Plasma dynamics

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Contents

Part I

Particle description

Chapter 1

Classical Mechanics

Table 1.1: Various coordinates of classical mechanics.

Classical coordinates	$\mathbf{x}(t)$	$\mathbf{v}(t)$
Generalized coordinates	$\mathbf{q}$	$\dot{\mathbf{q}}$
Canonical coordinates	$\mathbf{q}$	$\mathbf{p}$
Time-dependent canonical coordinates	$\tilde{\mathbf{q}}(t)$	$\tilde{\mathbf{p}}(t)$

1.1 Lagrangian mechanics

- Define the position $\mathbf{x} = \mathbf{x}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ of a particle.
- Define the Lagrangian as $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the generalized position and generalized velocity, respectively.
- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] = \left(\frac{\partial L}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}. \tag{1.1}$$

- For example, the Lagrangian of a particle in an electromagnetic field where $\phi = \phi(\mathbf{q}, t)$ is the electric potential and $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$ is the magnetic potential, is

$$L = \frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e\phi. \tag{1.2}$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e \dot{q}_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \tag{1.3}$$

$$\frac{\partial L}{\partial \dot{q}_i} = m \dot{q}_i + e A_i \tag{1.4}$$

$$\begin{aligned} \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] &= \frac{d}{dt}\left[m v_i + e A_i(\mathbf{x}, t)\right] \\ &= m \frac{d v_i}{dt} + e v_j \left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \tag{1.5}$$

Thus, the Euler-Lagrange equation becomes

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.6}$$

In vector notation, this is written as

$$m\frac{d\mathbf{v}}{dt} = \left(-e\mathbf{v}\cdot\nabla_q\mathbf{A} - e\frac{\partial \mathbf{A}}{\partial t} + e\nabla_q(\mathbf{v}\cdot\mathbf{A}) - e\nabla_q\phi\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.7}$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}, \tag{1.8}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$.

1.2 Hamiltonian mechanics

- Define the Hamiltonian as $H = H(\mathbf{q}, \mathbf{p}, t)$, where $\mathbf{q}$ and $\mathbf{p}$ are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.

- The Hamiltonian is obtained from the Lagrangian using

$$H = \left(\dot{\mathbf{q}}\cdot\mathbf{p} - L\right)_{\dot{\mathbf{q}}=\mathbf{f}(\mathbf{q},\mathbf{p})}, \tag{1.9}$$

  where the dependency of $\dot{\mathbf{q}}$ on $\mathbf{q}$ and $\mathbf{p}$ is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \tag{1.10}$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[\dot{q}_i p_i - \left(\frac{1}{2}m\dot{q}_i\dot{q}_i + e\dot{q}_iA_i - e\phi\right)\right]_{\dot{\mathbf{q}}=f(\mathbf{q},\mathbf{p})}. \tag{1.11}$$

  Evaluating eq. (1.10) gives $p_i = m\dot{q}_i + eA_i$, which allows us to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$ and $\mathbf{p}$ as $\dot{q}_i = \frac{1}{m}(p_i - eA_i)$. Thus

$$\begin{aligned} H &= \frac{1}{m}(p_i - eA_i)p_i - \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) - e\frac{1}{m}(p_i - eA_i)A_i + e\phi \\ &= \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) + e\phi. \end{aligned} \tag{1.12}$$

- We introduce the variables $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$, which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \tag{1.13}$$

$$\tilde{\mathbf{p}} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}} \tag{1.14}$$

- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.15}$$

$$\frac{d\tilde{p}_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.16}$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x}, t) \tag{1.17}$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m\frac{dv_i}{dt} + ev_j \left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}} . \tag{1.18}$$

$$\begin{aligned}
\frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i}\left[\frac{1}{2m}(p_j - eA_j)(p_j - eA_j) + e\phi\right] \\
&= \frac{1}{m}(p_j - eA_j)\frac{\partial}{\partial q_i}(p_j - eA_j) + e\frac{\partial \phi}{\partial q_i} \\
&= -\frac{e}{m}(p_j - eA_j)\frac{\partial A_j}{\partial q_i} + e\frac{\partial \phi}{\partial q_i}
\end{aligned} \tag{1.19}$$

$$\begin{aligned}
\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m}\left[mv_j + eA_j(\mathbf{x}, t) - eA_j(\mathbf{x}, t)\right]\left(\frac{\partial A_j}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} \\
&= \left(-ev_j\frac{\partial A_j}{\partial q_i} + e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} .
\end{aligned} \tag{1.20}$$

Equation (1.16) thus leads to

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} . \tag{1.21}$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)_{\mathbf{q}=\mathbf{x}} . \tag{1.22}$$

Chapter 2

Single-particle motion—guiding center theory

The motion of single particles is obtained by solving eq. (1.22), which we re-write below

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}. \tag{2.1}$$

In the above, $\mathbf{v} = \mathbf{v}(t)$ is the particle velocity, $\mathbf{x} = \mathbf{x}(t)$ the particle position, $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ the electric field, and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$ the magnetic field. In the subsections that follow, we will solve this equation of motion for simplified forms of $\mathbf{E}$ and $\mathbf{B}$. The solutions for the velocity vector will typically be of the form

$$\mathbf{v} = \mathbf{v}^{(c)} + \mathbf{v}^{(g)} + v^{||}\mathbf{b}, \tag{2.2}$$

where $\mathbf{v}^{(c)} = \mathbf{v}^{(c)}(t)$ is the gyromotion (cyclotron) velocity, $\mathbf{v}^{(g)} = \mathbf{v}^{(g)}(t)$ is the gyrocenter velocity, $v^{||} = v^{||}(t)$ is the parallel velocity. Not all of the velocities will always be present. $\mathbf{b} = \mathbf{B}/B$ is the unit magnetic field vector. The position of the particle is governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}. \tag{2.3}$$

Using eq. (2.2), we integrate the above to obtain

$$\int_0^t d\mathbf{x}(t') = \int_0^t \mathbf{v}^{(c)}(t')\,dt' + \int_0^t \mathbf{v}^{(g)}(t')\,dt' + \int_0^t v^{||}(t)\mathbf{b}\,dt'. \tag{2.4}$$

We introduce the positions $\mathbf{x}^{(c)} = \mathbf{x}^{(c)}(t)$, $\mathbf{x}^{(g)} = \mathbf{x}^{(g)}(t)$, and $\mathbf{x}^{||} = \mathbf{x}^{||}(t)$, which are defined as follows

$$\mathbf{x}^{(c)} = \int \mathbf{v}^{(c)}\,dt, \tag{2.5}$$

$$\mathbf{x}^{(g)} = \int \mathbf{v}^{(g)}\,dt, \tag{2.6}$$

$$\mathbf{x}^{||} = \int v^{||}\mathbf{b}\,dt. \tag{2.7}$$

Thus, eq. (2.4) is now re-written as

$$\mathbf{x}(t) - \mathbf{x}(0) = \mathbf{x}^{(c)}(t) - \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(t) - \mathbf{x}^{(g)}(0) - \mathbf{x}^{||}(t) - \mathbf{x}^{||}(0). \tag{2.8}$$

Without loss of generality, we will assume that the initial condition is as follows

$$\mathbf{x}(0) = \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(0) + \mathbf{x}^{||}(0). \tag{2.9}$$

Thus, the particle position is finally expressed as

$$\mathbf{x} = \mathbf{x}^{(c)} + \mathbf{x}^{(g)} + \mathbf{x}^{||}. \tag{2.10}$$

2.1 Uniform E and B fields

2.1.1 Only E field

Let's orient our coordinate system such that $\mathbf{E}$ points in the $\mathbf{e}_z$ direction. Thus, the equations of motion are

$$
\begin{aligned}
\frac{dv_x}{dt} &= 0 & \qquad v_x(0) &= v_\perp \cos(\phi), \\
\frac{dv_y}{dt} &= 0 & \qquad v_y(0) &= v_\perp \sin(\phi), \\
\frac{dv_z}{dt} &= \frac{eE}{m} & \qquad v_z(0) &= v_{||}.
\end{aligned} \tag{2.11}
$$

The solution of the above is

$$
\begin{aligned}
v_x &= v_\perp \cos(\phi) \\
v_y &= v_\perp \sin(\phi) \\
v_z &= v_{||} + \frac{eE}{m} t.
\end{aligned} \tag{2.12}
$$

2.1.2 Only B field

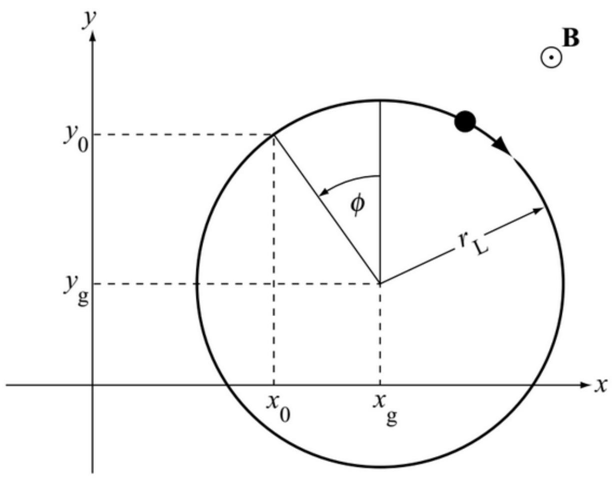

Figure 8.1 Gyro orbit of a positively charged particle in a magnetic field. Shown are the guiding center x_g, y_g and the initial position x_0, y_0.

Figure 2.1: Coordinates for gyromotion (extracted from Plasma Physics and Fusion Energy, J. P. Freidberg).

Let's orient our coordinate system such that $\mathbf{B}$ points in the $\mathbf{e}_z$ direction. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB}{m} v_y \qquad v_x(0) = v_\perp \cos(\phi), \tag{2.13a}$$

$$\frac{dv_y}{dt} = -\frac{eB}{m} v_x \qquad v_y(0) = v_\perp \sin(\phi), \tag{2.13b}$$

$$\frac{dv_z}{dt} = 0 \qquad v_z(0) = v_{||}. \tag{2.13c}$$

The z component is decoupled from the rest and has a trivial solution. For the other two components, we begin by taking the time derivative of eq. (2.13b). Thus

$$\frac{d^2 v_y}{dt^2} = -\frac{eB}{m}\frac{dv_x}{dt} = -w_c^2 v_y, \tag{2.14}$$

where $w_c = |e|B/m$ is the gyro frequency. We know that the general solution to the above is $v_y = c_1 \cos(w_c t) + c_2 \sin(w_c t)$. If we use the ICs and assume ions, we have

$$v_y = -v_\perp \sin(w_c t - \phi). \tag{2.15}$$

Integrating eq. (2.13a) then gives

$$v_x = v_\perp \cos(w_c t - \phi). \tag{2.16}$$

The final solution, for either positive or negative charges, can be written as

$$\begin{aligned} v_x^{(c)} &= v_\perp \cos(w_c t \pm \phi) \\ v_y^{(c)} &= \pm v_\perp \sin(w_c t \pm \phi), \end{aligned} \tag{2.17}$$

where upper signs correspond to a negative charge. Integrating the equations above leads to

$$\begin{aligned} x^{(c)} &= r_L \sin(w_c t \pm \phi) \\ y^{(c)} &= \mp r_L \cos(w_c t \pm \phi). \end{aligned} \tag{2.18}$$

where $r_L = v_\perp / w_c$ is the gyro radius.

2.1.3 Both E and B fields

Let's orient our coordinate system such that $\mathbf{B}$ still points along $\mathbf{e}_z$. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x}{m} + \frac{eB}{m} v_y \qquad v_x(0) = v_\perp \cos(\phi) + \frac{E_y}{B}, \tag{2.19a}$$

$$\frac{dv_y}{dt} = \frac{eE_y}{m} - \frac{eB}{m} v_x \qquad v_y(0) = v_\perp \sin(\phi) - \frac{E_x}{B}, \tag{2.19b}$$

$$\frac{dv_z}{dt} = \frac{eE_{||}}{m} \qquad v_z(0) = v_{||}, \tag{2.19c}$$

where we have chosen the given initial conditions simply to facilitate the math. Again, the z component is decoupled from the rest and has the trivial solution $v_z = v_{||} + (eE_{||}/m)t$. Thus, eq. (2.2) for the x and y components are

$$\begin{aligned} v_x &= v_x^{(c)} + v_x^{(g)}, \\ v_y &= v_y^{(c)} + v_y^{(g)}. \end{aligned} \tag{2.20}$$

We assume $v_x^{(g)}$ and $v_y^{(g)}$ are time independent. Using eq. (2.20) in eq. (2.19) we obtain

$$\begin{aligned} 0 &= \frac{eE_x}{m} + \frac{eB}{m} v_y^{(g)} \\ 0 &= \frac{eE_y}{m} - \frac{eB}{m} v_x^{(g)}. \end{aligned} \tag{2.21}$$

Thus, $v_x^{(g)} = E_y/B$ and $v_y^{(g)} = -E_x/B$, which in vector notation can be expressed as

$$\mathbf{v}_E^{(g)} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \tag{2.22}$$

2.2 Non-uniform B field

2.2.1 Change in magnitude along perpendicular directions

The magnetic field still points in the $\mathbf{e}_z$ direction, but its magnitude changes in directions parallel to $\mathbf{e}_z$: $B = B(q_x, q_y)$. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(x,y)}{m} v_y \qquad v_x(0) = v_\perp \cos(\phi) - \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} \frac{1}{B(x_g,y_g)}, \tag{2.23a}$$

$$\frac{dv_y}{dt} = -\frac{eB(x,y)}{m} v_x \qquad v_y(0) = v_\perp \sin(\phi) + \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} \frac{1}{B(x_g,y_g)}, \tag{2.23b}$$

$$\frac{dv_z}{dt} = 0 \qquad v_z(0) = v_{||}. \tag{2.23c}$$

In the above, the x and y in $B(x, y)$ are the perpendicular components of the particle's position. The v_z component is decoupled from the rest and has a trivial solution. Thus, eqs. (2.2) and (2.10) for the x and y components are

$$v_x = v_x^{(c)} + v_x^{(g)}, \tag{2.24}$$

$$v_y = v_y^{(c)} + v_y^{(g)}, \tag{2.25}$$

$$x = x^{(c)} + x^{(g)}, \tag{2.26}$$

$$y = y^{(c)} + y^{(g)}. \tag{2.27}$$

We begin by employing a Taylor-series expansion for the magnetic field

$$B(x,y) = B(x_g,y_g) + \left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} + ... \tag{2.28}$$

Thus, eqs. (2.23a) and (2.23b) are now

$$\frac{dv_x}{dt} = \frac{eB(x_g,y_g)}{m} v_y + \frac{e}{m}\left(\left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} \right) v_y \tag{2.29}$$

$$\frac{dv_y}{dt} = -\frac{eB(x_g,y_g)}{m} v_x + \frac{e}{m}\left(\left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} \right) v_x. \tag{2.30}$$

As before, we assume $v_x^{(g)}$, $v_y^{(g)}$ are time independent. Also, for simplicity we assume ions only. Plugging in eqs. (2.24) and (2.25) into the above, we get

$$0 = B(x_g,y_g) v_y^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} \right) \left(v_y^{(c)} + v_y^{(g)} \right), \tag{2.31}$$

$$0 = -B(x_g,y_g) v_x^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} \right) \left(v_x^{(c)} + v_x^{(g)} \right). \tag{2.32}$$

We assume $v_x^{(g)} \ll v_x^{(c)}$ and $v_y^{(g)} \ll v_y^{(c)}$. Thus, the above becomes

$$0 = B(x_g,y_g) v_y^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} \right) v_y^{(c)}, \tag{2.33}$$

$$0 = -B(x_g, y_g)v_x^{(g)} + \left(\left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} x^{(c)} + \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} y^{(c)} \right) v_x^{(c)}. \tag{2.34}$$

We now use the definitions in eq. (2.17) and eq. (2.18). For example, with those definitions we can show that

$$\begin{aligned} x^{(c)}v_y^{(c)} &= [r_L \sin(w_c t - \phi)]\,[-v_\perp \sin(w_c t - \phi)] \\ &= -\frac{v_\perp^2}{w_c} \sin^2(w_c t - \phi) \\ &= -\frac{v_\perp^2}{2w_c}\{1 - \cos[2(w_c t - \phi)]\} \end{aligned} \tag{2.35}$$

Similar derivations can be carried out for $y^{(c)}v_y^{(c)}$, $x^{(c)}v_x^{(c)}$, and $y^{(c)}v_x^{(c)}$. Thus, eqs. (2.33) and (2.34) become

$$\begin{aligned} 0 = B(x_g, y_g)v_y^{(g)} &- \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} \{1 - \cos[2(w_c t - \phi)]\} \\ &- \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} \sin[2(w_c t - \phi)], \end{aligned} \tag{2.36}$$

$$\begin{aligned} 0 = -B(x_g, y_g)v_x^{(g)} &- \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} \sin[2(w_c t - \phi)] \\ &- \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} \{1 + \cos[2(w_c t - \phi)]\}. \end{aligned} \tag{2.37}$$

It can be shown that the oscillatory terms containing the sines and cosines can be neglected. If it was not possible to neglect those terms, then the assumption that $v_x^{(g)}$, $v_y^{(g)}$ are time independent would not hold. Thus,

$$\begin{aligned} 0 &= B(x_g, y_g)v_y^{(g)} - \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} \\ 0 &= -B(x_g, y_g)v_x^{(g)} - \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g}. \end{aligned} \tag{2.38a}$$

Solving for the guiding center velocities, we finally have

$$\begin{aligned} v_x^{(g)} &= -\frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_y}\right|_{x_g,y_g} \frac{1}{B(x_g, y_g)} \\ v_y^{(g)} &= \frac{v_\perp^2}{2w_c} \left.\frac{\partial B}{\partial q_x}\right|_{x_g,y_g} \frac{1}{B(x_g, y_g)}. \end{aligned} \tag{2.39}$$

In vector notation, this is written as

$$\mathbf{v}_{\nabla B}^{(g)} = \frac{v_\perp^2}{2w_c} \frac{\mathbf{B} \times \nabla \mathbf{B}}{B^2}. \tag{2.40}$$

In the above, the fields and w_c are evaluated at (x_g, y_g).

