

# Plasma physics

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# Chapter 1

## Governing equations

### 1.1 Particle description

Table 1.1: Various coordinates of classical mechanics.

|                                      |                         |                         |
|--------------------------------------|-------------------------|-------------------------|
| Classical coordinates                | $\mathbf{x}(t)$         | $\mathbf{v}(t)$         |
| Generalized coordinates              | $\mathbf{q}$            | $\dot{\mathbf{q}}$      |
| Canonical coordinates                | $\mathbf{q}$            | $\mathbf{p}$            |
| Time-dependent canonical coordinates | $\tilde{\mathbf{q}}(t)$ | $\tilde{\mathbf{p}}(t)$ |

#### 1.1.1 Lagrangian mechanics

- Define the position  $\mathbf{x} = \mathbf{x}(t)$  and velocity  $\mathbf{v} = \mathbf{v}(t)$  of a particle.
- Define the Lagrangian as  $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ , where  $\mathbf{q}$  and  $\dot{\mathbf{q}}$  are the generalized position and generalized velocity, respectively.
- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt} \left[ \left( \frac{\partial L}{\partial \dot{q}_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \right] = \left( \frac{\partial L}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}}. \quad (1.1)$$

- For example, the Lagrangian of a particle in an electromagnetic field where  $\phi = \phi(\mathbf{q}, t)$  is the electric potential and  $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$  is the magnetic potential, is

$$L = \frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e \phi. \quad (1.2)$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e \dot{q}_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \quad (1.3)$$

$$\frac{\partial L}{\partial \dot{q}_i} = m \dot{q}_i + e A_i \quad (1.4)$$

$$\begin{aligned} \frac{d}{dt} \left[ \left( \frac{\partial L}{\partial \dot{q}_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \right] &= \frac{d}{dt} [mv_i + eA_i(\mathbf{x}, t)] \\ &= m \frac{dv_i}{dt} + ev_j \left( \frac{\partial A_i}{\partial q_j} \right)_{\mathbf{q}=\mathbf{x}} + e \left( \frac{\partial A_i}{\partial t} \right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \quad (1.5)$$

Thus, the Euler-Lagrange equation becomes

$$m \frac{dv_i}{dt} = \left( -ev_j \frac{\partial A_i}{\partial q_j} - e \frac{\partial A_i}{\partial t} + ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.6)$$

In vector notation, this is written as

$$m \frac{d\mathbf{v}}{dt} = \left( -e\mathbf{v} \cdot \nabla_q \mathbf{A} - e \frac{\partial \mathbf{A}}{\partial t} + e \nabla_q (\mathbf{v} \cdot \mathbf{A}) - e \nabla_q \phi \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.7)$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}, \quad (1.8)$$

where  $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$  and  $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$ .

### 1.1.2 Hamiltonian mechanics

- Define the Hamiltonian as  $H = H(\mathbf{q}, \mathbf{p}, t)$ , where  $\mathbf{q}$  and  $\mathbf{p}$  are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.
- The Hamiltonian is obtained from the Lagrangian using

$$H = (\dot{\mathbf{q}} \cdot \mathbf{p} - L)_{\dot{\mathbf{q}}=f(\mathbf{q}, \mathbf{p})}, \quad (1.9)$$

where the dependency of  $\dot{\mathbf{q}}$  on  $\mathbf{q}$  and  $\mathbf{p}$  is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \quad (1.10)$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[ \dot{q}_i p_i - \left( \frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e \phi \right) \right]_{\dot{\mathbf{q}}=f(\mathbf{q}, \mathbf{p})}. \quad (1.11)$$

Evaluating eq. (1.10) gives  $p_i = m \dot{q}_i + e A_i$ , which allows us to express  $\dot{\mathbf{q}}$  in terms of  $\mathbf{q}$  and  $\mathbf{p}$  as  $\dot{q}_i = \frac{1}{m} (p_i - e A_i)$ . Thus

$$\begin{aligned} H &= \frac{1}{m} (p_i - e A_i) p_i - \frac{1}{2m} (p_i - e A_i) (p_i - e A_i) - e \frac{1}{m} (p_i - e A_i) A_i + e \phi \\ &= \frac{1}{2m} (p_i - e A_i) (p_i - e A_i) + e \phi. \end{aligned} \quad (1.12)$$

- We introduce the variables  $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$  and  $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$ , which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \quad (1.13)$$

$$\tilde{\mathbf{p}} = \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \quad (1.14)$$

- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left( \frac{\partial H}{\partial p_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} \quad (1.15)$$

$$\frac{d\tilde{p}_i}{dt} = - \left( \frac{\partial H}{\partial q_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} \quad (1.16)$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x}, t) \quad (1.17)$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m \frac{dv_i}{dt} + ev_j \left( \frac{\partial A_j}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} + e \left( \frac{\partial A_i}{\partial t} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.18)$$

$$\begin{aligned} \frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i} \left[ \frac{1}{2m} (p_j - eA_j)(p_j - eA_j) + e\phi \right] \\ &= \frac{1}{m} (p_j - eA_j) \frac{\partial}{\partial q_i} (p_j - eA_j) + e \frac{\partial \phi}{\partial q_i} \\ &= -\frac{e}{m} (p_j - eA_j) \frac{\partial A_j}{\partial q_i} + e \frac{\partial \phi}{\partial q_i} \end{aligned} \quad (1.19)$$

$$\begin{aligned} \left( \frac{\partial H}{\partial q_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m} [mv_j + eA_j(\mathbf{x}, t) - eA_j(\mathbf{x}, t)] \left( \frac{\partial A_j}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} + e \left( \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} \\ &= \left( -ev_j \frac{\partial A_j}{\partial q_i} + e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \quad (1.20)$$

Equation (1.16) thus leads to

$$m \frac{dv_i}{dt} = \left( -ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial A_i}{\partial t} + ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.21)$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}. \quad (1.22)$$

## 1.2 Kinetic description

We denote the distribution function for a species  $s$  as  $f_s = f_s(\mathbf{r}, \mathbf{v}, t)$ , where  $\mathbf{r}$  and  $\mathbf{v}$  are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_s d\mathbf{r} d\mathbf{v} = N_s, \quad (1.23)$$

where  $N_s$  is the total number of particles corresponding to species  $s$ .

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution function of  $\mathcal{N}$  species along with Maxwell's equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = C_s + S_s \quad (1.24)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0} \quad (1.25)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.26)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.27)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1.28)$$

$$\mathbf{J} = \sum_s^{\mathcal{N}} Z_s e \int \mathbf{v} f_s d\mathbf{v} \quad (1.29)$$

$$\rho_q = \sum_s^{\mathcal{N}} Z_s e \int f_s d\mathbf{v}. \quad (1.30)$$

In the above,

- $m_s$  is the particle's mass
- $e$  is the elementary charge
- $Z_s$  is the charge number
- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$  the charge current
- $\rho_q = \rho_q(\mathbf{r}, t)$  the charge density
- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$  the electric field
- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$  the magnetic field.

The terms  $C_s$  and  $S_s$  represent collision and source terms.

If we express the collision term in the usual way, that is  $C_s = \sum_r^{\mathcal{N}} C_{sr}$ , then we can make the following statements:

1. Conservation of particles:

$$\int C_{ss} d\mathbf{v} = 0 \quad \forall s \quad \int C_{sr} d\mathbf{v} = 0 \quad \forall s, r | r \neq s. \quad (1.31)$$

2. Conservation of momentum:

$$\int m_s \mathbf{v} C_{ss} d\mathbf{v} = 0 \quad \forall s \quad \sum_s^{\mathcal{N}} \sum_{r, r \neq s}^{\mathcal{N}} \int m_s \mathbf{v} C_{sr} d\mathbf{v} = 0. \quad (1.32)$$

3. Conservation of energy:

$$\int \frac{1}{2} m_s v^2 C_{ss} d\mathbf{v} = 0 \quad \forall s \quad \sum_s^{\mathcal{N}} \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2} m_s v^2 C_{sr} d\mathbf{v} = 0. \quad (1.33)$$



### 1.3 Fluid description

We now define the particle density  $n_s = n_s(\mathbf{r}, t)$ , the fluid velocity  $\mathbf{u}_s = \mathbf{u}_s(\mathbf{r}, t)$  and the fluid energy per unit mass  $E_s = E_s(\mathbf{r}, t)$  as follows

$$n_s = \int f_s d\mathbf{v} \quad (1.34)$$

$$\mathbf{u}_s = \frac{1}{n_s} \int \mathbf{v} f_s d\mathbf{v} \quad (1.35)$$

$$E_s = \frac{1}{n_s} \int \frac{1}{2} v^2 f_s d\mathbf{v}. \quad (1.36)$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_s}{\partial t} + \nabla \cdot (\mathbf{v} f_s) + \nabla_v \cdot \left[ \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] = C_s + S_s \quad (1.37)$$

#### 1.3.1 Mass

Integrating eq. (1.37) over all  $\mathbf{v}$  we obtain

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = \hat{S}_s \quad (1.38)$$

where

$$\hat{S}_s = \int S_s d\mathbf{v} \quad (1.39)$$

is an external source of mass.

#### 1.3.2 Momentum

Multiplying eq. (1.37) by  $\mathbf{v}$  and then integrating over all  $\mathbf{v}$  leads to

$$\begin{aligned} \frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot \left( \int \mathbf{v} \mathbf{v} f_s d\mathbf{v} \right) + \int \nabla_v \cdot \left[ \mathbf{v} \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] - \nabla_v \mathbf{v} \cdot \left[ \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] d\mathbf{v} = \\ \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{v} C_{sr} d\mathbf{v} + \int \mathbf{v} S_s d\mathbf{v}. \end{aligned} \quad (1.40)$$

We note that the third term in eq. (1.40) is zero since we are integrating over all space, and that  $\nabla_v \mathbf{v}$  is the identity matrix. We thus have

$$\begin{aligned} \frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot \left( \int \mathbf{v} \mathbf{v} f_s d\mathbf{v} \right) - \frac{Z_s e n_s}{m_s} (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \\ \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{v} C_{sr} d\mathbf{v} + \int \mathbf{v} S_s d\mathbf{v}. \end{aligned} \quad (1.41)$$

To proceed, we decompose  $\mathbf{v}$  into a mean and a fluctuation, that is,  $\mathbf{v} = \mathbf{u}_s + \mathbf{w}_s$ . Using this decomposition

$$\int \mathbf{v} \mathbf{v} f_s d\mathbf{v} = \int (\mathbf{u}_s \mathbf{u}_s + 2\mathbf{u}_s \mathbf{w}_s + \mathbf{w}_s \mathbf{w}_s) f_s d\mathbf{v} = n_s \mathbf{u}_s \mathbf{u}_s + \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v}. \quad (1.42)$$

Thus, eq. (1.41) becomes

$$\frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s} (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} + \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{v} C_{sr} d\mathbf{v} + \int \mathbf{v} S_s d\mathbf{v}. \quad (1.43)$$

Conservation of particles is used to modify the collisional term as follows

$$\begin{aligned} \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{v} C_{sr} d\mathbf{v} &= \sum_{r, r \neq s}^{\mathcal{N}} \int (\mathbf{u}_s + \mathbf{w}_s) C_{sr} d\mathbf{v} \\ &= \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{w}_s C_{sr} d\mathbf{v} \end{aligned} \quad (1.44)$$

Thus, we now have

$$\frac{\partial n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s} (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} + \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{w}_s C_{sr} d\mathbf{v} + \int \mathbf{v} S_s d\mathbf{v}. \quad (1.45)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) - Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_s + \mathbf{R}_s + \hat{\mathbf{M}}_s, \quad (1.46)$$

where the stress tensor is

$$\boldsymbol{\sigma}_s = - \int m_s \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v}, \quad (1.47)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_s = \sum_{r, r \neq s}^{\mathcal{N}} \int m_s \mathbf{w}_s C_{sr} d\mathbf{v}, \quad (1.48)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_s = \int m_s \mathbf{v} S_s d\mathbf{v}. \quad (1.49)$$

The stress tensor is typically decomposed into isotropic  $p_s$  and anisotropic (shear)  $\mathbf{t}_s$  tensors as follows

$$\boldsymbol{\sigma}_s = -p_s \mathbf{I} + \mathbf{t}_s, \quad (1.50)$$

where  $p_s$  is given by

$$p_s = \frac{1}{3} \int m_s (\mathbf{w}_s \cdot \mathbf{w}_s) f_s d\mathbf{v}. \quad (1.51)$$

Thus, conservation of momentum becomes

$$\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) - Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = -\nabla p_s + \nabla \cdot \mathbf{t}_s + \mathbf{R}_s + \hat{\mathbf{M}}_s. \quad (1.52)$$

### 1.3.3 Energy

Multiplying eq. (1.37) by  $\frac{1}{2}v^2$  and then integrating over all  $\mathbf{v}$  leads to

$$\begin{aligned} \frac{\partial n_s E_s}{\partial t} + \nabla \cdot \left[ \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_s d\mathbf{v} \right] + \int \nabla_v \cdot \left[ \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] \\ - \nabla_v \left[ \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \right] \cdot \left[ \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right] d\mathbf{v} = \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{sr} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_s d\mathbf{v}. \end{aligned} \quad (1.53)$$

We note that the third term above is zero since we are integrating over all space, and that  $\nabla_v[1/2(\mathbf{v} \cdot \mathbf{v})] = \mathbf{v}$ . Thus, we have

$$\begin{aligned} \frac{\partial n_s E_s}{\partial t} + \nabla \cdot \left[ \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_s d\mathbf{v} \right] - \frac{Z_s e n_s}{m_s} \mathbf{E} \cdot \mathbf{u}_s = \\ \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{sr} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_s d\mathbf{v}. \end{aligned} \quad (1.54)$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_s d\mathbf{v} = \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) (\mathbf{u}_s + \mathbf{w}_s) f_s d\mathbf{v} = n_s E_s \mathbf{u}_s + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{w}_s f_s d\mathbf{v} \quad (1.55)$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{w}_s f_s d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_s \cdot \mathbf{u}_s + 2\mathbf{u}_s \cdot \mathbf{w}_s + \mathbf{w}_s \cdot \mathbf{w}_s) \mathbf{w}_s f_s d\mathbf{v} \quad (1.56)$$

$$= \mathbf{u}_s \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s) \mathbf{w}_s f_s d\mathbf{v}. \quad (1.57)$$

Using the expressions above, eq. (1.54) becomes

$$\begin{aligned} \frac{\partial n_s E_s}{\partial t} + \nabla \cdot (n_s E_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s} \mathbf{E} \cdot \mathbf{u}_s = -\nabla \cdot \left( \mathbf{u}_s \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s) \mathbf{w}_s f_s d\mathbf{v} \\ + \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{sr} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_s d\mathbf{v}. \end{aligned} \quad (1.58)$$

Conservation of particles is used to modify the collisional term as follows

$$\begin{aligned} \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{sr} d\mathbf{v} &= \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}[(\mathbf{u}_s + \mathbf{w}_s) \cdot (\mathbf{u}_s + \mathbf{w}_s)] C_{sr} d\mathbf{v} \\ &= \mathbf{u}_s \cdot \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{w}_s C_{sr} d\mathbf{v} + \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s) C_{sr} d\mathbf{v} \end{aligned} \quad (1.59)$$

Thus, we now have

$$\begin{aligned} \frac{\partial n_s E_s}{\partial t} + \nabla \cdot (n_s E_s \mathbf{u}_s) - \frac{Z_s e n_s}{m_s} \mathbf{E} \cdot \mathbf{u}_s = -\nabla \cdot \left( \mathbf{u}_s \cdot \int \mathbf{w}_s \mathbf{w}_s f_s d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s) \mathbf{w}_s f_s d\mathbf{v} \\ + \mathbf{u}_s \cdot \sum_{r, r \neq s}^{\mathcal{N}} \int \mathbf{w}_s C_{sr} d\mathbf{v} + \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2}(\mathbf{w}_s \cdot \mathbf{w}_s) C_{sr} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_s d\mathbf{v}. \end{aligned} \quad (1.60)$$

Multiplying by mass leads to the following equation

$$\begin{aligned} \frac{\partial m_s n_s E_s}{\partial t} + \nabla \cdot (m_s n_s E_s \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \nabla \cdot \mathbf{q}_s \\ + \mathbf{u}_s \cdot \mathbf{R}_s + Q_s + \hat{Q}_s, \end{aligned} \quad (1.61)$$

where heat flux due to random motion is

$$\mathbf{q}_s = \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) \mathbf{w}_s f_s d\mathbf{v}, \quad (1.62)$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_s = \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) C_{sr} d\mathbf{v}, \quad (1.63)$$

and the external source of energy is

$$\hat{Q}_s = \int \frac{1}{2} m_s (\mathbf{v} \cdot \mathbf{v}) S_s d\mathbf{v}. \quad (1.64)$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\begin{aligned} \frac{\partial m_s n_s E_s}{\partial t} + \nabla \cdot (m_s n_s E_s \mathbf{u}_s + p_s \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \nabla \cdot (\mathbf{u}_s \cdot \mathbf{t}_s) - \nabla \cdot \mathbf{q}_s \\ + \mathbf{u}_s \cdot \mathbf{R}_s + Q_s + \hat{Q}_s, \end{aligned} \quad (1.65)$$

We also note that the energy  $m_s n_s E_s$  can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (1.42) one obtains

$$\begin{aligned} m_s n_s E_s &= \int \frac{1}{2} m_s (\mathbf{v} \cdot \mathbf{v}) f_s d\mathbf{v} \\ &= \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) f_s d\mathbf{v} + \frac{1}{2} m_s n_s (\mathbf{u}_s \cdot \mathbf{u}_s) \\ &= \frac{3}{2} p_s + \frac{1}{2} m_s n_s (\mathbf{u}_s \cdot \mathbf{u}_s) \\ &= \frac{3}{2} p_s + m_s n_s K_s. \end{aligned} \quad (1.66)$$

where  $K_s = \frac{1}{2} \mathbf{u}_s \cdot \mathbf{u}_s$  is the kinetic energy of species  $s$ .

### 1.3.4 Kinetic and internal energies

The equation for the kinetic energy is obtained by dotting eq. (1.52) with  $\mathbf{u}_s$ . For this, we first show that

$$\mathbf{u}_s \cdot \left[ \frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) \right] \quad (1.67)$$

$$= \mathbf{u}_s \cdot \left\{ \left[ \frac{\partial m_s n_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s) \right] \mathbf{u}_s + m_s n_s \left( \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) \right\} \quad (1.68)$$

$$= \mathbf{u}_s \cdot \left[ m_s \hat{S}_s \mathbf{u}_s + m_s n_s \left( \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) \right] \quad (1.69)$$

$$= 2m_s \hat{S}_s K_s + m_s n_s \left( \frac{\partial K_s}{\partial t} + \mathbf{u}_s \cdot \nabla K_s \right) \quad (1.70)$$

$$= m_s \hat{S}_s K_s + \left[ \frac{\partial m_s n_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s) \right] K_s + m_s n_s \left( \frac{\partial K_s}{\partial t} + \mathbf{u}_s \cdot \nabla K_s \right) \quad (1.71)$$

$$= m_s \hat{S}_s K_s + \frac{\partial m_s n_s K_s}{\partial t} + \nabla \cdot (m_s n_s K \mathbf{u}_s). \quad (1.72)$$

Thus, the equation for the turbulent kinetic energy is

$$\begin{aligned} \frac{\partial m_s n_s K_s}{\partial t} + \nabla \cdot (m_s n_s K \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \\ \nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s + \mathbf{u}_s \cdot \mathbf{R}_s + \mathbf{u}_s \cdot \hat{\mathbf{M}}_s - m_s K_s \hat{S}_s. \end{aligned} \quad (1.73)$$

Subtracting the above equation from eq. (1.65) leads to

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_s \right) + \nabla \cdot \left( \frac{3}{2} p_s \mathbf{u}_s \right) = \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s - \nabla \cdot \mathbf{q}_s + Q_s + \hat{Q}_s - \mathbf{u}_s \cdot \hat{\mathbf{M}}_s + m_s K_s \hat{S}_s. \quad (1.74)$$

### 1.3.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = \hat{S}_s, \quad (1.75)$$

- Momentum

$$\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \mathbf{u}_s) - Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_s + \mathbf{R}_s + \hat{\mathbf{M}}_s, \quad (1.76)$$

- Total Energy

$$\begin{aligned} \frac{\partial m_s n_s E_s}{\partial t} + \nabla \cdot (m_s n_s E_s \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \nabla \cdot \mathbf{q}_s \\ + \mathbf{u}_s \cdot \mathbf{R}_s + Q_s + \hat{Q}_s, \end{aligned} \quad (1.77)$$

- Kinetic Energy

$$\begin{aligned} \frac{\partial m_s n_s K_s}{\partial t} + \nabla \cdot (m_s n_s K \mathbf{u}_s) - Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s = \\ \nabla \cdot (\mathbf{u}_s \cdot \boldsymbol{\sigma}_s) - \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s + \mathbf{u}_s \cdot \mathbf{R}_s + \mathbf{u}_s \cdot \hat{\mathbf{M}}_s - m_s K_s \hat{S}_s. \end{aligned} \quad (1.78)$$

- Internal Energy

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_s \right) + \nabla \cdot \left( \frac{3}{2} p_s \mathbf{u}_s \right) = \boldsymbol{\sigma}_s : \nabla \mathbf{u}_s - \nabla \cdot \mathbf{q}_s + Q_s + \hat{Q}_s - \mathbf{u}_s \cdot \hat{\mathbf{M}}_s + m_s K_s \hat{S}_s. \quad (1.79)$$

## Chapter 2

# A simplified plasma fluid model

### 2.1 The two-fluid model

Consider a single ion species and a single electron species. Thus, the fluid equations in 1.3.5 without sources simplify to

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (2.1)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (2.2)$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Z e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_i + \mathbf{R}_i, \quad (2.3)$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e, \quad (2.4)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_i \right) + \nabla \cdot \left( \frac{3}{2} p_i \mathbf{u}_i \right) = \boldsymbol{\sigma}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \quad (2.5)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \nabla \cdot \left( \frac{3}{2} p_e \mathbf{u}_e \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \quad (2.6)$$

$$\mathbf{J} = e(Z n_i \mathbf{u}_i - n_e \mathbf{u}_e), \quad (2.7)$$

$$\rho_q = e(Z n_i - n_e). \quad (2.8)$$

$$p_i = n_i k_B T_i, \quad (2.9)$$

$$p_e = n_e k_B T_e. \quad (2.10)$$

These equations correspond to eq. (2.22) in Freidberg's Ideal MHD book, but for ions that are not singly charged.

### 2.2 The single-fluid model

A single-fluid formulation is defined as that for which there is a single velocity vector. For the single-fluid formulation, we'll use the following assumptions

1.  $m_e \approx 0$  since  $m_e \ll m_i$ .
2. Quasi-neutrality, i.e.  $Z_i n_i = n_e$ .

We also introduce some additional variables, namely the total density

$$\rho = m_i n_i + m_e n_e = m_i n_i, \quad (2.11)$$

and the total velocity

$$\mathbf{u} = \frac{m_i}{m_i + m_e} \mathbf{u}_i + \frac{m_e}{m_i + m_e} \mathbf{u}_e = \mathbf{u}_i. \quad (2.12)$$

Multiplying eq. (2.1) by  $m_i$  and gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (2.13)$$

Multiplying eq. (2.1) by  $Z$  and subtracting eq. (2.2) from it gives

$$\frac{\partial}{\partial t} (Z n_i - n_e) + \nabla \cdot (Z n_i \mathbf{u}_i - n_e \mathbf{u}_e) = 0.$$

Using the quasi-neutrality assumption in item 2 and the definition of the current density, we get

$$\nabla \cdot \mathbf{J} = 0. \quad (2.14)$$

Adding up eqs. (2.3) and (2.4), noting that  $\mathbf{R}_i = -\mathbf{R}_e$ , and using the assumption in item 1, we get

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - e (Z n_i - n_e) - e (Z n_i \mathbf{u}_i - n_e \mathbf{u}_e) \times \mathbf{B} = \nabla \cdot \boldsymbol{\sigma},$$

where  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e$ . Using the quasi-neutrality assumption in item 2 and the definition of the current density, we get

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \boldsymbol{\sigma}. \quad (2.15)$$

Using the assumption in item 1, eq. (2.4) becomes

$$e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e,$$

Using the definition of the current density and the quasi-neutrality assumption in item 2, the left-hand side can be re-written as

$$e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = e n_e \mathbf{E} + (e Z n_i \mathbf{u}_i - \mathbf{J}) \times \mathbf{B} = e n_e \mathbf{E} + e n_e \mathbf{u} \times \mathbf{B} - \mathbf{J} \times \mathbf{B}.$$

Thus, the electron momentum equation becomes

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{e n_e} (\mathbf{J} \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e). \quad (2.16)$$

Equation (2.5) is now written as

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_i \right) + \nabla \cdot \left( \frac{3}{2} p_i \mathbf{u} \right) = \boldsymbol{\sigma}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i. \quad (2.17)$$

Introducing the term  $\mathbf{w} = \mathbf{u}_e - \mathbf{u}$ , eq. (2.6) is now written as

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \nabla \cdot \left( \frac{3}{2} p_e \mathbf{u} \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot \left( \frac{3}{2} p_e \mathbf{w}_e \right) + \boldsymbol{\sigma}_e : \nabla \mathbf{w}_e. \quad (2.18)$$

Using the quasi-neutrality assumption in item 2, the current is expressed as follows

$$\mathbf{J} = e n_e (\mathbf{u} - \mathbf{u}_e). \quad (2.19)$$

Using the quasi-neutrality assumption in item 2, the charge becomes

$$\rho_q = 0, \quad (2.20)$$

Finally, no changes are made to the equations of state, namely

$$p_i = n_i k_B T_i, \quad (2.21)$$

$$p_e = n_e k_B T_e. \quad (2.22)$$

# Chapter 3

## Plasma waves

### 3.1 Electron-plasma and ion-acoustic waves

#### 3.1.1 Governing equations

The starting point are the equations given in section 2.1. We first introduce the following assumptions

1. No stresses  $\mathbf{t}_s$ , heat flux  $\mathbf{q}_s$ , and collisions  $\mathbf{R}_s$ ,  $Q_s$ . This in itself would make the flow isentropic.
2. The flow is not just isentropic but also homentropic. Thus eqs. (3.5) and (3.6) hold across all space, not just along streamlines.

