian counterpart $f = f(t, \mathbf{x})$ such that they satisfy

$$f^+(t, \mathbf{y}) = f(t, \mathbf{x}^+(t, \mathbf{y})). \tag{11}$$

The expansion of an Eulerian variable in terms of basis functions is as follows

$$f = \sum_i^n F_i \Phi_i, \tag{12}$$

where $F_i = F_i(t)$. Plugging in $\mathbf{x}^+$ for $\mathbf{x}$ in the above, and using eqs. (10) and (11) gives

$$f^+ = \sum_i^n F_i \Phi_i^+. \tag{13}$$

We note that both the Lagrangian and Eulerian variables share the same FE coefficients F_i.

As shown in my fluid mechanics notes, we also have

$$\frac{\partial \Phi_i^+}{\partial t} = \left(\frac{\partial \Phi_i}{\partial t} + \mathbf{u} \cdot \nabla \Phi_i \right)_{\mathbf{x}=\mathbf{x}^+}, \tag{14}$$

where $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the Eulerian counterpart to $\mathbf{u}^+$. We'll introduce the restriction that Φ_i^+ is constant in time, that is $\partial \Phi_i^+ / \partial t = 0$, which gives

$$\frac{\partial \Phi_i}{\partial t} + \mathbf{u} \cdot \nabla \Phi_i = 0. \tag{15}$$

Thus, F_i in eq. (13) accounts for the time dependence of F^+, whereas Φ_i^+ accounts for the dependence on $\mathbf{y}$.

3 Finite element expansion

We introduce the coefficients $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_i(t)$, $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(t)$ and $\hat{e}_i = \hat{e}_i(t)$, as well as the Lagrangian basis functions $\phi_i^+ = \phi_i^+(\mathbf{y}) \in L^2$, and $w_i^+ = w_i^+(\mathbf{y}) \in H^1$. We note that $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{u}}_i$ are each vectors, e.g., the components of $\hat{\mathbf{u}}_i$ are $\hat{u}_{i,\alpha} = \hat{u}_{i,\alpha}(t)$ for $\alpha = x, y, z$. We also note that ϕ_i^+ and w_i^+ have Eulerian counterparts $\phi_i = \phi_i(t, \mathbf{x})$ and $w_i = w_i(t, \mathbf{x})$, respectively (see more details in section on finite elements in my notes for numerical methods). The coefficients are used in the following expansions

$$\mathbf{x}^+ = \sum_j^{N_w} \hat{\mathbf{x}}_j w_j^+, \tag{16}$$

$$\mathbf{u}^+ = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+, \tag{17}$$

$$e^+ = \sum_j^{N_\phi} \hat{e}_j \phi_j^+. \tag{18}$$

We note that the expansion coefficients are the same for the Lagrangian and Eulerian variables. For example, for the Eulerian velocity, we have

$$\mathbf{u} = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j. \tag{19}$$

4 Semi-discrete equations for $\mathbf{x}^+$ and $\mathbf{J}^+$

Plugging in eqs. (16) and (17) in eq. (2) gives

$$\sum_j^{N_w} \frac{d\hat{\mathbf{x}}_j}{dt} w_j^+ = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+. \tag{20}$$

To satisfy the equation above, we'll require

$$\frac{d\hat{\mathbf{x}}_j^+}{dt} = \hat{\mathbf{u}}_j. \tag{21}$$

We now introduce the vectors $\mathbf{X}$ and $\mathbf{U}$, whose components are $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{u}}_i$, respectively. Thus, the above is written as

$$\frac{d\mathbf{X}}{dt} = \mathbf{U}. \tag{22}$$

To obtain $\mathbf{J}^+$ we plug in eq. (16) into its definition, that is

$$\mathbf{J}^+ = \frac{\partial}{\partial \mathbf{y}} \sum_j^{N_w} \hat{\mathbf{x}}_j w_j^+ = \sum_j^{N_w} \hat{\mathbf{x}}_j \nabla_{\mathbf{y}} w_j^+. \tag{23}$$

Note that for any function $\mathbf{x}^+$,whether it be an exact analytical expression or a finite-element expansion as given by eq. (16), the determinant of the Jacobian will satisfy

$$\frac{\partial J^+}{\partial t} = J^+ \left(\frac{\partial u_k}{\partial x_k} \right)_{\mathbf{x}=\mathbf{x}^+}, \tag{24}$$

where $\mathbf{u}$ is the Eulerian counterpart to $\mathbf{u}^+$, which is given by eq. (2).

