

Plasma Physics

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Contents

I	Particle Description	4
1	Particle equations	5
1.1	Lagrangian mechanics	5
1.2	Hamiltonian mechanics	6
2	Guiding center theory	8
2.1	Uniform E and B fields	9
2.1.1	Only E field	9
2.1.2	Only B field	9
2.1.3	Both E and B fields	10
2.2	Non-uniform B field	11
2.2.1	Change in magnitude along perpendicular directions	11
2.2.2	Change in magnitude along parallel directions	13
2.2.3	Change in direction	17
2.3	Non-uniform E field	19
2.4	Time-varying E field	19
2.5	Time-varying B field	20
3	Collisions	23
3.1	Cross section	23
3.2	Differential cross section	23
3.3	Mean free path & collision frequency	26
4	Coulomb scattering	27
4.1	Particle equations	27
4.2	Conservation of energy and momentum	28
4.3	Polar coordinates	29
4.4	Particle trajectory	30
4.5	The scattering angle	32
4.6	The differential cross section	33
4.7	Collision integral	33
II	Kinetic Description	34
5	Kinetic equations	35

III	Fluid Description	37
6	Fluid equations	38
6.1	Mass	38
6.2	Momentum	38
6.3	Energy	40
6.4	Kinetic and Internal Energies	41
6.5	Summary	42
7	Basic concepts	43
7.1	Debye length	43
7.2	Plasma frequency	43
7.3	The coupling parameter	43
7.4	The plasma parameter	44
7.5	Electron degeneracy	44
8	Magnetohydrodynamics	45
8.1	Two-fluid equations	45
8.2	Low-frequency, long-wavelength, asymptotic expansions	46
8.3	Single-fluid equations	46
8.4	Resistive MHD	48
8.5	Ideal MHD	48
9	Transport and coupling coefficients	50
9.1	Plasma diffusion	50
9.2	Plasma viscosity	50
9.3	Plasma heat conduction	50
9.4	Momentum exchange collision frequency ν_{ei}	50
9.5	Electron-ion thermal coupling w_{ei}	50
A	Lagrangian and Eulerian PDFs	51
A.1	Eulerian PDF	51
A.2	Lagrangian PDF	51
A.3	Relation between Lagrangian and Eulerian PDFs	52
A.4	Evolution equation for fine-grained Eulerian PDF	52
A.5	Evolution equation for fine-grained Lagrangian PDF	52
B	Electromagnetism	53
B.1	Electrostatics	53
B.2	Magnetostatics	55
B.3	Electric Fields in Matter	56
B.3.1	Dipole moments	56
B.3.2	Electric displacement	57
B.3.3	Linear dielectric	57
B.4	Magnetic Fields in Matter	58
B.5	Electrodynamics	58
B.5.1	Ohm's Law	58
B.5.2	Electromagnetic induction	58
B.6	Conservation Laws	60
B.6.1	Conservation of energy	60

B.6.2	Conservation of momentum	60
B.7	Electromagnetic waves	61
B.7.1	Simple waves	61
B.7.2	Electromagnetic waves in vacuum	62

Part I

Particle Description

Chapter 1

Particle equations

Table 1.1: Various coordinates of classical mechanics.

Classical coordinates	$\mathbf{x}(t)$	$\mathbf{v}(t)$
Generalized coordinates	\mathbf{q}	$\dot{\mathbf{q}}$
Canonical coordinates	\mathbf{q}	\mathbf{p}
Time-dependent canonical coordinates	$\tilde{\mathbf{q}}(t)$	$\tilde{\mathbf{p}}(t)$

1.1 Lagrangian mechanics

- Define the position $\mathbf{x} = \mathbf{x}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ of a particle.
- Define the Lagrangian as $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where \mathbf{q} and $\dot{\mathbf{q}}$ are the generalized position and generalized velocity, respectively.
- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \right] = \left(\frac{\partial L}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}}. \quad (1.1)$$

- For example, the Lagrangian of a particle in an electromagnetic field where $\phi = \phi(\mathbf{q}, t)$ is the electric potential and $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$ is the magnetic potential, is

$$L = \frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e \phi. \quad (1.2)$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e \dot{q}_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \quad (1.3)$$

$$\frac{\partial L}{\partial \dot{q}_i} = m \dot{q}_i + e A_i \quad (1.4)$$

$$\begin{aligned} \frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_i} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \right] &= \frac{d}{dt} [m v_i + e A_i(\mathbf{x}, t)] \\ &= m \frac{dv_i}{dt} + e v_j \left(\frac{\partial A_i}{\partial q_j} \right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial A_i}{\partial t} \right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \quad (1.5)$$

Thus, the Euler-Lagrange equation becomes

$$m \frac{dv_i}{dt} = \left(-ev_j \frac{\partial A_i}{\partial q_j} - e \frac{\partial A_i}{\partial t} + ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.6)$$

In vector notation, this is written as

$$m \frac{d\mathbf{v}}{dt} = \left(-e\mathbf{v} \cdot \nabla_q \mathbf{A} - e \frac{\partial \mathbf{A}}{\partial t} + e \nabla_q (\mathbf{v} \cdot \mathbf{A}) - e \nabla_q \phi \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.7)$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}, \quad (1.8)$$

where $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$.

1.2 Hamiltonian mechanics

- Define the Hamiltonian as $H = H(\mathbf{q}, \mathbf{p}, t)$, where \mathbf{q} and \mathbf{p} are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.
- The Hamiltonian is obtained from the Lagrangian using

$$H = (\dot{\mathbf{q}} \cdot \mathbf{p} - L)_{\dot{\mathbf{q}}=f(\mathbf{q}, \mathbf{p})}, \quad (1.9)$$

where the dependency of $\dot{\mathbf{q}}$ on \mathbf{q} and \mathbf{p} is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \quad (1.10)$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[\dot{q}_i p_i - \left(\frac{1}{2} m \dot{q}_i \dot{q}_i + e \dot{q}_i A_i - e \phi \right) \right]_{\dot{\mathbf{q}}=f(\mathbf{q}, \mathbf{p})}. \quad (1.11)$$

Evaluating eq. (1.10) gives $p_i = m \dot{q}_i + e A_i$, which allows us to express $\dot{\mathbf{q}}$ in terms of \mathbf{q} and \mathbf{p} as $\dot{q}_i = \frac{1}{m}(p_i - e A_i)$. Thus

$$\begin{aligned} H &= \frac{1}{m}(p_i - e A_i)p_i - \frac{1}{2m}(p_i - e A_i)(p_i - e A_i) - e \frac{1}{m}(p_i - e A_i)A_i + e \phi \\ &= \frac{1}{2m}(p_i - e A_i)(p_i - e A_i) + e \phi. \end{aligned} \quad (1.12)$$

- We introduce the variables $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$, which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \quad (1.13)$$

$$\tilde{\mathbf{p}} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right)_{\mathbf{q}=\mathbf{x}, \dot{\mathbf{q}}=\mathbf{v}} \quad (1.14)$$

- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left(\frac{\partial H}{\partial p_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} \quad (1.15)$$

$$\frac{d\tilde{p}_i}{dt} = - \left(\frac{\partial H}{\partial q_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} \quad (1.16)$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x}, t) \quad (1.17)$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m \frac{dv_i}{dt} + ev_j \left(\frac{\partial A_i}{\partial q_j} \right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial A_i}{\partial t} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.18)$$

$$\begin{aligned} \frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i} \left[\frac{1}{2m} (p_j - eA_j)(p_j - eA_j) + e\phi \right] \\ &= \frac{1}{m} (p_j - eA_j) \frac{\partial}{\partial q_i} (p_j - eA_j) + e \frac{\partial \phi}{\partial q_i} \\ &= -\frac{e}{m} (p_j - eA_j) \frac{\partial A_j}{\partial q_i} + e \frac{\partial \phi}{\partial q_i} \end{aligned} \quad (1.19)$$

$$\begin{aligned} \left(\frac{\partial H}{\partial q_i} \right)_{\mathbf{q}=\tilde{\mathbf{q}}, \mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m} [mv_j + eA_j(\mathbf{x}, t) - eA_j(\mathbf{x}, t)] \left(\frac{\partial A_j}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} + e \left(\frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}} \\ &= \left(-ev_j \frac{\partial A_j}{\partial q_i} + e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \quad (1.20)$$

Equation (1.16) thus leads to

$$m \frac{dv_i}{dt} = \left(-ev_j \frac{\partial A_i}{\partial q_j} - e \frac{\partial A_i}{\partial t} + ev_j \frac{\partial A_j}{\partial q_i} - e \frac{\partial \phi}{\partial q_i} \right)_{\mathbf{q}=\mathbf{x}}. \quad (1.21)$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}. \quad (1.22)$$

Chapter 2

Guiding center theory

We begin with the velocity equation for a particle under the action of electric and magnetic fields,

$$m \frac{d\mathbf{v}}{dt} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{q}=\mathbf{x}}. \quad (2.1)$$

In the above, $\mathbf{v} = \mathbf{v}(t)$ is the particle velocity, $\mathbf{x} = \mathbf{x}(t)$ the particle position, $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ the electric field, and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$ the magnetic field. In the subsections that follow, we will solve this equation of motion for simplified forms of \mathbf{E} and \mathbf{B} . The solutions for the velocity vector will typically be of the form

$$\mathbf{v} = \mathbf{v}^{(c)} + \mathbf{v}^{(g)} + v^{\parallel} \mathbf{b}, \quad (2.2)$$

where $\mathbf{v}^{(c)} = \mathbf{v}^{(c)}(t)$ is the gyromotion (cyclotron) velocity, $\mathbf{v}^{(g)} = \mathbf{v}^{(g)}(t)$ is the guiding center velocity, $v^{\parallel} = v^{\parallel}(t)$ is the parallel velocity. Not all of the velocities will always be present. $\mathbf{b} = \mathbf{B}/B$ is the unit magnetic field vector. The position of the particle is governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}. \quad (2.3)$$

Using eq. (2.2), we integrate the above to obtain

$$\int_0^t d\mathbf{x}(t') = \int_0^t \mathbf{v}^{(c)}(t') dt' + \int_0^t \mathbf{v}^{(g)}(t') dt' + \int_0^t v^{\parallel}(t) \mathbf{b} dt'. \quad (2.4)$$

We introduce the positions $\mathbf{x}^{(c)} = \mathbf{x}^{(c)}(t)$, $\mathbf{x}^{(g)} = \mathbf{x}^{(g)}(t)$, and $\mathbf{x}^{\parallel} = \mathbf{x}^{\parallel}(t)$, which are defined as follows

$$\mathbf{x}^{(c)} = \int \mathbf{v}^{(c)} dt, \quad (2.5)$$

$$\mathbf{x}^{(g)} = \int \mathbf{v}^{(g)} dt, \quad (2.6)$$

$$\mathbf{x}^{\parallel} = \int v^{\parallel} \mathbf{b} dt. \quad (2.7)$$

Thus, eq. (2.4) is now re-written as

$$\mathbf{x}(t) - \mathbf{x}(0) = \mathbf{x}^{(c)}(t) - \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(t) - \mathbf{x}^{(g)}(0) - \mathbf{x}^{\parallel}(t) - \mathbf{x}^{\parallel}(0). \quad (2.8)$$