With these, the governing equations become

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (3.1)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (3.2)$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Z e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i, \quad (3.3)$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e, \quad (3.4)$$

$$p_i = C_i n_i^{\gamma_i}, \quad (3.5)$$

$$p_e = C_e n_e^{\gamma_e}, \quad (3.6)$$

$$\mathbf{J} = e(Z n_i \mathbf{u}_i - n_e \mathbf{u}_e), \quad (3.7)$$

$$\rho_q = e(Z n_i - n_e), \quad (3.8)$$

$$p_i = n_i k_B T_i, \quad (3.9)$$

$$p_e = n_e k_B T_e. \quad (3.10)$$



### 3.1.2 Linearization

The following decompositions will be used in the derivation of electron-plasma and ion-acoustic waves:

$$\begin{aligned}
n_i &= n_{i0} + n_{i1}, \\
n_e &= n_{e0} + n_{e1}, \\
p_i &= p_{i0} + p_{i1}, \\
p_e &= p_{e0} + p_{e1}, \\
\mathbf{u}_i &= \mathbf{u}_{i0} + \mathbf{u}_{i1}, \\
\mathbf{u}_e &= \mathbf{u}_{e0} + \mathbf{u}_{e1}, \\
\mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_1, \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1.
\end{aligned} \tag{3.11}$$

For these decompositions, we'll assume

1. Terms with a subscript 1 are small and thus products of two small quantities can be neglected.
2.  $\mathbf{u}_{i0}$ ,  $\mathbf{u}_{e0}$ ,  $\mathbf{E}_0$ , and  $\mathbf{B}_0$  are zero.
3.  $n_{i0}$ ,  $n_{e0}$ ,  $p_{i0}$ , and  $p_{e0}$  are uniform in space and time.

Using the variable decompositions in the evolution equation for electron density eq. (3.2), we have

$$\frac{\partial n_{e0} + n_{e1}}{\partial t} + \nabla \cdot [(n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] = 0.$$

Using assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot (n_{e0}\mathbf{u}_{e1}) = 0. \tag{3.12}$$

Using the variable decompositions in the evolution equation for ion density eq. (3.1), we have

$$\frac{\partial n_{i0} + n_{i1}}{\partial t} + \nabla \cdot [(n_{i0} + n_{i1})(\mathbf{u}_{i0} + \mathbf{u}_{i1})] = 0.$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i1}}{\partial t} + \nabla \cdot (n_{i0}\mathbf{u}_{i1}) = 0. \tag{3.13}$$

Using the variable decompositions in the momentum equation for electrons eq. (3.4), we have

$$\begin{aligned}
\frac{\partial}{\partial t} [m_e (n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] + \nabla \cdot [m_e (n_{e0} + n_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})(\mathbf{u}_{e0} + \mathbf{u}_{e1})] \\
+ e (n_{e0} + n_{e1}) [(\mathbf{E}_0 + \mathbf{E}_1) + (\mathbf{u}_{e0} + \mathbf{u}_{e1}) \times (\mathbf{B}_0 + \mathbf{B}_1)] = -\nabla (p_{e0} + p_{e1}).
\end{aligned}$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{e0}\mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e}\mathbf{E}_1 = -\frac{1}{m_e}\nabla p_{e1}. \tag{3.14}$$

Using the variable decompositions in the momentum equation for ions eq. (3.3), we have

$$\begin{aligned} \frac{\partial}{\partial t} [m_i (n_{i0} + n_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1})] + \nabla \cdot [m_i (n_{i0} + n_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1}) (\mathbf{u}_{i0} + \mathbf{u}_{i1})] \\ - Ze (n_{i0} + n_{i1}) [(\mathbf{E}_0 + \mathbf{E}_1) + (\mathbf{u}_{i0} + \mathbf{u}_{i1}) \times (\mathbf{B}_0 + \mathbf{B}_1)] = -\nabla (p_{i0} + p_{i1}). \end{aligned}$$

Given the assumptions in items 1 to 3, the above simplifies to

$$\frac{\partial n_{i0} \mathbf{u}_{i1}}{\partial t} - \frac{Ze n_{i0}}{m_i} \mathbf{E}_1 = -\frac{1}{m_i} \nabla p_{i1}. \quad (3.15)$$

We'll often need to take the gradient of the ion and electron pressure. Given  $p_s = C_s n_s^{\gamma_s}$ , where  $s = i, e$ , we have

$$\nabla p_s = C_s \gamma_s n_s^{\gamma_s - 1} \nabla n_s = C_s \gamma_s \frac{n_s^{\gamma_s}}{n_s} \nabla n_s = \gamma_s \frac{p_s}{n_s} \nabla n_s,$$

Using the equation of state  $p_s = n_s k_B T_s$ , the above becomes

$$\nabla p_s = \gamma_s k_B T_s \nabla n_s. \quad (3.16)$$

Given the assumption in item 3, the above simplifies to

$$\nabla p_{s1} = \gamma_s k_B T_s \nabla n_{s1}. \quad (3.17)$$

### 3.1.3 Electron Plasma Waves

On top of the assumptions in section 3.1.2, we'll assume

1. Quasi-neutrality for the base flow,  $Zn_{i0} = n_{e0}$ .
2. Uniform ion density,  $n_{i1} = 0$ .

Combining Equation (3.14) with eq. (3.17) gives

$$\frac{\partial n_{e0} \mathbf{u}_{e1}}{\partial t} + \frac{en_{e0}}{m_e} \mathbf{E}_1 = -\frac{\gamma_e k_B T_e}{m_e} \nabla n_{e1}. \quad (3.18)$$

Taking the time derivative of eq. (3.12) and using eq. (3.18) leads to the the wave equation for electron density

$$\frac{\partial^2 n_{e1}}{\partial t^2} - \frac{en_{e0}}{m_e} \nabla \cdot \mathbf{E}_1 = \frac{\gamma_e k_B T_e}{m_e} \nabla^2 n_{e1}. \quad (3.19)$$

For electron plasma waves, we'll assume that  $n_i$  varies in space and time so slowly that it can be assumed to be constant. That is, we assume  $n_{i1} = 0$ . Thus, Gauss's law now takes the form

$$\nabla \cdot \mathbf{E}_1 = \frac{e}{\epsilon_0} (Zn_{i0} - n_{e0} - n_{e1}).$$

Using the quasi-neutrality assumption ( $Zn_{i0} = n_{e0}$ )

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_{e1}. \quad (3.20)$$

Plugging the above in the electron wave equation we obtain

$$\frac{\partial^2 n_{e1}}{\partial t^2} + \frac{e^2 n_{e0}}{m_e \epsilon_0} n_{e1} = \frac{\gamma_e k_B T_e}{m_e} \nabla^2 n_{e1},$$

We now introduce the electron plasma frequency

$$w_{pe} = \left( \frac{n_{e0} e^2}{m_e \epsilon_0} \right)^{1/2}, \quad (3.21)$$

and the thermal velocity

$$v_{Ts} = \sqrt{\frac{k_B T_s}{m_s}}. \quad (3.22)$$

These two allow us to write the equation for  $n_{e1}$  as

$$\frac{\partial^2 n_{e1}}{\partial t^2} + w_{pe}^2 n_{e1} - \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0. \quad (3.23)$$

Assuming a mode of the form  $n_{e1} = \tilde{n}_{e1} \exp(-i\omega t)$ , where  $\tilde{n}_{e1} = \tilde{n}_{e1}(\mathbf{x})$ , gives the following

$$(w^2 - w_{pe}^2) n_{e1} + \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0. \quad (3.24)$$

If we assume  $w_{pe} < w$ , then  $\tilde{n}_{e1} = \hat{n}_{e1} \exp(i\mathbf{k}_e \cdot \mathbf{x})$  is a solution to eq. (3.24). Plugging in this expression for  $\tilde{n}_{e1}$  in eq. (3.24) finally gives

$$w^2 - w_{pe}^2 - \gamma_e v_{Te}^2 k_e^2 = 0. \quad (3.25)$$

The above is the dispersion relation for electron plasma waves.

### 3.1.4 Ion Acoustic Waves

On top of the assumptions in section 3.1.2, we'll assume

1. Quasi-neutrality for the base flow,  $Zn_{i0} = n_{e0}$ .
2. Approximate quasi-neutrality for the fluctuations,  $Zn_{i1} \approx n_{e1}$ .
3. Negligible electron mass,  $m_e \rightarrow 0$ .

Combining eq. (3.15) with eq. (3.17) gives

$$\frac{\partial n_{i0} \mathbf{u}_{i1}}{\partial t} - \frac{Z n_{i0}}{m_i} \mathbf{E}_1 = -\frac{\gamma_i k_B T_i}{m_i} \nabla n_{i1}. \quad (3.26)$$

Taking the time derivative of eq. (3.13) and using eq. (3.26) leads to the wave equation for ion density

$$\frac{\partial^2 n_{i1}}{\partial t^2} + \frac{Z n_{i0}}{m_i} \nabla \cdot \mathbf{E}_1 = \frac{\gamma_i k_B T_i}{m_i} \nabla^2 n_{i1}. \quad (3.27)$$

For this case, we assume that the mass of the electron, which is significantly smaller than that of the ions, is negligible. Thus, eq. (3.14) simplifies to

$$e n_{e0} \mathbf{E}_1 = -\gamma_e k_B T_e \nabla n_{e1}. \quad (3.28)$$

Using the above in the ion wave equation we obtain

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \frac{Z n_{i0}}{n_{e0}} \frac{\gamma_e k_B T_e}{m_i} \nabla^2 n_{e1} + \frac{\gamma_i k_B T_i}{m_i} \nabla^2 n_{i1}.$$

Due to quasi-neutrality, we have  $Zn_{i0} = n_{e0}$  and  $Zn_{i1} \approx n_{e1}$ , which gives

$$\frac{\partial^2 n_{i1}}{\partial t^2} = \left( \frac{Z \gamma_e k_B T_e}{m_i} + \frac{\gamma_i k_B T_i}{m_i} \right) \nabla^2 n_{i1}.$$

We now introduce the following velocity

$$v_s = \sqrt{\frac{Z\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}}, \quad (3.29)$$

which allows us to write the equation for  $n_{i1}$  as

$$\frac{\partial^2 n_{i1}}{\partial t^2} - v_s^2 \nabla^2 n_{i1} = 0. \quad (3.30)$$

Assuming a mode of the form  $n_{i1} = \tilde{n}_{i1} \exp(-i\omega t)$ , where  $\tilde{n}_{i1} = \tilde{n}_{i1}(\mathbf{x})$ , gives the following

$$\omega^2 n_{i1} + v_s^2 \nabla^2 n_{i1} = 0. \quad (3.31)$$

Since  $\omega^2$  is always positive,  $\tilde{n}_{i1} = \hat{n}_{i1} \exp(i\mathbf{k}_i \cdot \mathbf{x})$  is a solution to eq. (3.31). Plugging in this expression for  $\tilde{n}_{i1}$  in eq. (3.31) finally gives

$$\omega^2 - k_i^2 v_s^2 = 0. \quad (3.32)$$

The above is the dispersion relation for ion-acoustic waves.

## Chapter 4

# Plasma parameters

### 4.1 Debye length

To describe the Debye length we'll use the governing equations from section 2.1 (without assuming isentropic/homentropic flow) as well as the following assumptions

1. Stationary plasma  $\mathbf{u}_i = \mathbf{u}_e = 0$ .
2. No shear stresses  $\mathbf{t}_s$ , heat flux  $\mathbf{q}_s$ , and collisions  $\mathbf{R}_s$ ,  $Q_s$ .
3. Effects of magnetic fields can be neglected, i.e.  $\mathbf{B} = 0$ .
4. Constant temperatures  $T_i$ ,  $T_e$ .

Thus we have

$$\begin{aligned} -Zen_i\mathbf{E} &= -\nabla p_i, \\ en_e\mathbf{E} &= -\nabla p_e, \\ \nabla \cdot \mathbf{E} &= \frac{\rho_q}{\epsilon_0}, \\ \nabla \times \mathbf{E} &= 0, \\ \rho_q &= e(Zn_i - n_e), \\ p_i &= n_i k_B T_i, \\ p_e &= n_e k_B T_e. \end{aligned}$$

We can simplify the above to obtain

$$\begin{aligned} Zen_i \nabla \phi &= -k_B T_i \nabla n_i, \\ -en_e \nabla \phi &= -k_B T_e \nabla n_e, \\ -\nabla^2 \phi &= \frac{e}{\epsilon_0} (Zn_i - n_e). \end{aligned}$$

The physical domain will be one dimensional with length  $L$ , namely,  $[-L/2, L/2]$ . Thus, we have

$$Zen_i \frac{d\phi}{dx} = -k_B T_i \frac{dn_i}{dx}, \quad (4.1)$$

$$-en_e \frac{d\phi}{dx} = -k_B T_e \frac{dn_e}{dx}, \quad (4.2)$$

$$-\frac{d^2\phi}{dx^2} = \frac{e}{\epsilon_0}(Zn_i - n_e). \quad (4.3)$$

A voltage  $V$  is applied across this domain and thus the boundary condition for the electric potential becomes  $\phi(-L/2) - \phi(L/2) = V$ . We'll also enforce the condition  $\phi(0) = 0$  to fix the arbitrary constant of the potential.

The simplest case to consider is the one for which the one-dimensional domain is empty. Gauss's law then gives

$$\frac{d^2\phi}{dx^2} = 0,$$

which, upon applying the boundary conditions gives the trivial solution  $\phi = -(V/L)x$  and a constant electric field  $E = V/L$  pointing from left to right.

Now let's fill up the physical space with a plasma consisting of electrons and ions. Integrating eqs. (4.1) and (4.2) we obtain

$$\begin{aligned} \ln n_i &= -\frac{Ze}{k_B T_i} \phi + C_1, \\ \ln n_e &= \frac{e}{k_B T_e} \phi + C_2, \end{aligned}$$

which we re-write as

$$\begin{aligned} n_i &= D_1 \exp\left(-\frac{Ze\phi}{k_B T_i}\right), \\ n_e &= D_2 \exp\left(\frac{e\phi}{k_B T_e}\right). \end{aligned}$$

We introduce  $n_{i0}$  and  $n_{e0}$  as the densities at  $x = 0$ . Since  $\phi = 0$  at  $x = 0$ , we have

$$\begin{aligned} n_i &= n_{i0} \exp\left(-\frac{Ze\phi}{k_B T_i}\right), \\ n_e &= n_{e0} \exp\left(\frac{e\phi}{k_B T_e}\right). \end{aligned}$$

A condition for the densities at the center is that they satisfy quasi-neutrality, that is,  $Zn_{i0} = n_{e0}$ . Note that this does not mean  $Zn_i = n_e$ . We'll assume the terms inside the exponential are small, so that we can use  $\exp \alpha = 1 + \alpha$ . Thus we have

$$\begin{aligned} n_i &= n_{i0} - \frac{n_{i0}Ze\phi}{k_B T_i}, \\ n_e &= n_{e0} + \frac{n_{e0}e\phi}{k_B T_e}. \end{aligned}$$

Plugin this into eq. (4.3) gives

$$\frac{d^2\phi}{dx^2} - \left( \frac{n_{i0}Z^2e^2}{\epsilon_0 k_B T_i} + \frac{n_{e0}e^2}{\epsilon_0 k_B T_e} \right) \phi = 0. \quad (4.4)$$

We now introduce the Debye length for a particular species  $s$  as

$$\lambda_{Ds} = \left( \frac{\epsilon_0 k_B T_s}{n_{s0} Z^2 e^2} \right)^{1/2}, \quad (4.5)$$

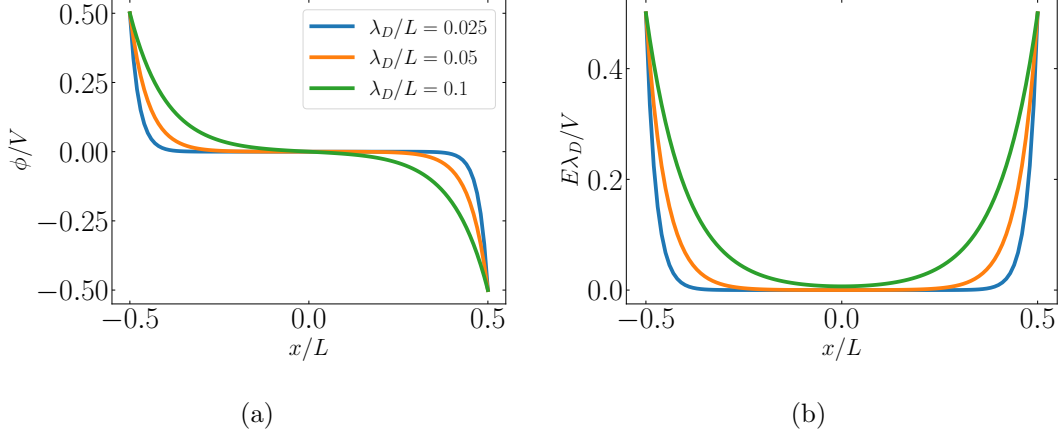


Figure 4.1: Electric potential and field for various values of  $\lambda_D/L$ .

and the total Debye length  $\lambda_D$  as

$$\frac{1}{\lambda_D} = \sum_s \frac{1}{\lambda_{Ds}}. \quad (4.6)$$

Using this, we can re-write eq. (4.4) simply as

$$\frac{d^2\phi}{dx^2} - \frac{1}{\lambda_D^2}\phi = 0. \quad (4.7)$$

The solution to the above that also conforms to the boundary conditions is

$$\phi = -\frac{V}{2} \frac{\sinh(x/\lambda_D)}{\sinh(L/2\lambda_D)}. \quad (4.8)$$

A plot of  $\phi$  and the corresponding  $E$  is shown in fig. 4.1 for various values of  $\lambda_D/L$ . As the figure shows, the presence of a plasma shields out the electric field, in contrast to the vacuum case where a constant electric field is present across the domain. Given a fixed domain size  $L$ , varying the parameter  $\lambda_D$  determines how well the plasma shields away the electric field.

We note that  $\lambda_{De}$ ,  $w_{pe}$ , and  $v_{Te}$  are all related to each other as shown below

$$\lambda_{De} w_{pe} = \left( \frac{\epsilon_0 k_B T_e}{n_{e0} e^2} \right)^{1/2} \left( \frac{n_{e0} e^2}{m_e \epsilon_0} \right)^{1/2} = \left( \frac{k_B T_e}{m_e} \right)^{1/2} = v_{Te}. \quad (4.9)$$

## 4.2 Plasma frequency

We'll use the same setting as that for the derivation of the Debye length. The only difference is that the potential drop across the domain now oscillates with a frequency of  $w$ , that is  $\phi_1(-L/2) - \phi_1(L/2) = V \exp(-iwt)$ .

The starting point is eq. (3.24), that is, the equation for electron-plasma waves after assuming  $n_{e1} = \tilde{n}_{e1} \exp(-iwt)$ , where  $\tilde{n}_{e1} = \tilde{n}_{e1}(\mathbf{x})$ . In one dimension this equation takes the form

$$(w^2 - w_{pe}^2) n_{e1} + \gamma_e v_{Te}^2 \frac{d^2 n_{e1}}{dx^2} = 0.$$

Plugging in eq. (3.20) in the above and using the definition of the electric potential, we get

$$(w^2 - w_{pe}^2) \frac{d^2 \phi_1}{dx^2} + \gamma_e v_{Te}^2 \frac{d^4 \phi_1}{dx^4} = 0,$$

which we re-write as

$$\frac{d^4\phi_1}{dx^4} + \frac{(w^2 - w_{pe}^2)}{\gamma_e v_{Te}^2} \frac{d^2\phi_1}{dx^2} = 0.$$

Using eq. (4.9) the above becomes

$$\frac{d^4\phi_1}{dx^4} + \frac{(w^2 - w_{pe}^2)}{\gamma_e (\lambda_{De}^2 w_{pe}^2)} \frac{d^2\phi_1}{dx^2} = 0,$$

or

$$\frac{d^4\phi_1}{dx^4} + \frac{1}{\gamma_e \lambda_{De}^2} \left( \frac{w^2}{w_{pe}^2} - 1 \right) \frac{d^2\phi_1}{dx^2} = 0.$$

Since we assumed electron number densities of the form  $n_{e1} = \tilde{n}_{e1} \exp(-i\omega t)$ , we choose  $\phi_1 = \tilde{\phi}_1 \exp(-i\omega t)$ , where  $\tilde{\phi}_1 = \tilde{\phi}_1(x)$ , so as to satisfy eq. (3.20). Thus, we get

$$\frac{d^4\tilde{\phi}_1}{dx^4} + \frac{1}{\gamma_e \lambda_{De}^2} \left( \frac{w^2}{w_{pe}^2} - 1 \right) \frac{d^2\tilde{\phi}_1}{dx^2} = 0.$$

Integrating twice we get

$$\frac{d^2\tilde{\phi}_1}{dx^2} + \frac{1}{\gamma_e \lambda_{De}^2} \left( \frac{w^2}{w_{pe}^2} - 1 \right) \tilde{\phi}_1 + C_1 x + C_2 = 0.$$

We only have one boundary condition, namely  $\phi_1(-L/2) - \phi_1(L/2) = V \exp(-i\omega t)$ , yet the equation for  $\phi$  is originally of fourth order. Thus, we prescribe additional boundary conditions such that  $C_1 = C_2 = 0$ . The equation for  $\tilde{\phi}$  then is simply

$$\frac{d^2\tilde{\phi}_1}{dx^2} + \frac{1}{\gamma_e \lambda_{De}^2} \left( \frac{w^2}{w_{pe}^2} - 1 \right) \tilde{\phi}_1 = 0. \quad (4.10)$$

### Case 1 $w_{pe} \leq w$

We introduce the variable  $\alpha$  so that it satisfies

$$\frac{1}{\alpha^2} = \frac{1}{\gamma_e \lambda_{De}^2} \left( \frac{w^2}{w_{pe}^2} - 1 \right). \quad (4.11)$$

Note that  $\alpha$  is always positive. Equation (4.10) becomes

$$\frac{d^2\tilde{\phi}_1}{dx^2} + \frac{1}{\alpha^2} \tilde{\phi}_1 = 0, \quad (4.12)$$

The general solution to the above is

$$\tilde{\phi}_1 = c_1 \exp(ix/\alpha) + c_2 \exp(-ix/\alpha).$$

We introduce a new boundary condition, namely  $\tilde{\phi}_1(0) = 0$ , which gives  $c_2 = -c_1$ . Thus, the solution takes the form

$$\tilde{\phi}_1 = c_1 [\exp(ix/\alpha) - \exp(-ix/\alpha)] = c_1 2i \sin(x/\alpha).$$

Applying the other boundary condition for  $\tilde{\phi}$ , namely,  $\tilde{\phi}_1(-L/2) - \tilde{\phi}_1(L/2) = V$ , we get

$$c_1 2i [-\sin(L/2\alpha) - \sin(L/2\alpha)] = V,$$



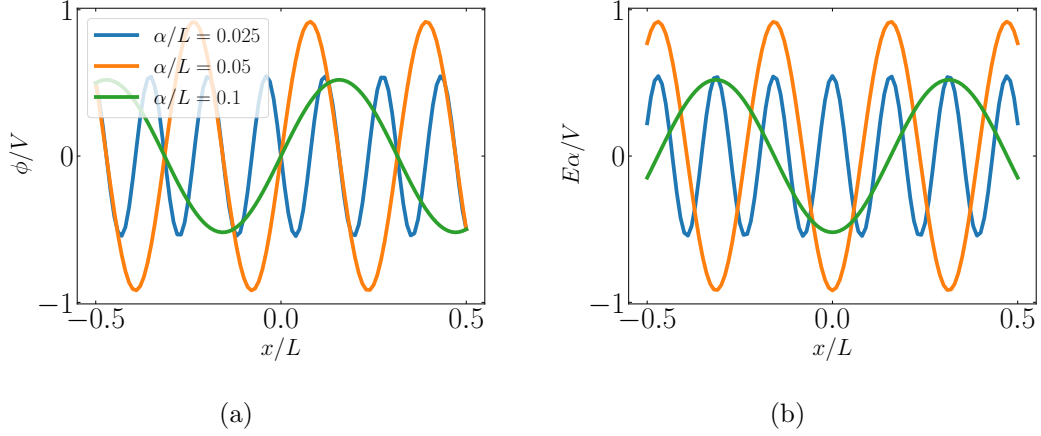


Figure 4.2: Electric potential and field for various values of  $\alpha/L$ .

or

$$c_1 = -\frac{V}{4i \sin(L/2\alpha)}.$$

Thus, the final solution for  $\phi_1$  is

$$\phi_1 = -\frac{V}{2} \frac{\sin(x/\alpha)}{\sin(L/2\alpha)} \exp(-i\omega t). \quad (4.13)$$

We plot the result above along with the corresponding electric field in fig. 4.2. Thus, as the ratio  $w/w_{pe}$  increases,  $\alpha$  decreases, and we go from green to orange to blue, that is, the frequency of oscillation increases.