2.2.2 Change in magnitude along parallel directions

Ideally, one would introduce a gradient along the parallel direction to the magnetic field, that is, one would have $\mathbf{B} = B(q_z)\mathbf{e}_z$. However, due to Gauss's law, this is too restrictive and instead we generalize and use $\mathbf{B} = B_x\mathbf{e}_x + B_z\mathbf{e}_z$, where $B_x = B_x(q_x, q_z)$ and $B_z = B_z(q_x, q_z)$. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{e}{m} v_y B_z(x, z), \tag{2.41}$$

$$\frac{dv_y}{dt} = -\frac{e}{m}[v_x B_z(x, z) - v_z B_x(x, z)], \tag{2.42}$$

$$\frac{dv_z}{dt} = -\frac{e}{m} v_y B_x(x, z). \tag{2.43}$$

However, the z direction no longer corresponds to the parallel direction, since the magnetic field also has a component along the x direction. Thus, we will introduce a rotating reference frame, in which one of the axis will always be aligned with the magnetic field vector, and thus would denote the parallel direction. This rotating reference frame is described by the rotation matrix

$$\mathbf{Q}(t) = \begin{bmatrix} b_x & 0 & b_z \\ 0 & 1 & 0 \\ b_z & 0 & -b_x \end{bmatrix}, \tag{2.44}$$

where $b_x = b_x(t)$ and $b_y = b_y(t)$ are given by

$$b_x = \frac{B_x(x, z)}{B(x, z)} \qquad b_z = \frac{B_z(x, z)}{B(x, z)} \tag{2.45}$$

In the above, $B(x, z) = [B_x^2(x, z) + B_y^2(x, z)]^{1/2}$. As an example, the transformation matrix above leads to the following unit vectors and velocities in the rotating reference frame

$$\mathbf{b} = b_x\mathbf{e}_x + b_z\mathbf{e}_z \tag{2.46}$$

$$\mathbf{e}_{\perp 2} = \mathbf{e}_y \tag{2.47}$$

$$\mathbf{e}_{\perp 1} = b_z\mathbf{e}_x - b_x\mathbf{e}_z = \mathbf{e}_2 \times \mathbf{b}, \tag{2.48}$$

$$v_{||} = b_x v_x + b_z v_z \tag{2.49}$$

$$v_{\perp 2} = v_y \tag{2.50}$$

$$v_{\perp 1} = b_z v_x - b_x v_z. \tag{2.51}$$

Using the transformation rule for the acceleration of a particle, but for some reason neglecting the coriollis and centrifugal forces, we obtain for the velocity derivatives

$$\frac{dv_{||}}{dt} = \frac{dv_x}{dt} b_x + \frac{dv_z}{dt} b_z - K v_{\perp 1} \tag{2.52}$$

$$\frac{dv_{\perp 2}}{dt} = \frac{dv_y}{dt} \tag{2.53}$$

$$\frac{dv_{\perp 1}}{dt} = \frac{dv_x}{dt} b_z - \frac{dv_z}{dt} b_x + K v_{||}, \tag{2.54}$$

where $K = K(t)$ is given by $K = b_x db_z/dt - b_z db_x/dt$. Using eqs. (2.41) to (2.43) in the above leads to

$$\frac{dv_{||}}{dt} = \frac{e}{m}v_y[B_z(x,z)b_x - B_x(x,z)b_z] - Kv_{\perp 1} = -Kv_{\perp 1}, \tag{2.55}$$

$$\frac{dv_{\perp 2}}{dt} = -\frac{eB}{m}(v_x b_z - v_z b_x) = -w_c v_{\perp 1}, \tag{2.56}$$

$$\frac{dv_{\perp 1}}{dt} = \frac{e}{m}v_y[B_z(x,z)b_z + B_x(x,z)b_x] + Kv_{||} = w_c v_{\perp 2} + Kv_{||}, \tag{2.57}$$

where $w_c = w_c(t)$ is given by $w_c = qB(x,z)/m$.

We now introduce a time transformation to simplify the equations above. To do so, we introduce the following variables

$$\hat{v}_{||} = \hat{v}_{||}(\tau) \qquad \hat{v}_{\perp 2} = \hat{v}_{\perp 2}(\tau) \qquad \hat{v}_{\perp 1} = \hat{v}_{\perp 1}(\tau) \tag{2.58}$$

$$\hat{x} = \hat{x}(\tau) \qquad \hat{z} = \hat{z}(\tau) \tag{2.59}$$

such that

$$v_{||} = \hat{v}_{||}(h(t)) \qquad v_{\perp 2} = \hat{v}_{\perp 2}(h(t)) \qquad v_{\perp 1} = \hat{v}_{\perp 1}(h(t)) \tag{2.60}$$

$$x = \hat{x}(h(t)) \qquad z = \hat{z}(h(t)). \tag{2.61}$$

The function $h(t)$ is given by

$$h(t) = \int_0^t w_c(t')dt'. \tag{2.62}$$

We also show that

$$b_x = \frac{B_x(x,z)}{B(x,z)} = \frac{B_x(\hat{x}(h(t)), \hat{z}(h(t)))}{B(\hat{x}(h(t)), \hat{z}(h(t)))}, \tag{2.63}$$

and thus

$$\frac{db_x}{dt} = \frac{dh(t)}{dt}\frac{d}{d\tau}\left[\frac{B_x(\hat{x},\hat{z})}{B(\hat{x},\hat{z})}\right]_{\tau=h(t)} = w_c \left.\frac{d\hat{b}_x}{d\tau}\right|_{\tau=h(t)}, \tag{2.64}$$

where $\hat{b}_x = \hat{b}_x(\tau)$ is given by $\hat{b}_x = B_x(\hat{x},\hat{z})/B(\hat{x},\hat{z})$. The analogous holds for b_z. This allows us to write

$$K = w_c \left(\hat{b}_x \frac{d\hat{b}_z}{dt} - \hat{b}_z \frac{d\hat{b}_x}{dt}\right)_{\tau=h(t)} = w_c \left.\hat{K}\right|_{\tau=h(t)}, \tag{2.65}$$

where $\hat{K} = \hat{K}(\tau)$ is given by $\hat{K} = \hat{b}_x d\hat{b}_z/d\tau - \hat{b}_z d\hat{b}_x/d\tau$. With these transformation, eqs. (2.55) to (2.57) are re-written as

$$\frac{d\hat{v}_{||}}{d\tau} = -\hat{K}\hat{v}_{\perp 1}, \tag{2.66}$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \tag{2.67}$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \hat{K}\hat{v}_{||}. \tag{2.68}$$

We now simplify B_z so that $B_z = B_z(q_z)$. To be consistent with Gauss's law, we require $B_x = B_x(q_x, q_y)$ where $B_x = -q_x dB_z/dq_z$. With these simplified forms, we have

$$\hat{K} = -\hat{b}_z^2 \frac{d}{d\tau}\left(\frac{\hat{b}_x}{\hat{b}_z}\right) \tag{2.69}$$

$$= -\frac{B_z^2(\hat{z})}{B^2(\hat{x},\hat{z})}\frac{d}{d\tau}\left(\frac{B_x(\hat{x},\hat{z})}{B_z(\hat{z})}\right) \tag{2.70}$$

$$= \frac{B_z^2(\hat{z})}{B^2(\hat{x},\hat{z})}\frac{d}{d\tau}\left[\hat{x}\left(\frac{1}{B_z}\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}\right]. \tag{2.71}$$

We now use the long-thin approximation. For this approximation, we assume that $B_x/B_z \ll 1$, and also that $\frac{1}{B_z}\frac{dB_z}{dq_z}$ changes very slowly. We thus have

$$\hat{K} \approx \frac{d\hat{x}}{d\tau}\left(\frac{1}{B_z}\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}. \tag{2.72}$$

Also, using the long-thin approximation in eqs. (2.49) and (2.51) allows us to write

$$v_{||} \approx v_z = \frac{dz}{dt} = \left(\frac{d\hat{z}}{d\tau}\right)_{\tau=h(t)} w_c \tag{2.73}$$

$$v_{\perp 1} \approx v_x = \frac{dx}{dt} = \left(\frac{d\hat{x}}{d\tau}\right)_{\tau=h(t)} w_c. \tag{2.74}$$

Evaluating the above at $t = h^{-1}(\tau)$, and defining $\hat{w}_c(\tau)$ from $w_c = \hat{w}_c(h(t))$, we obtain

$$\hat{v}_{||} \approx \frac{d\hat{z}}{d\tau}\hat{w}_c \tag{2.75}$$

$$\hat{v}_{\perp 1} \approx \frac{d\hat{x}}{d\tau}\hat{w}_c. \tag{2.76}$$

We also note that

$$\frac{dB_z(\hat{z})}{d\tau} = \left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}\frac{d\hat{z}}{d\tau}. \tag{2.77}$$

Using the expressions above in eq. (2.72), one can approximate $\hat{K}$ using either of the two forms below

$$\hat{K} \approx \frac{\hat{v}_{\perp 1}}{\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}} \approx \frac{\hat{v}_{\perp 1}}{\hat{v}_{||} B_z(\hat{z})}\frac{dB_z(\hat{z})}{d\tau}. \tag{2.78}$$

We thus write the governing equations for the velocities as

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_{\perp 1}^2}{\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}, \tag{2.79}$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \tag{2.80}$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \frac{\hat{v}_{\perp 1}}{B_z(\hat{z})}\frac{dB_z(\hat{z})}{d\tau}. \tag{2.81}$$

We now assume the solution for the perpendicular velocities is of the form

$$\hat{v}_{\perp 1} = \hat{v}_\perp \cos[\tau + \hat{\epsilon}] \tag{2.82}$$

$$\hat{v}_{\perp 2} = -\hat{v}_\perp \sin[\tau + \hat{\epsilon}], \tag{2.83}$$

where $\hat{v}_\perp = \hat{v}_\perp(\tau)$ and $\hat{\epsilon} = \hat{\epsilon}(\tau)$. Plugging these two assumed solutions into eqs. (2.80) and (2.81), and using some simple algebra, gives

$$\frac{d\hat{v}_\perp}{d\tau} = \frac{\hat{v}_\perp}{2B_z(\hat{z})}\frac{dB_z(\hat{z})}{d\tau}\left\{1+\cos[2(\tau+\hat{\epsilon})]\right\}. \tag{2.84}$$

The above can be re-arranged and expressed as

$$\frac{d\ln\hat{\mu}}{d\tau} = \frac{d\ln B_z(\hat{z})}{d\tau}\cos[2(\tau+\hat{\epsilon})], \tag{2.85}$$

where $\hat{\mu} = \hat{\mu}(\tau)$ is the adiabatic invariant, and is given by

$$\hat{\mu} = \frac{m\hat{v}_\perp^2}{2B_z(\hat{z})}. \tag{2.86}$$

Integrating eq. (2.85) from τ_1 to τ_2 such that $[\tau_2+\epsilon(\tau_2)]-[\tau_1+\epsilon(\tau_1)] = 2\pi$, gives

$$\ln\hat{\mu}(\tau_2) - \ln\hat{\mu}(\tau_1) = \ln[B_z(\hat{z})]\cos[2(\tau+\hat{\epsilon})]|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2\ln[B_z(\hat{z})]\sin[2(\tau+\hat{\epsilon})]d\tau. \tag{2.87}$$

Assuming $B(\hat{z})$ doesn't change significantly from τ_1 to τ_2, then we have $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$, that is, $\hat{\mu}$ is constant over one gyro-period. One can also define

$$\mu = \frac{mv_\perp^2}{2B_z(z)} \tag{2.88}$$

where $\mu = \mu(t)$ and $v_\perp = v_\perp(t)$. Given that $v_\perp = \hat{v}_\perp(h(t))$, we have $\mu = \hat{\mu}(h(t))$. Thus, $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ translates to $\mu(t_2) = \mu(t_1)$, where $t_2 = h^{-1}(\tau_2)$ and $t_1 = h^{-1}(\tau_1)$.

Finally, we focus not on the perpendicular velocities but the parallel velocity. Plugging-in the assumed solutions in the governing eq. (2.79) gives

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}\{1+\cos[2(\tau+\hat{\epsilon})]\}. \tag{2.89}$$

We now average the above while assuming $B(\hat{z})$, $d\hat{v}_{||}/d\tau$ and $\hat{v}_\perp^2$ do not change significantly from τ_1 to τ_2. Not that since this is an average, we are not just integrating from τ_1 to τ_2, but we are also dividing by $\tau_2 - \tau_1$. After averaging, we obtain

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}. \tag{2.90}$$

or

$$m\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{\mu}}{\hat{w}_c}\left(\frac{dB_z}{dq_z}\right)_{q_z=\hat{z}}. \tag{2.91}$$

Converting back to time t gives

$$m\frac{dv_{||}}{dt} = -\mu\left(\frac{dB_z}{dq_z}\right)_{q_z=z}. \tag{2.92}$$

2.2.3 Change in direction

Rather than writing eq. (2.1) in a Cartesian coordinate system as done in the previous sections, we begin by expressing the velocity as $\mathbf{v} = \mathbf{v}_\perp + v_{||}\mathbf{b}$. We assume there is no electric field, and thus write eq. (2.1) as

$$\frac{d}{dt}(\mathbf{v}_\perp + v_{||}\mathbf{b}) = \mp w_c(\mathbf{v}_\perp + v_{||}\mathbf{b}) \times \mathbf{b}, \tag{2.93}$$

where upper sign corresponds to negative charge and lower sign to positive charge. For simplicity we will assume positively charged particles only. We then cross both sides of the above by $\mathbf{b}$, that is

$$\mathbf{b} \times \left\{ \left[\frac{d}{dt}(\mathbf{v}_\perp + v_{||}\mathbf{b}) - w_c(\mathbf{v}_\perp + v_{||}\mathbf{b}) \times \mathbf{b} \right] \times \mathbf{b} \right\} = 0. \tag{2.94}$$

The above is simplified using the following three manipulations

$$\begin{aligned} \mathbf{b} \times \left\{ \left[w_c(\mathbf{v}_\perp + v_{||}\mathbf{b}) \times \mathbf{b} \right] \times \mathbf{b} \right\} &= \mathbf{b} \times \{ [w_c\mathbf{v}_\perp \times \mathbf{b}] \times \mathbf{b} \} \\ &= -\mathbf{b} \times \{ w_c\mathbf{v}_\perp(\mathbf{b}\cdot\mathbf{b}) - \mathbf{b}(\mathbf{b}\cdot w_c\mathbf{v}_\perp) \} \\ &= w_c\mathbf{v}_\perp \times \mathbf{b}. \end{aligned} \tag{2.95}$$

$$\mathbf{b} \times \left\{ \left[\frac{d\mathbf{v}_\perp}{dt} \right] \times \mathbf{b} \right\} = \frac{d\mathbf{v}_\perp}{dt}(\mathbf{b}\cdot\mathbf{b}) - \mathbf{b}\left(\mathbf{b}\cdot\frac{d\mathbf{v}_\perp}{dt}\right) = \left(\frac{d\mathbf{v}_\perp}{dt}\right)_\perp. \tag{2.96}$$

$$\begin{aligned} \mathbf{b} \times \left\{ \left[\frac{dv_{||}\mathbf{b}}{dt} \times \mathbf{b} \right] \right\} &= v_{||}\mathbf{b} \times \left\{ \left[\frac{d\mathbf{b}}{dt} \times \mathbf{b} \right] \right\} \\ &= v_{||}\left[\frac{d\mathbf{b}}{dt}(\mathbf{b}\cdot\mathbf{b}) - \mathbf{b}\left(\mathbf{b}\cdot\frac{d\mathbf{b}}{dt}\right) \right] \\ &= v_{||}\left[\frac{d\mathbf{b}}{dt} - \mathbf{b}\left(\frac{1}{2}\cdot\frac{d\mathbf{b}\cdot\mathbf{b}}{dt}\right) \right] \\ &= v_{||}\frac{d\mathbf{b}}{dt} \end{aligned} \tag{2.97}$$