Without loss of generality, we will assume that the initial condition is as follows

$$\mathbf{x}(0) = \mathbf{x}^{(c)}(0) + \mathbf{x}^{(g)}(0) + \mathbf{x}^{\parallel}(0). \quad (2.9)$$

Thus, the particle position is finally expressed as

$$\mathbf{x} = \mathbf{x}^{(c)} + \mathbf{x}^{(g)} + \mathbf{x}^{\parallel}. \quad (2.10)$$

2.1 Uniform \mathbf{E} and \mathbf{B} fields

2.1.1 Only \mathbf{E} field

Let's orient our coordinate system such that \mathbf{E} points in the \mathbf{e}_z direction. Thus, the equations of motion are

$$\begin{aligned}\frac{dv_x}{dt} &= 0 & v_x(0) &= v_{\perp} \cos(\phi), \\ \frac{dv_y}{dt} &= 0 & v_y(0) &= v_{\perp} \sin(\phi), \\ \frac{dv_z}{dt} &= \frac{eE}{m} & v_z(0) &= v_{\parallel}.\end{aligned}\tag{2.11}$$

The solution of the above is

$$\begin{aligned}v_x &= v_{\perp} \cos(\phi) \\ v_y &= v_{\perp} \sin(\phi) \\ v_z &= v_{\parallel} + \frac{eE}{m}t.\end{aligned}\tag{2.12}$$

2.1.2 Only \mathbf{B} field

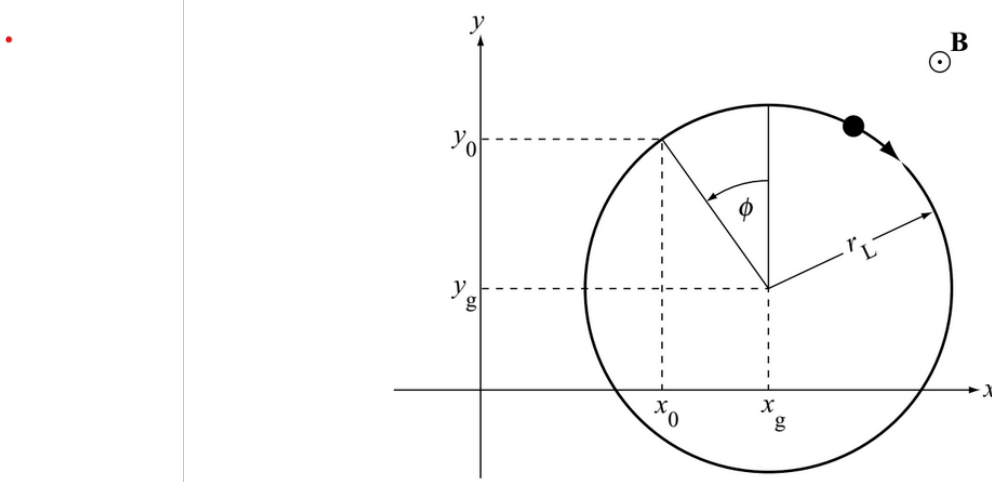


Figure 8.1 Gyro orbit of a positively charged particle in a magnetic field. Shown are the guiding center x_g, y_g and the initial position x_0, y_0 .

Figure 2.1: Coordinates for gyromotion (extracted from Plasma Physics and Fusion Energy, J. P. Freidberg).

Let's orient our coordinate system such that \mathbf{B} points in the \mathbf{e}_z direction. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB}{m}v_y \quad v_x(0) = v_{\perp} \cos(\phi), \tag{2.13a}$$

$$\frac{dv_y}{dt} = -\frac{eB}{m}v_x \quad v_y(0) = v_{\perp} \sin(\phi), \tag{2.13b}$$

$$\frac{dv_z}{dt} = 0 \quad v_z(0) = v_{\parallel}. \tag{2.13c}$$

The z component is decoupled from the rest and has a trivial solution. For the other two components, we begin by taking the time derivative of eq. (2.13b). Thus

$$\frac{d^2 v_y}{dt^2} = -\frac{eB}{m} \frac{dv_x}{dt} = -w_c^2 v_y, \quad (2.14)$$

where $w_c = |e|B/m$ is the gyro frequency. We know that the general solution to the above is $v_y = c_1 \cos(w_c t) + c_2 \sin(w_c t)$. If we use the ICs and assume ions, we have

$$v_y = -v_\perp \sin(w_c t - \phi). \quad (2.15)$$

Integrating eq. (2.13a) then gives

$$v_x = v_\perp \cos(w_c t - \phi). \quad (2.16)$$

The final solution, for either positive or negative charges, can be written as

$$\begin{aligned} v_x^{(c)} &= v_\perp \cos(w_c t \pm \phi) \\ v_y^{(c)} &= \pm v_\perp \sin(w_c t \pm \phi), \end{aligned} \quad (2.17)$$

where upper signs correspond to a negative charge. Integrating the equations above leads to

$$\begin{aligned} x^{(c)} &= r_L \sin(w_c t \pm \phi) \\ y^{(c)} &= \mp r_L \cos(w_c t \pm \phi). \end{aligned} \quad (2.18)$$

where $r_L = v_\perp/w_c$ is the gyro radius.

2.1.3 Both E and B fields

Let's orient our coordinate system such that \mathbf{B} still points along \mathbf{e}_z . The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x}{m} + \frac{eB}{m} v_y \quad v_x(0) = v_\perp \cos(\phi) + \frac{E_y}{B}, \quad (2.19a)$$

$$\frac{dv_y}{dt} = \frac{eE_y}{m} - \frac{eB}{m} v_x \quad v_y(0) = v_\perp \sin(\phi) - \frac{E_x}{B}, \quad (2.19b)$$

$$\frac{dv_z}{dt} = \frac{eE_\parallel}{m} \quad v_z(0) = v_\parallel, \quad (2.19c)$$

where we have chosen the given initial conditions simply to facilitate the math. Again, the z component is decoupled from the rest and has the trivial solution $v_z = v_\parallel + (eE_\parallel/m)t$. Thus, eq. (2.2) for the x and y components are

$$\begin{aligned} v_x &= v_x^{(c)} + v_x^{(g)}, \\ v_y &= v_y^{(c)} + v_y^{(g)}. \end{aligned} \quad (2.20)$$

We assume $v_x^{(g)}$ and $v_y^{(g)}$ are time independent. Using eq. (2.20) in eq. (2.19) we obtain

$$\begin{aligned} 0 &= \frac{eE_x}{m} + \frac{eB}{m} v_y^{(g)} \\ 0 &= \frac{eE_y}{m} - \frac{eB}{m} v_x^{(g)}. \end{aligned} \quad (2.21)$$

Thus, $v_x^{(g)} = E_y/B$ and $v_y^{(g)} = -E_x/B$, which in vector notation can be expressed as

$$\mathbf{v}_E^{(g)} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \quad (2.22)$$

2.2 Non-uniform B field

2.2.1 Change in magnitude along perpendicular directions

The magnetic field still points in the \mathbf{e}_z direction, but its magnitude changes in directions perpendicular to \mathbf{e}_z : $B = B(q_x, q_y)$. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(x, y)}{m} v_y \quad v_x(0) = v_\perp \cos(\phi) - \frac{v_\perp^2}{2w_c} \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}, \quad (2.23a)$$

$$\frac{dv_y}{dt} = -\frac{eB(x, y)}{m} v_x \quad v_y(0) = v_\perp \sin(\phi) + \frac{v_\perp^2}{2w_c} \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}, \quad (2.23b)$$

$$\frac{dv_z}{dt} = 0 \quad v_z(0) = v_\parallel. \quad (2.23c)$$

In the above, the x and y in $B(x, y)$ are the perpendicular components of the particle's position. The v_z component is decoupled from the rest and has a trivial solution. Thus, eqs. (2.2) and (2.10) for the x and y components are

$$v_x = v_x^{(c)} + v_x^{(g)}, \quad (2.24)$$

$$v_y = v_y^{(c)} + v_y^{(g)}, \quad (2.25)$$

$$x = x^{(c)} + x^{(g)}, \quad (2.26)$$

$$y = y^{(c)} + y^{(g)}. \quad (2.27)$$

We begin by employing a Taylor-series expansion for the magnetic field

$$B(x, y) = B(x^{(g)}, y^{(g)}) + \left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} + \dots \quad (2.28)$$

Thus, eqs. (2.23a) and (2.23b) are now

$$\frac{dv_x}{dt} = \frac{eB(x^{(g)}, y^{(g)})}{m} v_y + \frac{e}{m} \left(\left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_y \quad (2.29)$$

$$\frac{dv_y}{dt} = -\frac{eB(x^{(g)}, y^{(g)})}{m} v_x + \frac{e}{m} \left(\left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_x. \quad (2.30)$$

As before, we assume $v_x^{(g)}, v_y^{(g)}$ are time independent. Also, for simplicity we assume ions only. Plugging in eqs. (2.24) and (2.25) into the above, we get

$$0 = B(x^{(g)}, y^{(g)}) v_y^{(g)} + \left(\left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) (v_y^{(c)} + v_y^{(g)}), \quad (2.31)$$

$$0 = -B(x^{(g)}, y^{(g)}) v_x^{(g)} + \left(\left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) (v_x^{(c)} + v_x^{(g)}). \quad (2.32)$$

We assume $v_x^{(g)} \ll v_x^{(c)}$ and $v_y^{(g)} \ll v_y^{(c)}$. Thus, the above becomes

$$0 = B(x^{(g)}, y^{(g)}) v_y^{(g)} + \left(\left. \frac{\partial B}{\partial q_x} \right|_{x^{(g)}, y^{(g)}} x^{(c)} + \left. \frac{\partial B}{\partial q_y} \right|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_y^{(c)}, \quad (2.33)$$

$$0 = -B(x^{(g)}, y^{(g)})v_x^{(g)} + \left(\frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} x^{(c)} + \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} y^{(c)} \right) v_x^{(c)}. \quad (2.34)$$

We now use the definitions in eq. (2.17) and eq. (2.18). For example, with those definitions we can show that

$$\begin{aligned} x^{(c)}v_y^{(c)} &= [r_L \sin(w_c t - \phi)] [-v_\perp \sin(w_c t - \phi)] \\ &= -\frac{v_\perp^2}{w_c} \sin^2(w_c t - \phi) \\ &= -\frac{v_\perp^2}{2w_c} \{1 - \cos[2(w_c t - \phi)]\} \end{aligned} \quad (2.35)$$