### Case 2 $w_{pe} \geq w$

We introduce the variable  $\beta$  so that it satisfies

$$\frac{1}{\beta^2} = \frac{1}{\gamma_e \lambda_{De}^2} \left( 1 - \frac{w^2}{w_{pe}^2} \right). \quad (4.14)$$

Note that  $\beta$  is always positive. Equation (4.10) becomes

$$\frac{d^2 \tilde{\phi}_1}{dx^2} - \frac{1}{\beta^2} \tilde{\phi}_1 = 0, \quad (4.15)$$

The general solution to the above is

$$\tilde{\phi}_1 = c_1 \exp(x/\beta) + c_2 \exp(-x/\beta).$$

As before, we use the boundary condition  $\tilde{\phi}_1(0) = 0$ , which gives  $c_2 = -c_1$ . Thus, the solution takes the form

$$\tilde{\phi}_1 = c_1 [\exp(x/\beta) - \exp(-x/\beta)] = c_1 2 \sinh(x/\beta).$$

Applying the other boundary condition for  $\tilde{\phi}$ , namely,  $\tilde{\phi}_1(-L/2) - \tilde{\phi}_1(L/2) = V$ , we get

$$c_1 2 [-\sinh(L/2\beta) - \sinh(L/2\beta)] = V,$$

or

$$c_1 = -\frac{V}{4 \sinh(L/2\beta)}.$$

Thus, the final solution for  $\phi_1$  is

$$\phi_1 = -\frac{V}{2} \frac{\sinh(x/\beta)}{\sinh(L/2\beta)} \exp(-i\omega t). \quad (4.16)$$

Note that the spatial functional form of the above is equivalent to eq. (4.8) if  $\beta$  were to be equal to  $\lambda_D$ . Thus, as the ratio  $w/w_{pe}$  decreases,  $\beta$  decreases, and we go from green to orange to blue in fig. 4.1, that is, the shielding improves.

It is often important to know the plasma density at which the electron plasma frequency  $w_{pe}$  equals the external frequency  $w$ . This density is referred to as the critical density  $n_{cr}$ . Equating the electron plasma frequency with the external frequency we get

$$\frac{n_{cr}e^2}{m_e\epsilon_0} = w^2,$$

which leads to

$$n_{cr} = \frac{m_e\epsilon_0 w^2}{e^2} \quad (4.17)$$

The above can be re-written as

$$n_{cr} = \frac{m_e\epsilon_0(2\pi\nu)^2}{e^2} = \frac{4\pi^2 m_e\epsilon_0 c^2}{e^2} \frac{1}{\lambda^2} = 1.115 \times 10^{27} \frac{1}{\lambda_{\mu m}^2},$$

where  $\lambda_{\mu m}$  is in units of microns and  $n_{cr}$  in units of  $\#/m^3$ .

### 4.3 The coupling parameter

Coulomb interactions are those which occur when two charge particles head towards each other. We can define two types of Coulomb interactions: strong and weak. Strong Coulomb interactions are those for which the particle's Coulomb potential energy is larger than its kinetic energy, and vice versa for weak Coulomb interactions. Thus, we can also define two types of plasma regimes:

- Strongly-coupled plasmas: plasmas where the Coulomb interactions are mostly strong and thus drive the dynamics of its evolution. Coulomb interactions tend to be strong when the inter-particle distances are small, and thus this regime would be dominated by *short-range* interactions. These plasmas are also described as exhibiting *collisional* behavior, since a strong Coulomb interaction essentially means a collision has occurred.
- Weakly-coupled plasmas: plasmas where the Coulomb interactions are mostly weak, and as a result do not drive the dynamics of its evolution. The plasma dynamics are instead driven by *long-range* effects caused by smooth electromagnetic fields that result from integrating a large number of particles. These plasmas are also described as exhibiting *collective* behavior, since the long-range electromagnetic fields follow from the collective integration of many particles.

We describe an approximate Coulomb potential energy for particles in a plasma as

$$U = \frac{q_s^2}{4\pi\epsilon_0 a_s}. \quad (4.18)$$

The impact parameter that has been used above is  $a_s$ , the sphere radius. This provides a decent measure on the average spacing between particles in a plasma. Since the volume of a single

particle is  $1/n_s$ , and if we assume that this volume is given by  $4/3\pi a_s^3$ , then equating these two gives the expression for the sphere radius

$$a_s = \left( \frac{3}{4\pi n_s} \right)^{1/3}. \quad (4.19)$$

The thermal velocity of a particle is given by

$$v_{T_s} = \sqrt{\frac{k_B T_s}{m_s}} \quad (4.20)$$

A measure of the kinetic energy of a particle is given in terms of the thermal velocity as shown below

$$K = m_s v_{T_s}^2 = k_B T_s \quad (4.21)$$

The ratio of the particle's Coulomb potential energy and its kinetic energy is referred to as the coupling parameter  $\Gamma_s$ . That is

$$\Gamma_s = \frac{q_s^2}{4\pi\epsilon_0 a_s k_B T_s}. \quad (4.22)$$

$\Gamma_s > 1$  denotes a strongly coupled plasma, and  $\Gamma_s < 1$  denotes a weakly coupled plasma.

## 4.4 The plasma parameter

The standard plasma parameter  $\Lambda_s$  is defined as

$$\Lambda_s = \frac{4}{3}\pi\lambda_{D_s}^3 n_s. \quad (4.23)$$

There is a one to one relationship between the coupling parameter and the standard plasma parameter. Simple algebra shows that

$$\Gamma_s = (1/3)\Lambda_s^{-2/3}. \quad (4.24)$$

Thus, the coupling and plasma parameters are inversely proportional to each other.  $\Lambda_s < 1$  implies strongly-coupled plasmas, and  $\Lambda_s > 1$  weakly-coupled plasmas. Since  $\Lambda_s$  represents the number of particles per Debye sphere, it is interesting to see that a large number of particles within such a sphere is needed to be in the weakly-coupled-plasma regime. However, this does not correspond to a plasma with large density, in fact, it corresponds to the opposite. The explicit  $n_s$  term in the definition  $\Lambda_s = (4/3)\pi\lambda_{D_s}^3 n_s$  is dominated by the  $n_s$  in the denominator of  $\lambda_{D_s}$ . In other words, low plasma densities lead to large Debye spheres, which in turn leads to many particles per Debye sphere, and hence a weakly-coupled plasma.

## 4.5 Electron degeneracy

- DeBroglie wavelength

$$\lambda_{B_s} = \frac{h}{\sqrt{2\pi m_s v_{T_s}}} \quad (4.25)$$

- Quantum plasma parameter

$$\chi_s = \frac{4}{3}\pi\lambda_{B_s}^3 n_s \quad (4.26)$$

- Fermi energy:

$$E_{fs} = \frac{\hbar^2}{2m_s} (3\pi^2 n_s)^{2/3} \quad (4.27)$$

- The Fermi energy can be used to define the Fermi temperature  $T_{fs}$ , Fermi velocity  $v_{fs}$ , Fermi momentum  $p_{fs}$ , and Fermi wave vector  $k_{fs}$

$$E_{fs} = k_B T_{fs} = \frac{1}{2} m_s v_{fs}^2 = \frac{p_{fs}^2}{2m_s} = \frac{(\hbar k_{fs})^2}{2m_s} \quad (4.28)$$

- Degeneracy parameter:

$$\Theta_s = \frac{k_B T_s}{E_{fs}} = \left( \frac{2^{10} \pi}{3^4} \right)^{1/3} \chi_s^{-2/3}. \quad (4.29)$$

## Chapter 5

# Guiding center theory

We begin with the velocity equation for a particle under the action of electric and magnetic fields,

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}. \quad (5.1)$$

In the above,  $\mathbf{v} = \mathbf{v}(t)$  is the particle velocity,  $\mathbf{x} = \mathbf{x}(t)$  the particle position,  $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$  the electric field, and  $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$  the magnetic field. In the subsections that follow, we will solve this equation of motion for simplified forms of  $\mathbf{E}$  and  $\mathbf{B}$ . The solutions for the velocity vector will typically be of the form

$$\mathbf{v} = \mathbf{v}^{(c)} + \mathbf{v}^{(g)} + v^{\parallel} \mathbf{b}, \quad (5.2)$$

where  $\mathbf{v}^{(c)} = \mathbf{v}^{(c)}(t)$  is the gyro-motion (cyclotron) velocity,  $\mathbf{v}^{(g)} = \mathbf{v}^{(g)}(t)$  is the guiding center velocity,  $v^{\parallel} = v^{\parallel}(t)$  is the parallel velocity. Not all of the velocities will always be present.  $\mathbf{b} = \mathbf{B}/B$  is the unit magnetic field vector. The position of the particle is governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}. \quad (5.3)$$

Using eq. (5.2), we integrate the above to obtain

$$\int_0^t d\mathbf{x}(t') = \int_0^t \mathbf{v}^{(c)}(t') dt' + \int_0^t \mathbf{v}^{(g)}(t') dt' + \int_0^t v^{\parallel}(t) \mathbf{b} dt'. \quad (5.4)$$

We introduce the positions  $\mathbf{x}^{(c)} = \mathbf{x}^{(c)}(t)$ ,  $\mathbf{x}^{(g)} = \mathbf{x}^{(g)}(t)$ , and  $\mathbf{x}^{\parallel} = \mathbf{x}^{\parallel}(t)$ , which are defined as follows

$$\mathbf{x}^{(c)} = \int \mathbf{v}^{(c)} dt, \quad (5.5)$$

$$\mathbf{x}^{(g)} = \int \mathbf{v}^{(g)} dt, \quad (5.6)$$

$$\mathbf{x}^{\parallel} = \int v^{\parallel} \mathbf{b} dt. \quad (5.7)$$

Thus, eq. (5.4) is now re-written as

$$\mathbf{x}(t) - \mathbf{x}(0) = \mathbf{x}^{(c)}(t) - \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(t) - \mathbf{x}^{(g)}(0) + \mathbf{x}^{\parallel}(t) - \mathbf{x}^{\parallel}(0). \quad (5.8)$$

Without loss of generality, we will assume that the initial condition is as follows

$$\mathbf{x}(0) = \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(0) + \mathbf{x}^{\parallel}(0). \quad (5.9)$$

Thus, the particle position is finally expressed as

$$\mathbf{x} = \mathbf{x}^{(c)} + \mathbf{x}^{(g)} + \mathbf{x}^{\parallel}. \quad (5.10)$$

## 5.1 Uniform $\mathbf{E}$ and $\mathbf{B}$ fields

### 5.1.1 Only $\mathbf{E}$ field

Let's orient our coordinate system such that  $\mathbf{E}$  points in the  $\mathbf{e}_z$  direction. Thus, the equations of motion are

$$\begin{aligned}\frac{dv_x}{dt} &= 0 & v_x(0) &= v_{\perp} \cos(\phi), \\ \frac{dv_y}{dt} &= 0 & v_y(0) &= v_{\perp} \sin(\phi), \\ \frac{dv_z}{dt} &= \frac{eE}{m} & v_z(0) &= v_{\parallel}.\end{aligned}\tag{5.11}$$

The solution of the above is

$$\begin{aligned}v_x &= v_{\perp} \cos(\phi) \\ v_y &= v_{\perp} \sin(\phi) \\ v_z &= v_{\parallel} + \frac{eE}{m}t.\end{aligned}\tag{5.12}$$

### 5.1.2 Only $\mathbf{B}$ field

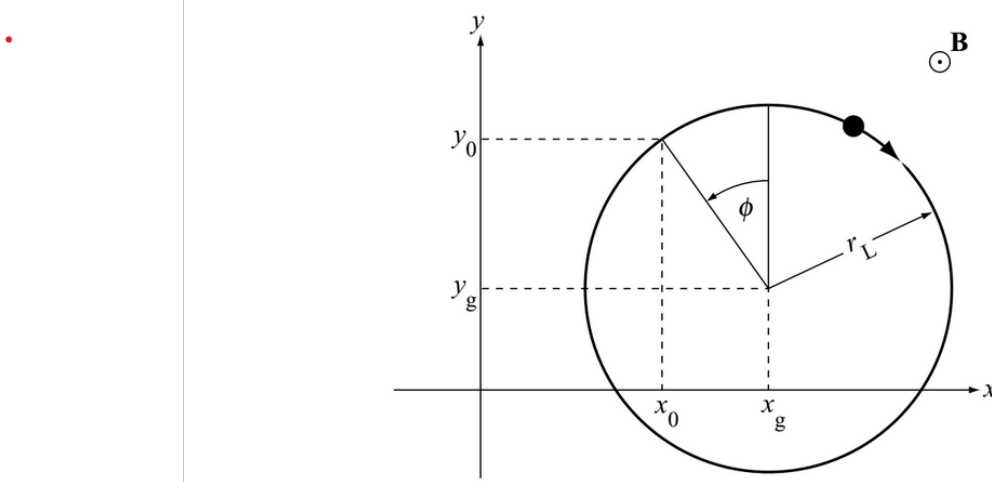


Figure 8.1 Gyro orbit of a positively charged particle in a magnetic field. Shown are the guiding center  $x_g, y_g$  and the initial position  $x_0, y_0$ .

Figure 5.1: Coordinates for gyro-motion (extracted from Plasma Physics and Fusion Energy, J. P. Freidberg).

Let's orient our coordinate system such that  $\mathbf{B}$  points in the  $\mathbf{e}_z$  direction. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB}{m}v_y \quad v_x(0) = v_{\perp} \cos(\phi), \tag{5.13a}$$

$$\frac{dv_y}{dt} = -\frac{eB}{m}v_x \quad v_y(0) = v_{\perp} \sin(\phi), \tag{5.13b}$$

$$\frac{dv_z}{dt} = 0 \quad v_z(0) = v_{\parallel}. \tag{5.13c}$$

The  $z$  component is decoupled from the rest and has a trivial solution. For the other two components, we begin by taking the time derivative of eq. (5.13b). Thus

$$\frac{d^2 v_y}{dt^2} = -\frac{eB}{m} \frac{dv_x}{dt} = -w_c^2 v_y, \quad (5.14)$$

where  $w_c = |e|B/m$  is the gyro frequency. We know that the general solution to the above is  $v_y = c_1 \cos(w_c t) + c_2 \sin(w_c t)$ . If we use the ICs and assume ions, we have

$$v_y = -v_\perp \sin(w_c t - \phi). \quad (5.15)$$

Integrating eq. (5.13a) then gives

$$v_x = v_\perp \cos(w_c t - \phi). \quad (5.16)$$

The final solution, for either positive or negative charges, can be written as

$$\begin{aligned} v_x^{(c)} &= v_\perp \cos(w_c t \pm \phi) \\ v_y^{(c)} &= \pm v_\perp \sin(w_c t \pm \phi), \end{aligned} \quad (5.17)$$

where upper signs correspond to a negative charge. Integrating the equations above leads to

$$\begin{aligned} x^{(c)} &= r_L \sin(w_c t \pm \phi) \\ y^{(c)} &= \mp r_L \cos(w_c t \pm \phi). \end{aligned} \quad (5.18)$$

where  $r_L = v_\perp/w_c$  is the gyro radius.

### 5.1.3 Both E and B fields

Let's orient our coordinate system such that  $\mathbf{B}$  still points along  $\mathbf{e}_z$ . The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x}{m} + \frac{eB}{m} v_y \quad v_x(0) = v_\perp \cos(\phi) + \frac{E_y}{B}, \quad (5.19a)$$

$$\frac{dv_y}{dt} = \frac{eE_y}{m} - \frac{eB}{m} v_x \quad v_y(0) = v_\perp \sin(\phi) - \frac{E_x}{B}, \quad (5.19b)$$

$$\frac{dv_z}{dt} = \frac{eE_{||}}{m} \quad v_z(0) = v_{||}, \quad (5.19c)$$

where we have chosen the given initial conditions simply to facilitate the math. Again, the  $z$  component is decoupled from the rest and has the trivial solution  $v_z = v_{||} + (eE_{||}/m)t$ . Thus, eq. (5.2) for the  $x$  and  $y$  components are

$$\begin{aligned} v_x &= v_x^{(c)} + v_x^{(g)}, \\ v_y &= v_y^{(c)} + v_y^{(g)}. \end{aligned} \quad (5.20)$$

We assume  $v_x^{(g)}$  and  $v_y^{(g)}$  are time independent. Using eq. (5.20) in eq. (5.19) we obtain

$$\begin{aligned} 0 &= \frac{eE_x}{m} + \frac{eB}{m} v_y^{(g)} \\ 0 &= \frac{eE_y}{m} - \frac{eB}{m} v_x^{(g)}. \end{aligned} \quad (5.21)$$

Thus,  $v_x^{(g)} = E_y/B$  and  $v_y^{(g)} = -E_x/B$ , which in vector notation can be expressed as

$$\mathbf{v}_E^{(g)} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \quad (5.22)$$

## 5.2 Non-uniform B field

### 5.2.1 Change in magnitude along perpendicular directions

The magnetic field still points in the  $\mathbf{e}_z$  direction, but its magnitude changes in directions perpendicular to  $\mathbf{e}_z$ :  $B = B(q_x, q_y)$ . The equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(x, y)}{m} v_y \quad v_x(0) = v_\perp \cos(\phi) - \frac{v_\perp^2}{2w_c} \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}, \quad (5.23a)$$

$$\frac{dv_y}{dt} = -\frac{eB(x, y)}{m} v_x \quad v_y(0) = v_\perp \sin(\phi) + \frac{v_\perp^2}{2w_c} \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}, \quad (5.23b)$$

$$\frac{dv_z}{dt} = 0 \quad v_z(0) = v_\parallel. \quad (5.23c)$$

In the above, the  $x$  and  $y$  in  $B(x, y)$  are the perpendicular components of the particle's position. The  $v_z$  component is decoupled from the rest and has a trivial solution. Thus, eqs. (5.2) and (5.10) for the  $x$  and  $y$  components are

$$v_x = v_x^{(c)} + v_x^{(g)}, \quad (5.24)$$

$$v_y = v_y^{(c)} + v_y^{(g)}, \quad (5.25)$$

$$x = x^{(c)} + x^{(g)}, \quad (5.26)$$

$$y = y^{(c)} + y^{(g)}. \quad (5.27)$$

We begin by employing a Taylor-series expansion for the magnetic field

$$B(x, y) = B(x^{(g)}, y^{(g)}) + \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} + \dots \quad (5.28)$$

Thus, eqs. (5.23a) and (5.23b) are now

$$\frac{dv_x}{dt} = \frac{eB(x^{(g)}, y^{(g)})}{m} v_y + \frac{e}{m} \left( \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_y \quad (5.29)$$

$$\frac{dv_y}{dt} = -\frac{eB(x^{(g)}, y^{(g)})}{m} v_x + \frac{e}{m} \left( \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_x. \quad (5.30)$$

As before, we assume  $v_x^{(g)}, v_y^{(g)}$  are time independent. Also, for simplicity we assume ions only. Plugging in eqs. (5.24) and (5.25) into the above, we get

$$0 = B(x^{(g)}, y^{(g)}) v_y^{(g)} + \left( \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) (v_y^{(c)} + v_y^{(g)}), \quad (5.31)$$

$$0 = -B(x^{(g)}, y^{(g)}) v_x^{(g)} + \left( \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) (v_x^{(c)} + v_x^{(g)}). \quad (5.32)$$

We assume  $v_x^{(g)} \ll v_x^{(c)}$  and  $v_y^{(g)} \ll v_y^{(c)}$ . Thus, the above becomes

$$0 = B(x^{(g)}, y^{(g)}) v_y^{(g)} + \left( \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_y^{(c)}, \quad (5.33)$$



$$0 = -B(x^{(g)}, y^{(g)})v_x^{(g)} + \left( \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} x^{(c)} + \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_x^{(c)}. \quad (5.34)$$

We now use the definitions in eq. (5.17) and eq. (5.18). For example, with those definitions we can show that

$$\begin{aligned} x^{(c)}v_y^{(c)} &= [r_L \sin(w_c t - \phi)] [-v_\perp \sin(w_c t - \phi)] \\ &= -\frac{v_\perp^2}{w_c} \sin^2(w_c t - \phi) \\ &= -\frac{v_\perp^2}{2w_c} \{1 - \cos[2(w_c t - \phi)]\} \end{aligned} \quad (5.35)$$

Similar derivations can be carried out for  $y^{(c)}v_y^{(c)}$ ,  $x^{(c)}v_x^{(c)}$ , and  $y^{(c)}v_x^{(c)}$ . Thus, eqs. (5.33) and (5.34) become

$$\begin{aligned} 0 = B(x^{(g)}, y^{(g)})v_y^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \{1 - \cos[2(w_c t - \phi)]\} \\ - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} \sin[2(w_c t - \phi)], \end{aligned} \quad (5.36)$$

$$\begin{aligned} 0 = -B(x^{(g)}, y^{(g)})v_x^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \sin[2(w_c t - \phi)] \\ - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} \{1 + \cos[2(w_c t - \phi)]\}. \end{aligned} \quad (5.37)$$

We neglect the oscillatory terms containing the sines and cosines—if it was not possible to neglect them, then the assumption that  $v_x^{(g)}$ ,  $v_y^{(g)}$  are time independent would not hold. Thus,

$$\begin{aligned} 0 &= B(x^{(g)}, y^{(g)})v_y^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \\ 0 &= -B(x^{(g)}, y^{(g)})v_x^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}}. \end{aligned} \quad (5.38a)$$

Solving for the guiding center velocities, we finally have

$$\begin{aligned} v_x^{(g)} &= -\frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})} \\ v_y^{(g)} &= \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}. \end{aligned} \quad (5.39)$$

In vector notation, this is written as

$$\mathbf{v}_{\nabla B}^{(g)} = \mp \frac{v_\perp^2}{2w_c} \frac{\mathbf{B} \times \nabla B}{B^2}. \quad (5.40)$$

In the above, the fields and  $w_c$  are evaluated at  $(x^{(g)}, y^{(g)})$ .

### 5.2.2 Change in magnitude along parallel directions

Ideally, one would introduce a gradient only in the direction parallel to the magnetic field, that is, one would have  $\mathbf{B} = B(q_z)\mathbf{e}_z$ . However, due to Gauss's law, this is too restrictive and instead we generalize and use  $\mathbf{B} = B_x\mathbf{e}_x + B_z\mathbf{e}_z$ , where  $B_x = B_x(q_x, q_z)$  and  $B_z = B_z(q_x, q_z)$ . Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{e}{m}v_y B_z(x, z), \quad (5.41)$$

$$\frac{dv_y}{dt} = -\frac{e}{m}[v_x B_z(x, z) - v_z B_x(x, z)], \quad (5.42)$$

$$\frac{dv_z}{dt} = -\frac{e}{m}v_y B_x(x, z). \quad (5.43)$$

However, the  $z$  direction no longer corresponds to the parallel direction, since the magnetic field also has a component along the  $x$  direction. To account for this, we will introduce a rotating reference frame, in which one of the axis will always be aligned with the magnetic field vector, and thus would denote the parallel direction. In the original static reference frame the unit vectors are  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  and the velocity components are  $(v_x, v_y, v_z)$ , whereas in this new rotating reference frame the unit vectors are  $(\mathbf{e}_{\perp 1}, \mathbf{e}_{\perp 2}, \mathbf{b})$  and the velocity components are  $(v_{\perp 1}, v_{\perp 2}, v_{\parallel})$ .

The rotating reference frame is described by the rotation matrix

$$\mathbf{Q}(t) = \begin{bmatrix} b_x & 0 & b_z \\ 0 & 1 & 0 \\ b_z & 0 & -b_x \end{bmatrix}, \quad (5.44)$$

where  $b_x = b_x(t)$  and  $b_y = b_y(t)$  are given by

$$b_x = \frac{B_x(x, z)}{B(x, z)} \quad b_z = \frac{B_z(x, z)}{B(x, z)} \quad (5.45)$$

In the above,  $B(x, z) = [B_x^2(x, z) + B_z^2(x, z)]^{1/2}$ . As an example, the matrix above leads to the following transformations for the unit vectors and velocities in the rotating reference frame

$$\mathbf{b} = b_x\mathbf{e}_x + b_z\mathbf{e}_z \quad (5.46)$$

$$\mathbf{e}_{\perp 2} = \mathbf{e}_y \quad (5.47)$$

$$\mathbf{e}_{\perp 1} = b_z\mathbf{e}_x - b_x\mathbf{e}_z = \mathbf{e}_{\perp 2} \times \mathbf{b}, \quad (5.48)$$

$$v_{\parallel} = b_x v_x + b_z v_z \quad (5.49)$$

$$v_{\perp 2} = v_y \quad (5.50)$$

$$v_{\perp 1} = b_z v_x - b_x v_z. \quad (5.51)$$

Using the transformation rule for the acceleration of a particle, but for some reason neglecting the coriolis and centrifugal forces, we obtain for the velocity derivatives

$$\frac{dv_{\parallel}}{dt} = \frac{dv_x}{dt}b_x + \frac{dv_z}{dt}b_z - K v_{\perp 1} \quad (5.52)$$

$$\frac{dv_{\perp 2}}{dt} = \frac{dv_y}{dt} \quad (5.53)$$

$$\frac{dv_{\perp 1}}{dt} = \frac{dv_x}{dt}b_z - \frac{dv_z}{dt}b_x + K v_{\parallel}, \quad (5.54)$$

where  $K = K(t)$  is given by  $K = b_x db_z/dt - b_z db_x/dt$ . Using eqs. (5.41) to (5.43) in the above leads to

$$\frac{dv_{||}}{dt} = \frac{e}{m} v_y [B_z(x, z) b_x - B_x(x, z) b_z] - K v_{\perp 1} \quad (5.55)$$

$$\frac{dv_{\perp 2}}{dt} = -\frac{eB}{m} (v_x b_z - v_z b_x) \quad (5.56)$$

$$\frac{dv_{\perp 1}}{dt} = \frac{e}{m} v_y [B_z(x, z) b_z + B_x(x, z) b_x] + K v_{||} \quad (5.57)$$

Using the definitions for  $b_x$  and  $b_z$  in eq. (5.45), as well as the expressions for  $v_{\perp 1}$ ,  $v_{\perp 2}$  in eqs. (5.50) and (5.51), we get

$$\frac{dv_{||}}{dt} = -K v_{\perp 1}, \quad (5.58)$$

$$\frac{dv_{\perp 2}}{dt} = -w_c v_{\perp 1}, \quad (5.59)$$

$$\frac{dv_{\perp 1}}{dt} = w_c v_{\perp 2} + K v_{||}, \quad (5.60)$$

where  $w_c = w_c(t)$  is given by  $w_c = eB(x, z)/m$ .