Thus, we have

$$\left(\frac{d\mathbf{v}_\perp}{dt}\right)_\perp - w_c\mathbf{v}_\perp \times \mathbf{b} = -v_{||}\frac{d\mathbf{b}}{dt}. \tag{2.98}$$

As shown in Freidberg

$$\frac{d\mathbf{b}(\mathbf{x}(t))}{dt} = \frac{d\mathbf{x}(t)}{dt}\cdot\nabla\mathbf{b} = \mathbf{v}\cdot\nabla\mathbf{b} = \mathbf{v}_\perp\cdot\nabla\mathbf{b} + v_{||}\mathbf{b}\cdot\nabla\mathbf{b}, \tag{2.99}$$

where $\nabla\mathbf{b}$ is evaluated at $\mathbf{x} = \mathbf{x}(t)$. Thus, eq. (2.98) becomes

$$\left(\frac{d\mathbf{v}_\perp}{dt}\right)_\perp - w_c\mathbf{v}_\perp \times \mathbf{b} = -v_{||}\mathbf{v}_\perp\cdot\nabla\mathbf{b} - v_{||}^2\mathbf{b}\cdot\nabla\mathbf{b}. \tag{2.100}$$

As was done for the other drifts, we assume the solution is of the form $\mathbf{v}_\perp = \mathbf{v}_\perp^{(gm)} + \mathbf{V}_\kappa$, where we assume again $\mathbf{V}_\kappa$ is time independent. The term $\mathbf{v}_\perp^{(gm)}$ corresponds to gyromotion in a rotating reference frame, and is thus given by $\mathbf{v}_\perp^{(gm)} = v_{\perp 1}^{(gm)}\mathbf{e}_{\perp 1} + v_{\perp 2}^{(gm)}\mathbf{e}_{\perp 2}$, where $\mathbf{e}_{\perp 1}$ and $\mathbf{e}_{\perp 2}$ are orthogonal to $\mathbf{b}$ and thus rotate in time. $v_{\perp 1}^{(gm)}$ is given by eq. (2.16) and $v_{\perp 2}^{(gm)}$ by eq. (2.15). We note that, in the non-rotating reference frame, $\mathbf{v}_\perp^{(gm)}$ is expressed as $\mathbf{v}_\perp^{(gm)} = v_x^{(gm)}\mathbf{e}_x + v_y^{(gm)}\mathbf{e}_y + v_z^{(gm)}\mathbf{e}_z$. We now prove that $\mathbf{v}_\perp^{(gm)}$ is the solution to the two terms on the left-hand side of eq. (2.100). To show this we first use the transformation rule for the acceleration of a particle in a rotating reference frame, but for some reason ignore the coriollis and centrifugal terms. Thus

$$\begin{aligned}\frac{d\mathbf{v}_\perp^{(gm)}}{dt} &= \frac{dv_x^{(gm)}}{dt}\mathbf{e}_x + \frac{dv_y^{(gm)}}{dt}\mathbf{e}_y + \frac{dv_z^{(gm)}}{dt}\mathbf{e}_z \\ &= \frac{dv_{\perp 1}^{(gm)}}{dt}\mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(gm)}}{dt}\mathbf{e}_{\perp 2} + 2\Omega \times \mathbf{v}_\perp^{(gm)}. \end{aligned} \tag{2.101}$$

We do not allow the rotating reference frame to rotate about the $\mathbf{b}$ axis. Thus, $\Omega = \Omega_{\perp 1}\mathbf{e}_{\perp 1} + \Omega_{\perp 2}\mathbf{e}_{\perp 2}$. Given that Ω and $\mathbf{v}_\perp^{(gm)}$ are in the same plane, $\Omega \times \mathbf{v}_\perp^{(gm)}$ must point in the $\mathbf{b}$ direction. Thus,

$$\left(\frac{d\mathbf{v}_\perp^{(gm)}}{dt}\right)_\perp = \frac{dv_{\perp 1}^{(gm)}}{dt}\mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(gm)}}{dt}\mathbf{e}_{\perp 2}. \tag{2.102}$$

This allows us to show that

$$\left(\frac{d\mathbf{v}_\perp^{(gm)}}{dt}\right)_\perp - w_c\mathbf{v}_\perp^{(gm)} \times \mathbf{b} = \frac{dv_{\perp 1}^{(gm)}}{dt}\mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(gm)}}{dt}\mathbf{e}_{\perp 2} - w_c v_{\perp 2}^{(gm)}\mathbf{e}_{\perp 1} + w_c v_{\perp 1}^{(gm)}\mathbf{e}_{\perp 2} = 0. \tag{2.103}$$

We now plug in $\mathbf{v}_\perp = \mathbf{v}_\perp^{(gm)} + \mathbf{V}_\kappa$ in eq. (2.100) to obtain

$$-w_c\mathbf{V}_\kappa \times \mathbf{b} = -v_{||}\mathbf{v}_\perp \cdot \nabla\mathbf{b} - v_{||}^2\mathbf{b}\cdot\nabla\mathbf{b}. \tag{2.104}$$

As explained in Freidberg, the term $v_{||}\mathbf{v}_\perp \cdot \nabla\mathbf{b}$ leads to small modifications of the gyro motion, but does not lead to a drift of the particles, and thus is ignored. Taking the cross product of eq. (2.104) with $\mathbf{b}$ finally gives

$$\mathbf{V}_\kappa = -\frac{v_{||}^2}{w_c}\frac{(\mathbf{b}\cdot\nabla\mathbf{b})\times\mathbf{B}}{B}. \tag{2.105}$$

2.3 Non-uniform E field

2.4 Time-varying E field

Consider the scenario used in section 2.1.3, but with a time varying electric field. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x(t)}{m} + \frac{eB}{m}v_y \qquad v_x(0) = v_\perp\cos(\phi) + \frac{E_y(t)}{B} + \frac{m}{eB^2}\frac{dE_x(t)}{dt}, \tag{2.106a}$$

$$\frac{dv_y}{dt} = \frac{eE_y(t)}{m} - \frac{eB}{m}v_x \qquad v_y(0) = v_\perp\sin(\phi) - \frac{E_x(t)}{B} + \frac{m}{eB^2}\frac{dE_y(t)}{dt}, \tag{2.106b}$$

$$\frac{dv_z}{dt} = \frac{eE_{||}(t)}{m} \qquad v_z(0) = v_{||}, \tag{2.106c}$$

where again we chose the initial conditions simply to be consistent with the solution that we'll derive. The parallel velocity is independent of the perpendicular velocities, and we won't worry about it for now. To solve for the perpendicular velocities, we again assume the general solution is

$$\begin{aligned} v_x &= v_x^{(gm)} + V_x \\ v_y &= v_y^{(gm)} + V_y \end{aligned} \tag{2.107}$$

but now V_x and V_y are not time independent. We expand V_i as $V_i = V_i^{(1)} + V_i^{(2)} + ...$, where $V^{(\alpha)} \sim \epsilon V^{(\alpha-1)}$, and the small parameter ϵ follows from assuming

$$\frac{1}{V_i^{(\alpha)}}\frac{dV_i^{(\alpha)}}{dt} \sim \epsilon w_c. \tag{2.108}$$

That is, the time scale associated with the rate of change of all of the $V_i^{(\alpha)}$ components is much larger than the time scale of the gyromotion. In other words, we assume particles gyrate faster than how quickly their drift velocity changes. Using eq. (2.107) in eq. (2.106) leads to

$$\begin{aligned}\frac{dV_x^{(1)}}{dt} + \frac{dV_x^{(2)}}{dt} &= \frac{eE_x(t)}{m} + \frac{eB}{m}V_y^{(1)} + \frac{eB}{m}V_y^{(2)} \\ \frac{dV_y^{(1)}}{dt} + \frac{dV_y^{(2)}}{dt} &= \frac{eE_y(t)}{m} - \frac{eB}{m}V_x^{(1)} - \frac{eB}{m}V_x^{(2)}.\end{aligned} \tag{2.109}$$

Collecting lowest order terms

$$\begin{aligned}0 &= \frac{eE_x(t)}{m} + \frac{eB}{m}V_y^{(1)} \\ 0 &= \frac{eE_y(t)}{m} - \frac{eB}{m}V_x^{(1)},\end{aligned} \tag{2.110}$$

and thus $V_x^{(1)} = E_y(t)/B$ and $V_y^{(1)} = -E_x(t)/B$, which in vector notation is

$$\mathbf{V}^{(1)} = \frac{\mathbf{E}(t) \times \mathbf{B}}{B^2}. \tag{2.111}$$

Collecting first order terms

$$\begin{aligned}\frac{dV_x^{(1)}}{dt} &= \frac{eB}{m}V_y^{(2)} \\ \frac{dV_y^{(1)}}{dt} &= -\frac{eB}{m}V_x^{(2)},\end{aligned} \tag{2.112}$$

and thus $V_x^{(2)} = (m/eB^2)dE_x(t)/dt$ and $V_y^{(2)} = (m/eB^2)dE_y(t)/dt$, which in vector notation is

$$\mathbf{V}^{(2)} = \mp\frac{1}{w_c B}\frac{d\mathbf{E}_\perp}{dt}. \tag{2.113}$$

We note that, by looking at the solutions for $V_x^{(1)}$ and $V_y^{(1)}$, the assumption in eq. (2.108) is equivalent to stating that the electric field changes slowly.

2.5 Time-varying B field

Let's assume the magnetic field points in the z direction again. Using Faraday's law, we have

$$\left(\frac{\partial E_z}{\partial q_y} - \frac{\partial E_y}{\partial q_z}\right)\mathbf{e}_x - \left(\frac{\partial E_z}{\partial q_x} - \frac{\partial E_x}{\partial q_z}\right)\mathbf{e}_y + \left(\frac{\partial E_y}{\partial q_x} - \frac{\partial E_x}{\partial q_y}\right)\mathbf{e}_z = -\frac{\partial B}{\partial t}\mathbf{e}_z. \tag{2.114}$$

To satisfy the above, we set $E_z = 0$, and $E_x = E_x(q_x, q_y, t)$, $E_y = E_y(q_x, q_y, t)$. That is, a time varying magnetic field requires a time and spatially varying electric field.

We will further simplify our analysis by having $E_x = 0$ and $E_y = E_y(q_x, t)$. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(t)}{m} v_y \tag{2.115}$$
$$\frac{dv_y}{dt} = \frac{eE_y(x,t)}{m} - \frac{eB(t)}{m} v_x \tag{2.116}$$

The electric field is linearized using a Taylor-series expansion

$$\frac{dv_x}{dt} = \frac{eB(t)}{m} v_y \tag{2.117}$$
$$\frac{dv_y}{dt} = \frac{e}{m}\left(E_y(x_g, t) + \left.\frac{\partial E_y}{\partial q_x}\right|_{x_g} r_x \right) - \frac{eB(t)}{m} v_x, \tag{2.118}$$

where $r_x = x - x_g$. We assume positive ions for simplicity and re-write the above as

$$\frac{dv_x}{dt} = w_c v_y \tag{2.119}$$
$$\frac{dv_y}{dt} = \frac{w_c}{B(t)}\left(E_y(x_g, t) + \left.\frac{\partial E_y}{\partial q_x}\right|_{x_g} r_x \right) - w_c v_x. \tag{2.120}$$

where $w_c = w_c(t)$. We introduce new variables

$$\begin{gathered}\hat{v}_x = \hat{v}_x(\tau) \qquad \hat{v}_y = \hat{v}_y(\tau) \qquad \hat{x} = \hat{x}(\tau) \qquad \hat{x}_g = \hat{x}_g(\tau) \\ \hat{r}_x = \hat{r}_x(\tau) \qquad \hat{E}_y = \hat{E}_y(q_x, \tau) \qquad \hat{B} = \hat{B}(\tau)\end{gathered} \tag{2.121}$$

such that

$$\begin{gathered}v_x(t) = \hat{v}_x(h(t)) \qquad v_y(t) = \hat{v}_y(h(t)) \qquad x(t) = \hat{x}(h(t)) \qquad x_g(t) = \hat{x}_g(h(t)) \\ r_x(t) = \hat{r}_x(h(t)) \qquad E_y(q_x, t) = \hat{E}_y(q_x, h(t)) \qquad B(t) = \hat{B}(h(t)).\end{gathered} \tag{2.122}$$

For the above

$$h(t) = \int_0^t w_c(t')\, dt'. \tag{2.123}$$

The equations of motion then become

$$\begin{aligned}\frac{d\hat{v}_x}{d\tau} &= \hat{v}_y \\ \frac{d\hat{v}_y}{d\tau} &= \frac{1}{\hat{B}(\tau)}\left(\hat{E}_y(\hat{x}_g, \tau) + \left.\frac{\partial \hat{E}_y}{\partial q_x}\right|_{\hat{x}_g} \hat{r}_x \right) - \hat{v}_x.\end{aligned} \tag{2.124}$$

These are the equations we need to solve. To do so, we assume a form of the solution that is inspired by the previous section on time-dependent electric fields. That is, we assume the solution is of the form

$$\begin{aligned}\hat{v}_x &= \hat{v}_\perp \cos[\tau + \hat{\epsilon}] + \frac{\hat{E}_y(\hat{x}_g, \tau)}{\hat{B}(\tau)} \\ \hat{v}_y &= -\hat{v}_\perp \sin[\tau + \hat{\epsilon}] + \frac{d}{d\tau}\left(\frac{\hat{E}_y(\hat{x}_g, \tau)}{\hat{B}(\tau)} \right).\end{aligned} \tag{2.125}$$

where $\hat{v}_\perp = \hat{v}_\perp(\tau)$ and $\hat{\epsilon} = \hat{\epsilon}(\tau)$ are now time-dependent functions. As done before, we assume $\hat{r}_x = \hat{r}_x^{(gm)} + \hat{g}_x$, where $\hat{r}_x^{(gm)} = \hat{x}^{(gm)} - \hat{x}_g$, and $\hat{g}_x \sim \epsilon \hat{r}_x^{(gm)}$. The analogue of eq. (2.18) would be

$$\hat{x}^{(gm)} = \hat{x}_g + \hat{r}_L \sin[\tau + \hat{\epsilon}], \tag{2.126}$$

where $\hat{r}_L = \hat{r}_L(\tau)$ is given by $\hat{r}_L = \hat{v}_\perp / \hat{w}_c$, and $\hat{w}_c = q\hat{B}(\tau)/m$. Plugging in eqs. (2.125) and (2.126) in eq. (2.124) and using some algebra, leads to

$$\frac{d \ln \hat{\mu}}{d\tau} = \frac{d \ln \hat{B}(\tau)}{d\tau} \cos[2(\tau + \hat{\epsilon})], \tag{2.127}$$

where $\hat{\mu} = \hat{\mu}(\tau)$ is given by

$$\hat{\mu} = \frac{m\hat{v}_\perp^2}{2\hat{B}(\tau)}. \tag{2.128}$$

Integrating over one gyro-period, i.e. from τ_1 to τ_2 such that $[\tau_2 + \epsilon(\tau_2)] - [\tau_1 + \epsilon(\tau_1)] = 2\pi$, gives

$$\ln \hat{\mu}(\tau_2) - \ln \hat{\mu}(\tau_1) = \ln[\hat{B}(\tau)] \cos[2(\tau + \hat{\epsilon})] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2 \ln[\hat{B}(\tau)] \sin[2(\tau + \hat{\epsilon})] \, d\tau. \tag{2.129}$$

Assuming $\hat{B}(\tau)$ doesn't change significantly from τ_1 to τ_2, then we have $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$, that is, $\hat{\mu}$ is constant over one gyro-period. On can also define

$$\mu = \frac{mv_\perp^2}{2B(t)} \tag{2.130}$$

where $\mu = \mu(t)$ and $v_\perp = v_\perp(t)$. Given that $v_\perp = \hat{v}_\perp(h(t))$, we have $\mu = \hat{\mu}(h(t))$. Thus, $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ translates to $\mu(t_2) = \mu(t_1)$, where $t_2 = h^{-1}(\tau_2)$ and $t_1 = h^{-1}(\tau_1)$.

As shown in the analysis above, for a time dependent magnetic field a drift of the following form is introduced

$$\hat{V}_y = \frac{d}{d\tau} \left(\frac{\hat{E}_y(\hat{x}_g, \tau)}{\hat{B}(\tau)} \right). \tag{2.131}$$

Converting back to time t

$$V_y = \frac{1}{w_c} \frac{d}{dt} \left(\frac{E_y(x_g, t)}{B(t)} \right). \tag{2.132}$$

For the more general case where $E_x = E_x(q_x, q_y, t)$ and $E_y = E_y(q_x, q_y, t)$ then

$$\mathbf{V}_p = \mp \frac{1}{w_c} \frac{d}{dt} \left(\frac{\mathbf{E}_\perp}{B} \right), \tag{2.133}$$

where top sign is for electrons and bottom sign is for ions, and it is assumed that the electric field is evaluated at the gyro-center. For an even more general case where the magnetic field does not necessarily point in one direction,

$$\mathbf{V}_p = \mp \frac{1}{w_c} \mathbf{b} \times \frac{d\mathbf{V}_E}{dt}. \tag{2.134}$$

Chapter 3

Plasma parameters, time scales, and length scales

A summary of fundamental time and length scales of plasmas in given in tables 3.1 and 3.2

Table 3.1: Plasma time scales, for either electrons ($\alpha = e$) or ions ($\alpha = i$).