Similar derivations can be carried out for $y^{(c)}v_y^{(c)}$, $x^{(c)}v_x^{(c)}$, and $y^{(c)}v_x^{(c)}$. Thus, eqs. (2.33) and (2.34) become

$$\begin{aligned} 0 = B(x^{(g)}, y^{(g)})v_y^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \{1 - \cos[2(w_c t - \phi)]\} \\ - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} \sin[2(w_c t - \phi)], \end{aligned} \quad (2.36)$$

$$\begin{aligned} 0 = -B(x^{(g)}, y^{(g)})v_x^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \sin[2(w_c t - \phi)] \\ - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} \{1 + \cos[2(w_c t - \phi)]\}. \end{aligned} \quad (2.37)$$

We neglect the oscillatory terms containing the sines and cosines—if it was not possible to neglect them, then the assumption that $v_x^{(g)}$, $v_y^{(g)}$ are time independent would not hold. Thus,

$$\begin{aligned} 0 &= B(x^{(g)}, y^{(g)})v_y^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \\ 0 &= -B(x^{(g)}, y^{(g)})v_x^{(g)} - \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}}. \end{aligned} \quad (2.38a)$$

Solving for the guiding center velocities, we finally have

$$\begin{aligned} v_x^{(g)} &= -\frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_y} \Big|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})} \\ v_y^{(g)} &= \frac{v_\perp^2}{2w_c} \frac{\partial B}{\partial q_x} \Big|_{x^{(g)}, y^{(g)}} \frac{1}{B(x^{(g)}, y^{(g)})}. \end{aligned} \quad (2.39)$$

In vector notation, this is written as

$$\mathbf{v}_{\nabla B}^{(g)} = \mp \frac{v_\perp^2}{2w_c} \frac{\mathbf{B} \times \nabla B}{B^2}. \quad (2.40)$$

In the above, the fields and w_c are evaluated at $(x^{(g)}, y^{(g)})$.

2.2.2 Change in magnitude along parallel directions

Ideally, one would introduce a gradient only in the direction parallel to the magnetic field, that is, one would have $\mathbf{B} = B(q_z)\mathbf{e}_z$. However, due to Gauss's law, this is too restrictive and instead we generalize and use $\mathbf{B} = B_x\mathbf{e}_x + B_z\mathbf{e}_z$, where $B_x = B_x(q_x, q_z)$ and $B_z = B_z(q_x, q_z)$. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{e}{m}v_y B_z(x, z), \quad (2.41)$$

$$\frac{dv_y}{dt} = -\frac{e}{m}[v_x B_z(x, z) - v_z B_x(x, z)], \quad (2.42)$$

$$\frac{dv_z}{dt} = -\frac{e}{m}v_y B_x(x, z). \quad (2.43)$$

However, the z direction no longer corresponds to the parallel direction, since the magnetic field also has a component along the x direction. To account for this, we will introduce a rotating reference frame, in which one of the axis will always be aligned with the magnetic field vector, and thus would denote the parallel direction. In the original static reference frame the unit vectors are $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and the velocity components are (v_x, v_y, v_z) , whereas in this new rotating reference frame the unit vectors are $(\mathbf{e}_{\perp 1}, \mathbf{e}_{\perp 2}, \mathbf{b})$ and the velocity components are $(v_{\perp 1}, v_{\perp 2}, v_{\parallel})$.

The rotating reference frame is described by the rotation matrix

$$\mathbf{Q}(t) = \begin{bmatrix} b_x & 0 & b_z \\ 0 & 1 & 0 \\ b_z & 0 & -b_x \end{bmatrix}, \quad (2.44)$$

where $b_x = b_x(t)$ and $b_z = b_z(t)$ are given by

$$b_x = \frac{B_x(x, z)}{B(x, z)} \quad b_z = \frac{B_z(x, z)}{B(x, z)} \quad (2.45)$$

In the above, $B(x, z) = [B_x^2(x, z) + B_z^2(x, z)]^{1/2}$. As an example, the matrix above leads to the following transformations for the unit vectors and velocities in the rotating reference frame

$$\mathbf{b} = b_x\mathbf{e}_x + b_z\mathbf{e}_z \quad (2.46)$$

$$\mathbf{e}_{\perp 2} = \mathbf{e}_y \quad (2.47)$$

$$\mathbf{e}_{\perp 1} = b_z\mathbf{e}_x - b_x\mathbf{e}_z = \mathbf{e}_{\perp 2} \times \mathbf{b}, \quad (2.48)$$

$$v_{\parallel} = b_x v_x + b_z v_z \quad (2.49)$$

$$v_{\perp 2} = v_y \quad (2.50)$$

$$v_{\perp 1} = b_z v_x - b_x v_z. \quad (2.51)$$

Using the transformation rule for the acceleration of a particle, but for some reason neglecting the coriolis and centrifugal forces, we obtain for the velocity derivatives

$$\frac{dv_{\parallel}}{dt} = \frac{dv_x}{dt}b_x + \frac{dv_z}{dt}b_z - K v_{\perp 1} \quad (2.52)$$

$$\frac{dv_{\perp 2}}{dt} = \frac{dv_y}{dt} \quad (2.53)$$

$$\frac{dv_{\perp 1}}{dt} = \frac{dv_x}{dt}b_z - \frac{dv_z}{dt}b_x + K v_{\parallel}, \quad (2.54)$$

where $K = K(t)$ is given by $K = b_x db_z/dt - b_z db_x/dt$. Using eqs. (2.41) to (2.43) in the above leads to

$$\frac{dv_{||}}{dt} = \frac{e}{m} v_y [B_z(x, z) b_x - B_x(x, z) b_z] - K v_{\perp 1} \quad (2.55)$$

$$\frac{dv_{\perp 2}}{dt} = -\frac{eB}{m} (v_x b_z - v_z b_x) \quad (2.56)$$

$$\frac{dv_{\perp 1}}{dt} = \frac{e}{m} v_y [B_z(x, z) b_z + B_x(x, z) b_x] + K v_{||} \quad (2.57)$$

Using the definitions for b_x and b_z in eq. (2.45), as well as the expressions for $v_{\perp 1}$, $v_{\perp 2}$ in eqs. (2.50) and (2.51), we get

$$\frac{dv_{||}}{dt} = -K v_{\perp 1}, \quad (2.58)$$

$$\frac{dv_{\perp 2}}{dt} = -w_c v_{\perp 1}, \quad (2.59)$$

$$\frac{dv_{\perp 1}}{dt} = w_c v_{\perp 2} + K v_{||}, \quad (2.60)$$

where $w_c = w_c(t)$ is given by $w_c = eB(x, z)/m$.

We now introduce a time transformation to simplify the equations above. To do so, we introduce the following variables

$$\hat{v}_{||} = \hat{v}_{||}(\tau) \quad \hat{v}_{\perp 2} = \hat{v}_{\perp 2}(\tau) \quad \hat{v}_{\perp 1} = \hat{v}_{\perp 1}(\tau) \quad (2.61)$$

$$\hat{x} = \hat{x}(\tau) \quad \hat{z} = \hat{z}(\tau) \quad (2.62)$$

such that

$$v_{||} = \hat{v}_{||}(h(t)) \quad v_{\perp 2} = \hat{v}_{\perp 2}(h(t)) \quad v_{\perp 1} = \hat{v}_{\perp 1}(h(t)) \quad (2.63)$$

$$x = \hat{x}(h(t)) \quad z = \hat{z}(h(t)). \quad (2.64)$$

The function $h(t)$ is given by

$$h(t) = \int_0^t w_c(t') dt'. \quad (2.65)$$

We also show that

$$b_x = \frac{B_x(x, z)}{B(x, z)} = \frac{B_x(\hat{x}(h(t)), \hat{z}(h(t)))}{B(\hat{x}(h(t)), \hat{z}(h(t)))}, \quad (2.66)$$

and thus

$$\frac{db_x}{dt} = \frac{dh(t)}{dt} \frac{d}{d\tau} \left[\frac{B_x(\hat{x}, \hat{z})}{B(\hat{x}, \hat{z})} \right]_{\tau=h(t)} = w_c \left. \frac{d\hat{b}_x}{d\tau} \right|_{\tau=h(t)}, \quad (2.67)$$

where $\hat{b}_x = \hat{b}_x(\tau)$ is given by $\hat{b}_x = B_x(\hat{x}, \hat{z})/B(\hat{x}, \hat{z})$. The analogous holds for b_z . This allows us to write

$$K = w_c \left(\hat{b}_x \frac{d\hat{b}_z}{d\tau} - \hat{b}_z \frac{d\hat{b}_x}{d\tau} \right)_{\tau=h(t)} = w_c \left. \hat{K} \right|_{\tau=h(t)}, \quad (2.68)$$

where $\hat{K} = \hat{K}(\tau)$ is given by $\hat{K} = \hat{b}_x d\hat{b}_z/d\tau - \hat{b}_z d\hat{b}_x/d\tau$. With these transformation, eqs. (2.58) to (2.60) are re-written as

$$\frac{d\hat{v}_{||}}{d\tau} = -\hat{K} \hat{v}_{\perp 1}, \quad (2.69)$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \quad (2.70)$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \hat{K} \hat{v}_{||}. \quad (2.71)$$

We now simplify B_z so that $B_z = B_z(q_z)$. To be consistent with Gauss's law, we require $B_x = B_x(q_x, q_y)$ where $B_x = -q_x dB_z/dq_z$. With these simplified forms, we have

$$\hat{K} = -\hat{b}_z^2 \frac{d}{d\tau} \left(\frac{\hat{b}_x}{\hat{b}_z} \right) \quad (2.72)$$

$$= -\frac{B_z^2(\hat{z})}{B^2(\hat{x}, \hat{z})} \frac{d}{d\tau} \left(\frac{B_x(\hat{x}, \hat{z})}{B_z(\hat{z})} \right) \quad (2.73)$$

$$= \frac{B_z^2(\hat{z})}{B^2(\hat{x}, \hat{z})} \frac{d}{d\tau} \left[\hat{x} \left(\frac{1}{B_z} \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \right]. \quad (2.74)$$

We now use the long-thin approximation. For this approximation, we assume that $B_x/B_z \ll 1$, and also that $\frac{1}{B_z} \frac{dB_z}{dq_z}$ changes very slowly. We thus have

$$\hat{K} \approx \frac{d\hat{x}}{d\tau} \left(\frac{1}{B_z} \frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}. \quad (2.75)$$

Also, using the long-thin approximation in eqs. (2.49) and (2.51) allows us to write

$$v_{||} \approx v_z = \frac{dz}{dt} = \left(\frac{d\hat{z}}{d\tau} \right)_{\tau=h(t)} w_c \quad (2.76)$$

$$v_{\perp 1} \approx v_x = \frac{dx}{dt} = \left(\frac{d\hat{x}}{d\tau} \right)_{\tau=h(t)} w_c. \quad (2.77)$$