We now introduce a time transformation to simplify the equations above. To do so, we introduce the following variables

$$\hat{v}_{||} = \hat{v}_{||}(\tau) \quad \hat{v}_{\perp 2} = \hat{v}_{\perp 2}(\tau) \quad \hat{v}_{\perp 1} = \hat{v}_{\perp 1}(\tau) \quad (5.61)$$

$$\hat{x} = \hat{x}(\tau) \quad \hat{z} = \hat{z}(\tau) \quad (5.62)$$

such that

$$v_{||} = \hat{v}_{||}(h(t)) \quad v_{\perp 2} = \hat{v}_{\perp 2}(h(t)) \quad v_{\perp 1} = \hat{v}_{\perp 1}(h(t)) \quad (5.63)$$

$$x = \hat{x}(h(t)) \quad z = \hat{z}(h(t)). \quad (5.64)$$

The function  $h(t)$  is given by

$$h(t) = \int_0^t w_c(t') dt'. \quad (5.65)$$

We also show that

$$b_x = \frac{B_x(x, z)}{B(x, z)} = \frac{B_x(\hat{x}(h(t)), \hat{z}(h(t)))}{B(\hat{x}(h(t)), \hat{z}(h(t)))}, \quad (5.66)$$

and thus

$$\frac{db_x}{dt} = \frac{dh(t)}{dt} \frac{d}{d\tau} \left[ \frac{B_x(\hat{x}, \hat{z})}{B(\hat{x}, \hat{z})} \right]_{\tau=h(t)} = w_c \left. \frac{d\hat{b}_x}{d\tau} \right|_{\tau=h(t)}, \quad (5.67)$$

where  $\hat{b}_x = \hat{b}_x(\tau)$  is given by  $\hat{b}_x = B_x(\hat{x}, \hat{z})/B(\hat{x}, \hat{z})$ . The analogous holds for  $b_z$ . This allows us to write

$$K = w_c \left( \hat{b}_x \frac{d\hat{b}_z}{d\tau} - \hat{b}_z \frac{d\hat{b}_x}{d\tau} \right)_{\tau=h(t)} = w_c \left. \hat{K} \right|_{\tau=h(t)}, \quad (5.68)$$

where  $\hat{K} = \hat{K}(\tau)$  is given by  $\hat{K} = \hat{b}_x d\hat{b}_z/d\tau - \hat{b}_z d\hat{b}_x/d\tau$ . With these transformation, eqs. (5.58) to (5.60) are re-written as

$$\frac{d\hat{v}_{||}}{d\tau} = -\hat{K} \hat{v}_{\perp 1}, \quad (5.69)$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \quad (5.70)$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \hat{K} \hat{v}_{||}. \quad (5.71)$$

We now simplify  $B_z$  so that  $B_z = B_z(q_z)$ . To be consistent with Gauss's law, we require  $B_x = B_x(q_x, q_y)$  where  $B_x = -q_x dB_z/dq_z$ . With these simplified forms, we have

$$\hat{K} = -\hat{b}_z^2 \frac{d}{d\tau} \left( \frac{\hat{b}_x}{\hat{b}_z} \right) \quad (5.72)$$

$$= -\frac{B_z^2(\hat{z})}{B^2(\hat{x}, \hat{z})} \frac{d}{d\tau} \left( \frac{B_x(\hat{x}, \hat{z})}{B_z(\hat{z})} \right) \quad (5.73)$$

$$= \frac{B_z^2(\hat{z})}{B^2(\hat{x}, \hat{z})} \frac{d}{d\tau} \left[ \hat{x} \left( \frac{1}{B_z} \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \right]. \quad (5.74)$$

We now use the long-thin approximation. For this approximation, we assume that  $B_x/B_z \ll 1$ , and also that  $\frac{1}{B_z} \frac{dB_z}{dq_z}$  changes very slowly. We thus have

$$\hat{K} \approx \frac{d\hat{x}}{d\tau} \left( \frac{1}{B_z} \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}. \quad (5.75)$$

Also, using the long-thin approximation in eqs. (5.49) and (5.51) allows us to write

$$v_{||} \approx v_z = \frac{dz}{dt} = \left( \frac{d\hat{z}}{d\tau} \right)_{\tau=h(t)} w_c \quad (5.76)$$

$$v_{\perp 1} \approx v_x = \frac{dx}{dt} = \left( \frac{d\hat{x}}{d\tau} \right)_{\tau=h(t)} w_c. \quad (5.77)$$

Evaluating the above at  $t = h^{-1}(\tau)$ , and defining  $\hat{w}_c(\tau)$  from  $w_c = \hat{w}_c(h(t))$ , we obtain

$$\hat{v}_{||} \approx \frac{d\hat{z}}{d\tau} \hat{w}_c \quad (5.78)$$

$$\hat{v}_{\perp 1} \approx \frac{d\hat{x}}{d\tau} \hat{w}_c. \quad (5.79)$$

We also note that

$$\frac{dB_z(\hat{z})}{d\tau} = \left( \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \frac{d\hat{z}}{d\tau}. \quad (5.80)$$

Using the expressions above in eq. (5.75), one can approximate  $\hat{K}$  using either of the two forms below

$$\hat{K} \approx \frac{\hat{v}_{\perp 1}}{\hat{w}_c B_z(\hat{z})} \left( \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \approx \frac{\hat{v}_{\perp 1}}{\hat{v}_{||} B_z(\hat{z})} \frac{dB_z(\hat{z})}{d\tau}. \quad (5.81)$$

We thus write the governing equations for the velocities as

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_{\perp 1}^2}{\hat{w}_c B_z(\hat{z})} \left( \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}, \quad (5.82)$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \quad (5.83)$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \frac{\hat{v}_{\perp 1}}{B_z(\hat{z})} \frac{dB_z(\hat{z})}{d\tau}. \quad (5.84)$$

We now assume the solution for the perpendicular velocities is of the form

$$\hat{v}_{\perp 1} = \hat{v}_{\perp} \cos[\tau + \hat{\epsilon}] \quad (5.85)$$

$$\hat{v}_{\perp 2} = -\hat{v}_{\perp} \sin[\tau + \hat{\epsilon}], \quad (5.86)$$

where  $\hat{v}_\perp = \hat{v}_\perp(\tau)$  and  $\hat{e} = \hat{e}(\tau)$ . Plugging these two assumed solutions into eqs. (5.83) and (5.84), and using some simple algebra, gives

$$\frac{d\hat{v}_\perp}{d\tau} = \frac{\hat{v}_\perp}{2B_z(\hat{z})} \frac{dB_z(\hat{z})}{d\tau} \{1 + \cos[2(\tau + \hat{e})]\}. \quad (5.87)$$

The above can be re-arranged and expressed as

$$\frac{d \ln \hat{\mu}}{d\tau} = \frac{d \ln B_z(\hat{z})}{d\tau} \cos[2(\tau + \hat{e})], \quad (5.88)$$

where  $\hat{\mu} = \hat{\mu}(\tau)$  is the adiabatic invariant, and is given by

$$\hat{\mu} = \frac{m\hat{v}_\perp^2}{2B_z(\hat{z})}. \quad (5.89)$$

Integrating eq. (5.88) from  $\tau_1$  to  $\tau_2$  gives

$$\ln \hat{\mu}(\tau_2) - \ln \hat{\mu}(\tau_1) = \ln[B_z(\hat{z})] \cos[2(\tau + \hat{e})] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2 \ln[B_z(\hat{z})] \sin[2(\tau + \hat{e})] d\tau. \quad (5.90)$$

Picking  $\tau_1$  and  $\tau_2$  such that  $[\tau_2 + \hat{e}(\tau_2)] - [\tau_1 + \hat{e}(\tau_1)] = 2\pi$ , and assuming  $B(\hat{z})$  doesn't change significantly from  $\tau_1$  to  $\tau_2$ , gives  $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ , that is,  $\hat{\mu}$  is constant over one gyro-period. One can also define

$$\mu = \frac{mv_\perp^2}{2B_z(z)} \quad (5.91)$$

where  $\mu = \mu(t)$  and  $v_\perp = v_\perp(t)$ . Given that  $v_\perp = \hat{v}_\perp(h(t))$ , we have  $\mu = \hat{\mu}(h(t))$ . Thus,  $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$  translates to  $\mu(t_2) = \mu(t_1)$ , where  $t_2 = h^{-1}(\tau_2)$  and  $t_1 = h^{-1}(\tau_1)$ .

Finally, we focus not on the perpendicular velocities but the parallel velocity. Plugging-in the assumed solutions in the governing eq. (5.82) gives

$$\frac{d\hat{v}_\parallel}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})} \left( \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \{1 + \cos[2(\tau + \hat{e})]\}. \quad (5.92)$$

We now average the above from  $\tau_1$  to  $\tau_2$  while assuming  $B(\hat{z})$ ,  $d\hat{v}_\parallel/d\tau$  and  $\hat{v}_\perp^2$  do not change significantly during that time scale. Note that since this is an average, we are not just integrating from  $\tau_1$  to  $\tau_2$ , but we are also dividing by  $\tau_2 - \tau_1$ . After averaging, we obtain

$$\frac{d\hat{v}_\parallel}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})} \left( \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}. \quad (5.93)$$

or

$$m \frac{d\hat{v}_\parallel}{d\tau} = -\frac{\hat{\mu}}{\hat{w}_c} \left( \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}. \quad (5.94)$$

Converting back to time  $t$  gives

$$m \frac{dv_\parallel}{dt} = -\mu \left( \frac{dB_z}{dq_z} \right)_{q_z=z}. \quad (5.95)$$

### 5.2.3 Change in direction

Rather than writing eq. (5.1) in terms of its components as done in previous sections, we leave the equation in vector form. Expressing the velocity as  $\mathbf{v} = \mathbf{v}_\perp + v_\parallel \mathbf{b}$  and assuming no electric field, we write eq. (5.1) as

$$\frac{d}{dt}(\mathbf{v}_\perp + v_\parallel \mathbf{b}) = \mp w_c(\mathbf{v}_\perp + v_\parallel \mathbf{b}) \times \mathbf{b}, \quad (5.96)$$

where upper sign corresponds to negative charge and lower sign to positive charge. For simplicity we will assume positively charged particles only. We then cross both sides of the above by  $\mathbf{b}$ , that is

$$\mathbf{b} \times \left\{ \left[ \frac{d}{dt}(\mathbf{v}_\perp + v_\parallel \mathbf{b}) - w_c(\mathbf{v}_\perp + v_\parallel \mathbf{b}) \times \mathbf{b} \right] \times \mathbf{b} \right\} = 0. \quad (5.97)$$

The above is simplified using the following three manipulations

$$\begin{aligned} \mathbf{b} \times \left\{ [w_c(\mathbf{v}_\perp + v_\parallel \mathbf{b}) \times \mathbf{b}] \times \mathbf{b} \right\} &= \mathbf{b} \times \{ [w_c \mathbf{v}_\perp \times \mathbf{b}] \times \mathbf{b} \} \\ &= -\mathbf{b} \times \{ w_c \mathbf{v}_\perp (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{b} \cdot w_c \mathbf{v}_\perp) \} \\ &= w_c \mathbf{v}_\perp \times \mathbf{b}. \end{aligned} \quad (5.98)$$

$$\mathbf{b} \times \left\{ \left[ \frac{d\mathbf{v}_\perp}{dt} \right] \times \mathbf{b} \right\} = \frac{d\mathbf{v}_\perp}{dt} (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b} \left( \mathbf{b} \cdot \frac{d\mathbf{v}_\perp}{dt} \right) = \left( \frac{d\mathbf{v}_\perp}{dt} \right)_\perp. \quad (5.99)$$

$$\begin{aligned} \mathbf{b} \times \left\{ \left[ \frac{dv_\parallel \mathbf{b}}{dt} \times \mathbf{b} \right] \right\} &= v_\parallel \mathbf{b} \times \left\{ \left[ \frac{d\mathbf{b}}{dt} \times \mathbf{b} \right] \right\} \\ &= v_\parallel \left[ \frac{d\mathbf{b}}{dt} (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b} \left( \mathbf{b} \cdot \frac{d\mathbf{b}}{dt} \right) \right] \\ &= v_\parallel \left[ \frac{d\mathbf{b}}{dt} - \mathbf{b} \left( \frac{1}{2} \frac{d\mathbf{b} \cdot \mathbf{b}}{dt} \right) \right] \\ &= v_\parallel \frac{d\mathbf{b}}{dt} \end{aligned} \quad (5.100)$$

Thus, we have

$$\left( \frac{d\mathbf{v}_\perp}{dt} \right)_\perp - w_c \mathbf{v}_\perp \times \mathbf{b} = -v_\parallel \frac{d\mathbf{b}}{dt}. \quad (5.101)$$

As shown in Freidberg

$$\frac{d\mathbf{b}(\mathbf{x}(t))}{dt} = \frac{d\mathbf{x}(t)}{dt} \cdot \nabla \mathbf{b} = \mathbf{v} \cdot \nabla \mathbf{b} = \mathbf{v}_\perp \cdot \nabla \mathbf{b} + v_\parallel \mathbf{b} \cdot \nabla \mathbf{b}, \quad (5.102)$$

where  $\nabla \mathbf{b}$  is evaluated at  $\mathbf{x} = \mathbf{x}(t)$ . Thus, eq. (5.101) becomes

$$\left( \frac{d\mathbf{v}_\perp}{dt} \right)_\perp - w_c \mathbf{v}_\perp \times \mathbf{b} = -v_\parallel \mathbf{v}_\perp \cdot \nabla \mathbf{b} - v_\parallel^2 \mathbf{b} \cdot \nabla \mathbf{b}. \quad (5.103)$$

As was done for the other drifts, we assume the solution is of the form  $\mathbf{v}_\perp = \mathbf{v}^{(c)} + \mathbf{v}^{(g)}$ , where we assume again that  $\mathbf{v}^{(g)}$  is time independent. The term  $\mathbf{v}^{(c)}$  corresponds to gyro-motion in a rotating reference frame, and is thus given by

$$\mathbf{v}^{(c)} = v_{\perp 1}^{(c)} \mathbf{e}_{\perp 1} + v_{\perp 2}^{(c)} \mathbf{e}_{\perp 2}, \quad (5.104)$$

where  $\mathbf{e}_{\perp 1}$  and  $\mathbf{e}_{\perp 2}$  are orthogonal to  $\mathbf{b}$  and thus rotate in time.  $v_{\perp 1}^{(c)}$  is given by eq. (5.16) and  $v_{\perp 2}^{(c)}$  by eq. (5.15). We note that, in the non-rotating reference frame,  $\mathbf{v}^{(c)}$  is expressed as  $\mathbf{v}^{(c)} = v_x^{(c)} \mathbf{e}_x + v_y^{(c)} \mathbf{e}_y + v_z^{(c)} \mathbf{e}_z$ . We now prove that  $\mathbf{v}_{\perp}^{(c)}$  is the solution to the two terms on the left-hand side of eq. (5.103). To show this we first use the transformation rule for the acceleration of a particle in a rotating reference frame, but for some reason ignore the coriolis and centrifugal terms. Thus

$$\begin{aligned} \frac{d\mathbf{v}^{(c)}}{dt} &= \frac{dv_x^{(c)}}{dt} \mathbf{e}_x + \frac{dv_y^{(c)}}{dt} \mathbf{e}_y + \frac{dv_z^{(c)}}{dt} \mathbf{e}_z \\ &= \frac{dv_{\perp 1}^{(c)}}{dt} \mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt} \mathbf{e}_{\perp 2} + 2\Omega \times \mathbf{v}^{(c)}. \end{aligned} \quad (5.105)$$

We do not allow the rotating reference frame to rotate about the  $\mathbf{b}$  axis. Thus,  $\Omega = \Omega_{\perp 1} \mathbf{e}_{\perp 1} + \Omega_{\perp 2} \mathbf{e}_{\perp 2}$ . Given that  $\Omega$  and  $\mathbf{v}^{(c)}$  are in the same plane,  $\Omega \times \mathbf{v}^{(c)}$  must point in the  $\mathbf{b}$  direction. Thus,

$$\left( \frac{d\mathbf{v}^{(c)}}{dt} \right)_{\perp} = \frac{dv_{\perp 1}^{(c)}}{dt} \mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt} \mathbf{e}_{\perp 2}. \quad (5.106)$$

This allows us to show that

$$\left( \frac{d\mathbf{v}^{(c)}}{dt} \right)_{\perp} - w_c \mathbf{v}^{(c)} \times \mathbf{b} = \frac{dv_{\perp 1}^{(c)}}{dt} \mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt} \mathbf{e}_{\perp 2} - w_c v_{\perp 2}^{(c)} \mathbf{e}_{\perp 1} + w_c v_{\perp 1}^{(c)} \mathbf{e}_{\perp 2} = 0. \quad (5.107)$$

We now plug in  $\mathbf{v}_{\perp} = \mathbf{v}^{(c)} + \mathbf{v}^{(g)}$  in eq. (5.103) to obtain

$$-w_c \mathbf{v}^{(g)} \times \mathbf{b} = -v_{\parallel} \mathbf{v}_{\perp} \cdot \nabla \mathbf{b} - v_{\parallel}^2 \mathbf{b} \cdot \nabla \mathbf{b}. \quad (5.108)$$

As explained in Freidberg, the term  $v_{\parallel} \mathbf{v}_{\perp} \cdot \nabla \mathbf{b}$  leads to small modifications of the gyro motion, but does not lead to a drift of the particles, and thus is ignored. Taking the cross product of eq. (5.108) with  $\mathbf{b}$  finally gives the curvature drift

$$\mathbf{v}_{\kappa}^{(g)} = \pm \frac{v_{\parallel}^2}{w_c} \frac{(\mathbf{b} \cdot \nabla \mathbf{b}) \times \mathbf{B}}{B}. \quad (5.109)$$

We now show that, if we assume  $\nabla \times \mathbf{B} = 0$ , the grad-B drift

$$\mathbf{v}_{\nabla B}^{(g)} = \mp \frac{v_{\perp}^2}{2w_c} \frac{\mathbf{B} \times \nabla B}{B^2} \quad (5.110)$$

can be written in the same form as the curvature drift. We begin by showing that

$$\mathbf{B} \times \nabla B = \mathbf{B} \times \nabla (\mathbf{B} \cdot \mathbf{B})^{1/2} = \mathbf{B} \times \frac{1}{2B} \nabla (\mathbf{B} \cdot \mathbf{B}). \quad (5.111)$$

We now use the vector identity  $\nabla (\mathbf{B} \cdot \mathbf{B}) = 2\mathbf{B} \times (\nabla \times \mathbf{B}) + 2\mathbf{B} \cdot \nabla \mathbf{B}$ , and assume magnetic curl of zero to obtain

$$\begin{aligned} \mathbf{B} \times \nabla B &= \mathbf{B} \times \frac{1}{B} \mathbf{B} \cdot \nabla \mathbf{B} \\ &= \mathbf{B} \times \mathbf{b} \cdot \nabla (B\mathbf{b}) \\ &= \mathbf{B} \times (\mathbf{b} \cdot \nabla B) \mathbf{b} + \mathbf{B} \times B (\mathbf{b} \cdot \nabla \mathbf{b}) \\ &= -B (\mathbf{b} \cdot \nabla \mathbf{b}) \times \mathbf{B}. \end{aligned} \quad (5.112)$$

Thus, the grad-B drift can be written as

$$\mathbf{v}_{\nabla B}^{(g)} = \pm \frac{v_{\perp}^2}{2w_c} \frac{(\mathbf{b} \cdot \nabla \mathbf{b}) \times \mathbf{B}}{B}. \quad (5.113)$$

### 5.3 Non-uniform E field

### 5.4 Time-varying E field

Consider the scenario used in section 5.1.3, but with a time varying electric field. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x(t)}{m} + \frac{eB}{m}v_y \quad v_x(0) = v_\perp \cos(\phi) + \frac{E_y(t)}{B} + \frac{m}{eB^2} \frac{dE_x(t)}{dt}, \quad (5.114a)$$

$$\frac{dv_y}{dt} = \frac{eE_y(t)}{m} - \frac{eB}{m}v_x \quad v_y(0) = v_\perp \sin(\phi) - \frac{E_x(t)}{B} + \frac{m}{eB^2} \frac{dE_y(t)}{dt}, \quad (5.114b)$$

$$\frac{dv_z}{dt} = \frac{eE_{||}(t)}{m} \quad v_z(0) = v_{||}, \quad (5.114c)$$

where again we chose the initial conditions simply to be consistent with the solution that we'll derive. The parallel velocity is independent of the perpendicular velocities, and we won't worry about it for now. To solve for the perpendicular velocities, we again assume the general solution is

$$\begin{aligned} v_x &= v_x^{(c)} + v_x^{(g)} \\ v_y &= v_y^{(c)} + v_y^{(g)} \end{aligned} \quad (5.115)$$

but now do not assume  $v_x^{(g)}$  and  $v_y^{(g)}$  are time independent. We expand  $v_i^{(g)}$  as

$$v_i^{(g)} = v_i^{(g,1)} + v_i^{(g,2)} + \dots, \quad (5.116)$$

where  $v_i^{(g,\alpha)} \sim \epsilon v_i^{(g,\alpha-1)}$ , and the small parameter  $\epsilon$  follows from assuming

$$\frac{1}{v_i^{(g,\alpha)}} \frac{dv_i^{(g,\alpha)}}{dt} \sim \epsilon \omega_c. \quad (5.117)$$

That is, the time scale associated with the rate of change of all of the  $v_i^{(g,\alpha)}$  components is much larger than the time scale of the gyro-motion. In other words, we assume particles gyrate faster than how quickly their drift velocity changes. Using eq. (5.115) in eq. (5.114) leads to

$$\begin{aligned} \frac{dv_x^{(g,1)}}{dt} + \frac{dv_x^{(g,2)}}{dt} &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y^{(g,1)} + \frac{eB}{m}v_y^{(g,2)} \\ \frac{dv_y^{(g,1)}}{dt} + \frac{dv_y^{(g,2)}}{dt} &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x^{(g,1)} - \frac{eB}{m}v_x^{(g,2)}. \end{aligned} \quad (5.118)$$

Collecting lowest order terms

$$\begin{aligned} 0 &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y^{(g,1)} \\ 0 &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x^{(g,1)}, \end{aligned} \quad (5.119)$$

and thus  $v_x^{(g,1)} = E_y(t)/B$  and  $v_y^{(g,1)} = -E_x(t)/B$ , which in vector notation is

$$\mathbf{v}^{(g,1)} = \frac{\mathbf{E}(t) \times \mathbf{B}}{B^2}. \quad (5.120)$$



Collecting first order terms

$$\begin{aligned}\frac{dv_x^{(g,1)}}{dt} &= \frac{eB}{m}v_y^{(g,2)} \\ \frac{dv_y^{(g,1)}}{dt} &= -\frac{eB}{m}v_x^{(g,2)},\end{aligned}\tag{5.121}$$

and thus  $v_x^{(g,2)} = (m/eB^2)dE_x(t)/dt$  and  $v_y^{(g,2)} = (m/eB^2)dE_y(t)/dt$ , which in vector notation is

$$\mathbf{v}^{(g,2)} = \mp \frac{1}{w_c B} \frac{d\mathbf{E}_\perp}{dt}.\tag{5.122}$$

We note that, by looking at the solutions for  $v_x^{(g,1)}$  and  $v_y^{(g,1)}$ , the assumption in eq. (5.117) is equivalent to stating that the electric field changes slowly.

## 5.5 Time-varying B field

Let's assume the magnetic field points in the  $z$  direction again. Using Faraday's law, we have

$$\left(\frac{\partial E_z}{\partial q_y} - \frac{\partial E_y}{\partial q_z}\right)\mathbf{e}_x - \left(\frac{\partial E_z}{\partial q_x} - \frac{\partial E_x}{\partial q_z}\right)\mathbf{e}_y + \left(\frac{\partial E_y}{\partial q_x} - \frac{\partial E_x}{\partial q_y}\right)\mathbf{e}_z = -\frac{\partial B}{\partial t}\mathbf{e}_z.\tag{5.123}$$

To satisfy the above, we set  $E_z = 0$ , and  $E_x = E_x(q_x, q_y, t)$ ,  $E_y = E_y(q_x, q_y, t)$ . That is, a time varying magnetic field requires a time and spatially varying electric field.