Time scales	Formulas
Gyro period	$\tau_{c\alpha} = \dfrac{2\pi}{w_{c\alpha}} \quad w_{c\alpha} = \dfrac{q_\alpha B}{m_\alpha}$
Plasma period	$\tau_{p\alpha} = \dfrac{2\pi}{w_{p\alpha}} \quad w_{p\alpha} = \sqrt{\dfrac{n_\alpha q_\alpha^2}{m_\alpha \epsilon_0}}$

Table 3.2: Plasma length scales, for either electrons ($\alpha = e$) or ions ($\alpha = i$).

Length scales	Formulas
Gyro radius	$r_{c\alpha} = \dfrac{v_{T\alpha}}{w_{c\alpha}} = \dfrac{m v_{T\alpha}}{q_\alpha B}$
Debye length	$\lambda_{D\alpha} = \dfrac{v_{T\alpha}}{\sqrt{2} w_{p\alpha}} = \sqrt{\dfrac{\epsilon_0 k_B T_\alpha}{n_\alpha q_\alpha^2}}$
DeBroglie wave length	$\lambda_{B\alpha} = \dfrac{h}{\sqrt{\pi} m_\alpha v_{T\alpha}}$
Sphere radius	$a_\alpha = \left(\dfrac{3}{4\pi n_\alpha}\right)^{1/3}$

- Thermal velocity:

$$v_{T_\alpha} = \sqrt{\frac{2k_B T_\alpha}{m_\alpha}} \tag{3.1}$$

- Total Debye length:

$$\frac{1}{\lambda_D^2} = \sum_\alpha \frac{1}{\lambda_{D\alpha}^2}. \tag{3.2}$$

- Plasma parameter

$$\Lambda_\alpha = \frac{4}{3}\pi\lambda_{D\alpha}^3 n_\alpha \tag{3.3}$$

- Quantum plasma parameter

$$\chi_\alpha = \frac{4}{3}\pi\lambda_{B\alpha}^3 n_\alpha \tag{3.4}$$

- Coupling parameter:

$$\Gamma_\alpha = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha k_B T_\alpha} = \frac{1}{3}\Lambda_\alpha^{-2/3} \tag{3.5}$$

- Degeneracy parameter for electrons:

$$\Theta_e = \frac{k_B T_e}{E_{fe}} = \left(\frac{2^{10}\pi}{3^4}\right)^{1/3} \chi_e^{-2/3} \tag{3.6}$$

- Fermi energy for electrons:

$$E_{fe} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3} \tag{3.7}$$

Some notes on the coupling parameter We can define two types of Coulomb interactions: strong and weak. Strong Coulomb interactions are those for which the particle's Coulomb potential energy is larger than its kinetic energy, and viceversa for weak Coulomb interactions. Plasmas for which the Coulomb interactions are mostly strong are dominated by those Coulomb interactions and are referred to as strongly coupled plasmas. On the other hand, plasmas for which the Coulomb interactions are mostly weak are dominated by long-range collective effects instead, and are referred to as weakly coupled plasmas.

We describe an approximate Coulomb potential energy for particles in a plasma as

$$\mathrm{U} = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha}. \tag{3.8}$$

The impact parameter that has been used above is a_α, the sphere radius. This provides a decent measure on the average spacing between particles in a plasma. Since the volume of a single particle is $1/n_\alpha$, and if we assume that this volume is given by $4/3\pi a_\alpha^3$, then equating these two gives the expression for the sphere radius

$$a_\alpha = \left(\frac{3}{4\pi n_\alpha}\right)^{1/3}. \tag{3.9}$$

The kinetic energy of a particle in a plasma can be approximated by the thermal energy, thus

$$\mathrm{K} = \frac{1}{2}m_\alpha v_{T_\alpha}^2 = k_b T_\alpha. \tag{3.10}$$

The ratio of the particle's Coulomb potential energy and its kinetic energy is referred to as the coupling parameter Γ_α. That is

$$\Gamma_\alpha = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha k_b T_\alpha}. \tag{3.11}$$

$\Gamma_\alpha > 1$ denotes a strongly coupled plasma, and $\Gamma_\alpha < 1$ denotes a weakly coupled plasma.

Chapter 4

Single-particle motion—Collisions

4.1 Cross section

The cross section characterizes in a quantitative form the probability that two particles traveling towards each other will undergo an interaction (also sometimes referred to as a collision). For example, imagine an incident particle traveling towards a target particle. This target particle has a spherical force field, and it affects incident particles that come within this sphere. Projecting the spherical force field to a plane perpendicular to the velocity of the incident particle gives a circular cross section. If the incident particle path takes it within this cross section, then the incident particle feels the force field of the target particle, that is, they interact. If the incident particle path does not take it within the cross section, then the particles do not interact. This is an example of a finite cross section, there can also be infinite cross sections for which particles always interact, although the farther away they are the weaker the interaction (e.g. electromagnetic force fields).

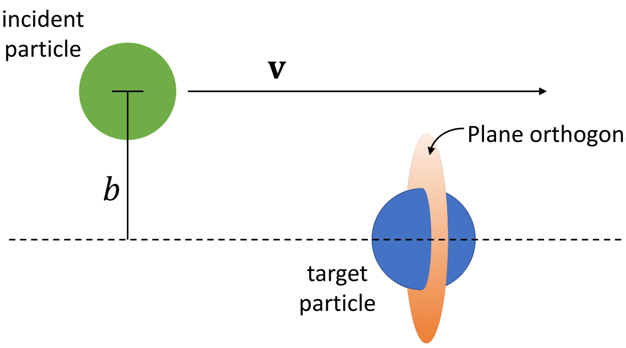

Figure 4.1: Cross section for particle interactions.

To quantify the above, the reader is referred to fig. 4.1. Imagine an incident particle traveling towards a stationary target particle with an impact parameter b and velocity $\mathbf{v}$ (if the target particle is not stationary, then $\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$, where $\mathbf{v}_1$ is the velocity of the target particle and $\mathbf{v}_2$ is the velocity of the incident particle.) As shown in fig. 4.1, the impact parameter is the perpendicular offset between the path of the incident particle, and the line parallel to the incident particle velocity that crosses the origin of the target particle (or the origin of the force field of the target particle). To determine the cross section, we ask the following question: for a particle with impact parameter b and velocity magnitude $v = |\mathbf{v}|$, does it interact with the target particle? Let's say it does not interact, then the infinitesimal surface located at b, i.e. $bdbd\phi$ for cylindrical coordinates, does not contribute to the cross section. If it does interact, then $bdbd\phi$ does contribute to the cross section. To obtain the total cross section $\sigma = \sigma(v)$, we sum over all infinitesimal areas $bdbd\phi$, but account whether a particle at a given b interacts or not with the target particle. This is expressed mathematically as

$$\sigma = \int_0^{2\pi} \int_0^{\infty} F(v,b)\, bdbd\phi. \tag{4.1}$$

In the above $F(v,b) = 1$ if an incident particle with impact parameter b and velocity v interacts with the target particle, and $F(v,b) = 0$ if it does not. However, in reality, $F(v,b)$ is not necessarily binary, and can take other values besides 0 and 1.

An example of $F(v,b)$ is that corresponding to particles that are hard spheres with radius R. For this case, $F(v,b) = H(2R-b)$, where H is the heaviside function. Thus,

$$\sigma = \int_0^{2\pi} \int_0^{\infty} H(2R-b)\, bdbd\phi = 2\pi \int_0^{2R} bdb = \pi(2R)^2, \tag{4.2}$$

as expected.

4.2 Mean free path, collision time, and collision frequency

The cross section then defines the mean free path λ_m, collision time τ_m and collision frequency ν_m. These are given by

$$\lambda_m = \frac{1}{n_1\sigma}, \tag{4.3}$$

$$\tau_m = \frac{\lambda_m}{v} = \frac{1}{n_1\sigma v}, \tag{4.4}$$

and

$$\nu_m = \frac{1}{\tau_m} = n_1\sigma v. \tag{4.5}$$

4.3 Coulomb scattering

4.3.1 Particle equations

Consider two particles, with positions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, velocities $\mathbf{v}_1 = \mathbf{v}_1(t)$ and $\mathbf{v}_2 = \mathbf{v}_2(t)$, charges q_1 and q_2, and masses m_1 and m_2, respectively. Their positions and velocities are governed by the following equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1, \tag{4.6}$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2, \tag{4.7}$$

$$m_1\frac{d\mathbf{v}_1}{dt} = -\frac{q_1q_2}{4\pi\epsilon}\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}, \tag{4.8}$$

$$m_2\frac{d\mathbf{v}_2}{dt} = -\frac{q_1q_2}{4\pi\epsilon}\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}. \tag{4.9}$$

We note that the above system consists of twelve equations for twelve unknowns. We now introduce the center-of-mass position $\mathbf{R} = \mathbf{R}(t)$, the center-of-mass velocity $\mathbf{V} = \mathbf{V}(t)$, the shifted position $\mathbf{r} = \mathbf{r}(t)$ and the shifted velocity $\mathbf{v} = \mathbf{v}(t)$ as follows

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{4.10}$$

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \tag{4.11}$$

Thus, in terms of these new four variables, the particle equations can be written as

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}, \tag{4.12}$$

$$\frac{d\mathbf{V}}{dt} = 0, \tag{4.13}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \tag{4.14}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{\mathbf{r}}{r^3}, \tag{4.15}$$

where the reduced mass m_r is given by

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4.16}$$

The first two equations above give the trivial solution $\mathbf{V} =$ constant and $\mathbf{R} = \mathbf{R}(0) + \mathbf{V}t$. Thus, we have reduced the problem from twelve unknowns to six unknowns, namely $\mathbf{r}$ and $\mathbf{v}$.

4.3.2 Conservation of energy

Dotting eq. (4.15) by $\mathbf{v}$ gives

$$\begin{aligned}
\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\mathbf{v}\cdot\frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{d\mathbf{r}}{dt}\cdot\frac{\mathbf{r}}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{2}\frac{dr^2}{dt}\frac{1}{r^3} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}\frac{dr}{dt} \\
&= -\frac{q_1 q_2}{4\pi\epsilon_0 m_r}\frac{d}{dt}\left(\frac{1}{r}\right).
\end{aligned} \tag{4.17}$$

For the left hand side above we have

$$\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{1}{2}\frac{dv^2}{dt}, \tag{4.18}$$

and thus we obtain the following expression for conservation of energy

$$\frac{d}{dt}\left(\frac{1}{2}m_r v^2 + \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r}\right) = 0. \tag{4.19}$$

4.3.3 Conservation of momentum

Crossing eq. (4.15) by $\mathbf{r}$ gives

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0, \tag{4.20}$$

and thus

$$\frac{d}{dt}\left[m_r \left(\mathbf{r} \times \mathbf{v}\right)\right] = 0. \tag{4.21}$$

That is, angular momentum is conserved. A consequence of this is that the vector $\mathbf{r} \times \mathbf{v}$ is always pointing in the same direction. Thus, if $\mathbf{r}(0)$ and $\mathbf{v}(0)$ form a plane, then $\mathbf{r}(t)$ and $\mathbf{v}(t)$ need to reside within that same plane for all times t so that $\mathbf{r}(t) \times \mathbf{v}(t)$ points in the same direction as $\mathbf{r}(0) \times \mathbf{v}(0)$. Therefore, the evolution of the position and velocity are confined to a plane and the problem can be reduced from six unknowns to four unknowns. This planar encounter is depicted in fig. 4.2.

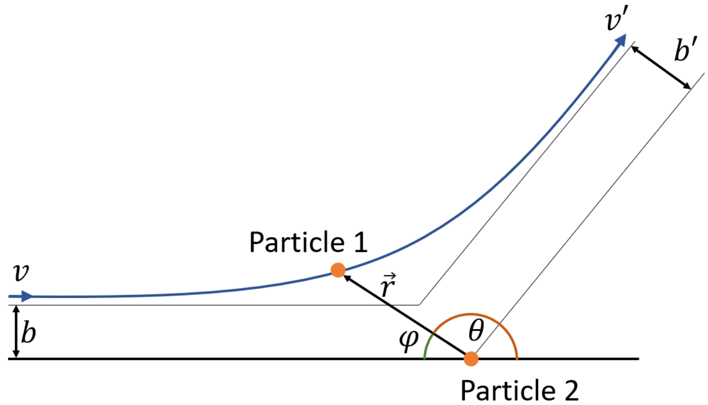

Figure 4.2: Depiction of Coulomb scattering.

4.3.4 Polar coordinates

We re-orient the plane of interaction (referred to above) so that it is orthogonal to the $\hat{\mathbf{z}}$ direction. Using polar coordinates, as shown in fig. 4.3, we get

$$r_x = r\cos\theta = r\cos(\pi - \varphi) = -r\cos\varphi, \tag{4.22}$$

$$r_y = r\sin\theta = r\sin(\pi - \varphi) = r\sin\varphi. \tag{4.23}$$

Also, since $\mathbf{r} = r\hat{\mathbf{r}}$, we have

$$\begin{aligned}\mathbf{v} =& \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},\end{aligned} \tag{4.24}$$

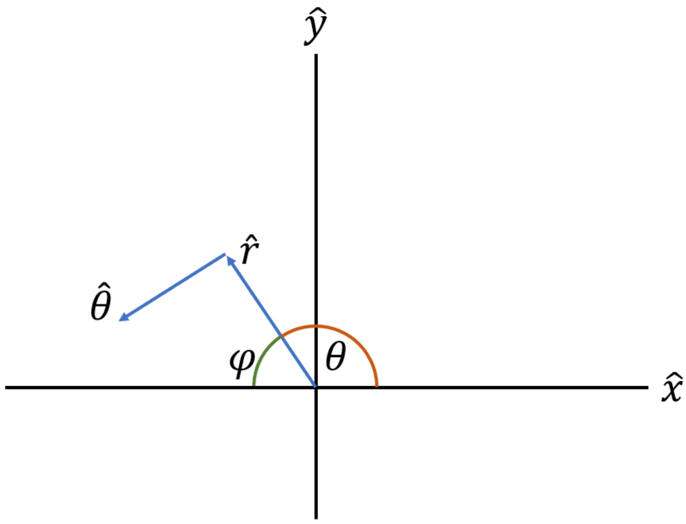

Figure 4.3: Polar coordinates in plane of interaction.

and

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\
&= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} \\
&= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}. \qquad (4.25)
\end{aligned}
$$

4.3.5 The force equation

The radial component of eq. (4.15) thus becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}. \qquad (4.26)$$

Since $\theta = \pi - \varphi$, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0 m_r}\frac{1}{r^2}. \qquad (4.27)$$

4.3.6 The angular momentum equation

Using polar coordinates, we obtain

$$m_r\mathbf{r}\times\mathbf{v} = m_r r\hat{\mathbf{r}}\times\left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right) = m_r r^2\frac{d\theta}{dt}\hat{\mathbf{z}} \qquad (4.28)$$

Since angular momentum $m_r\mathbf{r}\times\mathbf{v}$ is conserved, we have

$$m_r r^2\frac{d\varphi}{dt} = L = constant. \qquad (4.29)$$

We note here that L is positive since $d\varphi/dt$ is positive. Also, we have

$$m_r\mathbf{r}\times\mathbf{v} = -L\hat{\mathbf{z}}, \qquad (4.30)$$

that is, the angular momentum is in the negative $\hat{\mathbf{z}}$ direction.