Evaluating the above at $t = h^{-1}(\tau)$, and defining $\hat{w}_c(\tau)$ from $w_c = \hat{w}_c(h(t))$, we obtain

$$\hat{v}_{||} \approx \frac{d\hat{z}}{d\tau} \hat{w}_c \quad (2.78)$$

$$\hat{v}_{\perp 1} \approx \frac{d\hat{x}}{d\tau} \hat{w}_c. \quad (2.79)$$

We also note that

$$\frac{dB_z(\hat{z})}{d\tau} = \left(\frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \frac{d\hat{z}}{d\tau}. \quad (2.80)$$

Using the expressions above in eq. (2.75), one can approximate \hat{K} using either of the two forms below

$$\hat{K} \approx \frac{\hat{v}_{\perp 1}}{\hat{w}_c B_z(\hat{z})} \left(\frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \approx \frac{\hat{v}_{\perp 1}}{\hat{v}_{||} B_z(\hat{z})} \frac{dB_z(\hat{z})}{d\tau}. \quad (2.81)$$

We thus write the governing equations for the velocities as

$$\frac{d\hat{v}_{||}}{d\tau} = -\frac{\hat{v}_{\perp 1}^2}{\hat{w}_c B_z(\hat{z})} \left(\frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}, \quad (2.82)$$

$$\frac{d\hat{v}_{\perp 2}}{d\tau} = -\hat{v}_{\perp 1}, \quad (2.83)$$

$$\frac{d\hat{v}_{\perp 1}}{d\tau} = \hat{v}_{\perp 2} + \frac{\hat{v}_{\perp 1}}{B_z(\hat{z})} \frac{dB_z(\hat{z})}{d\tau}. \quad (2.84)$$

We now assume the solution for the perpendicular velocities is of the form

$$\hat{v}_{\perp 1} = \hat{v}_{\perp} \cos[\tau + \hat{\epsilon}] \quad (2.85)$$

$$\hat{v}_{\perp 2} = -\hat{v}_{\perp} \sin[\tau + \hat{\epsilon}], \quad (2.86)$$

where $\hat{v}_\perp = \hat{v}_\perp(\tau)$ and $\hat{\epsilon} = \hat{\epsilon}(\tau)$. Plugging these two assumed solutions into eqs. (2.83) and (2.84), and using some simple algebra, gives

$$\frac{d\hat{v}_\perp}{d\tau} = \frac{\hat{v}_\perp}{2B_z(\hat{z})} \frac{dB_z(\hat{z})}{d\tau} \{1 + \cos[2(\tau + \hat{\epsilon})]\}. \quad (2.87)$$

The above can be re-arranged and expressed as

$$\frac{d \ln \hat{\mu}}{d\tau} = \frac{d \ln B_z(\hat{z})}{d\tau} \cos[2(\tau + \hat{\epsilon})], \quad (2.88)$$

where $\hat{\mu} = \hat{\mu}(\tau)$ is the adiabatic invariant, and is given by

$$\hat{\mu} = \frac{m\hat{v}_\perp^2}{2B_z(\hat{z})}. \quad (2.89)$$

Integrating eq. (2.88) from τ_1 to τ_2 gives

$$\ln \hat{\mu}(\tau_2) - \ln \hat{\mu}(\tau_1) = \ln[B_z(\hat{z})] \cos[2(\tau + \hat{\epsilon})] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2 \ln[B_z(\hat{z})] \sin[2(\tau + \hat{\epsilon})] d\tau. \quad (2.90)$$

Picking τ_1 and τ_2 such that $[\tau_2 + \hat{\epsilon}(\tau_2)] - [\tau_1 + \hat{\epsilon}(\tau_1)] = 2\pi$, and assuming $B(\hat{z})$ doesn't change significantly from τ_1 to τ_2 , gives $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$, that is, $\hat{\mu}$ is constant over one gyro-period. One can also define

$$\mu = \frac{mv_\perp^2}{2B_z(z)} \quad (2.91)$$

where $\mu = \mu(t)$ and $v_\perp = v_\perp(t)$. Given that $v_\perp = \hat{v}_\perp(h(t))$, we have $\mu = \hat{\mu}(h(t))$. Thus, $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ translates to $\mu(t_2) = \mu(t_1)$, where $t_2 = h^{-1}(\tau_2)$ and $t_1 = h^{-1}(\tau_1)$.

Finally, we focus not on the perpendicular velocities but the parallel velocity. Plugging-in the assumed solutions in the governing eq. (2.82) gives

$$\frac{d\hat{v}_\parallel}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})} \left(\frac{dB_z}{dq_z} \right)_{q_z=\hat{z}} \{1 + \cos[2(\tau + \hat{\epsilon})]\}. \quad (2.92)$$

We now average the above from τ_1 to τ_2 while assuming $B(\hat{z})$, $d\hat{v}_\parallel/d\tau$ and \hat{v}_\perp^2 do not change significantly during that time scale. Note that since this is an average, we are not just integrating from τ_1 to τ_2 , but we are also dividing by $\tau_2 - \tau_1$. After averaging, we obtain

$$\frac{d\hat{v}_\parallel}{d\tau} = -\frac{\hat{v}_\perp^2}{2\hat{w}_c B_z(\hat{z})} \left(\frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}. \quad (2.93)$$

or

$$m \frac{d\hat{v}_\parallel}{d\tau} = -\frac{\hat{\mu}}{\hat{w}_c} \left(\frac{dB_z}{dq_z} \right)_{q_z=\hat{z}}. \quad (2.94)$$

Converting back to time t gives

$$m \frac{dv_\parallel}{dt} = -\mu \left(\frac{dB_z}{dq_z} \right)_{q_z=z}. \quad (2.95)$$

2.2.3 Change in direction

Rather than writing eq. (2.1) in terms of its components as done in previous sections, we leave the equation in vector form. Expressing the velocity as $\mathbf{v} = \mathbf{v}_\perp + v_\parallel \mathbf{b}$ and assuming no electric field, we write eq. (2.1) as

$$\frac{d}{dt}(\mathbf{v}_\perp + v_\parallel \mathbf{b}) = \mp w_c(\mathbf{v}_\perp + v_\parallel \mathbf{b}) \times \mathbf{b}, \quad (2.96)$$

where upper sign corresponds to negative charge and lower sign to positive charge. For simplicity we will assume positively charged particles only. We then cross both sides of the above by \mathbf{b} , that is

$$\mathbf{b} \times \left\{ \left[\frac{d}{dt}(\mathbf{v}_\perp + v_\parallel \mathbf{b}) - w_c(\mathbf{v}_\perp + v_\parallel \mathbf{b}) \times \mathbf{b} \right] \times \mathbf{b} \right\} = 0. \quad (2.97)$$

The above is simplified using the following three manipulations

$$\begin{aligned} \mathbf{b} \times \left\{ [w_c(\mathbf{v}_\perp + v_\parallel \mathbf{b}) \times \mathbf{b}] \times \mathbf{b} \right\} &= \mathbf{b} \times \{ [w_c \mathbf{v}_\perp \times \mathbf{b}] \times \mathbf{b} \} \\ &= -\mathbf{b} \times \{ w_c \mathbf{v}_\perp (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{b} \cdot w_c \mathbf{v}_\perp) \} \\ &= w_c \mathbf{v}_\perp \times \mathbf{b}. \end{aligned} \quad (2.98)$$

$$\mathbf{b} \times \left\{ \left[\frac{d\mathbf{v}_\perp}{dt} \right] \times \mathbf{b} \right\} = \frac{d\mathbf{v}_\perp}{dt} (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b} \left(\mathbf{b} \cdot \frac{d\mathbf{v}_\perp}{dt} \right) = \left(\frac{d\mathbf{v}_\perp}{dt} \right)_\perp. \quad (2.99)$$

$$\begin{aligned} \mathbf{b} \times \left\{ \left[\frac{dv_\parallel \mathbf{b}}{dt} \times \mathbf{b} \right] \right\} &= v_\parallel \mathbf{b} \times \left\{ \left[\frac{d\mathbf{b}}{dt} \times \mathbf{b} \right] \right\} \\ &= v_\parallel \left[\frac{d\mathbf{b}}{dt} (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b} \left(\mathbf{b} \cdot \frac{d\mathbf{b}}{dt} \right) \right] \\ &= v_\parallel \left[\frac{d\mathbf{b}}{dt} - \mathbf{b} \left(\frac{1}{2} \frac{d\mathbf{b} \cdot \mathbf{b}}{dt} \right) \right] \\ &= v_\parallel \frac{d\mathbf{b}}{dt} \end{aligned} \quad (2.100)$$

Thus, we have

$$\left(\frac{d\mathbf{v}_\perp}{dt} \right)_\perp - w_c \mathbf{v}_\perp \times \mathbf{b} = -v_\parallel \frac{d\mathbf{b}}{dt}. \quad (2.101)$$

As shown in Freidberg

$$\frac{d\mathbf{b}(\mathbf{x}(t))}{dt} = \frac{d\mathbf{x}(t)}{dt} \cdot \nabla \mathbf{b} = \mathbf{v} \cdot \nabla \mathbf{b} = \mathbf{v}_\perp \cdot \nabla \mathbf{b} + v_\parallel \mathbf{b} \cdot \nabla \mathbf{b}, \quad (2.102)$$

where $\nabla \mathbf{b}$ is evaluated at $\mathbf{x} = \mathbf{x}(t)$. Thus, eq. (2.101) becomes

$$\left(\frac{d\mathbf{v}_\perp}{dt} \right)_\perp - w_c \mathbf{v}_\perp \times \mathbf{b} = -v_\parallel \mathbf{v}_\perp \cdot \nabla \mathbf{b} - v_\parallel^2 \mathbf{b} \cdot \nabla \mathbf{b}. \quad (2.103)$$

As was done for the other drifts, we assume the solution is of the form $\mathbf{v}_\perp = \mathbf{v}^{(c)} + \mathbf{v}^{(g)}$, where we assume again that $\mathbf{v}^{(g)}$ is time independent. The term $\mathbf{v}^{(c)}$ corresponds to gyromotion in a rotating reference frame, and is thus given by

$$\mathbf{v}^{(c)} = v_{\perp 1}^{(c)} \mathbf{e}_{\perp 1} + v_{\perp 2}^{(c)} \mathbf{e}_{\perp 2}, \quad (2.104)$$

where $\mathbf{e}_{\perp 1}$ and $\mathbf{e}_{\perp 2}$ are orthogonal to \mathbf{b} and thus rotate in time. $v_{\perp 1}^{(c)}$ is given by eq. (2.16) and $v_{\perp 2}^{(c)}$ by eq. (2.15). We note that, in the non-rotating reference frame, $\mathbf{v}^{(c)}$ is expressed as $\mathbf{v}^{(c)} = v_x^{(c)} \mathbf{e}_x + v_y^{(c)} \mathbf{e}_y + v_z^{(c)} \mathbf{e}_z$. We now prove that $\mathbf{v}_{\perp}^{(c)}$ is the solution to the two terms on the left-hand side of eq. (2.103). To show this we first use the transformation rule for the acceleration of a particle in a rotating reference frame, but for some reason ignore the coriolis and centrifugal terms. Thus

$$\begin{aligned} \frac{d\mathbf{v}^{(c)}}{dt} &= \frac{dv_x^{(c)}}{dt} \mathbf{e}_x + \frac{dv_y^{(c)}}{dt} \mathbf{e}_y + \frac{dv_z^{(c)}}{dt} \mathbf{e}_z \\ &= \frac{dv_{\perp 1}^{(c)}}{dt} \mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt} \mathbf{e}_{\perp 2} + 2\Omega \times \mathbf{v}^{(c)}. \end{aligned} \quad (2.105)$$