5 Semi-discrete equation for ρ^+

Equation (3) allows us to write

$$\rho^+ = \frac{\rho_0^+}{J^+}, \tag{25}$$

where $\rho_0^+ = \rho^+(0, \mathbf{y})$.

6 Semi-discrete equation for $\mathbf{u}^+$

Plugging in eq. (25) in eq. (4) we get

$$\rho_0^+ \frac{\partial \mathbf{u}^+}{\partial t} = (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} J^+. \tag{26}$$

We then multiply both sides of the above by the basis functions for velocity and integrate over all space to obtain

$$\int_{\Omega_0} \rho_0^+ \frac{\partial \mathbf{u}^+}{\partial t} w_i^+ \, dV_y = \int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} w_i^+ J^+ \, dV_y. \tag{27}$$

For the left-hand side we have

$$\begin{aligned}
\int_{\Omega_0} \rho_0^+ \frac{\partial \mathbf{u}^+}{\partial t} w_i^+ \, dV_y &= \int_{\Omega_0} \rho_0^+ \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j^+ w_i^+ \, dV_y, \\
&= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} \int_{\Omega_0} \rho_0^+ w_i^+ w_j^+ \, dV_y, \\
&= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)},
\end{aligned} \tag{28}$$

where

$$m_{ij}^{(w)} = \int_{\Omega_0} \rho_0^+ w_i^+ w_j^+ \, dV_y \tag{29}$$

is a mass bilinear form (which is independent of time). For the right-hand side we have

$$\begin{aligned}
\int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} w_i^+ J^+ \, dV_y &= \int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma} w_i)_{\mathbf{x}=\mathbf{x}^+} J^+ \, dV_y \\
&= \int_{\Omega^+} \nabla \cdot \boldsymbol{\sigma} w_i \, dV_x \\
&= -\int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \, dV_x.
\end{aligned} \tag{30}$$

The second equality above follows from integration by substitution. Combining results we have

$$\sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)} = -\int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \, dV_x. \tag{31}$$

We introduce the matrix $\mathbf{M}^{(w)}$ whose components are $m_{ij}^{(w)}$. Thus, the left-hand side of eq. (31) can be written as $\mathbf{M}^{(w)} \, d\mathbf{U}/dt$. We also introduce the vector bilinear form

$$\mathbf{f}_{ij} = \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV_x. \tag{32}$$

This is a *vector* bilinear form since $\mathbf{f}_{ij}$ has components $f_{ij,\alpha} = f_{ij,\alpha}(t)$, for $\alpha = x, y, z$, where α denotes the first index of $\boldsymbol{\sigma}$. We introduce the matrix $\mathbf{F}$, whose components are $\mathbf{f}_{ij}$. We also expand the field with constant value of one as follows

$$1 = \sum_i^{N_\phi} \hat{c}_i \phi_i. \tag{33}$$

If we define the vector $\mathbf{C}$ as that with components $\hat{c}_i$, we can show that

$$\begin{aligned}
\mathbf{FC} &= \sum_j^{N_\phi} \mathbf{f}_{ij} \hat{c}_j \\
&= \sum_j^{N_\phi} \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV_x \hat{c}_j \\
&= \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \left(\sum_j^{N_\phi} \hat{c}_j \phi_j \right) \, dV_x \\
&= \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \, dV_x.
\end{aligned} \tag{34}$$

The above is the negative of the right-hand side of eq. (31). Thus, combining all together we get

$$\mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} = -\mathbf{FC}. \tag{35}$$

We note that since both the Lagrangian and Eulerian velocities share the same coefficients $\mathbf{U}$, we now have a solution for both.