4.3.7 Particle trajectory

The goal is to find the radial position of the particle as a function of its angular orientation. That is, we want to find $\tilde{r} = \tilde{r}(\tilde{\varphi})$ such that

$$r(t) = \tilde{r}(\varphi(t)). \tag{4.31}$$

To simplify the math, we introduce $\tilde{u} = \tilde{u}(\tilde{\varphi})$ such that $\tilde{u} = 1/\tilde{r}$. Thus

$$\frac{d\tilde{u}}{d\tilde{\varphi}} = -\frac{1}{\tilde{r}^2}\frac{d\tilde{r}}{d\tilde{\varphi}}, \tag{4.32}$$

or, after re-arranging

$$\frac{d\tilde{r}}{d\tilde{\varphi}} = -\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}. \tag{4.33}$$

We now proceed as follows. Taking the derivative of r, we get

$$\begin{aligned}
\frac{dr}{dt} &= \left(\frac{d\tilde{r}}{d\tilde{\varphi}}\right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} && [eq.\ (4.31)] \\
&= \left(-\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}\right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} && [eq.\ (4.33)] \\
&= \left(-\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}\right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} && [eq.\ (4.29)] \\
&= \left(-\frac{1}{\tilde{u}^2}\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r \tilde{r}^2}\right)_{\tilde{\varphi}=\varphi(t)} && [eq.\ (4.31)] \\
&= \left(-\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r}\right)_{\tilde{\varphi}=\varphi(t)} && (4.34)
\end{aligned}$$

Taking the derivative of the above, we get

$$\begin{aligned}
\frac{d}{dt}\frac{dr}{dt} &= \left[\frac{d}{d\tilde{\varphi}}\left(-\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r}\right)\right]_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L}{m_r}\right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} && [eq.\ (4.29)] \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L}{m_r}\frac{L}{m_r \tilde{r}^2}\right)_{\tilde{\varphi}=\varphi(t)} && [eq.\ (4.31)] \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L^2\tilde{u}^2}{m_r^2}\right)_{\tilde{\varphi}=\varphi(t)} && (4.35)
\end{aligned}$$

Plugging the last relation into eq. (4.27) gives

$$\left[-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}\frac{L^2\tilde{u}^2}{m_r^2} - \frac{1}{\tilde{u}}\left(\frac{L\tilde{u}^2}{m_r}\right)^2\right]_{\tilde{\varphi}=\varphi(t)} = \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r}\tilde{u}^2\right)_{\tilde{\varphi}=\varphi(t)}, \tag{4.36}$$

which, upon re-arranging and dropping the $\varphi(t)$ dependance, becomes

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2 m_r}{4\pi\epsilon_0 L^2} \tag{4.37}$$

Using eq. (4.30), we have

$$
\begin{aligned}
L &= -m_r(\mathbf{r}\times\mathbf{v})\cdot\hat{\mathbf{z}} \\
&= -m_r[\sin(-\theta)rv_0\hat{\mathbf{z}}]\cdot\hat{\mathbf{z}} \\
&= m_r\sin(\theta)rv_0 \\
&= m_r\sin(\pi-\varphi)rv_0 \\
&= m_r\sin\varphi rv_0 \\
&= m_r bv_0.
\end{aligned} \tag{4.38}
$$

Thus, we write the evolution equation for $\tilde{u}$ as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}+\tilde{u}=-\frac{q_1q_2}{4\pi\epsilon_0 m_r b^2 v_0^2}. \tag{4.39}$$

Introducing the notation

$$b_{90}=\frac{q_1q_2}{4\pi\epsilon_0 m_r v_0^2}, \tag{4.40}$$

the evolution equation for $\tilde{u}$ can be simply expressed as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2}+\tilde{u}=-\frac{b_{90}}{b^2}. \tag{4.41}$$

The boundary conditions for eq. (4.41) are as follows

$$\text{as } \varphi(t)\to 0,\quad r(t)\to\infty \tag{4.42}$$

$$\text{as } \varphi(t)\to 0,\quad \frac{dr(t)}{dt}\to -v_0 \tag{4.43}$$

Given eq. (4.31), eq. (4.42) can only be satisfied if as $\tilde{\varphi}\to 0$, $\tilde{r}\to\infty$. Thus, we also have, as $\tilde{\varphi}\to 0$, $\tilde{u}\to 0$. Similarly, given eq. (4.34), eq. (4.43) can only be satisfied if as $\tilde{\varphi}\to 0$

$$\frac{d\tilde{u}}{d\tilde{\varphi}}\frac{L}{m_r}\to v_0. \tag{4.44}$$

Using eq. (4.38) we rewrite the above as

$$\frac{d\tilde{u}}{d\tilde{\varphi}}\to\frac{1}{b}. \tag{4.45}$$

The general solution to eq. (4.41) is

$$\tilde{u}=A\cos\tilde{\varphi}+B\sin\tilde{\varphi}-\frac{b_{90}}{b^2}. \tag{4.46}$$

Applying the boundary conditions, we get

$$\tilde{u}=\frac{b_{90}}{b^2}\cos\tilde{\varphi}+\frac{1}{b}\sin\tilde{\varphi}-\frac{b_{90}}{b^2}, \tag{4.47}$$

which we finally re-write as

$$\frac{1}{\tilde{r}}=\frac{1}{b}\sin\tilde{\varphi}+\frac{b_{90}}{b^2}\left(\cos\tilde{\varphi}-1\right). \tag{4.48}$$

Part II

Kinetic description

Chapter 5

Governing equations

We denote the distribution function for a species α as $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where $\mathbf{r}$ and $\mathbf{v}$ are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_\alpha \, d\mathbf{r} d\mathbf{v} = N_\alpha, \tag{5.1}$$

where N_α is the total number of particles corresponding to species α.

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution along with Maxwell's equations

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = C_\alpha + S_\alpha \tag{5.2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{5.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{5.6}$$

$$\mathbf{J} = \sum_\alpha Z_\alpha e \int \mathbf{v} f_\alpha \, d\mathbf{v} \tag{5.7}$$

$$\rho_e = \sum_\alpha Z_\alpha e \int f_\alpha \, d\mathbf{v}. \tag{5.8}$$

In the above,

- m_α is the species mass
- e is the charge
- Z_α the charge number
- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ the charge current
- $\rho_e = \rho_e(\mathbf{r}, t)$ the charge density
- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the electric field

- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ the magnetic field.

The terms C_α and S_α represent collision and source terms.

If we express the collision term in the usual way, that is $C_\alpha = \sum_\beta C_{\alpha\beta}$, then we can make the following statements:

1. Conservation of particles:

$$\int C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall \alpha \qquad\qquad \int C_{\alpha\beta}\, d\mathbf{v} = 0 \quad \forall \alpha, \beta | \beta \neq \alpha. \tag{5.9}$$

2. Conservation of momentum:

$$\int m_\alpha \mathbf{v} C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall \alpha \qquad\qquad \sum_\alpha \sum_{\beta, \beta\neq\alpha} \int m_\alpha \mathbf{v} C_{\alpha\beta} d\mathbf{v} = 0. \tag{5.10}$$

3. Conservation of energy:

$$\int \frac{1}{2} m_\alpha v^2 C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall \alpha \qquad\qquad \sum_\alpha \sum_{\beta, \beta\neq\alpha} \int \frac{1}{2} m_\alpha v^2 C_{\alpha\beta}\, d\mathbf{v} = 0. \tag{5.11}$$

5.1 Fluid equations

We now define the particle density $n_\alpha = n_\alpha(\mathbf{r}, t)$, the fluid velocity $\mathbf{u}_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$ and the fluid energy per unit mass $E_\alpha = E_\alpha(\mathbf{r}, t)$ as follows

$$n_\alpha = \int f_\alpha\, d\mathbf{v} \tag{5.12}$$

$$\mathbf{u}_\alpha = \frac{1}{n_\alpha} \int \mathbf{v} f_\alpha\, d\mathbf{v} \tag{5.13}$$

$$E_\alpha = \frac{1}{n_\alpha} \int \frac{1}{2} v^2 f_\alpha\, d\mathbf{v}. \tag{5.14}$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_\alpha}{\partial t} + \nabla \cdot (\mathbf{v} f_\alpha) + \nabla_v \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] = C_\alpha + S_\alpha \tag{5.15}$$

5.1.1 Mass

Integrating eq. (5.15) over all $\mathbf{v}$ we obtain

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha \tag{5.16}$$

where

$$\hat{S}_\alpha = \int S_\alpha\, d\mathbf{v} \tag{5.17}$$

is an external source of mass.

5.1.2 Momentum

Multiplying eq. (5.15) by $\mathbf{v}$ and then integrating over all $\mathbf{v}$ leads to

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t}+\nabla\cdot\left(\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v}\right)+\int \nabla_v\cdot\left[\mathbf{v}\frac{Z_\alpha e}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_\alpha\right]-\nabla_v\mathbf{v}\cdot\left[\frac{Z_\alpha e}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_\alpha\right]\, d\mathbf{v}=$$
$$\sum_{\beta,\beta\neq\alpha}\int \mathbf{v} C_{\alpha\beta}\, d\mathbf{v}+\int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.18)$$

We note that the third term in eq. (5.18) is zero since we are integrating over all space, and that $\nabla_v \mathbf{v}$ is the identity matrix. We thus have

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t}+\nabla\cdot\left(\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v}\right)-\frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B})=$$
$$\sum_{\beta,\beta\neq\alpha}\int \mathbf{v} C_{\alpha\beta}\, d\mathbf{v}+\int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.19)$$

To proceed, we decompose $\mathbf{v}$ into a mean and a fluctuation, that is, $\mathbf{v}=\mathbf{u}_\alpha+\mathbf{w}_\alpha$. Using this decomposition

$$\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v}=\int(\mathbf{u}_\alpha\mathbf{u}_\alpha+2\mathbf{u}_\alpha\mathbf{w}_\alpha+\mathbf{w}_\alpha\mathbf{w}_\alpha)f_\alpha \, d\mathbf{v}=n_\alpha\mathbf{u}_\alpha\mathbf{u}_\alpha+\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}. \quad (5.20)$$

Thus, eq. (5.19) becomes

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t}+\nabla\cdot(n_\alpha\mathbf{u}_\alpha\mathbf{u}_\alpha)-\frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B})=-\nabla\cdot\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha d\mathbf{v}+$$
$$\sum_{\beta,\beta\neq\alpha}\int \mathbf{v} C_{\alpha\beta}\, d\mathbf{v}+\int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.21)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t}+\nabla\cdot(n_\alpha\mathbf{u}_\alpha\mathbf{u}_\alpha)-\frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B})=-\nabla\cdot\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha d\mathbf{v}+$$
$$\sum_{\beta,\beta\neq\alpha}\int \mathbf{w}_\alpha C_{\alpha\beta}\, d\mathbf{v}+\int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.22)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t}+\nabla\cdot(m_\alpha n_\alpha\mathbf{u}_\alpha\mathbf{u}_\alpha)-Z_\alpha e n_\alpha(\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B})=\nabla\cdot\boldsymbol{\sigma}_\alpha+\mathbf{R}_\alpha+\hat{\mathbf{M}}_\alpha, \quad (5.23)$$

where the stress tensor is

$$\boldsymbol{\sigma}_\alpha=-\int m_\alpha\mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}, \quad (5.24)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_\alpha=\sum_{\beta,\beta\neq\alpha}\int m_\alpha\mathbf{w}_\alpha C_{\alpha\beta}\, d\mathbf{v}, \quad (5.25)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_\alpha=\int m_\alpha\mathbf{v} S_\alpha \, d\mathbf{v}. \quad (5.26)$$

The stress tensor is typically decomposed into isotropic p_α and anisotropic (shear) $\mathbf{t}_\alpha$ tensors as follows

$$\boldsymbol{\sigma}_\alpha = -p_\alpha \mathbf{I} + \mathbf{t}_\alpha, \tag{5.27}$$

where P_α is given by

$$p_\alpha = \frac{1}{3}\int m_\alpha(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v}. \tag{5.28}$$

Thus, conservation of momentum becomes

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha. \tag{5.29}$$

5.1.3 Energy

Multiplying eq. (5.15) by $\frac{1}{2}v^2$ and then integrating over all $\mathbf{v}$ leads to

$$\begin{aligned}\frac{\partial n_\alpha E_\alpha}{\partial t} &+ \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v}\right] + \int \nabla_v \cdot \left[\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_\alpha\right] \\ &- \nabla_v\left[\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\right]\cdot\left[\frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_\alpha\right] d\mathbf{v} = \sum_{\beta,\beta\neq\alpha}\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta}\, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}.\end{aligned} \tag{5.30}$$

We note that the third term above is zero since we are integrating over all space, and that $\nabla_v[1/2(\mathbf{v}\cdot\mathbf{v})] = \mathbf{v}$. Thus, we have

$$\begin{aligned}\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v}\right] - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = \\ \sum_{\beta,\beta\neq\alpha}\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta}\, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}.\end{aligned} \tag{5.31}$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v} \;=\; \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}_\alpha + \mathbf{w}_\alpha) f_\alpha \, d\mathbf{v} \;=\; n_\alpha E_\alpha \mathbf{u}_\alpha + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \tag{5.32}$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha + 2\mathbf{u}_\alpha\cdot\mathbf{w}_\alpha + \mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \tag{5.33}$$

$$= \mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}. \tag{5.34}$$

Using the expressions above, eq. (5.31) becomes

$$\begin{aligned}\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla\cdot(n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = -\nabla\cdot\left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}\right) - \nabla\cdot\int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \\ + \sum_{\beta,\beta\neq\alpha}\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta}\, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}.\end{aligned} \tag{5.35}$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha E_\alpha}{\partial t}+\nabla\cdot(n_\alpha E_\alpha \mathbf{u}_\alpha)-\frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = -\nabla\cdot\left(\mathbf{u}_\alpha\cdot\int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha\,d\mathbf{v}\right)-\nabla\cdot\int\frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha\,d\mathbf{v}$$
$$+\,\mathbf{u}_\alpha\cdot\sum_{\beta,\beta\neq\alpha}\int \mathbf{w}_\alpha C_{\alpha\beta}\,d\mathbf{v}+\sum_{\beta,\beta\neq\alpha}\int\frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)C_{\alpha\beta}\,d\mathbf{v}+\int\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})S_\alpha\,d\mathbf{v}. \quad (5.36)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t}+\nabla\cdot(m_\alpha n_\alpha E_\alpha\mathbf{u}_\alpha)-Z_\alpha e n_\alpha\mathbf{E}\cdot\mathbf{u}_\alpha=\nabla\cdot(\mathbf{u}_\alpha\cdot\boldsymbol{\sigma}_\alpha)-\nabla\cdot\mathbf{q}_\alpha$$
$$+\,\mathbf{u}_\alpha\cdot\mathbf{R}_\alpha+Q_\alpha+\hat{Q}_\alpha, \quad (5.37)$$

where heat flux due to random motion is

$$\mathbf{q}_\alpha=\int\frac{1}{2}m_\alpha(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha\,d\mathbf{v}, \quad (5.38)$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_\alpha=\sum_{\beta,\beta\neq\alpha}\int\frac{1}{2}m_\alpha(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)C_{\alpha\beta}\,d\mathbf{v}, \quad (5.39)$$

and the external source of energy is

$$\hat{Q}_\alpha=\int\frac{1}{2}m_\alpha(\mathbf{v}\cdot\mathbf{v})S_\alpha\,d\mathbf{v}. \quad (5.40)$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t}+\nabla\cdot(m_\alpha n_\alpha E_\alpha\mathbf{u}_\alpha+p_\alpha\mathbf{u}_\alpha)-Z_\alpha e n_\alpha\mathbf{E}\cdot\mathbf{u}_\alpha=\nabla\cdot(\mathbf{u}_\alpha\cdot\mathbf{t}_\alpha)-\nabla\cdot\mathbf{q}_\alpha$$
$$+\,\mathbf{u}_\alpha\cdot\mathbf{R}_\alpha+Q_\alpha+\hat{Q}_\alpha, \quad (5.41)$$

We also note that the energy $m_\alpha n_\alpha E_\alpha$ can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (5.20) one obtains

$$\begin{aligned}
m_\alpha n_\alpha E_\alpha &= \int\frac{1}{2}m_\alpha(\mathbf{v}\cdot\mathbf{v})f_\alpha\,d\mathbf{v}\\
&= \int\frac{1}{2}m_\alpha(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)f_\alpha\,d\mathbf{v}+\frac{1}{2}m_\alpha n_\alpha(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha)\\
&= \frac{3}{2}P_\alpha+\frac{1}{2}m_\alpha n_\alpha(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha)\\
&= \frac{3}{2}P_\alpha+m_\alpha n_\alpha K_\alpha. \quad (5.42)
\end{aligned}$$

where $K_\alpha=\frac{1}{2}\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha$ is the kinetic energy of species α.