We do not allow the rotating reference frame to rotate about the \mathbf{b} axis. Thus, $\Omega = \Omega_{\perp 1} \mathbf{e}_{\perp 1} + \Omega_{\perp 2} \mathbf{e}_{\perp 2}$. Given that Ω and $\mathbf{v}^{(c)}$ are in the same plane, $\Omega \times \mathbf{v}^{(c)}$ must point in the \mathbf{b} direction. Thus,

$$\left(\frac{d\mathbf{v}^{(c)}}{dt} \right)_{\perp} = \frac{dv_{\perp 1}^{(c)}}{dt} \mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt} \mathbf{e}_{\perp 2}. \quad (2.106)$$

This allows us to show that

$$\left(\frac{d\mathbf{v}^{(c)}}{dt} \right)_{\perp} - w_c \mathbf{v}^{(c)} \times \mathbf{b} = \frac{dv_{\perp 1}^{(c)}}{dt} \mathbf{e}_{\perp 1} + \frac{dv_{\perp 2}^{(c)}}{dt} \mathbf{e}_{\perp 2} - w_c v_{\perp 2}^{(c)} \mathbf{e}_{\perp 1} + w_c v_{\perp 1}^{(c)} \mathbf{e}_{\perp 2} = 0. \quad (2.107)$$

We now plug in $\mathbf{v}_{\perp} = \mathbf{v}^{(c)} + \mathbf{v}^{(g)}$ in eq. (2.103) to obtain

$$-w_c \mathbf{v}^{(g)} \times \mathbf{b} = -v_{\parallel} \mathbf{v}_{\perp} \cdot \nabla \mathbf{b} - v_{\parallel}^2 \mathbf{b} \cdot \nabla \mathbf{b}. \quad (2.108)$$

As explained in Freidberg, the term $v_{\parallel} \mathbf{v}_{\perp} \cdot \nabla \mathbf{b}$ leads to small modifications of the gyro motion, but does not lead to a drift of the particles, and thus is ignored. Taking the cross product of eq. (2.108) with \mathbf{b} finally gives the curvature drift

$$\mathbf{v}_{\kappa}^{(g)} = \pm \frac{v_{\parallel}^2}{w_c} \frac{(\mathbf{b} \cdot \nabla \mathbf{b}) \times \mathbf{B}}{B}. \quad (2.109)$$

We now show that, if we assume $\nabla \times \mathbf{B} = 0$, the grad-B drift

$$\mathbf{v}_{\nabla B}^{(g)} = \mp \frac{v_{\perp}^2}{2w_c} \frac{\mathbf{B} \times \nabla B}{B^2} \quad (2.110)$$

can be written in the same form as the curvature drift. We begin by showing that

$$\mathbf{B} \times \nabla B = \mathbf{B} \times \nabla (\mathbf{B} \cdot \mathbf{B})^{1/2} = \mathbf{B} \times \frac{1}{2B} \nabla (\mathbf{B} \cdot \mathbf{B}). \quad (2.111)$$

We now use the vector identity $\nabla (\mathbf{B} \cdot \mathbf{B}) = 2\mathbf{B} \times (\nabla \times \mathbf{B}) + 2\mathbf{B} \cdot \nabla \mathbf{B}$, and assume magnetic curl of zero to obtain

$$\begin{aligned} \mathbf{B} \times \nabla B &= \mathbf{B} \times \frac{1}{B} \mathbf{B} \cdot \nabla \mathbf{B} \\ &= \mathbf{B} \times \mathbf{b} \cdot \nabla (B\mathbf{b}) \\ &= \mathbf{B} \times (\mathbf{b} \cdot \nabla B) \mathbf{b} + \mathbf{B} \times B (\mathbf{b} \cdot \nabla \mathbf{b}) \\ &= -B (\mathbf{b} \cdot \nabla \mathbf{b}) \times \mathbf{B}. \end{aligned} \quad (2.112)$$

Thus, the grad-B drift can be written as

$$\mathbf{v}_{\nabla B}^{(g)} = \pm \frac{v_{\perp}^2}{2w_c} \frac{(\mathbf{b} \cdot \nabla \mathbf{b}) \times \mathbf{B}}{B}. \quad (2.113)$$

2.3 Non-uniform E field

2.4 Time-varying E field

Consider the scenario used in section 2.1.3, but with a time varying electric field. The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE_x(t)}{m} + \frac{eB}{m}v_y \quad v_x(0) = v_\perp \cos(\phi) + \frac{E_y(t)}{B} + \frac{m}{eB^2} \frac{dE_x(t)}{dt}, \quad (2.114a)$$

$$\frac{dv_y}{dt} = \frac{eE_y(t)}{m} - \frac{eB}{m}v_x \quad v_y(0) = v_\perp \sin(\phi) - \frac{E_x(t)}{B} + \frac{m}{eB^2} \frac{dE_y(t)}{dt}, \quad (2.114b)$$

$$\frac{dv_z}{dt} = \frac{eE_{||}(t)}{m} \quad v_z(0) = v_{||}, \quad (2.114c)$$

where again we chose the initial conditions simply to be consistent with the solution that we'll derive. The parallel velocity is independent of the perpendicular velocities, and we won't worry about it for now. To solve for the perpendicular velocities, we again assume the general solution is

$$\begin{aligned} v_x &= v_x^{(c)} + v_x^{(g)} \\ v_y &= v_y^{(c)} + v_y^{(g)} \end{aligned} \quad (2.115)$$

but now do not assume $v_x^{(g)}$ and $v_y^{(g)}$ are time independent. We expand $v_i^{(g)}$ as

$$v_i^{(g)} = v_i^{(g,1)} + v_i^{(g,2)} + \dots, \quad (2.116)$$

where $v_i^{(g,\alpha)} \sim \epsilon v_i^{(g,\alpha-1)}$, and the small parameter ϵ follows from assuming

$$\frac{1}{v_i^{(g,\alpha)}} \frac{dv_i^{(g,\alpha)}}{dt} \sim \epsilon \omega_c. \quad (2.117)$$

That is, the time scale associated with the rate of change of all of the $v_i^{(g,\alpha)}$ components is much larger than the time scale of the gyromotion. In other words, we assume particles gyrate faster than how quickly their drift velocity changes. Using eq. (2.115) in eq. (2.114) leads to

$$\begin{aligned} \frac{dv_x^{(g,1)}}{dt} + \frac{dv_x^{(g,2)}}{dt} &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y^{(g,1)} + \frac{eB}{m}v_y^{(g,2)} \\ \frac{dv_y^{(g,1)}}{dt} + \frac{dv_y^{(g,2)}}{dt} &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x^{(g,1)} - \frac{eB}{m}v_x^{(g,2)}. \end{aligned} \quad (2.118)$$

Collecting lowest order terms

$$\begin{aligned} 0 &= \frac{eE_x(t)}{m} + \frac{eB}{m}v_y^{(g,1)} \\ 0 &= \frac{eE_y(t)}{m} - \frac{eB}{m}v_x^{(g,1)}, \end{aligned} \quad (2.119)$$

and thus $v_x^{(g,1)} = E_y(t)/B$ and $v_y^{(g,1)} = -E_x(t)/B$, which in vector notation is

$$\mathbf{v}^{(g,1)} = \frac{\mathbf{E}(t) \times \mathbf{B}}{B^2}. \quad (2.120)$$

Collecting first order terms

$$\begin{aligned}\frac{dv_x^{(g,1)}}{dt} &= \frac{eB}{m}v_y^{(g,2)} \\ \frac{dv_y^{(g,1)}}{dt} &= -\frac{eB}{m}v_x^{(g,2)},\end{aligned}\tag{2.121}$$

and thus $v_x^{(g,2)} = (m/eB^2)dE_x(t)/dt$ and $v_y^{(g,2)} = (m/eB^2)dE_y(t)/dt$, which in vector notation is

$$\mathbf{v}^{(g,2)} = \mp \frac{1}{w_c B} \frac{d\mathbf{E}_\perp}{dt}.\tag{2.122}$$

We note that, by looking at the solutions for $v_x^{(g,1)}$ and $v_y^{(g,1)}$, the assumption in eq. (2.117) is equivalent to stating that the electric field changes slowly.

2.5 Time-varying B field

Let's assume the magnetic field points in the z direction again. Using Faraday's law, we have

$$\left(\frac{\partial E_z}{\partial q_y} - \frac{\partial E_y}{\partial q_z}\right)\mathbf{e}_x - \left(\frac{\partial E_z}{\partial q_x} - \frac{\partial E_x}{\partial q_z}\right)\mathbf{e}_y + \left(\frac{\partial E_y}{\partial q_x} - \frac{\partial E_x}{\partial q_y}\right)\mathbf{e}_z = -\frac{\partial B}{\partial t}\mathbf{e}_z.\tag{2.123}$$

To satisfy the above, we set $E_z = 0$, and $E_x = E_x(q_x, q_y, t)$, $E_y = E_y(q_x, q_y, t)$. That is, a time varying magnetic field requires a time and spatially varying electric field.