We will further simplify our analysis by having  $E_x = 0$  and  $E_y = E_y(q_x, t)$ . Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(t)}{m}v_y,\tag{5.124}$$

$$\frac{dv_y}{dt} = \frac{eE_y(x, t)}{m} - \frac{eB(t)}{m}v_x,\tag{5.125}$$

with  $v_z$  constant. As done in previous sections, the velocities and positions are decomposed as follows

$$v_x = v_x^{(c)} + v_x^{(g)},\tag{5.126}$$

$$v_y = v_y^{(c)} + v_y^{(g)},\tag{5.127}$$

$$x = x^{(c)} + x^{(g)},\tag{5.128}$$

$$y = y^{(c)} + y^{(g)}.\tag{5.129}$$

The electric field is then linearized using a Taylor-series expansion about the guiding center,

$$\frac{dv_x}{dt} = \frac{eB(t)}{m}v_y\tag{5.130}$$

$$\frac{dv_y}{dt} = \frac{e}{m} \left[ E_y \left( x^{(g)}, t \right) + \left. \frac{\partial E_y}{\partial q_x} \right|_{x^{(g)}} x^{(c)} \right] - \frac{eB(t)}{m}v_x,\tag{5.131}$$

We assume positive ions for simplicity and re-write the above as

$$\frac{dv_x}{dt} = w_c v_y\tag{5.132}$$

$$\frac{dv_y}{dt} = \frac{w_c}{B(t)} \left[ E_y \left( x^{(g)}, t \right) + \left. \frac{\partial E_y}{\partial q_x} \right|_{x^{(g)}} x^{(c)} \right] - w_c v_x.\tag{5.133}$$

where  $w_c = w_c(t)$ . We introduce new variables

$$\begin{aligned}\hat{v}_x &= \hat{v}_x(\tau) & \hat{v}_y &= \hat{v}_y(\tau) & \hat{x}^{(c)} &= \hat{x}^{(c)}(\tau) & \hat{x}^{(g)} &= \hat{x}^{(g)}(\tau) \\ \hat{E}_y &= \hat{E}_y(q_x, \tau) & \hat{B} &= \hat{B}(\tau)\end{aligned}\quad (5.134)$$

such that

$$\begin{aligned}v_x(t) &= \hat{v}_x(h(t)) & v_y(t) &= \hat{v}_y(h(t)) & x^{(c)}(t) &= \hat{x}^{(c)}(h(t)) & x^{(g)}(t) &= \hat{x}^{(g)}(h(t)) \\ E_y(q_x, t) &= \hat{E}_y(q_x, h(t)) & B(t) &= \hat{B}(h(t)).\end{aligned}\quad (5.135)$$

For the above

$$h(t) = \int_0^t w_c(t') dt'. \quad (5.136)$$

The equations of motion then become

$$\begin{aligned}\frac{d\hat{v}_x}{d\tau} &= \hat{v}_y \\ \frac{d\hat{v}_y}{d\tau} &= \frac{1}{\hat{B}(\tau)} \left[ \hat{E}_y(\hat{x}^{(g)}, \tau) + \left. \frac{\partial \hat{E}_y}{\partial q_x} \right|_{\hat{x}^{(g)}} \hat{x}^{(c)} \right] - \hat{v}_x.\end{aligned}\quad (5.137)$$

For the gyro-motion quantities, we'll assume they are of the following form,

$$\hat{v}_x^{(c)} = \hat{v}_\perp \cos(\tau + \hat{\epsilon}), \quad (5.138)$$

$$\hat{v}_y^{(c)} = -\hat{v}_\perp \sin(\tau + \hat{\epsilon}), \quad (5.139)$$

$$\hat{x}^{(c)} = \hat{r}_L \sin(\tau + \hat{\epsilon}), \quad (5.140)$$

$$\hat{y}^{(c)} = \hat{r}_L \cos(\tau + \hat{\epsilon}), \quad (5.141)$$

where  $\hat{v}_\perp = \hat{v}_\perp(\tau)$ ,  $\hat{\epsilon} = \hat{\epsilon}(\tau)$ ,  $\hat{w}_c = \hat{w}_c(\tau) = e\hat{B}(\tau)/m$ , and  $\hat{r}_L = \hat{v}_\perp/\hat{w}_c$  are now time-dependent functions. Note that for this specific case, the  $\tau$ -derivatives of the positions above are not equal to their respective velocities, and instead the relationship holds only to leading order. For the guiding center velocities, we'll guess a given form and then check if it satisfies the governing equations. Thus, we guess

$$\begin{aligned}\hat{v}_x^{(g)} &= \frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \\ \hat{v}_y^{(g)} &= \frac{d}{d\tau} \left( \frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \right).\end{aligned}\quad (5.142)$$

Plugging in all of these expressions in the evolution equations given by eq. (5.137), and using a bit of algebra, leads to

$$\frac{d \ln \hat{\mu}}{d\tau} = \frac{d \ln \hat{B}(\tau)}{d\tau} \cos[2(\tau + \hat{\epsilon})], \quad (5.143)$$

where  $\hat{\mu} = \hat{\mu}(\tau)$  is given by

$$\hat{\mu} = \frac{m \hat{v}_\perp^2}{2 \hat{B}(\tau)}. \quad (5.144)$$

Integrating over one gyro-period, i.e. from  $\tau_1$  to  $\tau_2$  such that  $[\tau_2 + \epsilon(\tau_2)] - [\tau_1 + \epsilon(\tau_1)] = 2\pi$ , gives

$$\ln \hat{\mu}(\tau_2) - \ln \hat{\mu}(\tau_1) = \ln[\hat{B}(\tau)] \cos[2(\tau + \hat{\epsilon})] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2 \ln[\hat{B}(\tau)] \sin[2(\tau + \hat{\epsilon})] d\tau. \quad (5.145)$$

Assuming  $\hat{B}(\tau)$  doesn't change significantly from  $\tau_1$  to  $\tau_2$ , then we have  $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ , that is,  $\hat{\mu}$  is constant over one gyro-period. One can also define

$$\mu = \frac{mv_{\perp}^2}{2B(t)} \quad (5.146)$$

where  $\mu = \mu(t)$  and  $v_{\perp} = v_{\perp}(t)$ . Given that  $v_{\perp} = \hat{v}_{\perp}(h(t))$ , we have  $\mu = \hat{\mu}(h(t))$ . Thus,  $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$  translates to  $\mu(t_2) = \mu(t_1)$ , where  $t_2 = h^{-1}(\tau_2)$  and  $t_1 = h^{-1}(\tau_1)$ .

As shown in the analysis above, for a time dependent magnetic field a drift of the following form is introduced

$$\hat{v}_y^{(g)} = \frac{d}{d\tau} \left( \frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \right). \quad (5.147)$$

Converting back to time  $t$

$$v_y^{(g)} = \frac{1}{w_c} \frac{d}{dt} \left( \frac{E_y(x^{(g)}, t)}{B(t)} \right). \quad (5.148)$$

For the more general case where  $E_x = E_x(q_x, q_y, t)$  and  $E_y = E_y(q_x, q_y, t)$  then

$$\mathbf{v}_p^{(g)} = \mp \frac{1}{w_c} \frac{d}{dt} \left( \frac{\mathbf{E}_{\perp}}{B} \right), \quad (5.149)$$

where top sign is for electrons and bottom sign is for ions, and it is assumed that the electric field is evaluated at the guiding center. For an even more general case where the magnetic field does not necessarily point in one direction,

$$\mathbf{v}_p^{(g)} = \mp \frac{1}{w_c} \mathbf{b} \times \frac{d\mathbf{v}_E^{(g)}}{dt}. \quad (5.150)$$

# Chapter 6

## Collisions

### 6.1 Cross section

#### 6.1.1 General definition

Two particles traveling towards each other can undergo an interaction. Types of interactions include fission and fusion reactions, as well as photon-matter phenomena such as the photoelectric effect, Coulomb collisions, Compton scattering, and pair production.

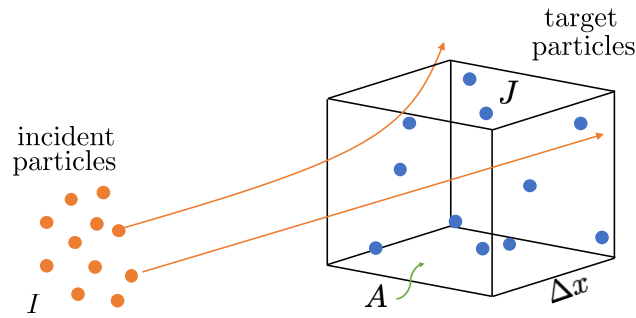


Figure 6.1: Interaction of incident and target particles.

To define the cross section, we'll consider  $I$  incident particles heading towards  $J$  stationary target particles (see fig. 6.1). Not all of the incident particles will interact with the target particles, some will instead continue to travel in a uniform trajectory. The number of incident particles that do interact with the target particles is labeled as  $N$ . The cross section  $\sigma$  is then a constant of proportionality defined by the following equation

$$N = \sigma I n_A. \quad (6.1)$$

In the above,  $n_A$  is the areal number density, that is,  $n_A = J/A = n\Delta x$ , where  $n$  is the volume number density.

#### 6.1.2 Mean free path & collision frequency

Collisions alter a beam of particles passing through a material. The cross sections of each process, which depend on properties of the incoming particles (such as frequency when the

incoming particles are photons), can be added up to obtain a total cross section. From the definition of a cross section, we can determine how many particles keep their course and how many do not as they traverse through a material. Consider the material under consideration to have the shape of a thick slab, which starts at  $x = 0$  and continues on for a definite length along  $x > 0$ . We want to know how  $I(x)$ , the number of particles at any location  $x$  moving along the  $x$  direction, decreases as  $x$  increases. Let's focus on an infinitesimal thin lamina within the slab, of width  $dx$  and located at some arbitrary location  $x$ . The number of target particles per unit area in that lamina will be  $ndx$ , where  $n$  is the number volume density of particles in the target material. Then, the number of incident particles after crossing the lamina would be

$$I(x + dx) = I(x) - \sigma I(x)ndx. \quad (6.2)$$

This leads to the ODE  $dI(x)/dx = -\sigma I(x)n$ , which has as solution

$$I(x) = I(0) \exp(-\sigma nx). \quad (6.3)$$

$\Lambda = 1/\sigma n$  is referred to as the mean free path. If the incident particle is traveling with a velocity  $v$ , then the collision time is defined as follows

$$\tau = \frac{\lambda}{v} = \frac{1}{\sigma nv}. \quad (6.4)$$

The collision frequency is simply the inverse of the collision time, that is

$$\nu = \frac{1}{\tau} = \sigma nv. \quad (6.5)$$

When the incident particles are photons that get absorbed when a collision occurs, the mean free path is referred to as the attenuation length. It's inverse, that is  $\sigma n$  [1/cm], is referred to as the attenuation coefficient. The mass attenuation coefficient, also referred to as opacity, is given by  $\kappa = \sigma n/\rho$  [cm<sup>2</sup>/g], where  $\rho$  is the mass density.

### 6.1.3 Differential cross section

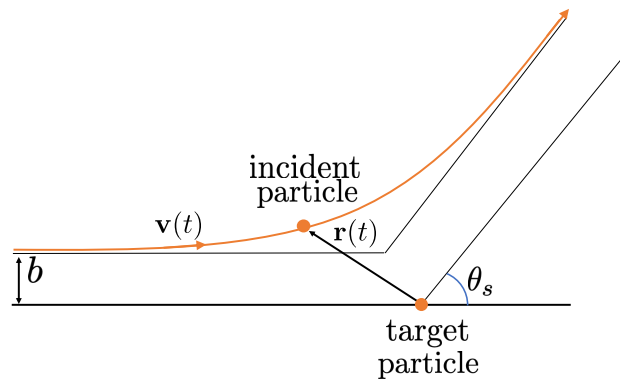


Figure 6.2: Depiction of particle scattering.

Consider the scattering of two particles: an incident and a target particle. If we fix the reference frame to follow the target particle, then the scattering can be depicted as shown in fig. 6.2. The displacement parameter is labelled as  $b$ , and the scattering angle as  $\theta_s$ . For three

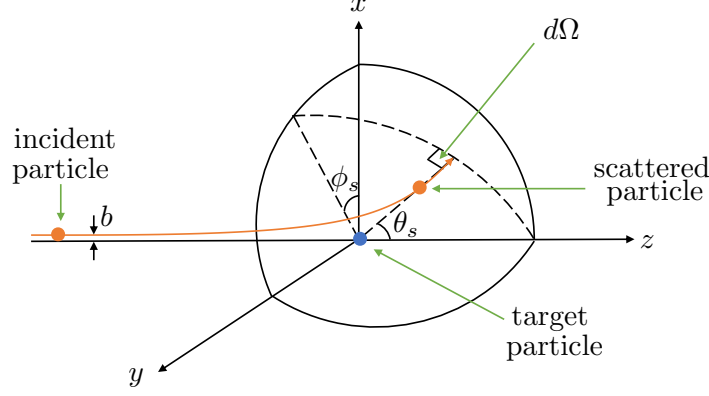


Figure 6.3: Depiction of particle scattering in 3D.

dimensional scattering, the encounter is as shown in fig. 6.3. Note that in that figure the incident particle starts within the  $x - z$  plane, and after scattering the particle is confined to a plane that is tilted an angle  $\phi_s$  from the  $x - z$  plane.

From the entire set of incident particles  $N$  that interact with the target particles, one can define an infinitesimal subset  $N_{\theta,\phi}d\Omega$  as the number of particles scattered into an infinitesimal solid angle  $d\Omega = \sin\theta_s d\theta_s d\phi_s$ , as shown in fig. 6.3. We note that  $N_{\theta,\phi} = N_{\theta,\phi}(\theta_s, \phi_s)$ . We introduce the differential cross section

$$\frac{d\sigma_{\theta,\phi}}{d\Omega} = \frac{d\sigma_{\theta,\phi}}{d\Omega}(\theta_s, \phi_s), \quad (6.6)$$

which is defined by the following expression in an analogous manner to eq. (6.1),

$$N_{\theta,\phi}d\Omega = \left( \frac{d\sigma_{\theta,\phi}}{d\Omega} d\Omega \right) I n_A. \quad (6.7)$$

It is best to not think of it as a derivative (what does a derivative with respect to solid angle mean?) and instead to simply think of it as a function that depends on  $\theta_s$  and  $\phi_s$ . Integrating over all  $\theta_s$  and  $\phi_s$ , i.e.

$$\int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} N_{\theta,\phi} d\Omega = \int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} \frac{d\sigma_{\theta,\phi}}{d\Omega} d\Omega I n_A$$

gives eq. (6.1).

For various cases the scattering is axi-symmetric, that is, it is independent of  $\phi_s$ . Thus

$$N_{\theta,\phi} \rightarrow N_{\theta} \quad \frac{d\sigma_{\theta,\phi}}{d\Omega} \rightarrow \frac{d\sigma_{\theta}}{d\Omega},$$

where

$$N_{\theta} = N_{\theta}(\theta_s),$$

and

$$\frac{d\sigma_{\theta}}{d\Omega} = \frac{d\sigma_{\theta}}{d\Omega}(\theta_s).$$

Integrating eq. (6.7) from  $\phi_s = 0$  to  $\phi_s = 2\pi$  gives

$$N_{\theta} d\Omega_{\theta} = \frac{d\sigma_{\theta}}{d\Omega} d\Omega_{\theta} I n_A, \quad (6.8)$$

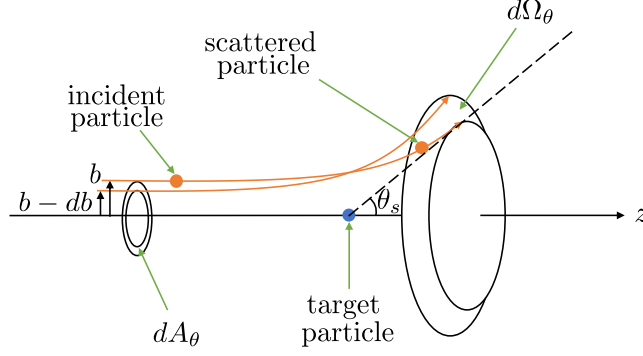


Figure 6.4: Depiction of particle scattering in for axi-symmetric interactions.

where  $d\Omega_\theta = 2\pi \sin \theta_s d\theta_s$ .  $N_\theta d\Omega_\theta$  thus represents the number of particles that are scattered into the infinitesimal band  $d\Omega_\theta$  on a sphere, where  $d\Omega_\theta$  is defined by scattering angle  $\theta_s$  (see fig. 6.4). We will note that there is a one-to-one correspondence between the impact parameter  $b$  and the scattering angle  $\theta_s$ , that is,  $b = b(\theta_s)$ . In other words, any incident particle scattered out through the infinitesimal band  $d\Omega_\theta$  would have approached the target-particle through the infinitesimal ring  $dA_\theta$  that corresponds to  $d\Omega_\theta$ . Since there are many target particles, there are many  $dA_\theta$ 's that correspond to the same  $d\Omega_\theta$ . Thus, the total number of particles scattered through  $d\Omega_\theta$  is given by all the incident particles that cross through the  $dA_\theta$ 's of all the target particles.

The number of incident particles crossing all the infinitesimal rings  $dA_\theta$  is equal to the total number of incident particles  $I$  times the probability  $P$  that any single incident particle will cross one of those rings. Thus, we can write

$$N_\theta d\Omega_\theta = IP.$$

The probability that an incident particle will cross one of those rings is simply the ratio of the surface area covered by all the rings in a section of the target material to the total area of that section. The surface area covered by all the rings in a section of area  $S$  is given by  $(n_A S) dA_\theta$ . Thus,  $P = n_A dA_\theta$  and

$$N_\theta d\Omega_\theta = I n_A dA_\theta.$$

We now introduce the differential

$$db = \frac{db}{d\theta_s} d\theta_s. \quad (6.9)$$

We note that by definition  $d\theta_s$  is positive but  $db$  can be either positive or negative depending on the sign of  $db/d\theta_s$ . The infinitesimal area  $dA_\theta$  is then given by

$$dA_\theta = 2\pi b |db| = 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s. \quad (6.10)$$

Thus,

$$N_\theta d\Omega_\theta = n_A I 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s.$$

Using eq. (6.8) in the above, we get

$$\frac{d\sigma_\theta}{d\Omega} d\Omega_\theta I n_A = I n_A 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s,$$

or

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin \theta_s} \left| \frac{db}{d\theta_s} \right|. \quad (6.11)$$

## 6.2 Coulomb collisions

### 6.2.1 Particle equations

Consider two particles, with positions  $\mathbf{r}_1 = \mathbf{r}_1(t)$  and  $\mathbf{r}_2 = \mathbf{r}_2(t)$ , velocities  $\mathbf{v}_1 = \mathbf{v}_1(t)$  and  $\mathbf{v}_2 = \mathbf{v}_2(t)$ , charges  $q_1$  and  $q_2$ , and masses  $m_1$  and  $m_2$ , respectively. Their positions and velocities are governed by the following equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1, \quad (6.12)$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2, \quad (6.13)$$

$$m_1 \frac{d\mathbf{v}_1}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \quad (6.14)$$

$$m_2 \frac{d\mathbf{v}_2}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \quad (6.15)$$

We note that the above system consists of twelve equations for twelve unknowns. We now introduce the center-of-mass position  $\mathbf{R} = \mathbf{R}(t)$ , the center-of-mass velocity  $\mathbf{V} = \mathbf{V}(t)$ , the shifted position  $\mathbf{r} = \mathbf{r}(t)$  and the shifted velocity  $\mathbf{v} = \mathbf{v}(t)$  as follows

$$\begin{aligned} \mathbf{R} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} & \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2, \\ \mathbf{V} &= \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} & \mathbf{v} &= \mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

Thus, in terms of these new four variables, the particle equations can be written as

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}, \quad (6.16)$$

$$\frac{d\mathbf{V}}{dt} = 0, \quad (6.17)$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad (6.18)$$

$$\frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r}}{r^3}, \quad (6.19)$$

where the reduced mass  $m_r$  is given by

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}. \quad (6.20)$$

The first two equations above give the trivial solution  $\mathbf{V} = \text{constant}$  and  $\mathbf{R} = \mathbf{R}(0) + \mathbf{V}t$ . Thus, we have reduced the problem from twelve unknowns to six unknowns, namely  $\mathbf{r}$  and  $\mathbf{v}$ .



### 6.2.2 Conservation of energy and momentum

Dotting eq. (6.19) by  $\mathbf{v}$  gives

$$\begin{aligned}
\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \mathbf{v} \cdot \frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{1}{2} \frac{dr^2}{dt} \frac{1}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{1}{r^2} \frac{dr}{dt} \\
&= -\frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{d}{dt} \left( \frac{1}{r} \right).
\end{aligned}$$

For the left hand side above we have

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{1}{2} \frac{dv^2}{dt},$$

and thus we obtain the following expression for conservation of energy

$$\frac{d}{dt} \left( \frac{1}{2} m_r v^2 + \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r} \right) = 0.$$

Crossing eq. (6.19) by  $\mathbf{r}$  gives

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0,$$

and thus

$$\frac{d}{dt} [m_r (\mathbf{r} \times \mathbf{v})] = 0.$$

That is, angular momentum is conserved. A consequence of this is that the vector  $\mathbf{r} \times \mathbf{v}$  is always pointing in the same direction. Thus, if  $\mathbf{r}(0)$  and  $\mathbf{v}(0)$  form a plane, then  $\mathbf{r}(t)$  and  $\mathbf{v}(t)$  need to reside within that same plane for all times  $t$  so that  $\mathbf{r}(t) \times \mathbf{v}(t)$  points in the same direction as  $\mathbf{r}(0) \times \mathbf{v}(0)$ . Therefore, the evolution of the position and velocity are confined to a plane and the problem can be reduced from six unknowns to four unknowns. This planar encounter is depicted in fig. 6.5.

If we refer to the plane shown in fig. 6.5 as the  $x - y$  plane, then one can tell that the angular-momentum vector points in the negative  $z$  direction. We will denote the magnitude of the conserved angular momentum by  $L$ , and thus we can write

$$m_r (\mathbf{r} \times \mathbf{v}) = -L \hat{\mathbf{z}}. \quad (6.21)$$

A consequence of both conservation of energy and momentum is as follows. Consider the two limiting states of particle 1—the initial state  $v_i, b_i$  and the final state  $v_f, b_f$ . Assuming the potential energy is very low at sufficiently early and late times, conservation of energy gives

$$\frac{1}{2} m_r v_i^2 = \frac{1}{2} m_r v_f^2, \quad (6.22)$$

that is,  $v_i = v_f$  (note that for other scattering processes, e.g. Compton scattering, this is not necessarily the case). For the angular momentum of the initial state we have

$$\begin{aligned}
m_r (\mathbf{r}_i \times \mathbf{v}_i) &= m_r \sin(-\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\pi - \varphi_i) r_i v_i \hat{\mathbf{z}} \\
&= -m_r \sin(\varphi_i) r_i v_i \hat{\mathbf{z}} = -m_r b_i v_i \hat{\mathbf{z}} \quad (6.23)
\end{aligned}$$

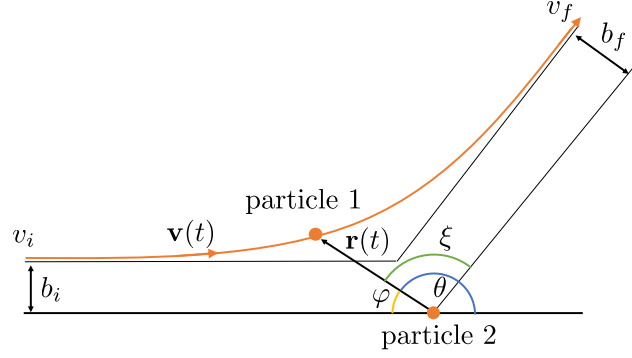


Figure 6.5: Depiction of Coulomb scattering.

Similarly, for the angular momentum of the final state we have

$$m_r (\mathbf{r}_f \times \mathbf{v}_f) = m_r \sin(-\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r \sin(\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r b_f v_f \hat{\mathbf{z}}. \quad (6.24)$$

Equating the last two relationships gives  $m_r b_i v_i = m_r b_f v_f$ . Since  $v_i = v_f$ , we finally have  $b_i = b_f = b$ . Using eq. (6.21) in eq. (6.23), we can also write

$$L = m_r b v_i. \quad (6.25)$$

### 6.2.3 Polar coordinates

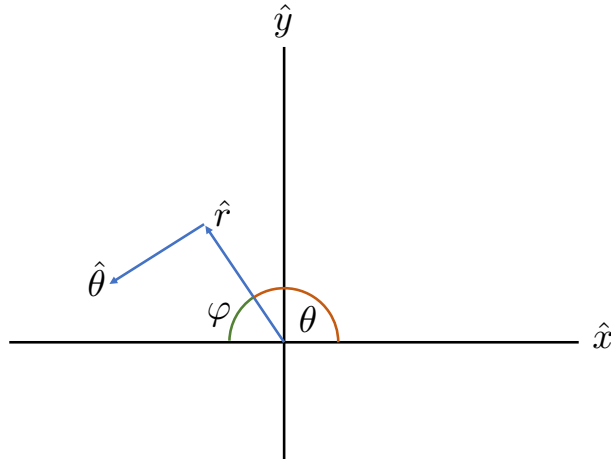


Figure 6.6: Polar coordinates in plane of interaction.