5.1.4 Kinetic and Internal Energies

The equation for the kinetic energy is obtained by dotting eq. (5.29) with $\mathbf{u}_\alpha$. For this, we first show that

$$\begin{aligned}
&\mathbf{u}_\alpha \cdot \left[\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha)\right] &(5.43)\\
&\quad = \mathbf{u}_\alpha \cdot \left\{\left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha)\right]\mathbf{u}_\alpha + m_\alpha n_\alpha\left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha\cdot\nabla\mathbf{u}_\alpha\right)\right\} &(5.44)\\
&\quad = \mathbf{u}_\alpha \cdot \left[m_\alpha \hat{S}_\alpha \mathbf{u}_\alpha + m_\alpha n_\alpha\left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha\cdot\nabla\mathbf{u}_\alpha\right)\right] &(5.45)\\
&\quad = 2m_\alpha \hat{S}_\alpha K_\alpha + m_\alpha n_\alpha\left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha\cdot\nabla K_\alpha\right) &(5.46)\\
&\quad = m_\alpha \hat{S}_\alpha K_\alpha + \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha)\right]K_\alpha + m_\alpha n_\alpha\left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha\cdot\nabla K_\alpha\right) &(5.47)\\
&\quad = m_\alpha \hat{S}_\alpha K_\alpha + \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha K \mathbf{u}_\alpha). &(5.48)
\end{aligned}$$

Thus, the equation for the turbulent kinetic energy is

$$\begin{aligned}
&\frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha K\mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E}\cdot\mathbf{u}_\alpha = \\
&-\nabla\cdot(\mathbf{u}_\alpha p_\alpha) + \nabla\cdot(\mathbf{u}_\alpha\cdot\mathbf{t}_\alpha) + p_\alpha\nabla\cdot\mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla\mathbf{u}_\alpha + \mathbf{u}_\alpha\cdot\mathbf{R}_\alpha + \mathbf{u}_\alpha\cdot\hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \quad (5.49)
\end{aligned}$$

Subtracting the above equation from eq. (5.41) leads to

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha\right) + \nabla\cdot\left(\frac{3}{2}p_\alpha\mathbf{u}_\alpha\right) = -p_\alpha\nabla\cdot\mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla\mathbf{u}_\alpha - \nabla\cdot\mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha\cdot\hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \quad (5.50)$$

5.1.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha, \quad (5.51)$$

- Momentum

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha\mathbf{u}_\alpha) - Z_\alpha e n_\alpha(\mathbf{E} + \mathbf{u}_\alpha\times\mathbf{B}) = -\nabla p_\alpha + \nabla\cdot\mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (5.52)$$

- Total Energy

$$\begin{aligned}
\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha\mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E}\cdot\mathbf{u}_\alpha &= \nabla\cdot(\mathbf{u}_\alpha\cdot\mathbf{t}_\alpha) - \nabla\cdot\mathbf{q}_\alpha \\
&\quad + \mathbf{u}_\alpha\cdot\mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \quad (5.53)
\end{aligned}$$

- Kinetic Energy

$$\begin{aligned}
&\frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha K\mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E}\cdot\mathbf{u}_\alpha = \\
&-\nabla\cdot(\mathbf{u}_\alpha p_\alpha) + \nabla\cdot(\mathbf{u}_\alpha\cdot\mathbf{t}_\alpha) + p_\alpha\nabla\cdot\mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla\mathbf{u}_\alpha + \mathbf{u}_\alpha\cdot\mathbf{R}_\alpha + \mathbf{u}_\alpha\cdot\hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \\
&\hfill (5.54)
\end{aligned}$$

- Internal Energy

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha\right)+\nabla\cdot\left(\frac{3}{2}p_\alpha\mathbf{u}_\alpha\right)=-p_\alpha\nabla\cdot\mathbf{u}_\alpha+\mathbf{t}_\alpha:\nabla\mathbf{u}_\alpha-\nabla\cdot\mathbf{q}_\alpha+Q_\alpha+\hat{Q}_\alpha-\mathbf{u}_\alpha\cdot\hat{\mathbf{M}}_\alpha+m_\alpha K_\alpha\hat{S}_\alpha. \tag{5.55}$$

Chapter 6

Transport coefficients

Collison integral

$$\Omega_{\alpha\beta}^{(lk)} = \sqrt{\frac{k_B T}{2\pi M_{\alpha\beta}}} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} \, dg. \tag{6.1}$$

In the above $M_{\alpha\beta}$ is the reduced mass, given by

$$M_{\alpha\beta} = \frac{M_\alpha M_\beta}{M_\alpha + M_\beta}, \tag{6.2}$$

and $\phi_{\alpha\beta}^{(l)}$ is the collision cross section for a given velocity, and is computed as

$$\phi_{\alpha\beta}^{(l)} = 2\pi \int_0^\infty \left(1 - \cos^l \chi_{\alpha\beta}\right) b \, db. \tag{6.3}$$

The scattering angle $\chi_{\alpha\beta}$ is given by

$$\chi_{\alpha\beta} = \pi - 2 \int_{r_{\alpha\beta}^{\min}}^\infty \frac{b}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{\alpha\beta(r)}}{g^2 k_B T}\right]^{1/2}} \, dr. \tag{6.4}$$

For a Coulombic interaction between ions, we can define the natural scale fore the cross-sectional area as

$$\phi_{\alpha\beta}^{(0)} = \frac{\pi \left(Z_\alpha Z_\beta e^2\right)^2}{\left(2k_B T\right)^2}. \tag{6.5}$$

Given this definition, we express the collision integral as

$$\Omega_{\alpha\beta} = \sqrt{\frac{\pi}{M_{\alpha\beta}}} \frac{(Z_\alpha Z_\beta e^2)^2}{(2k_B T)^{3/2}} \mathcal{F}_{\alpha\beta}^{lk}, \tag{6.6}$$

where

$$\mathcal{F}_{\alpha\beta}^{(lk)} = \frac{1}{2\phi_0} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} \, dg \tag{6.7}$$

We note that $\mathcal{F}_{\alpha\beta}^{(lk)} = 4\mathcal{K}_{lk}(g_{\alpha\beta})$, where $\mathcal{K}_{lk}(g_{\alpha\beta})$ is the notation from the Stanton-Murillo paper.

Part III

Fluid description

Chapter 7

Magnetohydrodynamics

7.1 Two-fluid equations

The starting point are the multi-fluid conservation laws eqs. (5.51), (5.52) and (5.55) and the Maxwell equations eqs. (5.3) to (5.6). We assume there are two species: electrons and singly-charged ions. Additionally, we assume no sources. Thus, the starting governing equations are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{7.1}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{7.2}$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i + \nabla \cdot \mathbf{t}_i + \mathbf{R}_i, \tag{7.3}$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e, \tag{7.4}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} p_i\right) + \nabla \cdot \left(\frac{3}{2} p_i \mathbf{u}_i\right) = -p_i \nabla \cdot \mathbf{u}_i + \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \tag{7.5}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} p_e\right) + \nabla \cdot \left(\frac{3}{2} p_e \mathbf{u}_e\right) = -p_e \nabla \cdot \mathbf{u}_e + \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \tag{7.6}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{7.7}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.9}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7.10}$$

$$\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e) \tag{7.11}$$

$$\rho_e = e(n_i - n_e) \tag{7.12}$$

These equations correspond to eq. (2.22) in Freidberg's Ideal MHD book.

7.2 Low-frequency, long-wavelength, asymptotic expansions

Two assumptions:

1. Transform full Maxwell's equations to low-frequency pre-Maxwell's equations. Formally achieved with $\epsilon_0 \to 0$. This has two consequences:
   - $\epsilon_0 \partial \mathbf{E}/\partial t \to 0$
     For this to be achieved it is required that $w/k \ll c$ and $V_{Ti}, V_{Te} \ll c$.
   - $\epsilon_0 \nabla \cdot \mathbf{E} \to 0$
     For this to be achieved it is required that $w \ll w_{pe}$ and $a \gg \lambda_D$.
2. Neglect electron inertia in the electron momentum equations. Formally achieved with $m_e \to 0$.

Due to the first assumption, the Maxwell equations eqs. (7.7) to (7.12) are now written as

$$n_i - n_e = 0 \tag{7.13}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.14}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.15}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{7.16}$$

7.3 Single-fluid equations

We define single-fluid variables as

$$\rho = m_i n_i + m_e n_e = m_i n \tag{7.17}$$

$$\mathbf{v} = \frac{m_i n_i \mathbf{u}_i + m_e n_e \mathbf{u}_e}{m_i n_i + m_e n_e} = \mathbf{u}_i \tag{7.18}$$

$$p = p_i + p_e = n(T_i + T_e) \tag{7.19}$$

$$T = \frac{T_i + T_e}{2}. \tag{7.20}$$

The two conservation of mass equations eqs. (7.1) and (7.2) will lead to two single-fluid equations. The first is obtained by multiplying eq. (7.1) by m_i, and the second is obtained by multiplying the ion and electron mass equations by e and then subtracting. The results are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7.21}$$

$$\nabla \cdot \mathbf{J} = 0. \tag{7.22}$$

Note that the second equation above is superfluous since it also follows from taking the divergence of eq. (7.16)

The two conservation of momentum equations will also lead to two single-fluid equations. The first is obtained by adding the ion and electron conservation of momentum equations, and the second is obtained by re-writing the electron conservation of momentum equations using the single-fluid variables. The results are

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u}\right) - \mathbf{J} \times \mathbf{B} + \nabla p = \nabla \cdot \mathbf{t}_i + \nabla \cdot \mathbf{t}_e, \tag{7.23}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en}(\mathbf{J} \times \mathbf{B} - \nabla p_e + \nabla \cdot \mathbf{T}_e + \mathbf{R}_e). \tag{7.24}$$

The two conservation of energy equations will also lead to two single-fluid equations. Each is evaluated using the single-fluid variables. As part of this derivation, we first rewrite the ion and electron internal energy eqs. (7.5) and (7.6) as

$$\frac{1}{\gamma - 1}\left(\frac{\partial p_i}{\partial t} + \mathbf{u}_i \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{u}_i\right) = \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \tag{7.25}$$

$$\frac{1}{\gamma - 1}\left(\frac{\partial p_e}{\partial t} + \mathbf{u}_e \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u}_e\right) = \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \tag{7.26}$$

where we have used $\gamma = 5/3$ (the ratio of specific heats for monoatomic systems). We then note that

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t} - \frac{1}{\rho}\nabla \rho \cdot \mathbf{v} = -\frac{\partial \ln \rho}{\partial t} - \nabla \ln \rho \cdot \mathbf{v}, \tag{7.27}$$

and thus

$$\gamma \nabla \cdot \mathbf{v} = -\frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial t} - \frac{1}{\rho^\gamma}\nabla \rho^\gamma \cdot \mathbf{v}. \tag{7.28}$$

The result above allows us to write

$$\begin{aligned} \frac{\partial p_\alpha}{\partial t} + \mathbf{v} \cdot \nabla p_\alpha + \gamma p_\alpha \nabla \cdot \mathbf{v} &= \frac{\partial p_\alpha}{\partial t} - p_\alpha \frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial t} + \mathbf{v} \cdot \nabla p_\alpha - p_\alpha \frac{1}{\rho^\gamma}\nabla \rho^\gamma \cdot \mathbf{v} \\ &= \rho^\gamma \left[\frac{\partial}{\partial t}\left(\frac{p_\alpha}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p_\alpha}{\rho^\gamma}\right)\right]. \end{aligned} \tag{7.29}$$

Thus, the ion energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{p_i}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p_i}{\rho^\gamma}\right) = \frac{\gamma - 1}{\rho^\gamma}\left(\mathbf{t}_i : \nabla \mathbf{v} - \nabla \cdot \mathbf{q}_i + Q_i\right), \tag{7.30}$$

and the electron energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{p_e}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p_e}{\rho^\gamma}\right) = \frac{\gamma - 1}{\rho^\gamma}\left[\mathbf{t}_e : \nabla \left(\mathbf{v} - \frac{\mathbf{J}}{en}\right) - \nabla \cdot \mathbf{q}_e + Q_e\right] + \frac{1}{en}\mathbf{J} \cdot \nabla \left(\frac{p_e}{\rho^\gamma}\right). \tag{7.31}$$

7.4 Resisitive MHD

7.5 Ideal MHD

The ideal MHD equations are obtained by neglecting the right-hand sides of eqs. (7.23), (7.24), (7.30) and (7.31). Summing the two pressure equations, the resulting equations would be

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7.32}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{7.33}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \tag{7.34}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma}\right) = 0, \tag{7.35}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.36}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7.37}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{7.38}$$

Given the vector identity

$$\frac{1}{2}\nabla\left(B^2\right) = \mathbf{B} \times (\nabla \times \mathbf{B}) + (\mathbf{B} \cdot \nabla)\,\mathbf{B}, \tag{7.39}$$

we can use Ampere's law to re-write the $\mathbf{J} \times \mathbf{B}$ term in the velocity equation as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0}\left[(\mathbf{B} \cdot \nabla)\,\mathbf{B} - \frac{1}{2}\nabla\left(B^2\right)\right]. \tag{7.40}$$

Similarly, given the vector identity

$$\nabla \times (\mathbf{B} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\,\mathbf{B} - (\mathbf{B} \cdot \nabla)\,\mathbf{v} + \mathbf{B}\,(\nabla \cdot \mathbf{v}) - \mathbf{v}\,(\nabla \cdot \mathbf{B}), \tag{7.41}$$

we can use Ohm's law to re-write the $\nabla \times \mathbf{E}$ term in Faraday's law as

$$\nabla \times \mathbf{E} = \nabla \times (-\mathbf{v} \times \mathbf{B}) = (\mathbf{v} \cdot \nabla)\,\mathbf{B} - (\mathbf{B} \cdot \nabla)\,\mathbf{v} + \mathbf{B}\,(\nabla \cdot \mathbf{v}). \tag{7.42}$$

Thus, the ideal MHD equations can be summarized as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7.43}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.44}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \frac{1}{\mu_0}\left[(\mathbf{B} \cdot \nabla)\,\mathbf{B} - \frac{1}{2}\nabla\left(B^2\right)\right] \tag{7.45}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{B} = (\mathbf{B} \cdot \nabla)\,\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) \tag{7.46}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{7.47}$$

If we assume incompressibility, then the above simplifies to

$$\nabla \cdot \mathbf{v} = 0, \tag{7.48}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7.49}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \frac{1}{\mu_0}\left[(\mathbf{B} \cdot \nabla)\,\mathbf{B} - \frac{1}{2}\nabla\left(B^2\right)\right] \tag{7.50}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{B} = (\mathbf{B} \cdot \nabla)\,\mathbf{v} \tag{7.51}$$

Chapter 8

Waves in plasmas

Chapter 9

Simple models

9.1 Hasegawa-Mima

9.1.1 Assumptions

1. Singly-charged ions.
2. No shear stresses, sources, or collisions.
3. Cold ion approximation, i.e. $T_e \gg T_i$ and thus $\nabla p_i \approx 0$ [Hasegawa and Mima, 1977]. .
4. Electrostatic field, i.e. $\mathbf{E} = -\nabla\phi$.
5. Magnetic field is constant.
6. Neglect parallel ion inertia, i.e. $\nabla\cdot(\alpha\mathbf{u}_i) \approx \nabla\cdot(\alpha\mathbf{u}_{i,\perp})$, or in other words $(\mathbf{u}_i\cdot\nabla) \approx (\mathbf{u}_{i,\perp}\cdot\nabla)$ and $\nabla\cdot\mathbf{u}_i \approx \nabla\cdot\mathbf{u}_{i,\perp}$ [Hasegawa and Mima, 1977].
7. Adiabatic electrons, i.e. $n_e = n_0 \exp(e\phi/T_e)$, where $n_0 = n_0(\mathbf{x})$.
8. Quasi-neutrality, i.e. $n_i \approx n_e$.

9.1.2 Derivation

Using the assumptions in items 1 and 2, the momentum eq. (5.52) becomes

$$m_i n_i\left(\frac{\partial\mathbf{u}_i}{\partial t} + \mathbf{u}_i\cdot\nabla\mathbf{u}_i\right) = en_i(\mathbf{E} + \mathbf{u}_i\times\mathbf{B}) - \nabla p_i. \tag{9.1}$$

Using the assumptions in items 3 and 4, the above becomes

$$\frac{\partial\mathbf{u}_i}{\partial t} + \mathbf{u}_i\cdot\nabla\mathbf{u}_i = -\frac{e}{m_i}\nabla\phi + \frac{e}{m_i}\mathbf{u}_i\times\mathbf{B}. \tag{9.2}$$

Introduce the coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and assume $\mathbf{B}$ points in the $\mathbf{e}_3$ direction. Thus, in terms of its components, the above is

$$\begin{aligned}
\frac{\partial u_{i,1}}{\partial t} + \mathbf{u}_i\cdot\nabla u_{i,1} &= -\frac{e}{m_i}\frac{\partial\phi}{\partial x_1} + \frac{e}{m_i}(\mathbf{u}_i\times\mathbf{B})_1, \\
\frac{\partial u_{i,2}}{\partial t} + \mathbf{u}_i\cdot\nabla u_{i,2} &= -\frac{e}{m_i}\frac{\partial\phi}{\partial x_2} + \frac{e}{m_i}(\mathbf{u}_i\times\mathbf{B})_2, \\
\frac{\partial u_{i,3}}{\partial t} + \mathbf{u}_i\cdot\nabla u_{i,3} &= -\frac{e}{m_i}\frac{\partial\phi}{\partial x_3}.
\end{aligned} \tag{9.3}$$

Defining the perpendicular velocity as $\mathbf{u}_{i,\perp} = [u_{i,1}, u_{i,2}, 0]^T$ and the perpendicular gradient as $\nabla_\perp = [\partial_1, \partial_2, 0]^T$, we have

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_{i,\perp} = -\frac{e}{m_i}\nabla_\perp \phi + \frac{e}{m_i}\mathbf{u}_i \times \mathbf{B}. \tag{9.4}$$

Using the assumption in item 6 and noting that $\mathbf{u}_i \times \mathbf{B} = \mathbf{u}_{i,\perp} \times \mathbf{B}$, we obtain

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} + \mathbf{u}_{i,\perp} \cdot \nabla_\perp \mathbf{u}_{i,\perp} = -\frac{e}{m_i}\nabla_\perp \phi + \frac{e}{m_i}\mathbf{u}_{i,\perp} \times \mathbf{B}. \tag{9.5}$$

We now introduce the scalings for a characteristic frequency w and length scale r

$$\frac{w}{w_{c,i}} \sim \epsilon \qquad \frac{r_s}{r} \sim \epsilon, \tag{9.6}$$

where $w_{c,i} = eB/m_i$ is the cyclotron frequency, $r_s = v_s/w_{c,i}$ is a reference length scale, and $v_s = \sqrt{T_e/m_i}$ a reference velocity scale. Given these variables, we assume

$$\frac{\partial \mathbf{u}_{i,\perp}}{\partial t} \sim \mathbf{u}_{i,\perp} w \qquad \nabla_\perp \mathbf{u}_{i,\perp} \sim \frac{\mathbf{u}_{i,\perp}}{r} \qquad \mathbf{E} \sim \mathbf{u}_{i,\perp} B. \tag{9.7}$$

Finally, we introduce the decomposition $\mathbf{u}_{i,\perp} = \mathbf{u}^{(0)}_{i,\perp} + \mathbf{u}^{(1)}_{i,\perp}$, where $\mathbf{u}^{(0)}_{i,\perp} \sim v_s$ and $\mathbf{u}^{(1)}_{i,\perp} \sim \epsilon v_s$. We use this decomposition in eq. (9.5) and then divide the PDE by $w_{c,i}v_s$. The order of each element in the resulting equation is as follows

1. $\frac{\partial \mathbf{u}^{(0)}_{i,\perp}}{\partial t} \sim \epsilon$.
2. $\frac{\partial \mathbf{u}^{(1)}_{i,\perp}}{\partial t} \sim \epsilon^2$.
3. $\mathbf{u}^{(0)}_{i,\perp} \cdot \nabla_\perp \mathbf{u}^{(0)}_{i,\perp} \sim \epsilon$.
4. $\mathbf{u}^{(0)}_{i,\perp} \cdot \nabla_\perp \mathbf{u}^{(1)}_{i,\perp} \sim \epsilon^2$.
5. $\mathbf{u}^{(1)}_{i,\perp} \cdot \nabla_\perp \mathbf{u}^{(0)}_{i,\perp} \sim \epsilon^2$.
6. $\mathbf{u}^{(1)}_{i,\perp} \cdot \nabla_\perp \mathbf{u}^{(1)}_{i,\perp} \sim \epsilon^3$.
7. $-\frac{e}{m_i}\nabla_\perp \phi \sim 1$.
8. $\frac{e}{m_i}\mathbf{u}^{(0)}_{i,\perp} \times \mathbf{B} \sim 1$.
9. $\frac{e}{m_i}\mathbf{u}^{(1)}_{i,\perp} \times \mathbf{B} \sim \epsilon$.