We will further simplify our analysis by having $E_x = 0$ and $E_y = E_y(q_x, t)$. Thus, the equations of motion are

$$\frac{dv_x}{dt} = \frac{eB(t)}{m}v_y,\tag{2.124}$$

$$\frac{dv_y}{dt} = \frac{eE_y(x, t)}{m} - \frac{eB(t)}{m}v_x,\tag{2.125}$$

with v_z constant. As done in previous sections, the velocities and positions are decomposed as follows

$$v_x = v_x^{(c)} + v_x^{(g)},\tag{2.126}$$

$$v_y = v_y^{(c)} + v_y^{(g)},\tag{2.127}$$

$$x = x^{(c)} + x^{(g)},\tag{2.128}$$

$$y = y^{(c)} + y^{(g)}.\tag{2.129}$$

The electric field is then linearized using a Taylor-series expansion about the guiding center,

$$\frac{dv_x}{dt} = \frac{eB(t)}{m}v_y\tag{2.130}$$

$$\frac{dv_y}{dt} = \frac{e}{m} \left[E_y \left(x^{(g)}, t \right) + \left. \frac{\partial E_y}{\partial q_x} \right|_{x^{(g)}} x^{(c)} \right] - \frac{eB(t)}{m}v_x,\tag{2.131}$$

We assume positive ions for simplicity and re-write the above as

$$\frac{dv_x}{dt} = w_c v_y\tag{2.132}$$

$$\frac{dv_y}{dt} = \frac{w_c}{B(t)} \left[E_y \left(x^{(g)}, t \right) + \left. \frac{\partial E_y}{\partial q_x} \right|_{x^{(g)}} x^{(c)} \right] - w_c v_x.\tag{2.133}$$

where $w_c = w_c(t)$. We introduce new variables

$$\begin{aligned}\hat{v}_x &= \hat{v}_x(\tau) & \hat{v}_y &= \hat{v}_y(\tau) & \hat{x}^{(c)} &= \hat{x}^{(c)}(\tau) & \hat{x}^{(g)} &= \hat{x}^{(g)}(\tau) \\ \hat{E}_y &= \hat{E}_y(q_x, \tau) & \hat{B} &= \hat{B}(\tau)\end{aligned}\quad (2.134)$$

such that

$$\begin{aligned}v_x(t) &= \hat{v}_x(h(t)) & v_y(t) &= \hat{v}_y(h(t)) & x^{(c)}(t) &= \hat{x}^{(c)}(h(t)) & x^{(g)}(t) &= \hat{x}^{(g)}(h(t)) \\ E_y(q_x, t) &= \hat{E}_y(q_x, h(t)) & B(t) &= \hat{B}(h(t)).\end{aligned}\quad (2.135)$$

For the above

$$h(t) = \int_0^t w_c(t') dt'. \quad (2.136)$$

The equations of motion then become

$$\begin{aligned}\frac{d\hat{v}_x}{d\tau} &= \hat{v}_y \\ \frac{d\hat{v}_y}{d\tau} &= \frac{1}{\hat{B}(\tau)} \left[\hat{E}_y(\hat{x}^{(g)}, \tau) + \left. \frac{\partial \hat{E}_y}{\partial q_x} \right|_{\hat{x}^{(g)}} \hat{x}^{(c)} \right] - \hat{v}_x.\end{aligned}\quad (2.137)$$

For the gyromotion quantities, we'll assume they are of the following form,

$$\hat{v}_x^{(c)} = \hat{v}_\perp \cos(\tau + \hat{\epsilon}), \quad (2.138)$$

$$\hat{v}_y^{(c)} = -\hat{v}_\perp \sin(\tau + \hat{\epsilon}), \quad (2.139)$$

$$\hat{x}^{(c)} = \hat{r}_L \sin(\tau + \hat{\epsilon}), \quad (2.140)$$

$$\hat{y}^{(c)} = \hat{r}_L \cos(\tau + \hat{\epsilon}), \quad (2.141)$$

where $\hat{v}_\perp = \hat{v}_\perp(\tau)$, $\hat{\epsilon} = \hat{\epsilon}(\tau)$, $\hat{w}_c = \hat{w}_c(\tau) = e\hat{B}(\tau)/m$, and $\hat{r}_L = \hat{v}_\perp/\hat{w}_c$ are now time-dependent functions. Note that for this specific case, the τ -derivatives of the positions above are not equal to their respective velocities, and instead the relationship holds only to leading order. For the guiding center velocities, we'll guess a given form and then check if it satisfies the governing equations. Thus, we guess

$$\begin{aligned}\hat{v}_x^{(g)} &= \frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \\ \hat{v}_y^{(g)} &= \frac{d}{d\tau} \left(\frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \right).\end{aligned}\quad (2.142)$$

Plugging in all of these expressions in the evolution equations given by eq. (2.137), and using a bit of algebra, leads to

$$\frac{d \ln \hat{\mu}}{d\tau} = \frac{d \ln \hat{B}(\tau)}{d\tau} \cos[2(\tau + \hat{\epsilon})], \quad (2.143)$$

where $\hat{\mu} = \hat{\mu}(\tau)$ is given by

$$\hat{\mu} = \frac{m \hat{v}_\perp^2}{2 \hat{B}(\tau)}. \quad (2.144)$$

Integrating over one gyro-period, i.e. from τ_1 to τ_2 such that $[\tau_2 + \epsilon(\tau_2)] - [\tau_1 + \epsilon(\tau_1)] = 2\pi$, gives

$$\ln \hat{\mu}(\tau_2) - \ln \hat{\mu}(\tau_1) = \ln[\hat{B}(\tau)] \cos[2(\tau + \hat{\epsilon})] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2 \ln[\hat{B}(\tau)] \sin[2(\tau + \hat{\epsilon})] d\tau. \quad (2.145)$$

Assuming $\hat{B}(\tau)$ doesn't change significantly from τ_1 to τ_2 , then we have $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$, that is, $\hat{\mu}$ is constant over one gyro-period. One can also define

$$\mu = \frac{mv_{\perp}^2}{2B(t)} \quad (2.146)$$

where $\mu = \mu(t)$ and $v_{\perp} = v_{\perp}(t)$. Given that $v_{\perp} = \hat{v}_{\perp}(h(t))$, we have $\mu = \hat{\mu}(h(t))$. Thus, $\hat{\mu}(\tau_2) = \hat{\mu}(\tau_1)$ translates to $\mu(t_2) = \mu(t_1)$, where $t_2 = h^{-1}(\tau_2)$ and $t_1 = h^{-1}(\tau_1)$.

As shown in the analysis above, for a time dependent magnetic field a drift of the following form is introduced

$$\hat{v}_y^{(g)} = \frac{d}{d\tau} \left(\frac{\hat{E}_y(\hat{x}^{(g)}, \tau)}{\hat{B}(\tau)} \right). \quad (2.147)$$

Converting back to time t

$$v_y^{(g)} = \frac{1}{w_c} \frac{d}{dt} \left(\frac{E_y(x^{(g)}, t)}{B(t)} \right). \quad (2.148)$$

For the more general case where $E_x = E_x(q_x, q_y, t)$ and $E_y = E_y(q_x, q_y, t)$ then

$$\mathbf{v}_p^{(g)} = \mp \frac{1}{w_c} \frac{d}{dt} \left(\frac{\mathbf{E}_{\perp}}{B} \right), \quad (2.149)$$

where top sign is for electrons and bottom sign is for ions, and it is assumed that the electric field is evaluated at the guiding center. For an even more general case where the magnetic field does not necessarily point in one direction,

$$\mathbf{v}_p^{(g)} = \mp \frac{1}{w_c} \mathbf{b} \times \frac{d\mathbf{v}_E^{(g)}}{dt}. \quad (2.150)$$

Chapter 3

Collisions

3.1 Cross section

that two particles traveling towards each other will undergo an interaction. Types of interactions include Coulomb collisions between two charged particles, fusion reactions between ions, and photon-matter phenomena such as Compton scattering, the photoelectric effect, and pair production.

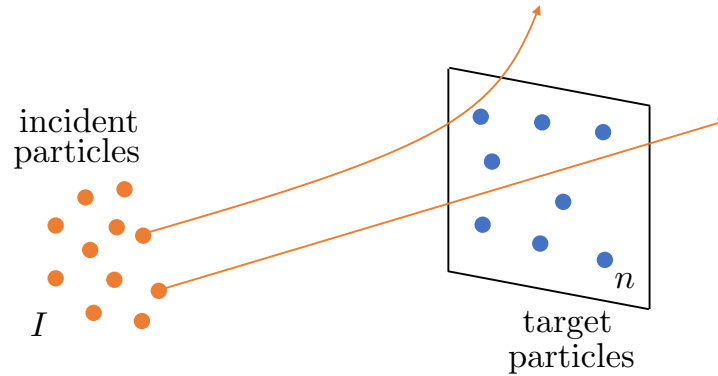


Figure 3.1: Interaction of incident and target particles.

To define the cross section, we'll consider various incident particles heading towards a stationary set of target particles (see fig. 3.1). The number of incident particles is denoted as I , and the areal density of target particles is denoted by n_A . Not all of the incident particles will interact with the target particles, some will instead continue to travel in a uniform trajectory. The number of incident particles that do interact with the target particles is labeled as N . The cross section σ is then a constant of proportionality defined by the following equation

$$N = \sigma I n_A. \quad (3.1)$$

3.2 Differential cross section

Consider the scattering of two particles: an incident and a target particle. If we fix the reference frame to follow the target particle, then the scattering can be depicted as shown in fig. 3.2. The displacement parameter is labelled as b , and the scattering angle as θ_s . For three dimensional scattering, the encounter is as shown in fig. 3.3. Not that in that figure the incident particle

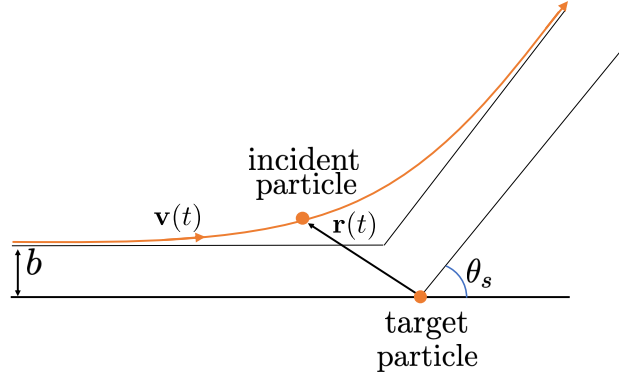


Figure 3.2: Depiction of particle scattering.

starts within the $x - z$ plane, and after scattering the particle is confined to a plane that is tilted an angle ϕ_s from the $x - z$ plane.

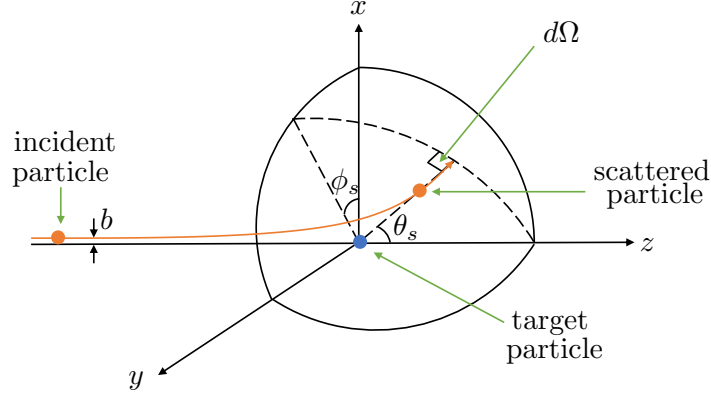


Figure 3.3: Depiction of particle scattering in 3D.

From the entire set of incident particles N that interact with the target particles, one can define an infinitesimal subset $N_{\theta,\phi}d\Omega$ as the number of particles scattered into an infinitesimal solid angle $d\Omega = \sin\theta_s d\theta_s d\phi_s$, as shown in fig. 3.3. We note that $N_{\theta,\phi} = N_{\theta,\phi}(\theta_s, \phi_s)$. The subset of particles $N_{\theta,\phi}d\Omega$ is now given by

$$N_{\theta,\phi}d\Omega = \left(\frac{d\sigma_{\theta,\phi}}{d\Omega} d\Omega \right) I n_A. \quad (3.2)$$

In the above,

$$\frac{d\sigma_{\theta,\phi}}{d\Omega} = \frac{d\sigma_{\theta,\phi}}{d\Omega}(\theta_s, \phi_s) \quad (3.3)$$

is the differential cross section. It is best to not think of it as a derivative (what does a derivative with respect to solid angle mean?) and instead to simply think of it as a function that depends on θ_s and ϕ_s . Integrating over all θ_s and ϕ_s , i.e.

$$\int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} N_{\theta,\phi} d\Omega = \int_{\theta_s=0}^{\pi} \int_{\phi_s=0}^{2\pi} \frac{d\sigma_{\theta,\phi}}{d\Omega} d\Omega I n_A \quad (3.4)$$

gives eq. (3.1).