Using polar coordinates, as shown in fig. 6.6, we get

$$r_x = r \cos \theta = r \cos(\pi - \varphi) = -r \cos \varphi,$$

$$r_y = r \sin \theta = r \sin(\pi - \varphi) = r \sin \varphi.$$

Also, since  $\mathbf{r} = r\hat{\mathbf{r}}$ , we have

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},\end{aligned}$$

and

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}.\end{aligned}$$

The radial component of eq. (6.19) thus becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0m_r}\frac{1}{r^2}.$$

Since  $\theta = \pi - \varphi$ , we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0m_r}\frac{1}{r^2}. \quad (6.26)$$

For the angular momentum we have

$$m_r\mathbf{r} \times \mathbf{v} = m_r r\hat{\mathbf{r}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right) = m_r r^2 \frac{d\theta}{dt}\hat{\mathbf{z}}$$

Using eq. (6.21), we can write the above as

$$m_r r^2 \frac{d\varphi}{dt} = L. \quad (6.27)$$

#### 6.2.4 Particle trajectory

The goal is to find the radial position of the particle as a function of its angular orientation. That is, we want to find  $\tilde{r} = \tilde{r}(\tilde{\varphi})$  such that

$$r(t) = \tilde{r}(\varphi(t)). \quad (6.28)$$

To simplify the math, we introduce  $\tilde{u} = \tilde{u}(\tilde{\varphi})$  such that  $\tilde{u} = 1/\tilde{r}$ . Thus

$$\frac{d\tilde{u}}{d\tilde{\varphi}} = -\frac{1}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{\varphi}},$$

or, after re-arranging

$$\frac{d\tilde{r}}{d\tilde{\varphi}} = -\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}. \quad (6.29)$$

We now proceed as follows. Taking the derivative of  $r$ , we get

$$\begin{aligned}
\frac{dr}{dt} &= \left( \frac{d\tilde{r}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} & [eq. (6.28)] \\
&= \left( -\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} & [eq. (6.29)] \\
&= \left( -\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} & [eq. (6.27)] \\
&= \left( -\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} & [eq. (6.28)] \\
&= \left( -\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} & (6.30)
\end{aligned}$$

Taking the derivative of the above, we get

$$\begin{aligned}
\frac{d}{dt} \frac{dr}{dt} &= \left[ \frac{d}{d\tilde{\varphi}} \left( -\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right) \right]_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \\
&= \left( -\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} & [eq. (6.27)] \\
&= \left( -\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} & [eq. (6.28)] \\
&= \left( -\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2 \tilde{u}^2}{m_r^2} \right)_{\tilde{\varphi}=\varphi(t)} & (6.31)
\end{aligned}$$

Plugging the last relation into eq. (6.26) gives

$$\left[ -\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2 \tilde{u}^2}{m_r^2} - \frac{1}{\tilde{u}} \left( \frac{L \tilde{u}^2}{m_r} \right)^2 \right]_{\tilde{\varphi}=\varphi(t)} = \left( \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \tilde{u}^2 \right)_{\tilde{\varphi}=\varphi(t)},$$

which, upon re-arranging and dropping the  $\varphi(t)$  dependance, becomes

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2 m_r}{4\pi\epsilon_0 L^2} \quad (6.32)$$

Using eq. (6.25) we write the evolution equation for  $\tilde{u}$  as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2}{4\pi\epsilon_0 m_r b^2 v_i^2}. \quad (6.33)$$

Introducing the notation

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r v_i^2}, \quad (6.34)$$

the evolution equation for  $\tilde{u}$  can be simply expressed as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{b_{90}}{b^2}. \quad (6.35)$$

The boundary conditions for eq. (6.35) are as follows

$$\text{as } \varphi(t) \rightarrow 0, \quad r(t) \rightarrow \infty \quad (6.36)$$

$$\text{as } \varphi(t) \rightarrow 0, \quad \frac{dr(t)}{dt} \rightarrow -v_i \quad (6.37)$$

Given eq. (6.28), eq. (6.36) can only be satisfied if as  $\tilde{\varphi} \rightarrow 0$ ,  $\tilde{r} \rightarrow \infty$ . Thus, we also have, as  $\tilde{\varphi} \rightarrow 0$ ,  $\tilde{u} \rightarrow 0$ . Similarly, given eq. (6.30), eq. (6.37) can only be satisfied if as  $\tilde{\varphi} \rightarrow 0$

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \rightarrow v_i.$$

Using eq. (6.25) we rewrite the above as

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \rightarrow \frac{1}{b}.$$

The general solution to eq. (6.35) is

$$\tilde{u} = A \cos \tilde{\varphi} + B \sin \tilde{\varphi} - \frac{b_{90}}{b^2}.$$

Applying the boundary conditions, we get

$$\tilde{u} = \frac{b_{90}}{b^2} \cos \tilde{\varphi} + \frac{1}{b} \sin \tilde{\varphi} - \frac{b_{90}}{b^2},$$

which we finally re-write as

$$\frac{1}{\tilde{r}} = \frac{1}{b} \sin \tilde{\varphi} + \frac{b_{90}}{b^2} (\cos \tilde{\varphi} - 1). \quad (6.38)$$

### 6.2.5 The scattering angle

We now drop the tilde notation for the sake of simplicity. That is, for the radial location of an incident particle, we have

$$\frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{b_{90}}{b^2} (\cos \varphi - 1), \quad (6.39)$$

where  $\varphi$  is the independent variable and  $r = r(\varphi)$ . We want to know the value of  $\varphi$  as  $r$  goes to infinity. Using eq. (6.39), and labeling this angle as  $\varphi_s$ , we have

$$0 = \sin \varphi_s + \frac{b_{90}}{b} (\cos \varphi_s - 1).$$

We express the above in terms of the scattering angle  $\theta_s = \pi - \varphi_s$ ,

$$0 = \sin(\pi - \theta_s) + \frac{b_{90}}{b} [\cos(\pi - \theta_s) - 1].$$

or

$$0 = \sin \theta_s + \frac{b_{90}}{b} (-\cos \theta_s - 1).$$

Re-writing the above as

$$\frac{\cos \theta_s + 1}{\sin \theta_s} = \frac{b}{b_{90}},$$

and using the trig identity  $\cot(\theta/2) = (\cos \theta + 1)/\sin \theta$ , we get

$$\cot\left(\frac{\theta_s}{2}\right) = \frac{b}{b_{90}}. \quad (6.40)$$

### 6.2.6 The differential cross section

The differential cross section for Coulomb scattering can be computed by making use of eq. (6.11), which is repeated below

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin \theta_s} \left| \frac{db}{d\theta_s} \right|. \quad (6.11)$$

From eq. (6.40) we get,

$$\frac{db}{d\theta} = -\frac{b_{90}}{2} \frac{1}{\sin^2(\theta_s/2)}, \quad (6.41)$$

which, plugging in eq. (6.11), gives

$$\frac{d\sigma_\theta}{d\Omega} = \left[ b_{90} \frac{\cot(\theta_s/2)}{\sin \theta_s} \right] \left[ \frac{b_{90}}{2} \frac{1}{\sin^2(\theta_s/2)} \right].$$

Using the trig identities  $\cot(\theta) = \cos(\theta)/\sin(\theta)$  and  $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$  we get

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b_{90}^2}{4} \frac{1}{\sin^4(\theta_s/2)}. \quad (6.42)$$

### 6.2.7 Collision integral

$$\Omega_{\alpha\beta}^{(lk)} = \sqrt{\frac{k_B T}{2\pi M_{\alpha\beta}}} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} dg. \quad (6.43)$$

In the above  $M_{\alpha\beta}$  is the reduced mass, given by

$$M_{\alpha\beta} = \frac{M_\alpha M_\beta}{M_\alpha + M_\beta}, \quad (6.44)$$

and  $\phi_{\alpha\beta}^{(l)}$  is the collision cross section for a given velocity, and is computed as

$$\phi_{\alpha\beta}^{(l)} = 2\pi \int_0^\infty \left(1 - \cos^l \chi_{\alpha\beta}\right) b db. \quad (6.45)$$

The scattering angle  $\chi_{\alpha\beta}$  is given by

$$\chi_{\alpha\beta} = \pi - 2 \int_{r_{\alpha\beta}^{\min}}^\infty \frac{b}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{\alpha\beta}(r)}{g^2 k_B T}\right]^{1/2}} dr. \quad (6.46)$$

For a Coulombic interaction between ions, we can define the natural scale for the cross-sectional area as

$$\phi_{\alpha\beta}^{(0)} = \frac{\pi (Z_\alpha Z_\beta e^2)^2}{(2k_B T)^2}. \quad (6.47)$$

Given this definition, we express the collision integral as

$$\Omega_{\alpha\beta} = \sqrt{\frac{\pi}{M_{\alpha\beta}}} \frac{(Z_\alpha Z_\beta e^2)^2}{(2k_B T)^{3/2}} \mathcal{F}_{\alpha\beta}^{lk}, \quad (6.48)$$

where

$$\mathcal{F}_{\alpha\beta}^{(lk)} = \frac{1}{2\phi_0} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} dg \quad (6.49)$$

We note that  $\mathcal{F}_{\alpha\beta}^{(lk)} = 4\mathcal{K}_{lk}(g_{\alpha\beta})$ , where  $\mathcal{K}_{lk}(g_{\alpha\beta})$  is the notation from the Stanton-Murillo paper.

# Chapter 7

## Multi-material formulation

### 7.1 Multi-material vs. multi-fluid vs. multi-species

Multi-material, multi-fluid, and multi-species refer to different aspects of a fluid model. These are not mutually exclusive, for example, one can have a multi-material multi-fluid model, or a multi-material single-fluid model. Below are some defining characteristics:

1. Multi-species: there is more than one species.
2. Multi-fluid: a velocity equation is solved for each species.
3. Multi-material: the species are grouped into bunches that share some general properties, such as EOS and thermal conductivity. Both a multi-fluid and a single-fluid formulation can each be multi-material.

We note that multi-fluid or multi-material problems are always multi-species. Single-fluid or single-material problems can also be multi-species, but are not necessarily always multi-species. Thus, we use the multi-species label only when dealing with the single-material single-fluid model, to clarify whether we are using that model for cases where there are multiple species or not.

### 7.2 Definition of multi-material variables

We define a material  $k$  as a collection of species  $s$ . Rather than labeling species using the single subscript  $s$ , we'll use the subscript  $k, i, s$  and  $k, e$ , where the former denotes the ion species  $s$  that belongs to material  $k$ , and the latter denotes the electron species that belongs to material  $k$ . The number of ion species within a material is denoted as  $\mathcal{N}_k$ , and the total number of materials as  $\mathcal{M}$ .

For multi-material problems there is a large number of new variables that are used. These are listed below.

#### Mass densities

$$\rho = \frac{M}{V}. \quad (7.1)$$

$$\rho_k = \frac{M_k}{V_k}. \quad (7.2)$$

$$\rho_{k,i,s} = \frac{M_{k,i,s}}{V_{k,i,s}}. \quad (7.3)$$

$$\rho_{k,e} = \frac{M_{k,e}}{V_k}. \quad (7.4)$$

### Number densities

$$n = \frac{N}{V}. \quad (7.5)$$

$$n_k = \frac{N_k}{V_k}. \quad (7.6)$$

$$n_{k,i,s} = \frac{N_{k,i,s}}{V_{k,i,s}}. \quad (7.7)$$

$$n_{k,e} = \frac{N_{k,e}}{V_k}. \quad (7.8)$$

### Relation between mass and number densities

$$m_{k,i,s} n_{k,i,s} = \rho_{k,i,s}. \quad (7.9)$$

$$m_e n_{k,e} = \rho_{k,e}. \quad (7.10)$$

### Material volume and mass fractions

$$\eta_k = \frac{V_k}{V}, \quad (7.11)$$

$$Y_k = \frac{M_k}{M}. \quad (7.12)$$

$$\rho_k = \rho \frac{Y_k}{\eta_k}. \quad (7.13)$$

### Species volume and mass fractions

$$\eta_{k,i,s} = \frac{V_{k,i,s}}{V_k}. \quad (7.14)$$

$$Y_{k,i,s} = \frac{M_{k,i,s}}{M_k}. \quad (7.15)$$

$$\rho_{k,i,s} = \rho_k \frac{Y_{k,i,s}}{\eta_{k,i,s}}. \quad (7.16)$$

### Energies

$$e = \frac{IE}{M}. \quad (7.17)$$

$$e_k = \frac{IE_k}{M_k}. \quad (7.18)$$

$$e_{k,i,s} = \frac{IE_{k,i,s}}{M_{k,i,s}}. \quad (7.19)$$

$$e_{k,e} = \frac{IE_{k,e}}{M_k}. \quad (7.20)$$



### Total ion variables

$$\eta_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \approx 1, \quad (7.21)$$

$$Y_{k,i} = \sum_s^{\mathcal{N}_k} Y_{k,i,s} \approx 1, \quad (7.22)$$

$$n_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} n_{k,i,s}. \quad (7.23)$$

$$e_{k,i} = \sum_s^{\mathcal{N}_k} Y_{k,i,s} e_{k,i,s}. \quad (7.24)$$

$$h_{k,i} = \sum_s^{\mathcal{N}_k} Y_{k,i,s} h_{k,i,s}. \quad (7.25)$$

$$p_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} p_{k,i,s}. \quad (7.26)$$

$$\rho_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \rho_{k,i,s}. \quad (7.27)$$

$$\mathbf{t}_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \mathbf{t}_{k,i,s}. \quad (7.28)$$

$$\boldsymbol{\sigma}_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s}. \quad (7.29)$$

$$\mathbf{R}_{k,i} = \sum_s^{\mathcal{N}_k} \mathbf{R}_{k,i,s}. \quad (7.30)$$

$$\mathbf{q}_{k,i} = \sum_s^{\mathcal{N}_k} \mathbf{q}_{k,i,s}. \quad (7.31)$$

$$Q_{k,i} = \sum_s^{\mathcal{N}_k} Q_{k,i,s}. \quad (7.32)$$

### 7.3 Interpretation of multi-material variables

Fluid variables are defined at every point  $\mathbf{x}$  within the domain containing the fluid. Thus, in the fluid approach we do not account for the microscopic distribution of the fluid's atoms and molecules, which are not necessarily present at every point  $\mathbf{x}$ . That being said, we can establish a correlation between the fluid variables and atomistic variables, which define the underlying microscopic structure. Examples of these atomistic variables include  $m_i$ ,  $\mathbf{v}_i$ , and  $\epsilon_i$ , which are the mass, velocity, and internal energy of the  $i^{\text{th}}$  atom or molecule in the system.

Consider the total mass  $M_\Omega$  in some subdomain  $\Omega$ . This mass can be computed by summing up the masses  $m_i$  of the atoms or molecules in this subdomain, or by using the Eulerian field variable for density, as shown below

$$M_\Omega = \sum_{i \in \Omega} m_i = \int_\Omega \rho \, d\mathbf{x}. \quad (7.33)$$

In the above,  $i \in \Omega$  denotes the atoms or molecules within the subdomain  $\Omega$ . Now consider the total momentum  $P_\Omega$  in some subdomain  $\Omega$ . This momentum can be computed by summing up the momentum  $m_i \mathbf{u}_i$  of the atoms or molecules in this subdomain, or by using the Eulerian field variable for velocity, as shown below

$$P_\Omega = \sum_{i \in \Omega} m_i \mathbf{v}_i = \int_\Omega \rho \mathbf{u} \, d\mathbf{x}. \quad (7.34)$$

Finally, consider the internal energy  $IE_\Omega$  in some subdomain  $\Omega$ . This internal energy can be computed by summing up the internal energies  $e_i$  of the atoms or molecules in this subdomain, or by using the Eulerian field variable for the specific internal energy, as shown below

$$IE_\Omega = \sum_{i \in \Omega} \epsilon_i = \int_\Omega \rho e \, d\mathbf{x}. \quad (7.35)$$

Denote the volume of domain  $\Omega$  as  $V_\Omega$ , which is given by

$$V_\Omega = \int_\Omega d\mathbf{x}. \quad (7.36)$$

We can now write

$$\rho = \lim_{\Omega \rightarrow \epsilon} \frac{1}{V_\Omega} \int_\Omega \rho \, d\mathbf{x} = \lim_{\Omega \rightarrow \epsilon} \frac{1}{V_\Omega} \sum_{i \in \Omega} m_i = \lim_{\Omega \rightarrow \epsilon} \frac{M_\Omega}{V_\Omega}. \quad (7.37)$$

Thus, the density is the mass contained in a small domain divided by the volume of that small domain as the domain becomes sufficiently small. Similarly for velocity, we have

$$\mathbf{u} = \frac{\rho \mathbf{v}}{\rho} = \lim_{\Omega \rightarrow \epsilon} \frac{\frac{1}{V_\Omega} \int_\Omega \rho \mathbf{u} \, d\mathbf{x}}{\frac{1}{V_\Omega} \int_\Omega \rho \, d\mathbf{x}} = \lim_{\Omega \rightarrow \epsilon} \frac{\sum_{i \in \Omega} m_i v^{(i)}}{\sum_{i \in \Omega} m_i} = \lim_{\Omega \rightarrow \epsilon} \frac{P_\Omega}{M_\Omega}. \quad (7.38)$$

That is, the velocity is the the momentum contained in a small domain divided by the mass contained in that small domain as the domain becomes sufficiently small. Finally, for energy we have

$$e = \frac{\rho e}{\rho} = \lim_{\Omega \rightarrow \epsilon} \frac{\frac{1}{V_\Omega} \int_\Omega \rho e \, d\mathbf{x}}{\frac{1}{V_\Omega} \int_\Omega \rho \, d\mathbf{x}} = \lim_{\Omega \rightarrow \epsilon} \frac{\sum_{i \in \Omega} e_i}{\sum_{i \in \Omega} m_i} = \lim_{\Omega \rightarrow \epsilon} \frac{IE_\Omega}{M_\Omega}. \quad (7.39)$$

That is, the specific internal energy is the internal energy contained in a small subdomain divided by the mass contained in that small subdomain as the subdomain becomes sufficiently small.

In the above, a sufficiently small subdomain is one that is small compared to the smallest length scale in the flow field but large enough that it contains a sufficiently large number of atoms or molecules.

Now consider the volume  $V_{\Omega,k}$  occupied by material  $k$  in some subdomain  $\Omega$ . This can be computed by using the volume fraction  $\eta_k = \eta_k(t, \mathbf{x})$  as shown below

$$V_{\Omega,k} = \int_\Omega \eta_k \, d\mathbf{x}. \quad (7.40)$$

Now consider the mass  $M_{\Omega,k}$  of material  $k$  in some subdomain  $\Omega$ . This can be computed by summing up the masses  $m_i$  of the atoms or molecules of material  $k$  in this subdomain, or by using the mass fraction  $Y_k = Y_k(t, \mathbf{x})$ , as shown below

$$M_{\Omega,k} = \sum_{i \in \Omega, i \in k} m_i = \int_{\Omega} \rho Y_k d\mathbf{x}. \quad (7.41)$$

In the above  $i \in \Omega, i \in k$  denotes the atoms or molecules of material  $k$  located in the subdomain  $\Omega$ . Finally, consider the internal energy  $IE_{\Omega,k}$  of material  $k$  in some subdomain  $\Omega$ . This internal energy can be computed by summing up the internal energies  $\epsilon_i$  of the atoms or molecules of material  $k$  in this subdomain, or by using the specific internal energy  $e_k = e_k(t, \mathbf{x})$ , as shown below

$$IE_{\Omega,k} = \sum_{i \in \Omega, i \in k} \epsilon_i = \int_{\Omega} \rho Y_k e_k d\mathbf{x}. \quad (7.42)$$

We can now write

$$\eta_k = \lim_{\Omega \rightarrow \epsilon} \frac{1}{V_{\Omega}} \int_{\Omega} \eta_k d\mathbf{x} = \lim_{\Omega \rightarrow \epsilon} \frac{V_{\Omega,k}}{V_{\Omega}}. \quad (7.43)$$

That is, the volume fraction is the volume occupied by material  $k$  within a small domain divided by the volume of that small domain as the domain becomes sufficiently small. For mass fraction, we have

$$Y_k = \frac{\rho Y_k}{\rho} = \lim_{\Omega \rightarrow \epsilon} \frac{\frac{1}{V_{\Omega}} \int_{\Omega} \rho Y_k d\mathbf{x}}{\frac{1}{V_{\Omega}} \int_{\Omega} \rho d\mathbf{x}} = \lim_{\Omega \rightarrow \epsilon} \frac{\sum_{i \in \Omega, i \in k} m_i}{\sum_{i \in \Omega} m_i} = \lim_{\Omega \rightarrow \epsilon} \frac{M_{\Omega,k}}{M_{\Omega}}. \quad (7.44)$$

That is, the mass fraction is the mass of material  $k$  contained in a small domain divided by the mass contained in that small domain as the domain becomes sufficiently small. Similarly for specific internal energy of an material, we have

$$e_k = \frac{\rho Y_k e_k}{\rho Y_k} = \lim_{\Omega \rightarrow \epsilon} \frac{\frac{1}{V_{\Omega}} \int_{\Omega} \rho Y_k e_k d\mathbf{x}}{\frac{1}{V_{\Omega}} \int_{\Omega} \rho Y_k d\mathbf{x}} = \lim_{\Omega \rightarrow \epsilon} \frac{\sum_{i \in \Omega, i \in k} e_i}{\sum_{i \in \Omega, i \in k} m_i} = \lim_{\Omega \rightarrow \epsilon} \frac{IE_{\Omega,k}}{M_{\Omega,k}}. \quad (7.45)$$

That is, the specific internal energy of an material is the internal energy of material  $k$  contained in a small domain divided by the mass of material  $k$  in that small domain as the domain becomes sufficiently small.

## 7.4 Multi-fluid formulation

The following assumptions are used:

1. No sources ( $\hat{S}_s = \hat{\mathbf{M}}_s = \hat{Q}_s = 0$  for all  $s$ ).

### 7.4.1 Multi-material multi-fluid model

We note that the number densities in section 1.3 use as their denominator the total volume. We would instead like to express densities using variables such as that defined in eq. (7.7), where the denominator is only the volume of the species under consideration. The correspondence between these two densities is  $n_s \rightarrow \eta_k \eta_{k,i,s} n_{k,i,s}$ . Using an ideal equation of state, we have

$$p_i = n_i k_B T_i \rightarrow \eta_k \eta_{k,i,s} n_{k,i,s} k_B T_{k,i,s} = \eta_k \eta_{k,i,s} p_{k,i,s}, \quad (7.46)$$

where  $p_{k,i,s} = n_{k,i,s} k_B T_{k,i,s}$ . The same argument above applies to the stress tensor since it is defined in terms of the pressure. Finally, the analogous is used for electron quantities as well. Thus, the governing equations in section 1.3 are now written as

$$\frac{\partial \eta_k \eta_{k,i,s} n_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s}) = 0, \quad (7.47)$$

$$\frac{\partial \eta_k n_{k,e}}{\partial t} + \nabla \cdot (\eta_k n_{k,e} \mathbf{u}_{k,e}) = 0, \quad (7.48)$$

$$\begin{aligned} \frac{\partial m_{k,i,s} \eta_k \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (m_{k,i,s} \eta_k \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s}) \\ - Z_{k,i,s} e \eta_k \eta_{k,i,s} n_{k,i,s} (\mathbf{E} + \mathbf{u}_{k,i,s} \times \mathbf{B}) = \nabla \cdot (\eta_k \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s}) + \mathbf{R}_{k,i,s}, \end{aligned} \quad (7.49)$$

$$\begin{aligned} \frac{\partial m_{k,e} \eta_k n_{k,e} \mathbf{u}_{k,e}}{\partial t} + \nabla \cdot (m_{k,e} \eta_k n_{k,e} \mathbf{u}_{k,e} \mathbf{u}_{k,e}) \\ + e \eta_k n_{k,e} (\mathbf{E} + \mathbf{u}_{k,e} \times \mathbf{B}) = \nabla \cdot (\eta_k \boldsymbol{\sigma}_{k,e}) + \mathbf{R}_{k,e}, \end{aligned} \quad (7.50)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k \eta_{k,i,s} p_{k,i,s} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k \eta_{k,i,s} p_{k,i,s} \mathbf{u}_{k,i,s} \right) = \eta_k \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} : \nabla \mathbf{u}_{k,i,s} - \nabla \cdot \mathbf{q}_{k,i,s} + Q_{k,i,s}, \quad (7.51)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k p_{k,e} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,e} \mathbf{u}_{k,e} \right) = \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u}_{k,e} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e}, \quad (7.52)$$

$$\mathbf{J}_k = e \eta_k \left( \sum_s \mathcal{N}_k Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e} \right), \quad (7.53)$$

$$\mathbf{J} = \sum_k^{\mathcal{M}} \mathbf{J}_k \quad (7.54)$$

$$\rho_{q,k} = e \eta_k \left( \sum_s \mathcal{N}_k Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} - n_{k,e} \right), \quad (7.55)$$

$$\rho_q = \sum_k^{\mathcal{M}} \rho_{q,k}, \quad (7.56)$$

$$p_{k,i,s} = n_{k,i,s} k_B T_{k,i,s}, \quad (7.57)$$

$$p_{k,e} = n_{k,e} k_B T_{k,e}. \quad (7.58)$$

#### 7.4.2 Single-material multi-fluid model

The equations are the same as the multi-material formulation, but there is only one material now and hence the  $k$  subscript can be dropped and all  $\eta_k = 1$ . Thus we have

$$\frac{\partial \eta_{i,s} n_{i,s}}{\partial t} + \nabla \cdot (\eta_{i,s} n_{i,s} \mathbf{u}_{i,s}) = 0, \quad (7.59)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (7.60)$$

$$\frac{\partial m_{i,s} \eta_{i,s} n_{i,s} \mathbf{u}_{i,s}}{\partial t} + \nabla \cdot (m_{i,s} \eta_{i,s} n_{i,s} \mathbf{u}_{i,s} \mathbf{u}_{i,s}) - Z_{i,s} e \eta_{i,s} n_{i,s} (\mathbf{E} + \mathbf{u}_{i,s} \times \mathbf{B}) = \nabla \cdot (\eta_{i,s} \boldsymbol{\sigma}_{i,s}) + \mathbf{R}_{i,s}, \quad (7.61)$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e, \quad (7.62)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} \eta_{i,s} p_{i,s} \right) + \nabla \cdot \left( \frac{3}{2} \eta_{i,s} p_{i,s} \mathbf{u}_{i,s} \right) = \eta_{i,s} \boldsymbol{\sigma}_{i,s} : \nabla \mathbf{u}_{i,s} - \nabla \cdot \mathbf{q}_{i,s} + Q_{i,s}, \quad (7.63)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \nabla \cdot \left( \frac{3}{2} p_e \mathbf{u}_e \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \quad (7.64)$$

$$\mathbf{J} = e \left( \sum_s^{\mathcal{N}} Z_{i,s} \eta_{i,s} n_{i,s} \mathbf{u}_{i,s} - n_e \mathbf{u}_e \right), \quad (7.65)$$

$$\rho_q = e \left( \sum_s^{\mathcal{N}} Z_{i,s} \eta_{i,s} n_{i,s} - n_e \right), \quad (7.66)$$

$$p_{i,s} = n_{i,s} k_B T_{i,s}, \quad (7.67)$$

$$p_e = n_e k_B T_e. \quad (7.68)$$

### 7.4.3 Single-material two-fluid model

If there is only one ion species, then we have the two-fluid model. For this case  $\eta_{i,s} n_{i,s}$ ,  $\eta_{i,s} p_{i,s}$ , and  $\eta_{i,s} \boldsymbol{\sigma}_{i,s}$  simplify to  $\eta_i n_i$ ,  $\eta_i p_i$ , and  $\eta_i \boldsymbol{\sigma}_i$ , respectively. Since  $\eta_i \approx 1$ , we have the equations given in section 2.1.

## 7.5 Single-fluid formulation

A single-fluid formulation is defined as that for which there is a single velocity vector. For the single-fluid formulation, we'll use the following assumptions

1.  $m_{k,e} \approx 0$  since  $m_{k,e} \ll m_{k,i,s}$ .
2. Quasi-neutrality, i.e.  $\sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} = n_{k,e}$ .