Combining the first order terms we obtain

$$0 = -\nabla_\perp \phi + \mathbf{u}^{(0)}_{i,\perp} \times \mathbf{B}, \tag{9.8}$$

which, upon crossing by $\mathbf{B}$, gives

$$\mathbf{u}^{(0)}_{i,\perp} = -\nabla_\perp \phi \times \frac{\mathbf{b}}{B}. \tag{9.9}$$

Combining the terms of order ϵ we obtain

$$\frac{\partial \mathbf{u}^{(0)}_{i,\perp}}{\partial t} + \mathbf{u}^{(0)}_{i,\perp} \cdot \nabla_\perp \mathbf{u}^{(0)}_{i,\perp} = \frac{e}{m_i}\mathbf{u}^{(1)}_{i,\perp} \times \mathbf{B}, \tag{9.10}$$

which, upon crossing by $\mathbf{B}$, gives

$$\mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B}\left[\frac{\partial \nabla_\perp \phi}{\partial t} + \left(\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp\right)\nabla_\perp \phi\right]. \tag{9.11}$$

The velocity given by eq. (9.9) is referred to as the $E \times B$ drift, and the velocity given by eq. (9.11) as the polarization drift.

Using the assumption in item 2, the continuity equation for ions is

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0. \tag{9.12}$$

Using the assumption in item 6 the above becomes

$$\frac{\partial n_i}{\partial t} + \nabla_\perp \cdot (n_i \mathbf{u}_{i,\perp}) = 0, \tag{9.13}$$

which is re-written as

$$\frac{1}{n_i}\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp} \cdot \left(\frac{1}{n_i}\nabla_\perp n_i\right) + \nabla_\perp \cdot \mathbf{u}_{i,\perp} = 0. \tag{9.14}$$

We now perform a similar scaling analysis as with the momentum equation, assuming that

$$\frac{\partial n_i}{\partial t} \sim n_i w \qquad \nabla_\perp n_i \sim \frac{n_i}{r}. \tag{9.15}$$

The resulting order of each element after dividing by w_c is as follows

1. $\frac{1}{n_i}\frac{\partial n_i}{\partial t} \sim \epsilon$.
2. $\mathbf{u}_{i,\perp}^{(0)} \cdot \left(\frac{1}{n_i}\nabla_\perp n_i\right) \sim \epsilon$.
3. $\mathbf{u}_{i,\perp}^{(1)} \cdot \left(\frac{1}{n_i}\nabla_\perp n_i\right) \sim \epsilon^2$.
4. $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(0)} \sim \epsilon$.
5. $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} \sim \epsilon^2$.

However, this case reveals the limits of a scaling analysis. Using a vector identity, we show that

$$\nabla_\perp \cdot (\nabla_\perp \phi \times \mathbf{B}) = (\nabla_\perp \times \nabla_\perp \phi) \cdot \mathbf{B} - \nabla_\perp \phi \cdot (\nabla_\perp \times \mathbf{B}) = 0, \tag{9.16}$$

Given the definition in eq. (9.9) we conclude that $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(0)} = 0$. Now, since scaling analysis indicates $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)}$ is smaller than $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(0)}$ by a factor of ϵ, one would conclude $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)}$ has to be zero as well. However, given the definition of $\mathbf{u}_{i,\perp}^{(1)}$, its divergence is not zero. Thus, for this case, we'll retain all terms up to order ϵ, but since $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(0)}$ is zero, we'll also retain the term $\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)}$. The governing equation is thus

$$\frac{1}{n_i}\frac{\partial n_i}{\partial t} + \mathbf{u}_{i,\perp}^{(0)} \cdot \left(\frac{1}{n_i}\nabla_\perp n_i\right) + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0, \tag{9.17}$$

which can be re-written as

$$\frac{w_{c,i}}{n_i}\frac{\partial}{\partial t}\left(\frac{n_i}{w_{c,i}}\right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \left[\frac{w_{c,i}}{n_i}\nabla_\perp \left(\frac{n_i}{w_{c,i}}\right)\right] + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0, \tag{9.18}$$

or

$$\frac{\partial}{\partial t}\left[\ln\left(\frac{n_i}{w_{c,i}}\right)\right] + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left[\ln\left(\frac{n_i}{w_{c,i}}\right)\right] + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{9.19}$$

Using the assumptions in items 7 and 8, one obtains

$$\ln\left(\frac{n_i}{w_{c,i}}\right) = \ln\left[\frac{n_0}{w_{c,i}}\exp\left(\frac{e\phi}{T_e}\right)\right] = \ln\left(\frac{n_0}{w_{c,i}}\right) + \frac{e\phi}{T_e}. \tag{9.20}$$

Thus, the governing equation for density becomes

$$\frac{\partial}{\partial t}\left(\frac{e\phi}{T_e}\right) + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp \left[\ln\left(\frac{n_0}{w_{c,i}}\right)\right] + \mathbf{u}_{i,\perp}^{(0)} \cdot \frac{e\nabla_\perp \phi}{T_e} + \nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = 0. \tag{9.21}$$

We note that the third term above is zero, and the fourth term is

$$\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B}\left\{\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \nabla_\perp \cdot \left[\left(\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp\right)\nabla_\perp \phi\right]\right\}. \tag{9.22}$$

The second term above is best computed using tensor notation, and we'll use u_j to denote the components of $\mathbf{u}_{i,\perp}^{(0)}$. Thus,

$$\frac{\partial}{\partial x_i}\left[\left(u_j \frac{\partial}{\partial x_j}\right)\frac{\partial \phi}{\partial x_i}\right] = \frac{\partial u_j}{\partial x_i}\frac{\partial^2 \phi}{\partial x_j \partial x_i} + u_j \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \phi}{\partial x_i \partial x_i}\right). \tag{9.23}$$

Using the definition of $\mathbf{u}_{i,\perp}^{(0)}$, the first term on the right-hand side above can be expressed as

$$\begin{aligned}
\frac{\partial u_j}{\partial x_i}\frac{\partial^2 \phi}{\partial x_j \partial x_i} &= -\frac{1}{B^2}\epsilon_{jpq}\frac{\partial^2 \phi}{\partial x_p \partial x_i} B_q \frac{\partial^2 \phi}{\partial x_j \partial x_i} \\
&= -\frac{1}{B^2}\epsilon_{qjp}\frac{\partial^2 \phi}{\partial x_j \partial x_i}\frac{\partial^2 \phi}{\partial x_p \partial x_i} B_q \\
&= -\frac{1}{B^2}\epsilon_{qjp}\left(\frac{\partial^2 \phi}{\partial x_j \partial x_1}\frac{\partial^2 \phi}{\partial x_p \partial x_1} + \frac{\partial^2 \phi}{\partial x_j \partial x_2}\frac{\partial^2 \phi}{\partial x_p \partial x_2}\right) B_q.
\end{aligned} \tag{9.24}$$

Since $\epsilon_{qjp}\partial_j a \partial_p a \to \nabla a \times \nabla a = 0$ for any scalar a, the term above is identically zero. Thus, we have

$$\nabla_\perp \cdot \mathbf{u}_{i,\perp}^{(1)} = -\frac{1}{w_{c,i}B}\left[\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \left(\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp\right)\nabla_\perp^2 \phi\right], \tag{9.25}$$

and eq. (9.21) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - \frac{e\phi}{T_e}\right) + \left(\mathbf{u}_{i,\perp}^{(0)} \cdot \nabla_\perp\right)\left[\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - \ln\left(\frac{n_0}{w_{c,i}}\right)\right] = 0. \tag{9.26}$$

Plugging in for $\mathbf{u}_{i,\perp}^{(0)}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - \frac{e\phi}{T_e}\right) - \left[\left(\nabla_\perp \phi \times \frac{\mathbf{b}}{B}\right) \cdot \nabla_\perp\right]\left[\frac{1}{w_{c,i}B}\nabla_\perp^2 \phi - \ln\left(\frac{n_0}{w_{c,i}}\right)\right] = 0. \tag{9.27}$$

We now introduce the normalizations

$$\phi(t, \mathbf{x}) = \frac{T_e}{e}\hat{\phi}(\hat{t}, \hat{\mathbf{x}}) \qquad n_0(\mathbf{x}) = \hat{n}_0(\hat{\mathbf{x}}), \tag{9.28}$$

where $\hat{t} = tw_{c,i}$ and $\hat{\mathbf{x}} = \mathbf{x}/r_s$. Neglecting the hat notation for the sake of simplicity, eq. (9.27) finally becomes

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2\phi - \phi\right) - \left[(\nabla_\perp\phi \times \mathbf{b}) \cdot \nabla_\perp\right]\left[\nabla_\perp^2\phi - \ln\left(\frac{n_0}{w_{c,i}}\right)\right] = 0. \tag{9.29}$$

Using the following expansion

$$(\nabla_\perp\phi \times \mathbf{b}) \cdot \nabla_\perp = \frac{\partial\phi}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial\phi}{\partial x_1}\frac{\partial}{\partial x_2}, \tag{9.30}$$

The Hasegawa-Mima equation can be written as

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2\phi - \phi\right) - \frac{\partial\phi}{\partial x_2}\frac{\partial\nabla_\perp^2\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_1}\frac{\partial\nabla_\perp^2\phi}{\partial x_2} + \beta\frac{\partial\phi}{\partial x_2} = 0, \tag{9.31}$$

where

$$\beta = \frac{\partial}{\partial x_1}\ln\left(\frac{n_0}{w_{c,i}}\right). \tag{9.32}$$

9.1.3 Spectral space

In this section we derive the equation for the Fourier coefficient $\hat{\phi}_\mathbf{n} = \hat{\phi}(t)_\mathbf{n}$, which relates to the potential through the following

$$\phi(t,x) = \sum_{\mathbf{n}=-\infty}^{\infty} \hat{\phi}_\mathbf{n}(t)e^{i\mathbf{k}_\mathbf{n}\cdot\mathbf{x}}, \tag{9.33}$$

$$\hat{\phi}_\mathbf{n}(t) = \frac{1}{L^2}\int_{L^2}\phi(t,\mathbf{x})e^{-i\mathbf{k}_\mathbf{n}\cdot\mathbf{x}}\,d\mathbf{x}. \tag{9.34}$$

We introduce the operator $\mathcal{F}\{\}_\mathbf{n}$, which is defined by

$$\mathcal{F}\{\phi(t,\mathbf{x})\}_\mathbf{n} = \frac{1}{L^2}\int_{L^2}\phi(t,\mathbf{x})e^{-i\mathbf{k}_\mathbf{n}\cdot\mathbf{x}}\,d\mathbf{x}. \tag{9.35}$$

The equation for $\hat{\phi}_\mathbf{n}$ is obtained by applying this operator to eq. (9.29). Thus, the time derivative term in that equation becomes

$$\mathcal{F}\left\{\frac{\partial}{\partial t}\left(\nabla_\perp^2 - \phi\right)\right\}_\mathbf{n} = \frac{\partial}{\partial t}\mathcal{F}\left\{\nabla_\perp^2\phi - \phi\right\}_\mathbf{n} = \frac{\partial}{\partial t}\left(-k_\mathbf{n}^2\hat{\phi}_\mathbf{n} - \hat{\phi}_\mathbf{n}\right) = -\left(1 + k_\mathbf{n}^2\right)\frac{\partial\hat{\phi}_\mathbf{n}}{\partial t}. \tag{9.36}$$

We assume $\nabla\ln(n_o/w_{ci})$ is constant in space. Thus, the term containing the inhomogeneity becomes

$$\begin{aligned}\mathcal{F}&\left\{(\nabla_\perp\phi \times \mathbf{b}) \cdot \nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right)\right\}_\mathbf{n} \\ &= \left(\mathcal{F}\left\{\nabla_\perp\phi\right\}_\mathbf{n} \times \mathbf{b}\right) \cdot \nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right) = i\left(\mathbf{k}_\mathbf{n} \times \mathbf{b}\right) \cdot \nabla_\perp \ln\left(\frac{n_o}{w_{ci}}\right)\hat{\phi}_\mathbf{n}.\end{aligned} \tag{9.37}$$

The remaining term is computed as follows

$$
\begin{aligned}
&\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}} \\
&\qquad=\mathcal{F}\left\{-\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left[\hat{\phi}_{\mathbf{n}'}i\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{b}\right)e^{i\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{x}}\right]\cdot\left[\hat{\phi}_{\mathbf{n}''}\left(-ik_{\mathbf{n}''}^2\mathbf{k}_{\mathbf{n}''}\right)e^{i\mathbf{k}_{\mathbf{n}''}\cdot\mathbf{x}}\right]\right\}_{\mathbf{n}} \\
&\qquad=-\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{b}\right)\cdot\mathbf{k}_{\mathbf{n}''}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\mathcal{F}\left\{e^{i\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{x}}e^{i\mathbf{k}_{\mathbf{n}''}\cdot\mathbf{x}}\right\} \\
&\qquad=\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''}
\end{aligned}
\tag{9.38}
$$

Since $\mathbf{n}'$ and $\mathbf{n}''$ are just symbolic variables for the summation, we can write the above as follows

$$
\begin{aligned}
&\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}} \\
&\qquad=\frac{1}{2}\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}k_{\mathbf{n}''}^2\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''} \\
&\qquad\quad+\frac{1}{2}\sum_{\mathbf{n}''=-\infty}^{\infty}\sum_{\mathbf{n}'=-\infty}^{\infty}\left(\mathbf{k}_{\mathbf{n}''}\times\mathbf{k}_{\mathbf{n}'}\right)\cdot\mathbf{b}k_{\mathbf{n}'}^2\hat{\phi}_{\mathbf{n}''}\hat{\phi}_{\mathbf{n}'}\delta_{\mathbf{n},\mathbf{n}''+\mathbf{n}'} \\
&\qquad=\sum_{\mathbf{n}'=-\infty}^{\infty}\sum_{\mathbf{n}''=-\infty}^{\infty}\frac{1}{2}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}\delta_{\mathbf{n},\mathbf{n}'+\mathbf{n}''}
\end{aligned}
\tag{9.39}
$$

Thus, we finally have

$$
\mathcal{F}\left\{-\left(\nabla_\perp\phi\times\mathbf{b}\right)\cdot\nabla_\perp^3\phi\right\}_{\mathbf{n}}=\sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''}\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''}
\tag{9.40}
$$

Combining the results above, we obtain

$$
\frac{\partial\hat{\phi}_{\mathbf{n}}}{\partial t}+iw_{\mathbf{n}}\hat{\phi}_{\mathbf{n}}=\sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''}\Lambda_{\mathbf{n}',\mathbf{n}''}^{\mathbf{n}}\hat{\phi}_{\mathbf{n}'}\hat{\phi}_{\mathbf{n}''},
\tag{9.41}
$$

where

$$
w_{\mathbf{n}}=-\frac{\left(\mathbf{k}_{\mathbf{n}}\times\mathbf{b}\right)}{1+k_{\mathbf{n}}^2}\cdot\nabla_\perp\ln\left(\frac{n_o}{w_{ci}}\right),
\tag{9.42}
$$

and

$$
\Lambda_{\mathbf{n}',\mathbf{n}''}^{\mathbf{n}}=\frac{1}{2}\frac{\left(\mathbf{k}_{\mathbf{n}'}\times\mathbf{k}_{\mathbf{n}''}\right)\cdot\mathbf{b}\left(k_{\mathbf{n}''}^2-k_{\mathbf{n}'}^2\right)}{1+k_{\mathbf{n}}^2}.
\tag{9.43}
$$