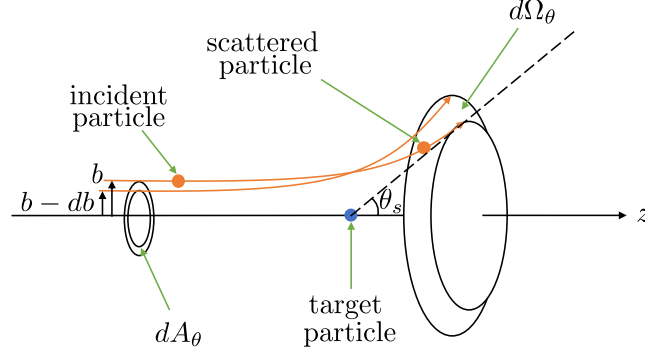


Figure 3.4: Depiction of particle scattering in for axisymmetric interactions.

For various cases the scattering is axisymmetric, that is, it is independent of ϕ_s . Thus

$$N_{\theta,\phi} \rightarrow N_{\theta} \quad \frac{d\sigma_{\theta,\phi}}{d\Omega} \rightarrow \frac{d\sigma_{\theta}}{d\Omega}, \quad (3.5)$$

where

$$N_{\theta} = N_{\theta}(\theta_s), \quad (3.6)$$

and

$$\frac{d\sigma_{\theta}}{d\Omega} = \frac{d\sigma_{\theta}}{d\Omega}(\theta_s). \quad (3.7)$$

Integrating eq. (3.2) from $\phi_s = 0$ to $\phi_s = 2\pi$ gives

$$N_{\theta}d\Omega_{\theta} = \frac{d\sigma_{\theta}}{d\Omega}d\Omega_{\theta}In_A, \quad (3.8)$$

where $d\Omega_{\theta} = 2\pi \sin \theta_s d\theta_s$. $N_{\theta}d\Omega_{\theta}$ thus represents the number of particles that are scattered into the infinitesimal band $d\Omega_{\theta}$ on a sphere, where $d\Omega_{\theta}$ is defined by scattering angle θ_s (see fig. 3.4). We will note that there is a one-to-one correspondence between the impact parameter b and the scattering angle θ_s , that is, $b = b(\theta_s)$. Thus, any incident particle scattered out through the infinitesimal band $d\Omega_{\theta}$ would have approached the target-particle through the infinitesimal ring dA_{θ} that corresponds to $d\Omega_{\theta}$. Since there are many target particles, there are many dA_{θ} 's that correspond to the same $d\Omega_{\theta}$. Thus, the total number of particles scattered through $d\Omega_{\theta}$ is given by all the incident particles that cross through the dA_{θ} 's of all the target particles.

The number of incident particles crossing all the infinitesimal rings dA_{θ} is equal to the total number of incident particles I times the probability P that a single incident particle will cross one of those rings. Thus, we can write

$$N_{\theta}d\Omega_{\theta} = IP.$$

The probability that an incident particle will cross one of those rings is simply the ratio of the surface area covered by all the rings in a section of the target material to the total area of that section. The surface area covered by all the rings in a section of area S is given by $(n_AS)dA_{\theta}$. Thus, $P = n_AdA_{\theta}$ and

$$N_{\theta}d\Omega_{\theta} = In_AdA_{\theta}.$$

We now introduce the differential

$$db = \frac{db}{d\theta_s}d\theta_s. \quad (3.9)$$

We note that by definition $d\theta_s$ is positive but db can be either positive or negative depending on the sign of $db/d\theta_s$. The infinitesimal area dA_θ is then given by

$$dA_\theta = 2\pi b|db| = 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s. \quad (3.10)$$

Thus,

$$N_\theta d\Omega_\theta = n_A I 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s.$$

Using eq. (3.8) in the above, we get

$$\frac{d\sigma_\theta}{d\Omega} d\Omega_\theta I n_A = I n_A 2\pi b \left| \frac{db}{d\theta_s} \right| d\theta_s,$$

or

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin \theta_s} \left| \frac{db}{d\theta_s} \right|. \quad (3.11)$$

3.3 Mean free path & collision frequency

The mean free path can be expressed in terms of the cross section as

$$\lambda_m = \frac{1}{n_1 \sigma}. \quad (3.12)$$

Given the particle's speed v , one can also define the collision time as follows

$$\tau_m = \frac{\lambda_m}{v} = \frac{1}{n_1 \sigma v}. \quad (3.13)$$

Finally, the collision frequency is simply the inverse of the collision time, that is

$$\nu_m = \frac{1}{\tau_m} = n_1 \sigma v. \quad (3.14)$$

Chapter 4

Coulomb scattering

4.1 Particle equations

Consider two particles, with positions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, velocities $\mathbf{v}_1 = \mathbf{v}_1(t)$ and $\mathbf{v}_2 = \mathbf{v}_2(t)$, charges q_1 and q_2 , and masses m_1 and m_2 , respectively. Their positions and velocities are governed by the following equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1, \quad (4.1)$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2, \quad (4.2)$$

$$m_1 \frac{d\mathbf{v}_1}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \quad (4.3)$$

$$m_2 \frac{d\mathbf{v}_2}{dt} = -\frac{q_1 q_2}{4\pi\epsilon} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \quad (4.4)$$

We note that the above system consists of twelve equations for twelve unknowns. We now introduce the center-of-mass position $\mathbf{R} = \mathbf{R}(t)$, the center-of-mass velocity $\mathbf{V} = \mathbf{V}(t)$, the shifted position $\mathbf{r} = \mathbf{r}(t)$ and the shifted velocity $\mathbf{v} = \mathbf{v}(t)$ as follows

$$\begin{aligned} \mathbf{R} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} & \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2, \\ \mathbf{V} &= \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} & \mathbf{v} &= \mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

Thus, in terms of these new four variables, the particle equations can be written as

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}, \quad (4.5)$$

$$\frac{d\mathbf{V}}{dt} = 0, \quad (4.6)$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad (4.7)$$

$$\frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r}}{r^3}, \quad (4.8)$$

where the reduced mass m_r is given by

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}. \quad (4.9)$$

The first two equations above give the trivial solution $\mathbf{V} = \text{constant}$ and $\mathbf{R} = \mathbf{R}(0) + \mathbf{V}t$. Thus, we have reduced the problem from twelve unknowns to six unknowns, namely \mathbf{r} and \mathbf{v} .

4.2 Conservation of energy and momentum

Dotting eq. (4.8) by \mathbf{v} gives

$$\begin{aligned}\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \mathbf{v} \cdot \frac{\mathbf{r}}{r^3} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{r}}{r^3} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{1}{2} \frac{dr^2}{dt} \frac{1}{r^3} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{1}{r^2} \frac{dr}{dt} \\ &= -\frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{d}{dt} \left(\frac{1}{r} \right).\end{aligned}$$

For the left hand side above we have

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{1}{2} \frac{dv^2}{dt},$$

and thus we obtain the following expression for conservation of energy

$$\frac{d}{dt} \left(\frac{1}{2} m_r v^2 + \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r} \right) = 0.$$

Crossing eq. (4.8) by \mathbf{r} gives

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r} \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0,$$

and thus

$$\frac{d}{dt} [m_r (\mathbf{r} \times \mathbf{v})] = 0.$$

That is, angular momentum is conserved. A consequence of this is that the vector $\mathbf{r} \times \mathbf{v}$ is always pointing in the same direction. Thus, if $\mathbf{r}(0)$ and $\mathbf{v}(0)$ form a plane, then $\mathbf{r}(t)$ and $\mathbf{v}(t)$ need to reside within that same plane for all times t so that $\mathbf{r}(t) \times \mathbf{v}(t)$ points in the same direction as $\mathbf{r}(0) \times \mathbf{v}(0)$. Therefore, the evolution of the position and velocity are confined to a plane and the problem can be reduced from six unknowns to four unknowns. This planar encounter is depicted in fig. 4.1.

If we refer to the plane shown in fig. 4.1 as the $x - y$ plane, then one can tell that the angular-momentum vector points in the negative z direction. We will denote the magnitude of the conserved angular momentum by L , and thus we can write

$$m_r (\mathbf{r} \times \mathbf{v}) = -L \hat{\mathbf{z}}. \quad (4.10)$$

A consequence of both conservation of energy and momentum is as follows. Consider the two limiting states of particle 1—the initial state v_i, b_i and the final state v_f, b_f . Assuming the potential energy is very low at sufficiently early and late times, conservation of energy gives

$$\frac{1}{2} m_r v_i^2 = \frac{1}{2} m_r v_f^2, \quad (4.11)$$

that is, $v_i = v_f$ (note that for other scattering processes, e.g. Compton scattering, this is not necessarily the case). For the angular momentum of the initial state we have

$$\begin{aligned}m_r (\mathbf{r}_i \times \mathbf{v}_i) &= m_r \sin(-\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\theta_i) r_i v_i \hat{\mathbf{z}} = -m_r \sin(\pi - \varphi_i) r_i v_i \hat{\mathbf{z}} \\ &= -m_r \sin(\varphi_i) r_i v_i \hat{\mathbf{z}} = -m_r b_i v_i \hat{\mathbf{z}}\end{aligned} \quad (4.12)$$

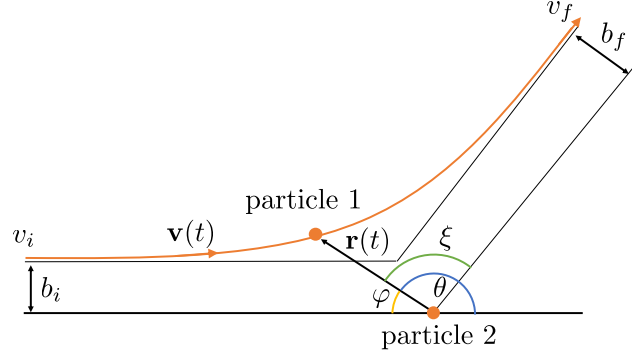


Figure 4.1: Depiction of Coulomb scattering.