7 Semi-discrete equation for e^+

Plugging in eq. (25) in eq. (5) we get

$$\rho_0^+ \frac{\partial e^+}{\partial t} = (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} J^+. \tag{36}$$

We then multiply both sides of the above by the basis functions for energy and integrate over all space to obtain

$$\int_{\Omega_0} \rho_0^+ \frac{\partial e^+}{\partial t} \phi_i^+ \, dV_y = \int_{\Omega_0} (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} \phi_i^+ J^+ \, dV_y. \tag{37}$$

For the left-hand side we have

$$\begin{aligned}
\int_{\Omega_0} \rho_0^+ \frac{\partial e^+}{\partial t} \phi_i^+ \, dV_y &= \int_{\Omega_0} \rho_0^+ \sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} \phi_j^+ \phi_i^+ \, dV_y, \\
&= \sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} \int_{\Omega_0} \rho_0^+ \phi_j^+ \phi_i^+ \, dV_y, \\
&= \sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)}
\end{aligned} \tag{38}$$

where

$$m_{ij}^{(\phi)} = \int_{\Omega_0} \rho_0^+ \phi_j^+ \phi_i^+ \, dV_y \tag{39}$$

is a mass bilinear form (which is independent of time). For the right-hand side we have

$$\begin{aligned} \int_{\Omega_0} (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} \phi_i^+ J^+ \, dV_y &= \int_{\Omega_0} (\boldsymbol{\sigma} : \nabla \mathbf{u} \phi_i)_{\mathbf{x}=\mathbf{x}^+} J^+ \, dV_y \\ &= \int_{\Omega^+} \boldsymbol{\sigma} : \nabla \mathbf{u} \phi_i \, dV_x. \end{aligned} \tag{40}$$

Combining results we have

$$\sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)} = \int_{\Omega^+} \boldsymbol{\sigma} : \nabla \mathbf{u} \phi_i \, dV_x. \tag{41}$$

We no show that

$$\boldsymbol{\sigma} : \nabla \mathbf{u} = \boldsymbol{\sigma} : \nabla \left(\sum_k^{N_w} \hat{\mathbf{u}}_k w_k \right) = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k, \tag{42}$$

and hence the previous result is written as

$$\sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_k \phi_i \, dV_x. \tag{43}$$

The above is finally re-written as

$$\sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki}. \tag{44}$$

Note that in the above there is a dot product in the right-hand side, that is, the right-hand side expanded out is

$$\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} = \sum_k^{N_w} \sum_{\alpha=x,y,z} \hat{u}_{k,\alpha} f_{ki,\alpha}. \tag{45}$$

We now introduce the vector $\mathbf{E}$ whose components are $\hat{e}_i$. We also introduce the matrix $\mathbf{M}^{(\phi)}$ whose components are $m_{ij}^{(\phi)}$. Thus, eq. (44) can be succinctly written as

$$\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{U}. \tag{46}$$

Note again that on the right-hand side above there is a matrix-vector product *and* a dot product. We also note that since both the Lagrangian and Eulerian internal energies share the same coefficients $\mathbf{E}$, we now have a solution for both.

8 Momentum and energy conservation

$$
\begin{aligned}
IE(t) &= \int_{\Omega^+} \rho e \, dV_x \\
&= \int_{\Omega_0} \rho^+ e^+ J^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ e^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ \sum_j^{N_\phi} \hat{e}_j \phi_j^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ \sum_j^{N_\phi} \hat{e}_j \phi_j^+ \left(\sum_i^{N_\phi} \hat{c}_i \phi_i^+ \right) \, dV_y \\
&= \sum_i^{N_\phi} \sum_j^{N_\phi} \hat{c}_i \int_{\Omega_0} \rho_0^+ \phi_i^+ \phi_j^+ \, dV_y \hat{e}_j \\
&= \mathbf{C} \cdot \mathbf{M}^{(\phi)} \cdot \mathbf{E}
\end{aligned} \tag{47}
$$