### 7.5.1 Single-fluid variables

We now introduce the following single-fluid variables

$$\mathbf{u}_{k,i} = \sum_s^{\mathcal{N}_k} Y_{k,i,s} \mathbf{u}_{k,i,s} \quad (7.69)$$

$$\mathbf{u} = \sum_k^{\mathcal{M}} Y_k \mathbf{u}_{k,i}. \quad (7.70)$$

$$e_k = e_{k,i} + e_{k,e}. \quad (7.71)$$

$$e = \sum_k^{\mathcal{M}} Y_k e_k. \quad (7.72)$$

$$h_k = h_{k,i} + h_{k,e}. \quad (7.73)$$

$$h = \sum_k^{\mathcal{M}} Y_k h_k. \quad (7.74)$$

### 7.5.2 Total sum of collisional terms

In this section we derive the total sum across all species (ions and electrons) of the collisional terms  $\mathbf{R}_s$  and  $Q_s$ . Use eq. (1.44) to re-write  $\mathbf{R}_s$  as

$$\mathbf{R}_s = \sum_{r, r \neq s}^{\mathcal{N}} \int m_s \mathbf{w}_s C_{sr} d\mathbf{v} = \sum_{r, r \neq s}^{\mathcal{N}} \int m_s \mathbf{v} C_{sr} d\mathbf{v},$$

where  $\mathcal{N}$  is the total number of species. Using eq. (1.32) this leads to

$$\sum_s^{\mathcal{N}} \mathbf{R}_s = 0, \quad (7.75)$$

which, using our current notation, can be expressed as

$$\sum_k^{\mathcal{M}} \left( \sum_s^{\mathcal{N}_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e} \right) = 0. \quad (7.76)$$

Similarly, we use eq. (1.59) to re-write  $Q_s$  as

$$\begin{aligned} Q_s &= \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2} m_s (\mathbf{w}_s \cdot \mathbf{w}_s) C_{sr} d\mathbf{v} \\ &= \sum_{r, r \neq s}^{\mathcal{N}} \int \frac{1}{2} m_s (\mathbf{v} \cdot \mathbf{v}) C_{sr} d\mathbf{v} - \mathbf{u}_s \cdot \sum_{r, r \neq s}^{\mathcal{N}} \int m_s \mathbf{w}_s C_{sr} d\mathbf{v} \end{aligned}$$

Summing over all species and using eq. (1.33) leads to

$$\sum_s^{\mathcal{N}} Q_s = - \sum_s^{\mathcal{N}} \mathbf{u}_s \cdot \mathbf{R}_s, \quad (7.77)$$

which, using our current notation, can be expressed as

$$\sum_k^{\mathcal{M}} \left( \sum_s^{\mathcal{N}_k} Q_{k,i,s} + Q_{k,e} \right) = - \sum_k^{\mathcal{M}} \left( \sum_s^{\mathcal{N}_k} \mathbf{u}_{k,i,s} \cdot \mathbf{R}_{k,i,s} + \mathbf{u}_{k,e} \cdot \mathbf{R}_{k,e} \right). \quad (7.78)$$

### 7.5.3 The general model

#### Mass

Multiplying eq. (7.47) by  $m_{k,i,s}$  and using eq. (7.9) gives

$$\frac{\partial \eta_k \eta_{k,i,s} \rho_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \eta_{k,i,s} \rho_{k,i,s} \mathbf{u}_{k,i,s}) = 0.$$

Using eq. (7.16), the above becomes

$$\frac{\partial \eta_k \rho_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s}) = 0.$$

We introduce the species diffusion vector

$$\mathbf{j}_{k,i,s} = \eta_k \rho_k Y_{k,i,s} \mathbf{w}_{k,i,s}, \quad (7.79)$$

where

$$\mathbf{w}_{k,i,s} = \mathbf{u}_{k,i,s} - \mathbf{u}. \quad (7.80)$$

The mass-conservation equation is then written as

$$\frac{\partial \eta_k \rho_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \rho_k Y_{k,i,s} \mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i,s}.$$

Note that adding over all  $s$  of a material gives

$$\frac{\partial \eta_k \rho_k}{\partial t} + \nabla \cdot (\eta_k \rho_k \mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i},$$

where

$$\mathbf{j}_{k,i} = \eta_k \rho_k \mathbf{w}_{k,i}, \quad (7.81)$$

and

$$\mathbf{w}_{k,i} = \mathbf{u}_{k,i} - \mathbf{u}. \quad (7.82)$$

Multiplying eq. (7.47) by  $Z_{k,i,s}$  and summing over all  $s$  of a material gives

$$\frac{\partial}{\partial t} \left( \eta_k \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \right) + \nabla \cdot \left( \eta_k \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \right) = 0.$$

Subtracting eq. (7.48) from the above and using the assumption of quasi-neutrality in item 2 gives

$$\nabla \cdot \left( \eta_k \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e} \right) = 0,$$

which is equivalent to

$$\nabla \cdot \mathbf{J}_k = 0.$$

Summing over all  $k$  gives

$$\nabla \cdot \mathbf{J} = 0.$$

## Momentum

We now add eqs. (7.49) and (7.50) and use item 1 to obtain

$$\begin{aligned} & \frac{\partial m_{k,i,s} \eta_k \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (m_{k,i,s} \eta_k \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s}) - e \eta_k (Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} - n_{k,e}) \mathbf{E} \\ & - e \eta_k (Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e}) \times \mathbf{B} = \nabla \cdot [\eta_k (\eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e})] + \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}. \end{aligned}$$

Using eq. (7.9) the above becomes

$$\begin{aligned} & \frac{\partial \eta_k \eta_{k,i,s} \rho_{k,i,s} \mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \eta_{k,i,s} \rho_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s}) - e \eta_k (Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} - n_{k,e}) \mathbf{E} \\ & - e \eta_k (Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e}) \times \mathbf{B} = \nabla \cdot [\eta_k (\eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e})] + \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}. \end{aligned}$$

Using eq. (7.16) we get

$$\begin{aligned} & \frac{\partial \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s}) - e \eta_k (Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} - n_{k,e}) \mathbf{E} \\ & - e \eta_k (Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - n_{k,e} \mathbf{u}_{k,e}) \times \mathbf{B} = \nabla \cdot [\eta_k (\eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e})] + \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}. \end{aligned}$$

Summing over all  $s$  of a material, using the assumption of item 2 and the definition of the current in eq. (7.53) the above simplifies to

$$\begin{aligned} \frac{\partial}{\partial t} \sum_s^{\mathcal{N}_k} \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_s^{\mathcal{N}_k} \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}_{k,i,s} - \mathbf{J}_k \times \mathbf{B} = \\ \nabla \cdot \left[ \eta_k \left( \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) \right] + \sum_s^{\mathcal{N}_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}. \end{aligned}$$

We now write the momentum equation as

$$\begin{aligned} \frac{\partial}{\partial t} \sum_s^{\mathcal{N}_k} \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_s^{\mathcal{N}_k} [\eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u} + \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} (\mathbf{u}_{k,i,s} - \mathbf{u})] - \mathbf{J}_k \times \mathbf{B} = \\ \nabla \cdot \left[ \eta_k \left( \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) \right] + \sum_s^{\mathcal{N}_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}. \end{aligned}$$

Using the definition of  $\mathbf{j}_{k,i,s}$  the above becomes

$$\begin{aligned} \frac{\partial}{\partial t} \sum_s^{\mathcal{N}_k} \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_s^{\mathcal{N}_k} (\eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}) - \mathbf{J}_k \times \mathbf{B} = \\ \nabla \cdot \left[ \eta_k \left( \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) \right] - \nabla \cdot \sum_s^{\mathcal{N}_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s} + \sum_s^{\mathcal{N}_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e}. \end{aligned}$$

We now sum over all materials to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} + \nabla \cdot \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} (\eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \\ \nabla \cdot \sum_k^{\mathcal{M}} \left( \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) - \nabla \cdot \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s} + \sum_k^{\mathcal{M}} \left( \sum_s^{\mathcal{N}_k} \mathbf{R}_{k,i,s} + \mathbf{R}_{k,e} \right). \end{aligned}$$

We note that

$$\sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \eta_k \rho_k Y_{k,i,s} \mathbf{u}_{k,i,s} = \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \rho Y_k Y_{k,i,s} \mathbf{u}_{k,i,s} = \rho \mathbf{u}. \quad (7.83)$$

Using the above and eq. (7.76) we get

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k^{\mathcal{M}} \left[ \eta_k \left( \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} + \boldsymbol{\sigma}_{k,e} \right) \right] - \nabla \cdot \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s}.$$

Finally, we write the above as

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k^{\mathcal{M}} [\eta_k (\boldsymbol{\sigma}_{k,i} + \boldsymbol{\sigma}_{k,e})] - \nabla \cdot \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s},$$

or

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k^{\mathcal{M}} \eta_k \boldsymbol{\sigma}_k - \nabla \cdot \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \mathbf{j}_{k,i,s} \mathbf{u}_{k,i,s},$$



where  $\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_{k,i} + \boldsymbol{\sigma}_{k,e}$ .

Given the assumption in item 1, eq. (7.50) for electron momentum becomes

$$e\eta_k n_{k,e}(\mathbf{E} + \mathbf{u}_{k,e} \times \mathbf{B}) = \nabla \cdot (\eta_k \boldsymbol{\sigma}_{k,e}) + \mathbf{R}_{k,e},$$

Using the definition of the current in eq. (7.53), the left-hand side can be re-written as

$$e\eta_k n_{k,e}(\mathbf{E} + \mathbf{u}_{k,e} \times \mathbf{B}) = e\eta_k n_{k,e} \mathbf{E} + \left( e\eta_k \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} - \mathbf{J}_k \right) \times \mathbf{B},$$

and thus the electron momentum equation becomes

$$e\eta_k n_{k,e} \mathbf{E} + e\eta_k \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \times \mathbf{B} = \mathbf{J}_k \times \mathbf{B} + \nabla \cdot (\eta_k \boldsymbol{\sigma}_{k,e}) + \mathbf{R}_{k,e},$$

Using the definition of  $\mathbf{j}_{k,i,s}$ , we can write

$$\begin{aligned} \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \times \mathbf{B} &= \sum_s^{\mathcal{N}_k} [Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u} + Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} (\mathbf{u}_{k,i,s} - \mathbf{u})] \times \mathbf{B} \\ &= \sum_s^{\mathcal{N}_k} \left( Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u} + Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \frac{\mathbf{j}_{k,i,s}}{\eta_k \rho_k Y_{k,i,s}} \right) \times \mathbf{B}. \end{aligned}$$

Using eq. (7.9) we get

$$\begin{aligned} \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u}_{k,i,s} \times \mathbf{B} &= \sum_s^{\mathcal{N}_k} \left( Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u} + Z_{k,i,s} \eta_{k,i,s} \frac{\rho_{k,i,s}}{m_{k,i,s}} \frac{\mathbf{j}_{k,i,s}}{\eta_k \rho_k Y_{k,i,s}} \right) \times \mathbf{B} \\ &= \sum_s^{\mathcal{N}_k} \left( Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u} + \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s} \eta_k} \right) \times \mathbf{B} \\ &= \sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} \mathbf{u} \times \mathbf{B} + \sum_s^{\mathcal{N}_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s} \eta_k} \times \mathbf{B} \\ &= n_{k,e} \mathbf{u} \times \mathbf{B} + \sum_s^{\mathcal{N}_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s} \eta_k} \times \mathbf{B}. \end{aligned}$$

Thus, the electron momentum equation becomes

$$e\eta_k n_{k,e} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{J}_k \times \mathbf{B} + \nabla \cdot (\eta_k \boldsymbol{\sigma}_{k,e}) + \mathbf{R}_{k,e} - e \sum_s^{\mathcal{N}_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \times \mathbf{B}.$$

We define the total electron density as

$$n_e = \sum_k^{\mathcal{M}} \eta_k n_{k,e}.$$

Thus, summing over all  $k$  gives

$$en_e (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k^{\mathcal{M}} \eta_k \boldsymbol{\sigma}_{k,e} + \sum_k^{\mathcal{M}} \mathbf{R}_{k,e} - e \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{m_{k,i,s}} \times \mathbf{B}.$$

or

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} \left( \mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k^{\mathcal{M}} \eta_k \boldsymbol{\sigma}_{k,e} + \sum_k^{\mathcal{M}} \mathbf{R}_{k,e} \right) - \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{n_e m_{k,i,s}} \times \mathbf{B}. \quad (7.84)$$

## Internal energy

Use the definition of  $\mathbf{w}_{k,i,s}$  to re-write eq. (7.51) and eq. (7.52) as follows

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k \eta_{k,i,s} p_{k,i,s} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k \eta_{k,i,s} p_{k,i,s} \mathbf{u} \right) = \\ \eta_k \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i,s} + Q_{k,i,s} - \nabla \cdot \left( \frac{3}{2} \eta_k \eta_{k,i,s} p_{k,i,s} \mathbf{w}_{k,i,s} \right) + \eta_k \eta_{k,i,s} \boldsymbol{\sigma}_{k,i,s} : \nabla \mathbf{w}_{k,i,s}. \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k p_{k,e} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,e} \mathbf{u} \right) = \\ \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} - \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,e} \mathbf{w}_{k,e} \right) + \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{w}_{k,e}. \end{aligned}$$

### 7.5.4 Multi-material single-fluid model

We add two extra assumptions

1. All ions of a material  $k$  move with the same ion velocity  $\mathbf{u}_{k,i}$ .
2. All ions of a material  $k$  have the same temperature  $T_{k,i}$ .

If all ions of a material move with a single ion velocity, then the definition of  $\mathbf{u}_k$  in eq. (7.69) gives

$$\mathbf{u}_{k,i,s} = \mathbf{u}_{k,i}, \quad (7.85)$$

which gives for the species diffusion vector

$$\mathbf{j}_{k,i,s} = \rho Y_k Y_{k,i,s} \mathbf{w}_{k,i,s} = \rho Y_k Y_{k,i,s} \mathbf{w}_{k,i}. \quad (7.86)$$

Adding over all ion species of a material and using expressions for total ion quantities, the ion pressure equation becomes

$$\frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k p_{k,i} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,i} \mathbf{u} \right) = \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} - \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,i} \mathbf{w}_{k,i} \right) + \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{w}_{k,i}.$$

Similarly, the ion EOS is multiplied by  $\eta_{k,i,s}$  on both sides and then added over all materials, to obtain

$$p_{k,i} = n_{k,i} k_B T_{k,i}. \quad (7.87)$$

To summarize, we have

$$\frac{\partial \eta_k \rho_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \rho_k Y_{k,i,s} \mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i,s}, \quad (7.88)$$

$$\nabla \cdot \mathbf{J} = 0, \quad (7.89)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k \eta_k \boldsymbol{\sigma}_k - \nabla \cdot \sum_k \mathbf{j}_{k,i} \mathbf{u}_{k,i}, \quad (7.90)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} \left( \mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k \eta_k \boldsymbol{\sigma}_{k,e} + \sum_k \mathbf{R}_{k,e} \right) - \sum_k \sum_s \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{n_e m_{k,i,s}} \times \mathbf{B}, \quad (7.91)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k p_{k,i} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,i} \mathbf{u} \right) = \\ \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} - \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,i} \mathbf{w}_{k,i} \right) + \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{w}_{k,i}, \end{aligned} \quad (7.92)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{3}{2} \eta_k p_{k,e} \right) + \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,e} \mathbf{u} \right) = \\ \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} - \nabla \cdot \left( \frac{3}{2} \eta_k p_{k,e} \mathbf{w}_{k,e} \right) + \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{w}_{k,e}, \end{aligned} \quad (7.93)$$

$$\mathbf{J}_k = e \eta_k n_{k,e} (\mathbf{u}_{k,i} - \mathbf{u}_{k,e}), \quad (7.94)$$

$$\mathbf{J} = \sum_k^{\mathcal{M}} \mathbf{J}_k, \quad (7.95)$$

$$\rho_{q,k} = 0, \quad (7.96)$$

$$\rho_q = \sum_k^{\mathcal{M}} \rho_{q,k}, \quad (7.97)$$

$$p_{k,i} = n_{k,i} k_B T_{k,i}, \quad (7.98)$$

$$p_{k,e} = n_{k,e} k_B T_{k,e}. \quad (7.99)$$

Note that eq. (7.88) gives  $\eta_k \rho_k Y_{k,i,s}$ . Summing over all  $s$  gives  $\eta_k \rho_k$ . Summing over all  $k$  gives  $\rho$ . Dividing  $\eta_k \rho_k Y_{k,i,s}$  by  $\eta_k \rho_k$  gives  $Y_{k,i,s}$ . Dividing  $\eta_k \rho_k$  by  $\rho$  gives  $Y_k$ .

### 7.5.5 Single-material single-fluid multi-species model

Assuming there is only one material, then  $\mathbf{w}_{k,i} = \mathbf{u}_{k,i} - \mathbf{u} = 0$  and thus eq. (7.86) gives  $\mathbf{j}_{k,i,s} = 0$ . Summing over all  $s$  would then give  $\mathbf{j}_{k,i} = 0$ . Additionally,  $\eta_k = 1$  and  $\rho_k = \rho$ . Thus, the model in section 7.5.4 becomes

$$\frac{\partial \rho Y_{i,s}}{\partial t} + \nabla \cdot (\rho Y_{i,s} \mathbf{u}) = 0, \quad (7.100)$$

$$\nabla \cdot \mathbf{J} = 0, \quad (7.101)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \boldsymbol{\sigma}, \quad (7.102)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} (\mathbf{J} \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e), \quad (7.103)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_i \right) + \nabla \cdot \left( \frac{3}{2} p_i \mathbf{u} \right) = \boldsymbol{\sigma}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i, \quad (7.104)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \nabla \cdot \left( \frac{3}{2} p_e \mathbf{u} \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot \left( \frac{3}{2} p_e \mathbf{w}_e \right) + \boldsymbol{\sigma}_e : \nabla \mathbf{w}_e, \quad (7.105)$$

$$\mathbf{J} = en_e (\mathbf{u} - \mathbf{u}_e), \quad (7.106)$$

$$\rho_q = 0, \quad (7.107)$$

$$p_i = n_i k_B T_i, \quad (7.108)$$

$$p_e = n_e k_B T_e. \quad (7.109)$$

### 7.5.6 Single-material single-fluid two-species model

If there is only one ion species, along with the electron species, then the model in section 7.5.5 simplifies to the model in section 2.2.

### 7.5.7 Alternate forms of the energy equations

The enthalpies are given by

$$h_{k,i,s} = e_{k,i,s} + \frac{p_{k,i,s}}{\rho_{k,i,s}}, \quad (7.110)$$

$$h_{k,e} = e_{k,e} + \frac{p_{k,e}}{\rho_k}. \quad (7.111)$$

Using expressions from section 7.5.1, we can show that

$$h_{k,i} = \sum_s \mathcal{N}_k Y_{k,i,s} h_{k,i,s} = \sum_s \mathcal{N}_k Y_{k,i,s} e_{k,i,s} + \sum_s \mathcal{N}_k \frac{\eta_{k,i,s} p_{k,i,s}}{\rho_k} = e_{k,i} + \frac{p_{k,i}}{\rho_k}. \quad (7.112)$$

Thus far we've been working with plasmas that behave as ideal gasses with three degrees of freedom (the three translational degrees of freedom). For ideal gasses we also have

$$p_{k,i,s} = (\gamma - 1) \rho_{k,i,s} e_{k,i,s}, \quad (7.113)$$

$$p_{k,e} = (\gamma - 1) \rho_k e_{k,e}. \quad (7.114)$$

Multiplying both sides of eq. (7.113) and summing over all  $s$  gives

$$p_{k,i} = (\gamma - 1) \rho_k e_{k,i}. \quad (7.115)$$

Since  $\gamma = 5/3$  for the plasmas under consideration, the internal energy equation for ions given by eq. (7.92) can now be written as

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,i}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,i} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} \\ &\quad - \nabla \cdot (\eta_k \rho_k e_{k,i} \mathbf{w}_{k,i}) + \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{w}_{k,i}. \end{aligned}$$

Similarly, the internal energy equation for electrons given by eq. (7.93) becomes

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,e}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,e} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} \\ &\quad - \nabla \cdot (\eta_k \rho_k e_{k,e} \mathbf{w}_{k,e}) + \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{w}_{k,e}. \end{aligned}$$

We re-write the last term in the two equations above as

$$\begin{aligned} \eta_k \boldsymbol{\sigma}_{k,s} : \nabla \mathbf{w}_{k,s} &= \nabla \cdot (\mathbf{w}_{k,s} \cdot \eta_k \boldsymbol{\sigma}_{k,s}) - \mathbf{w}_{k,s} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,s}) \\ &= -\nabla \cdot (\eta_k p_{k,s} \mathbf{w}_{k,s}) + \nabla \cdot (\mathbf{w}_{k,s} \cdot \eta_k \mathbf{t}_{k,s}) - \mathbf{w}_{k,s} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,s}), \end{aligned}$$

where  $s$  is equal to  $i$  or  $e$ . Thus, the ion and electron internal energy equations become

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,i}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,i} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} \\ &\quad - \nabla \cdot \left[ \left( e_{k,i} + \frac{p_{k,i}}{\rho_k} \right) \eta_k \rho_k \mathbf{w}_{k,i} \right] + \nabla \cdot (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i}) - \mathbf{w}_{k,i} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,i}). \end{aligned}$$

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,e}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,e} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} \\ &\quad - \nabla \cdot \left[ \left( e_{k,e} + \frac{p_{k,e}}{\rho_k} \right) \eta_k \rho_k \mathbf{w}_{k,e} \right] + \nabla \cdot (\mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,e}). \end{aligned}$$

Using eqs. (7.111) and (7.112) for  $h_{k,i}$  and  $h_{k,e}$  the above two become

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,i}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,i} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} \\ &\quad - \nabla \cdot (h_{k,i} \eta_k \rho_k \mathbf{w}_{k,i}) + \nabla \cdot (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i}) - \mathbf{w}_{k,i} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,i}). \end{aligned}$$

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,e}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,e} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} \\ &\quad - \nabla \cdot (h_{k,e} \eta_k \rho_k \mathbf{w}_{k,e}) + \nabla \cdot (\mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,e}). \end{aligned}$$

Using eq. (7.81) for  $\mathbf{j}_{k,i}$  and  $\mathbf{j}_{k,e} = \eta_k \rho_k \mathbf{w}_{k,e}$  the internal energy equations become

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,i}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,i} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} \\ &\quad - \nabla \cdot (h_{k,i} \mathbf{j}_{k,i}) + \nabla \cdot (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i}) - \mathbf{w}_{k,i} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,i}). \end{aligned}$$

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,e}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,e} \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} \\ &\quad - \nabla \cdot (h_{k,e} \mathbf{j}_{k,e}) + \nabla \cdot (\mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,e}). \end{aligned}$$

Finally, adding up the ion and electron internal energy equations and using eq. (7.71) gives

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_k}{\partial t} + \nabla \cdot (\eta_k \rho_k e_k \mathbf{u}) &= \eta_k \boldsymbol{\sigma}_k : \nabla \mathbf{u} - \nabla \cdot (\mathbf{q}_{k,i} + \mathbf{q}_{k,e}) + Q_{k,i} + Q_{k,e} \\ &\quad - \nabla \cdot (h_{k,i} \mathbf{j}_{k,i} + h_{k,e} \mathbf{j}_{k,e}) + \nabla \cdot (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i} + \mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) - \mathbf{w}_{k,i} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,i}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,e}). \end{aligned} \tag{7.116}$$

Summing eq. (7.116) over all materials and using eq. (7.72) gives

$$\begin{aligned} \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) &= \sum_k^{\mathcal{M}} \eta_k \boldsymbol{\sigma}_k : \nabla \mathbf{u} - \nabla \cdot \sum_k^{\mathcal{M}} (\mathbf{q}_{k,i} + \mathbf{q}_{k,e}) + \sum_k^{\mathcal{M}} (Q_{k,i} + Q_{k,e}) \\ &\quad - \nabla \cdot \sum_k^{\mathcal{M}} (h_{k,i} \mathbf{j}_{k,i} + h_{k,e} \mathbf{j}_{k,e}) + \nabla \cdot \sum_k^{\mathcal{M}} (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i} + \mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) \\ &\quad - \sum_k^{\mathcal{M}} [\mathbf{w}_{k,i} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,i}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \boldsymbol{\sigma}_{k,e})]. \end{aligned}$$

We can now re-write eq. (7.78) as

$$\sum_k^{\mathcal{M}} (Q_{k,i} + Q_{k,e}) = - \sum_k^{\mathcal{M}} (\mathbf{u}_{k,i} \cdot \mathbf{R}_{k,i} + \mathbf{u}_{k,e} \cdot \mathbf{R}_{k,e}). \tag{7.117}$$

Thus, the total internal energy equation becomes

$$\begin{aligned} \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) &= \sum_k^{\mathcal{M}} \eta_k \sigma_k : \nabla \mathbf{u} - \nabla \cdot \sum_k^{\mathcal{M}} (\mathbf{q}_{k,i} + \mathbf{q}_{k,e}) - \sum_k^{\mathcal{M}} (\mathbf{u}_{k,i} \cdot \mathbf{R}_{k,i} + \mathbf{u}_{k,e} \cdot \mathbf{R}_{k,e}) \\ &\quad - \nabla \cdot \sum_k^{\mathcal{M}} (h_{k,i} \mathbf{j}_{k,i} + Y_{k,e} h_{k,e} \bar{\mathbf{j}}_{k,e}) + \nabla \cdot \sum_k^{\mathcal{M}} (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i} + \mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) \\ &\quad - \sum_k^{\mathcal{M}} [\mathbf{w}_{k,i} \cdot \nabla (\eta_k \sigma_{k,i}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \sigma_{k,e})]. \end{aligned}$$