Note that $w_{\mathbf{n}}$ can also be written as

$$
w_{\mathbf{n}}=-\frac{k_{2,\mathbf{n}}\mathbf{e}_1-k_{1,\mathbf{n}}\mathbf{e}_2}{1+k_{\mathbf{n}}^2}\cdot\beta\mathbf{e}_1=-\frac{k_{2,\mathbf{n}}\beta}{1+k_{\mathbf{n}}^2}.
\tag{9.44}
$$

9.2 Hasegawa-Wakatani

Part IV

Appendices

Appendix A

Electromagnetism

A.1 Electrostatics

- Coulomb's Law

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}} \tag{A.1}$$

- Electric Field $\mathbf{E}$ derived from $\mathbf{F} = Q\mathbf{E}$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{A.2}$$

- If there are multiple point charges

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i \tag{A.3}$$

- **Charge distributions and fields:** if the charges are so small and so numerous that they can be described using a continuous distribution (i.e. $q_i \to dq = \rho d\tau$, where ρ is a charge density and $d\tau$ and infinitesimal volume)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{r^2}\hat{\mathbf{r}}d\tau' \tag{A.4}$$

If the charge distribution is localized to a surface or a line, then the analogous of the above is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{r}')}{r^2}\hat{\mathbf{r}}da' \qquad \text{or} \qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{r^2}\hat{\mathbf{r}}dl' \tag{A.5}$$

Taking the divergence and curl of eq. (A.4):

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{A.6}$$

$$\nabla\times\mathbf{E} = 0 \tag{A.7}$$

- **Fields and potentials**
Since $\nabla\times\mathbf{E} = 0$ we have

$$\mathbf{E} = -\boldsymbol{\nabla}V. \tag{A.8}$$

where V is the electric potential. Fundamental theorem of calculus can be used to express the potential $V(\mathbf{r})$ as

$$V(\mathbf{r}) - V(\mathcal{O}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{A.9}$$

where $\mathcal{O}$ is the reference point, at which one usually defines $V(\mathcal{O}) = 0$ (e.g. sea-level as the altitude at which height is equal to zero).

- **Charge distributions and potentials**
  Divergence of eq. (A.8) gives

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{A.10}$$

  whose solution is

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{r} d\tau'. \tag{A.11}$$

- Define potential energy U as the negative of the work required to move charge Q from $\mathbf{a}$ to $\mathbf{b}$.

$$U = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})] \tag{A.12}$$

  If the reference point is infinity, then $U(\mathbf{r}) = QV(\mathbf{r})$.

- Potential energy of a set of charges q_i

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{A.13}$$

  where $V(\mathbf{r}_i)$ is the potential due to all charges except the one at $\mathbf{r}_i$. The continuous form is

$$U = \frac{1}{2}\int \rho V \, d\tau = \frac{\epsilon_0}{2}\int E^2 \, d\tau \tag{A.14}$$

  were now V represents the potential due to all charges. Thus, if ρ is such that it defines a set of point charges (e.g. $\delta(\mathbf{r})$), then eq. (A.14) would be equal to eq. (A.13) plus the additional terms corresponding to $i = j$. Those additional terms correspond to the energy required to create point charges, which is infinity.

- Electrostatic conductors: materials whose charges are free to move but are in a state of electrostatic equilibrium. $\mathbf{E} = 0$ inside, since if it were not, then charges would move and the material would not be in electrostatic equilibrium. As a consequence, $\rho = 0$ inside, all the charge is on the surface, and $\mathbf{E}$ is perpendicular to the outer surface.

- If there is a cavity within the conductor, and within the cavity a charge q, an amount $-q$ of charge will reside in the inner surface, and an amount q on the outer surface, and that configuration will lead to $\mathbf{E} = 0$ inside the conductor.

- Faraday cage: if there are no charges within such cavity, then $\mathbf{E} = 0$ within the cavity as well, regardless of how many charges are outside the conductor. If $\mathbf{E}$ was not zero inside the cavity, then its field lines would start and end on the cavity walls. Letting the field lines be part of a closed loop, the rest of which is inside the conductor, then the line integral along the closed loop would be positive, in violation of $\nabla \times \mathbf{E} = 0$.

- A capacitor consists of two conductors, one with charge Q and the other with charge $-Q$. The constant of proportionality between Q and the voltage difference between the two conductors is the capacitance $C = Q/V$. The energy stored in a capacitor is $W = \frac{1}{2}CV^2$.

A.2 Electric Fields in Matter

A.3 Magnetostatiscs

- Lorentz force law: $\mathbf{F} = Q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$
- Given the charge densities λ, σ, and ρ
  - Current [Amperes]: the amount of charge that passes a point in a small amount of time.
$$\mathbf{I} = \lambda \mathbf{v} \tag{A.15}$$
  - Surface current density: the amount of charge that passes a line in a small amount of time.
$$\mathbf{K} = \sigma \mathbf{v} \tag{A.16}$$
  - Volume current density: the amount of charge that passes an area in a small amount of time.
$$\mathbf{J} = \rho \mathbf{v} \tag{A.17}$$
- Magnetic component of Lorentz force
$$\mathbf{F}_{\text{mag}} = \int \mathbf{I} \times \mathbf{B}\, dl = \int \mathbf{K} \times \mathbf{B}\, da = \int \mathbf{J} \times \mathbf{B}\, d\tau \tag{A.18}$$
- Conservation of current
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{A.19}$$
- **Charge currents and fields**
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, dl' \tag{A.20}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, da' \tag{A.21}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2}\, d\tau' \tag{A.22}$$
  Taking the divergence and curl of eq. (A.22):
$$\nabla \cdot \mathbf{B} = 0 \tag{A.23}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{A.24}$$
- A steady straight-line current leads to a circular magnetic field around it. A steady circular current leads to a straight magnetic field line along the axis of the circle.
- **Fields and potentials**
  Since $\nabla \cdot \mathbf{B} = 0$ we have
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{A.25}$$
  where $\mathbf{A}$ is the magnetic vector potential.

- **Charge currents and potentials**
  The magnetic field is not altered if a function whose curl vanishes (that is $\nabla\lambda$) is added to **A**. Thus, λ can be picked to make **A** divergence-less. Taking the curl of **B** then leads to
  $$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}, \tag{A.26}$$
  whose solution is
  $$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{r}d\tau'. \tag{A.27}$$

A.4 Magnetic Fields in Matter

A.5 Electrodynamics

A.5.1 Ohm's Law

- Ohm's law refers to the proportionality between the force per unit charge applied to charged elements and the resulting volume current that occurs. That is,
  $$\mathbf{J} = \sigma\mathbf{f}, \tag{A.28}$$
  where **f** is the force per unit charge, and the proportionality σ is the conductivity. If one neglects the magnetic contribution to **f**, which is typically done for non-plasmas, then
  $$\mathbf{J} = \sigma\mathbf{E}. \tag{A.29}$$
  For steady currents ($\partial\rho/\partial t = 0$) and uniform conductivity
  $$\nabla\cdot\mathbf{E} = \frac{1}{\sigma}\nabla\cdot\mathbf{J} = 0 \tag{A.30}$$
  and thus, the charge density is zero. This is similar to a conductor, but now we have charges moving.
- Similarly, given an applied voltage, a current will result. The constant of proportionality R, known as the resistance, is given by
  $$V = IR. \tag{A.31}$$

A.5.2 Electromagnetic induction

- Defined the electromotive force (emf) as
  $$\mathcal{E} = \oint\mathbf{f}\cdot d\mathbf{l} \tag{A.32}$$
- The universal flux rule states: whenever the magnetic flux through a loop
  $$\Phi = \int\mathbf{B}\cdot d\mathbf{a} \tag{A.33}$$
  changes, an emf
  $$\mathcal{E} = -\frac{d\Phi}{dt} \tag{A.34}$$
  will appear in the loop. This can occur in two ways:

1. Magnetic field doesn't change, loop changes:
   For example, a loop of wire is pulled to the right through a constant magnetic field. In this case the emf is magnetic.
2. Magnetic field changes, loop doesn't change:
   There is a stationary loop (any loop, not necessarily a physical loop of wire), and the magnetic field through it changes. In this case, the **changing magnetic field induces an electric field** and thus the emf is electric. Using eq. (A.34) we get **Faraday's law**
   $$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}, \tag{A.35}$$
   which, in differential form is
   $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{A.36}$$

- Lenz's law: Nature abhors a change in flux. Thus, as the magnetic flux changes and it induces an electric field over a loop, the resulting current goes in a direction such that it would create an opposing flux that tries to cancel the original change in magnetic flux.

- Mutual inductance:
  If there is a steady current going through a wire loop, this will create a magnetic field and thus a magnetic flux through another wire loop close by. The constant of proportionality between the flux through the second loop and the current in the first is the mutual inductance M. That is
  $$\Phi_2 = M_{21} I \tag{A.37}$$
  Note: If I ran the same current on loop two, then the flux in loop one would be $\Phi_1 = M_{12} I$. However, it can be shown that $M_{21} = M_{12}$ and thus $\Phi_1 = \Phi_2$.

  Now, imagine the current in loop one changes in time. The magnetic field associated with that current changes in time, and thus the magnetic flux through loop two changes as well. That is,
  $$\Phi_2(t) = M I_1(t). \tag{A.38}$$
  Due to Faraday's law an induced emf would be created in the second loop,
  $$\mathcal{E}_2(t) = -M \frac{dI(t)}{dt}. \tag{A.39}$$
  This emf creates a current $I_2(t)$ in the second loop.

- Self inductance:
  The changing magnetic field associated with the changing current in loop one also creates a changing flux within this loop. This is given by
  $$\Phi_1(t) = L I_1(t), \tag{A.40}$$
  where L is the self-inductance. Again, the changing flux leads to an emf within loop one, called the back emf
  $$\mathcal{E}(t) = -L \frac{dI(t)}{dt}. \tag{A.41}$$
  This emf drives a new current in loop one that opposes the original current change.

- The energy stored in magnetic fields is given by
  $$W = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{J} \, d\tau = \frac{1}{2\mu_0} \int B^2 \, d\tau. \tag{A.42}$$

- Ampere's law eq. (A.24) was derived using assumptions of magnetostatics. Maxwell extended Ampere's law to work for magnetodynamics, so that the divergence of eq. (A.24) would actually give zero on both sides. Thus, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{A.43}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{A.44}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{A.45}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{A.46}$$

- As shown earlier, Faraday's law indicates that a changing magnetic field induces an electric field. Maxwell's correction to Ampere's law then indicates that a changing electric field induces a magnetic field.

A.6 Conservation Laws

A.7 Electromagnetic waves

A.7.1 Simple waves

- The simplest kind of waves can be written as

$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi\right) \tag{A.47}$$

where

$$\begin{aligned} A &: \text{magnitude} \\ \lambda &: \text{wavelength} \\ T &: \text{period} \\ \phi &: \text{phase constant} \end{aligned}$$

Thus, as x goes from zero to λ, for example, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in space. Similarly, as t goes from zero to T, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in time.

- Defining the wavevector and angular frequency as

$$k = \frac{2\pi}{\lambda} \qquad w = \frac{2\pi}{T}, \tag{A.48}$$

then

$$u(x,t) = A \sin(kx - wt + \phi), \tag{A.49}$$

- The frequency ν is the inverse of the period, $\nu = 1/T$.

- By inspecting the form of the simple sinusoidal wave above, it is clear that the velocity of the wave is

$$v = \frac{w}{k} = \frac{\lambda}{T} = \lambda\nu. \tag{A.50}$$

Appendix B

Nuclear Fusion

B.1 Basic definitions

- Atomic number (Z): # of protons
- Mass number (A): # of protons + # of neutrons
- Atomic mass (m_a): mass of a particular isotope of an element.
- Relative atomic mass (A_r): (also known as atomic weight). Average of the atomic masses of all the different isotopes in a sample, with each isotope's contribution to the average determined by how big a fraction of the sample it makes up.
- Atomic mass unit (u): unit of mass, equivalent to $\frac{1}{12}$ the mass of a carbon-12 atom. That is
$$1u = \frac{m_c}{12}. \tag{B.1}$$
where m_c is the mass of a carbon-12 atom, in grams. Think of u as similar to a microgram.
- Mole: # of elementary entities equal to # of atoms in 12 grams of carbon-12. That is,
$$1mol = \frac{12g}{m_c} \tag{B.2}$$
Using eq. (B.1), we get
$$1u = \frac{1}{mol} g. \tag{B.3}$$
The value of the mole is $6.02214086 \times 10^{23}$.
- Molar mass (M):
  - If it is an atom (e.g. Carbon, C), then it is its atomic weight, but one uses eq. (B.3) to express the value in g/mol.
  - If it is a compound (e.g. Methane, CH_4), simply add up the atomic weights of each atom in the molecule, and again, express the result in g/mol.
  - If it is a mixture (e.g. air, $N_2, O_2, Ar, CO_2, ...$), then it is the weighted average of the atomic weights of the constituents, and the result again is expressed in g/mol.
- Avogadro's number (N_a): a conversion factor so that things can be measured in terms of moles.
$$N_a = \frac{6.02214086 \times 10^{23}}{mol} \tag{B.4}$$

B.2 The fusion reaction

- The fundamental relation for nuclear reactions is $E = mc^2$. A mass m can be transformed into energy E, and viceversa. Two examples for m are the following:
  - Defect mass (Dm): the difference in mass between the atom and the sum of its constituents,
$$Dm = Nm_n + Zm_p - m_a. \tag{B.5}$$
For carbon
$$Dm = 6 \times 1.008664u + 6 \times 1.007276u - 12u = 0.09564u. \tag{B.6}$$
For fluorine
$$Dm = 10 \times 1.008664u + 9 \times 1.007276u - 18.998403u = 0.154u. \tag{B.7}$$
The binding energy is then the energy corresponding to the mass defect as given by $E = (Dm)c^2$.
  - Mass change of a fusion reaction:
$$m = \text{mass of particles before reaction} - \text{mass of particles after reaction} \tag{B.8}$$
Consider the DT reaction as an example, then we have
$$m = 2.013553u\ (D) + 3.015501u\ (T) - 4.001503u\ (\alpha) - 1.008665u\ (n) = 0.018886u \tag{B.9}$$
The above mass translates to $E_f = mc^2 = 17.6MeV$.

- Momentum conservation:
Lets assume the particles before a fusion reaction move sufficiently slow that their velocities can be neglected. Conservation of momentum thus gives
$$0 = m_1v_1 + m_2v_2, \tag{B.10}$$
where m_1, m_2, v_1, v_2 are the mass and velocity of particles after the reaction.

- Energy conservation:
Energy is not conserved since some of the mass is converted to energy. The energy balance can be written as $E_{after} - E_{before} = E_f$. Assuming again that the particles before a fusion reaction move sufficiently slow, then
$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = E_f, \tag{B.11}$$
where E_f is obtained from Einstein's equation.

B.3 Fusion power density

The fusion power density S_f is the fusion energy produced per unit volume per unit time. Label the energy generated by each fusion collision between particles 1 and 2 by E_f, and the number of those fusion collisions per unit volume per unit time (also known as reaction rate) as R_{12}. Then the fusion power density is given by

$$S_f = E_f R_{12}. \tag{B.12}$$

We note that E_f is an energy released by the reaction (it can either be the total energy, the energy carried out by the alpha particles only, the energy carried out by the neutrons only, etc.).

The reaction rate between two distinct particle is given by

$$R_{12} = n_1 n_2 \langle \sigma v \rangle, \tag{B.13}$$

where n_1 and n_2 are the number densities of particles 1 and 2, respectively. The expected value $\langle \sigma v \rangle$ is given by

$$\langle \sigma v \rangle = \frac{1}{n_1 n_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) \sigma(v) v \, d\mathbf{v}_1 d\mathbf{v}_2. \tag{B.14}$$

Thus, the fusion power density can be expressed as

$$S_f = E_f \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) \sigma(v) v \, d\mathbf{v}_1 d\mathbf{v}_2. \tag{B.15}$$

Using the definition of the cross-section eq. (4.1), the above becomes

$$S_f = E_f \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\infty} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) F(v,b) vb \, dbd\phi d\mathbf{v}_1 d\mathbf{v}_2. \tag{B.16}$$

For cases in which we are not interested in the energy generated by the collision, but instead on some other physical property associated with the collision (for example change in momentum rather than change in energy) then the above needs to be generalized. Thus, we would use

$$S = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\infty} f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) E(v,b) F(v,b) vb \, dbd\phi d\mathbf{v}_1 d\mathbf{v}_2, \tag{B.17}$$

where $E(v, b)$ is the physical property associated with the collision.

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