To summarize we have

$$\frac{\partial \eta_k \rho_k Y_{k,i,s}}{\partial t} + \nabla \cdot (\eta_k \rho_k Y_{k,i,s} \mathbf{u}) = -\nabla \cdot \mathbf{j}_{k,i,s}, \quad (7.118)$$

$$\nabla \cdot \mathbf{J} = 0, \quad (7.119)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot \sum_k^{\mathcal{M}} \eta_k \sigma_k - \nabla \cdot \sum_k^{\mathcal{M}} \mathbf{j}_{k,i} \mathbf{u}_{k,i}, \quad (7.120)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} \left( \mathbf{J} \times \mathbf{B} + \nabla \cdot \sum_k^{\mathcal{M}} \eta_k \sigma_{k,e} + \sum_k^{\mathcal{M}} \mathbf{R}_{k,e} \right) - \sum_k^{\mathcal{M}} \sum_s^{\mathcal{N}_k} \frac{Z_{k,i,s} \mathbf{j}_{k,i,s}}{n_e m_{k,i,s}} \times \mathbf{B}, \quad (7.121)$$

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,i}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,i} \mathbf{u}) &= \eta_k \sigma_{k,i} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,i} + Q_{k,i} \\ &\quad - \nabla \cdot (h_{k,i} \mathbf{j}_{k,i}) + \nabla \cdot (\mathbf{w}_{k,i} \cdot \eta_k \mathbf{t}_{k,i}) - \mathbf{w}_{k,i} \cdot \nabla (\eta_k \sigma_{k,i}). \end{aligned} \quad (7.122)$$

$$\begin{aligned} \frac{\partial \eta_k \rho_k e_{k,e}}{\partial t} + \nabla \cdot (\eta_k \rho_k e_{k,e} \mathbf{u}) &= \eta_k \sigma_{k,e} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_{k,e} + Q_{k,e} \\ &\quad - \nabla \cdot (h_{k,e} \mathbf{j}_{k,e}) + \nabla \cdot (\mathbf{w}_{k,e} \cdot \eta_k \mathbf{t}_{k,e}) - \mathbf{w}_{k,e} \cdot \nabla (\eta_k \sigma_{k,e}). \end{aligned} \quad (7.123)$$

$$\mathbf{J}_k = e \eta_k n_{k,e} (\mathbf{u}_{k,i} - \mathbf{u}_{k,e}), \quad (7.124)$$

$$\mathbf{J} = \sum_k^{\mathcal{M}} \mathbf{J}_k, \quad (7.125)$$

$$\rho_{q,k} = 0, \quad (7.126)$$

$$\rho_q = \sum_k^{\mathcal{M}} \rho_{q,k}, \quad (7.127)$$

$$p_{k,i} = n_{k,i} k_B T_{k,i}, \quad (7.128)$$

$$p_{k,e} = n_{k,e} k_B T_{k,e}, \quad (7.129)$$

$$p_{k,i} = (\gamma - 1) \rho_k e_{k,i}, \quad (7.130)$$

$$p_{k,e} = (\gamma - 1) \rho_k e_{k,e}. \quad (7.131)$$

Some extra notes on this model:

1. As before the mass transport equation gives  $\eta_k \rho_k Y_{k,i,s}$ . Summing over all  $s$  gives  $\eta_k \rho_k$ . Dividing  $\eta_k \rho_k Y_{k,i,s}$  by  $\eta_k \rho_k$  gives  $Y_{k,i,s}$ . Summing  $\eta_k \rho_k$  over all  $k$  gives  $\rho$ . Dividing  $\eta_k \rho_k$  by  $\rho$  gives  $Y_k$ .
2. The ion internal energy equation gives  $e_{k,i}$ .
3. The electron internal energy equation gives  $e_{k,e}$ .
4. If  $\eta_k$  is known, we can compute  $\rho_k$  from  $\rho_k = \rho Y_k / \eta_k$ .
5. Using  $\rho_k$  and  $e_{k,i}$  in eq. (7.130) we get  $p_{k,i}$ .
6. Using  $\rho_k$  and  $e_{k,e}$  in eq. (7.131) we get  $p_{k,e}$ .
7. We can compute  $h_{k,i}$  using eq. (7.112), which we repeat below

$$h_{k,i} = e_{k,i} + \frac{p_{k,i}}{\rho_k}.$$

8. We can compute  $h_{k,e}$  using eq. (7.111), which we repeat below

$$h_{k,e} = e_{k,e} + \frac{p_{k,e}}{\rho_k}.$$

9. We can compute  $n_{k,i,s}$  using eq. (7.9), which we repeat below

$$m_{k,i,s} n_{k,i,s} = \rho_k Y_{k,i,s}.$$

10. We can compute  $n_{k,i}$  using eqs. (7.9) and (7.23), as shown below

$$n_{k,i} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} n_{k,i,s} = \sum_s^{\mathcal{N}_k} \eta_{k,i,s} \frac{\rho_{k,i,s}}{m_{k,i,s}} = \sum_s^{\mathcal{N}_k} \frac{\rho_k Y_{k,i,s}}{m_{k,i,s}}. \quad (7.132)$$

11. We can compute  $n_{k,e}$  from quasi-neutrality, which we repeat below

$$\sum_s^{\mathcal{N}_k} Z_{k,i,s} \eta_{k,i,s} n_{k,i,s} = n_{k,e}, \quad (7.133)$$

12. Using  $p_{k,i}$  and  $n_{k,i}$  in eq. (7.128) we get  $T_{k,i}$ .
13. Using  $p_{k,e}$  and  $n_{k,e}$  in eq. (7.129) we get  $T_{k,e}$ .

Note that this model requires knowing the value of  $\eta_k$ . Thus the following equation is also solved

$$\frac{\partial \eta_k}{\partial t} + \mathbf{u} \cdot \nabla (\eta_k) = \alpha_k, \quad (7.134)$$

where  $\alpha_k$  is a closure model.

### 7.5.8 Generic EOS

Rather than assuming ideal equations of state for each species, we can instead account for generic equations of state. To show this, we first start with

## Chapter 8

# Magnetohydrodynamics

### 8.1 Single-fluid equations

The starting point are the single-material single-fluid multi-species equations of section 7.5.5, along with Maxwell's equations, which are summarized below

$$\frac{\partial \rho Y_{i,s}}{\partial t} + \nabla \cdot (\rho Y_{i,s} \mathbf{u}) = 0, \quad (7.100)$$

$$\nabla \cdot \mathbf{J} = 0, \quad (7.101)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = \nabla \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e), \quad (7.102)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} (\mathbf{J} \times \mathbf{B} + \nabla \cdot \boldsymbol{\sigma}_e + \mathbf{R}_e), \quad (7.103)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_i \right) + \nabla \cdot \left( \frac{3}{2} p_i \mathbf{u} \right) = \boldsymbol{\sigma}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i, \quad (7.104)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \nabla \cdot \left( \frac{3}{2} p_e \mathbf{u} \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e - \nabla \cdot \left( \frac{3}{2} p_e \mathbf{w}_e \right) + \boldsymbol{\sigma}_e : \nabla \mathbf{w}_e, \quad (7.105)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \quad (8.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (8.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (8.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (8.4)$$

$$\mathbf{J} = en_e (\mathbf{u} - \mathbf{u}_e), \quad (7.106)$$

$$\rho_q = 0, \quad (7.107)$$

$$p_i = n_i k_B T_i, \quad (7.108)$$

$$p_e = n_e k_B T_e. \quad (7.109)$$



### 8.1.1 Mass

Summing the species mass density over all ions gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (8.5)$$

We then note that

$$\nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{\partial \rho}{\partial t} - \frac{1}{\rho} \nabla \rho \cdot \mathbf{u} = -\frac{\partial \ln \rho}{\partial t} - \nabla \ln \rho \cdot \mathbf{u},$$

and thus

$$\gamma \nabla \cdot \mathbf{u} = -\frac{1}{\rho^\gamma} \frac{\partial \rho^\gamma}{\partial t} - \frac{1}{\rho^\gamma} \nabla \rho^\gamma \cdot \mathbf{u}. \quad (8.6)$$

### 8.1.2 Momentum

Expanding the expressions for  $\sigma_i$  and  $\sigma_e$ , the momentum equation becomes

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = -\nabla (p_i + p_e) + \nabla \cdot (\mathbf{t}_i + \mathbf{t}_e),$$

If we define the total pressure  $p$  as  $p = p_i + p_e$  we get

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \mathbf{J} \times \mathbf{B} = -\nabla p + \nabla \cdot (\mathbf{t}_i + \mathbf{t}_e),$$

### 8.1.3 Ion internal energy

The second term on the left-hand side of the ion internal energy equation can be expanded to obtain

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_i \right) + \mathbf{u} \cdot \nabla \left( \frac{3}{2} p_i \right) + \frac{3}{2} p_i \nabla \cdot \mathbf{u} = \sigma_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i,$$

Expanding the stress tensor we get

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_i \right) + \mathbf{u} \cdot \nabla \left( \frac{3}{2} p_i \right) + \frac{5}{2} p_i \nabla \cdot \mathbf{u} = \mathbf{t}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i,$$

Since for a plasma with three translational degrees of freedom  $\gamma = 5/3$ , we re-write the above as

$$\frac{1}{\gamma - 1} \left( \frac{\partial p_i}{\partial t} + \mathbf{u} \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{u} \right) = \mathbf{t}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i,$$

or

$$\frac{\partial p_i}{\partial t} + \mathbf{u} \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{u} = (\gamma - 1) (\mathbf{t}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i).$$

Using eq. (8.6) we can show that

$$\begin{aligned} \frac{\partial p_i}{\partial t} + \mathbf{u} \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{u} &= \frac{\partial p_i}{\partial t} - p_i \frac{1}{\rho^\gamma} \frac{\partial \rho^\gamma}{\partial t} + \mathbf{u} \cdot \nabla p_i - p_i \frac{1}{\rho^\gamma} \nabla \rho^\gamma \cdot \mathbf{u} \\ &= \rho^\gamma \left[ \frac{\partial}{\partial t} \left( \frac{p_i}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p_i}{\rho^\gamma} \right) \right]. \end{aligned}$$

Thus, the ion internal energy equation can finally be written as

$$\frac{\partial}{\partial t} \left( \frac{p_i}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p_i}{\rho^\gamma} \right) = \frac{\gamma - 1}{\rho^\gamma} (\mathbf{t}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i).$$

### 8.1.4 Electron internal energy

Since  $\mathbf{J} = en_e(\mathbf{u} - \mathbf{u}_e)$  and  $\mathbf{w}_e = \mathbf{u}_e - \mathbf{u}$ , we have

$$\mathbf{w}_e = -\frac{\mathbf{J}}{en_e} \quad (8.7)$$

Thus, the electron internal energy equation becomes

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \nabla \cdot \left( \frac{3}{2} p_e \mathbf{u} \right) = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e + \nabla \cdot \left( \frac{3}{2} p_e \frac{\mathbf{J}}{en_e} \right) - \boldsymbol{\sigma}_e : \nabla \left( \frac{\mathbf{J}}{en_e} \right).$$

The second term on the left-hand side of the electron internal energy equation is expanded to obtain

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \mathbf{u} \cdot \nabla \left( \frac{3}{2} p_e \right) + \frac{3}{2} p_e \nabla \cdot \mathbf{u} = \boldsymbol{\sigma}_e : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_e + Q_e + \nabla \cdot \left( \frac{3}{2} p_e \frac{\mathbf{J}}{en_e} \right) - \boldsymbol{\sigma}_e : \nabla \left( \frac{\mathbf{J}}{en_e} \right).$$

And expanding the stress tensor we get

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{3}{2} p_e \right) + \mathbf{u} \cdot \nabla \left( \frac{3}{2} p_e \right) + \frac{5}{2} p_e \nabla \cdot \mathbf{u} = \\ \mathbf{t}_e : \nabla \left( \mathbf{u} - \frac{\mathbf{J}}{en_e} \right) - \nabla \cdot \mathbf{q}_e + Q_e + \nabla \cdot \left( \frac{3}{2} p_e \frac{\mathbf{J}}{en_e} \right) + p_e \nabla \cdot \left( \frac{\mathbf{J}}{en_e} \right). \end{aligned}$$

The last two terms above can be re-written as

$$\begin{aligned} \nabla \cdot \left( \frac{3}{2} p_e \frac{\mathbf{J}}{en_e} \right) + p_e \nabla \cdot \left( \frac{\mathbf{J}}{en_e} \right) &= \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{3}{2} p_e \right) + \frac{3}{2} p_e \nabla \cdot \left( \frac{\mathbf{J}}{en_e} \right) + p_e \nabla \cdot \left( \frac{\mathbf{J}}{en_e} \right) \\ &= \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{3}{2} p_e \right) + \frac{5}{2} p_e \nabla \cdot \left( \frac{\mathbf{J}}{en_e} \right) \\ &= \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{3}{2} p_e \right) + \frac{5}{2} p_e \left( \frac{\nabla \cdot \mathbf{J}}{en_e} - \frac{\mathbf{J}}{(en_e)^2} \cdot \nabla(en_e) \right) \\ &= \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{3}{2} p_e \right) - \frac{5}{2} p_e \frac{\mathbf{J}}{(en_e)^2} \cdot \nabla(en_e) \\ &= \frac{\mathbf{J}}{en_e} \cdot \left[ \nabla \left( \frac{3}{2} p_e \right) - \frac{5}{2} p_e \frac{1}{n_e} \nabla n_e \right]. \end{aligned}$$

Since for a plasma with three translational degrees of freedom we have  $\gamma = 5/3$ , we re-write the electron internal energy equation as

$$\begin{aligned} \frac{1}{\gamma - 1} \left( \frac{\partial p_e}{\partial t} + \mathbf{u} \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u} \right) = \\ \mathbf{t}_e : \nabla \left( \mathbf{u} - \frac{\mathbf{J}}{en_e} \right) - \nabla \cdot \mathbf{q}_e + Q_e + \frac{1}{\gamma - 1} \frac{\mathbf{J}}{en_e} \cdot \left[ \nabla p_e - \gamma p_e \frac{1}{n_e} \nabla n_e \right], \end{aligned}$$

or

$$\frac{\partial p_e}{\partial t} + \mathbf{u} \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u} = (\gamma - 1) \left[ \mathbf{t}_e : \nabla \left( \mathbf{u} - \frac{\mathbf{J}}{en_e} \right) - \nabla \cdot \mathbf{q}_e + Q_e \right] + \frac{\mathbf{J}}{en_e} \cdot \left[ \nabla p_e - \gamma p_e \frac{1}{n_e} \nabla n_e \right],$$

The last term above can be written as

$$\begin{aligned}
\frac{\mathbf{J}}{en_e} \cdot \left[ \nabla p_e - \gamma p_e \frac{1}{n_e} \nabla n_e \right] &= \frac{\mathbf{J}}{en_e} \cdot \{ \nabla p_e - p_e \nabla [\gamma \ln(n_e)] \} \\
&= \frac{\mathbf{J}}{en_e} \cdot [\nabla p_e - p_e \nabla \ln(n_e^\gamma)] \\
&= \frac{\mathbf{J}}{en_e} \cdot \left[ \nabla p_e - \frac{p_e}{n_e^\gamma} \nabla n_e^\gamma \right] \\
&= n_e^\gamma \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{p_e}{n_e^\gamma} \right).
\end{aligned}$$

Also, using eq. (8.6) we can show that

$$\begin{aligned}
\frac{\partial p_e}{\partial t} + \mathbf{u} \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u} &= \frac{\partial p_e}{\partial t} - p_e \frac{1}{\rho^\gamma} \frac{\partial \rho^\gamma}{\partial t} + \mathbf{u} \cdot \nabla p_e - p_e \frac{1}{\rho^\gamma} \nabla \rho^\gamma \cdot \mathbf{u} \\
&= \rho^\gamma \left[ \frac{\partial}{\partial t} \left( \frac{p_e}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p_e}{\rho^\gamma} \right) \right].
\end{aligned}$$

Thus, the electron internal energy equation can finally be written as

$$\frac{\partial}{\partial t} \left( \frac{p_e}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p_e}{\rho^\gamma} \right) = \frac{\gamma - 1}{\rho^\gamma} \left[ \mathbf{t}_e : \nabla \left( \mathbf{u} - \frac{\mathbf{J}}{en_e} \right) - \nabla \cdot \mathbf{q}_e + Q_e \right] + \left( \frac{n_e}{\rho} \right)^\gamma \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{p_e}{n_e^\gamma} \right).$$

In the specific case where there is only one ion species, then quasi-neutrality gives  $n_e = n_i = n$ . Additionally,  $\rho = M_i n_i$ . Thus, the last term in the electron internal energy equation can be written as

$$\left( \frac{n_e}{\rho} \right)^\gamma \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{p_e}{n_e^\gamma} \right) = \left( \frac{n_i}{\rho} \right)^\gamma \frac{\mathbf{J}}{en} \cdot \nabla \left( \frac{p_e}{n_i^\gamma} \right) = \frac{\mathbf{J}}{en} \cdot \nabla \left( \frac{p_e}{\rho^\gamma} \right) \quad (8.8)$$

### 8.1.5 Maxwell's equations

We assume that the electromagnetic-wave velocity and the ion and electron thermal velocities are much smaller than the speed of light, that is,  $w/k \ll c$  and  $V_{Ti}, V_{Te} \ll c$ . This allows us to neglect Maxwell's correction in Ampere's law, so as to obtain

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \quad (8.9)$$

### 8.1.6 Summary

Combining all of the results above, we now have

$$\frac{\partial \rho Y_{i,s}}{\partial t} + \nabla \cdot (\rho Y_{i,s} \mathbf{u}) = 0. \quad (8.10)$$

$$\nabla \cdot \mathbf{J} = 0. \quad (8.11)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \cdot \mathbf{u} \right) - \mathbf{J} \times \mathbf{B} + \nabla p = \nabla \cdot (\mathbf{t}_i + \mathbf{t}_e), \quad (8.12)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} (\mathbf{J} \times \mathbf{B} - \nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e). \quad (8.13)$$

$$\frac{\partial}{\partial t} \left( \frac{p_i}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p_i}{\rho^\gamma} \right) = \frac{\gamma - 1}{\rho^\gamma} (\mathbf{t}_i : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}_i + Q_i). \quad (8.14)$$

$$\frac{\partial}{\partial t} \left( \frac{p_e}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p_e}{\rho^\gamma} \right) = \frac{\gamma-1}{\rho^\gamma} \left[ \mathbf{t}_e : \nabla \left( \mathbf{u} - \frac{\mathbf{J}}{en_e} \right) - \nabla \cdot \mathbf{q}_e + Q_e \right] + \left( \frac{n_e}{\rho} \right)^\gamma \frac{\mathbf{J}}{en_e} \cdot \nabla \left( \frac{p_e}{n_e^\gamma} \right). \quad (8.15)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (8.16)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (8.17)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (8.18)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (8.19)$$

$$\mathbf{J} = en_e (\mathbf{u} - \mathbf{u}_e), \quad (8.20)$$

$$\rho_q = 0, \quad (8.21)$$

$$p_i = n_i k_B T_i, \quad (8.22)$$

$$p_e = n_e k_B T_e. \quad (8.23)$$

## 8.2 Resistive MHD

The electron collision term is modeled as

$$\mathbf{R}_e = m_e n_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e), \quad (8.24)$$

where  $\nu_{ei}$  is the momentum exchange collision frequency. Using the expression for current in eq. (8.20) we get

$$\mathbf{R}_e = \frac{m_e \nu_{ei}}{e} \mathbf{J}. \quad (8.25)$$

Neglecting all terms on the right-hand side of eq. (8.13) except for the electron collision term, we have

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en_e} \mathbf{R}_e. \quad (8.26)$$

Using eq. (8.25) in the above, we have

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{m_e \nu_{ei}}{e^2 n_e} \mathbf{J}, \quad (8.27)$$

which we re-write as

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}, \quad (8.28)$$

where

$$\eta = \frac{m_e \nu_{ei}}{e^2 n_e} \quad (8.29)$$

is the resistivity.

### 8.3 Ideal MHD

To derive the ideal MHD equations we neglect the right-hand sides of eqs. (8.12) to (8.15) and assume a single ion species plus a single electron species. Summing the two pressure equations shown in section 8.1.6, the resulting equations would be

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (8.30)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \quad (8.31)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0, \quad (8.32)$$

$$\frac{\partial}{\partial t} \left( \frac{p}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p}{\rho^\gamma} \right) = 0, \quad (8.33)$$

$$\nabla \cdot \mathbf{E} = 0. \quad (8.34)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.35)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (8.36)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (8.37)$$

Given the vector identity

$$\frac{1}{2} \nabla (B^2) = \mathbf{B} \times (\nabla \times \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (8.38)$$

we can use Ampere's law to re-write the  $\mathbf{J} \times \mathbf{B}$  term in the velocity equation as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} \left[ (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (B^2) \right]. \quad (8.39)$$

Similarly, given the vector identity

$$\nabla \times (\mathbf{B} \times \mathbf{u}) = (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} + \mathbf{B} (\nabla \cdot \mathbf{u}) - \mathbf{u} (\nabla \cdot \mathbf{B}), \quad (8.40)$$

we can use Ohm's law to re-write the  $\nabla \times \mathbf{E}$  term in Faraday's law as

$$\nabla \times \mathbf{E} = \nabla \times (-\mathbf{u} \times \mathbf{B}) = (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} + \mathbf{B} (\nabla \cdot \mathbf{u}). \quad (8.41)$$

Thus, the ideal MHD equations can be summarized as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (8.42)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.43)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{1}{\mu_0} \left[ (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (B^2) \right] \quad (8.44)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} - \mathbf{B} (\nabla \cdot \mathbf{u}) \quad (8.45)$$

$$\frac{\partial}{\partial t} \left( \frac{p}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left( \frac{p}{\rho^\gamma} \right) = 0, \quad (8.46)$$

If we assume incompressibility, then the above simplifies to

$$\nabla \cdot \mathbf{u} = 0, \quad (8.47)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.48)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{1}{\mu_0} \left[ (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (B^2) \right] \quad (8.49)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} \quad (8.50)$$

# Appendix A

## Lagrangian and Eulerian PDFs

### A.1 Eulerian PDF

Consider an Eulerian velocity field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ . The Eulerian PDF  $f = f(\mathbf{V}; \mathbf{x}, t)$  gives the probability that the velocity field will have a value of  $\mathbf{V}$  at location  $\mathbf{x}$  and at time  $t$ . We'll also introduce the fine-grained Eulerian PDF  $f' = f'(\mathbf{V}; \mathbf{x}, t)$ , which is defined as

$$f'(\mathbf{V}; \mathbf{x}, t) = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}). \quad (\text{A.1})$$

Note: a delta function of a 3D argument means the following  $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$ . The Eulerian PDF can be obtained from the fine-grained Eulerian PDF using

$$f(\mathbf{V}; \mathbf{x}, t) = \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle. \quad (\text{A.2})$$

The proof is as follows,

$$\begin{aligned} \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle &= \langle \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) f(\mathbf{V}'; \mathbf{x}, t) d\mathbf{V}' \\ &= f(\mathbf{V}; \mathbf{x}, t). \end{aligned} \quad (\text{A.3})$$

### A.2 Lagrangian PDF

Consider a Lagrangian particle with velocity  $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$  and position  $\mathbf{x}^+(t, \mathbf{y})$ . The Lagrangian PDF  $f_L = f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$  gives the probability that the particle that started at location  $\mathbf{y}$  at the reference time  $t_0$  will have a velocity  $\mathbf{V}$  and position  $\mathbf{x}$  at time  $t$ . We'll also introduce the fine-grained Eulerian PDF  $f'_L = f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$ , which is defined as

$$f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}). \quad (\text{A.4})$$

Note: a delta function of a 3D argument means the following  $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$ . The Lagrangian PDF can be obtained from the fine-grained Lagrangian PDF using

$$f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle. \quad (\text{A.5})$$

The proof is as follows,

$$\begin{aligned} \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle &= \langle \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{V}', \mathbf{x}'; t|\mathbf{y}) d\mathbf{V}' d\mathbf{x}' \\ &= f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}). \end{aligned} \quad (\text{A.6})$$

### A.3 Relation between Lagrangian and Eulerian PDFs

As a quick side note, we mention that the inverse of  $\mathbf{x}^+$  is  $\mathbf{y}^+ = \mathbf{y}^+(t, \mathbf{z})$ , which gives the initial location of a fluid particle that at time  $t$  is located at position  $\mathbf{z}$ . Thus,  $\mathbf{x}^+(t, \mathbf{y}^+(t, \mathbf{z})) = \mathbf{z}$ .

We begin as follows

$$\begin{aligned} \int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} &= \int \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) |\det D\mathbf{x}^+| d\mathbf{y}, \end{aligned} \quad (\text{A.7})$$

where we have introduced  $|\det D\mathbf{x}^+|$ , which is the absolute value of the determinant of the Jacobean  $\partial\mathbf{x}^+/\partial\mathbf{y}$ , and is equal to one for incompressible flows. Using integration by substitution we obtain

$$\int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = \int \delta(\mathbf{u}(\mathbf{z}, t) - \mathbf{V}) \delta(\mathbf{z} - \mathbf{x}) d\mathbf{z} = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \quad (\text{A.8})$$

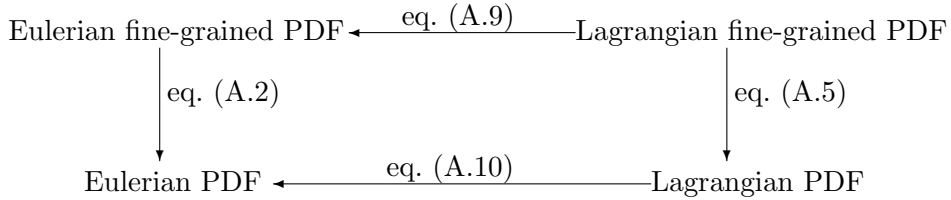
Given the definition of  $f'(\mathbf{V}; \mathbf{x}, t)$ , we have

$$\int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = f'(\mathbf{V}; \mathbf{x}, t). \quad (\text{A.9})$$

Taking the expectation of the above we obtain

$$\int f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = f(\mathbf{V}; \mathbf{x}, t). \quad (\text{A.10})$$

A summary of all of the relations derived thus far is given by the following graph



### A.4 Evolution equation for fine-grained Eulerian PDF

### A.5 Evolution equation for fine-grained Lagrangian PDF