Similarly, for the angular momentum of the final state we have

$$m_r (\mathbf{r}_f \times \mathbf{v}_f) = m_r \sin(-\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r \sin(\xi_f) r_f v_f \hat{\mathbf{z}} = -m_r b_f v_f \hat{\mathbf{z}}. \quad (4.13)$$

Equating the last two relationships gives $m_r b_i v_i = m_r b_f v_f$. Since $v_i = v_f$, we finally have $b_i = b_f = b$. Using eq. (4.10) in eq. (4.12), we can also write

$$L = m_r b v_i. \quad (4.14)$$

4.3 Polar coordinates

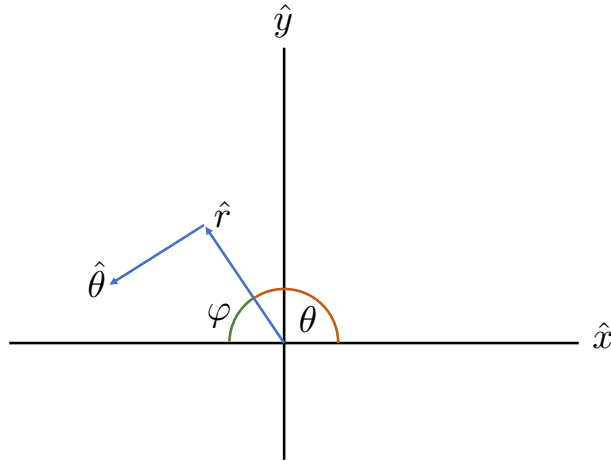


Figure 4.2: Polar coordinates in plane of interaction.

Using polar coordinates, as shown in fig. 4.2, we get

$$r_x = r \cos \theta = r \cos(\pi - \varphi) = -r \cos \varphi,$$

$$r_y = r \sin \theta = r \sin(\pi - \varphi) = r \sin \varphi.$$

Also, since $\mathbf{r} = r\hat{\mathbf{r}}$, we have

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},\end{aligned}$$

and

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}.\end{aligned}$$

The radial component of eq. (4.8) thus becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0m_r}\frac{1}{r^2}.$$

Since $\theta = \pi - \varphi$, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = \frac{q_1q_2}{4\pi\epsilon_0m_r}\frac{1}{r^2}. \quad (4.15)$$

For the angular momentum we have

$$m_r\mathbf{r} \times \mathbf{v} = m_r r\hat{\mathbf{r}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right) = m_r r^2 \frac{d\theta}{dt} \hat{\mathbf{z}}$$

Using eq. (4.10), we can write the above as

$$m_r r^2 \frac{d\varphi}{dt} = L. \quad (4.16)$$

4.4 Particle trajectory

The goal is to find the radial position of the particle as a function of its angular orientation. That is, we want to find $\tilde{r} = \tilde{r}(\tilde{\varphi})$ such that

$$r(t) = \tilde{r}(\varphi(t)). \quad (4.17)$$

To simplify the math, we introduce $\tilde{u} = \tilde{u}(\tilde{\varphi})$ such that $\tilde{u} = 1/\tilde{r}$. Thus

$$\frac{d\tilde{u}}{d\tilde{\varphi}} = -\frac{1}{\tilde{r}^2} \frac{d\tilde{r}}{d\tilde{\varphi}},$$

or, after re-arranging

$$\frac{d\tilde{r}}{d\tilde{\varphi}} = -\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}}. \quad (4.18)$$

We now proceed as follows. Taking the derivative of r , we get

$$\begin{aligned}
\frac{dr}{dt} &= \left(\frac{d\tilde{r}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} & [eq. (4.17)] \\
&= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} & [eq. (4.18)] \\
&= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} & [eq. (4.16)] \\
&= \left(-\frac{1}{\tilde{u}^2} \frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} & [eq. (4.17)] \\
&= \left(-\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} & (4.19)
\end{aligned}$$

Taking the derivative of the above, we get

$$\begin{aligned}
\frac{d}{dt} \frac{dr}{dt} &= \left[\frac{d}{d\tilde{\varphi}} \left(-\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \right) \right]_{\tilde{\varphi}=\varphi(t)} \frac{d\varphi}{dt} \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \right)_{\tilde{\varphi}=\varphi(t)} \frac{L}{m_r r^2} & [eq. (4.16)] \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L}{m_r} \frac{L}{m_r \tilde{r}^2} \right)_{\tilde{\varphi}=\varphi(t)} & [eq. (4.17)] \\
&= \left(-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2 \tilde{u}^2}{m_r^2} \right)_{\tilde{\varphi}=\varphi(t)} & (4.20)
\end{aligned}$$

Plugging the last relation into eq. (4.15) gives

$$\left[-\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} \frac{L^2 \tilde{u}^2}{m_r^2} - \frac{1}{\tilde{u}} \left(\frac{L \tilde{u}^2}{m_r} \right)^2 \right]_{\tilde{\varphi}=\varphi(t)} = \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r} \tilde{u}^2 \right)_{\tilde{\varphi}=\varphi(t)},$$

which, upon re-arranging and dropping the $\varphi(t)$ dependance, becomes

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2 m_r}{4\pi\epsilon_0 L^2} \quad (4.21)$$

Using eq. (4.14) we write the evolution equation for \tilde{u} as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{q_1 q_2}{4\pi\epsilon_0 m_r b^2 v_i^2}. \quad (4.22)$$

Introducing the notation

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 m_r v_i^2}, \quad (4.23)$$

the evolution equation for \tilde{u} can be simply expressed as

$$\frac{d^2\tilde{u}}{d\tilde{\varphi}^2} + \tilde{u} = -\frac{b_{90}}{b^2}. \quad (4.24)$$

The boundary conditions for eq. (4.24) are as follows

$$\text{as } \varphi(t) \rightarrow 0, \quad r(t) \rightarrow \infty \quad (4.25)$$

$$\text{as } \varphi(t) \rightarrow 0, \quad \frac{dr(t)}{dt} \rightarrow -v_i \quad (4.26)$$

Given eq. (4.17), eq. (4.25) can only be satisfied if as $\tilde{\varphi} \rightarrow 0$, $\tilde{r} \rightarrow \infty$. Thus, we also have, as $\tilde{\varphi} \rightarrow 0$, $\tilde{u} \rightarrow 0$. Similarly, given eq. (4.19), eq. (4.26) can only be satisfied if as $\tilde{\varphi} \rightarrow 0$

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \frac{L}{m_r} \rightarrow v_i.$$

Using eq. (4.14) we rewrite the above as

$$\frac{d\tilde{u}}{d\tilde{\varphi}} \rightarrow \frac{1}{b}.$$

The general solution to eq. (4.24) is

$$\tilde{u} = A \cos \tilde{\varphi} + B \sin \tilde{\varphi} - \frac{b_{90}}{b^2}.$$

Applying the boundary conditions, we get

$$\tilde{u} = \frac{b_{90}}{b^2} \cos \tilde{\varphi} + \frac{1}{b} \sin \tilde{\varphi} - \frac{b_{90}}{b^2},$$

which we finally re-write as

$$\frac{1}{\tilde{r}} = \frac{1}{b} \sin \tilde{\varphi} + \frac{b_{90}}{b^2} (\cos \tilde{\varphi} - 1). \quad (4.27)$$

4.5 The scattering angle

We now drop the tilde notation for the sake of simplicity. That is, for the radial location of an incident particle, we have

$$\frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{b_{90}}{b^2} (\cos \varphi - 1), \quad (4.28)$$

where φ is the independent variable and $r = r(\varphi)$. We want to know the value of φ as r goes to infinity. Using eq. (4.28), and labeling this angle as φ_s , we have

$$0 = \sin \varphi_s + \frac{b_{90}}{b} (\cos \varphi_s - 1).$$

We express the above in terms of the scattering angle $\theta_s = \pi - \varphi_s$,

$$0 = \sin(\pi - \theta_s) + \frac{b_{90}}{b} [\cos(\pi - \theta_s) - 1].$$

or

$$0 = \sin \theta_s + \frac{b_{90}}{b} (-\cos \theta_s - 1).$$

Re-writing the above as

$$\frac{\cos \theta_s + 1}{\sin \theta_s} = \frac{b}{b_{90}},$$

and using the trig identity $\cot(\theta/2) = (\cos \theta + 1)/\sin \theta$, we get

$$\cot\left(\frac{\theta_s}{2}\right) = \frac{b}{b_{90}}. \quad (4.29)$$

4.6 The differential cross section

The differential cross section for Coulomb scattering can be computed by making use of a formula derived in my introductory notes, namely

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b}{\sin\theta_s} \left| \frac{db}{d\theta_s} \right|. \quad (4.30)$$

From eq. (4.29) we get,

$$\frac{db}{d\theta} = -\frac{b_{90}}{2} \frac{1}{\sin^2(\theta_s/2)}, \quad (4.31)$$

which, plugging in eq. (4.30), gives

$$\frac{d\sigma_\theta}{d\Omega} = \left[b_{90} \frac{\cot(\theta_s/2)}{\sin\theta_s} \right] \left[\frac{b_{90}}{2} \frac{1}{\sin^2(\theta_s/2)} \right].$$

Using the trig identities $\cot(\theta) = \cos(\theta)/\sin(\theta)$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$ we get

$$\frac{d\sigma_\theta}{d\Omega} = \frac{b_{90}^2}{4} \frac{1}{\sin^4(\theta_s/2)}. \quad (4.32)$$

4.7 Collision integral

$$\Omega_{\alpha\beta}^{(lk)} = \sqrt{\frac{k_B T}{2\pi M_{\alpha\beta}}} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} dg. \quad (4.33)$$

In the above $M_{\alpha\beta}$ is the reduced mass, given by

$$M_{\alpha\beta} = \frac{M_\alpha M_\beta}{M_\alpha + M_\beta}, \quad (4.34)$$

and $\phi_{\alpha\beta}^{(l)}$ is the collision cross section for a given velocity, and is computed as

$$\phi_{\alpha\beta}^{(l)} = 2\pi \int_0^\infty \left(1 - \cos^l \chi_{\alpha\beta}\right) b db. \quad (4.35)$$

The scattering angle $\chi_{\alpha\beta}$ is given by

$$\chi_{\alpha\beta} = \pi - 2 \int_{r_{\alpha\beta}^{\min}}^\infty \frac{b}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{\alpha\beta}(r)}{g^2 k_B T}\right]^{1/2}} dr. \quad (4.36)$$

For a Coulombic interaction between ions, we can define the natural scale for the cross-sectional area as

$$\phi_{\alpha\beta}^{(0)} = \frac{\pi (Z_\alpha Z_\beta e^2)^2}{(2k_B T)^2}. \quad (4.37)$$

Given this definition, we express the collision integral as

$$\Omega_{\alpha\beta} = \sqrt{\frac{\pi}{M_{\alpha\beta}}} \frac{(Z_\alpha Z_\beta e^2)^2}{(2k_B T)^{3/2}} \mathcal{F}_{\alpha\beta}^{lk}, \quad (4.38)$$

where

$$\mathcal{F}_{\alpha\beta}^{(lk)} = \frac{1}{2\phi_0} \int_0^\infty e^{-g^2} g^{2k+3} \phi_{\alpha\beta}^{(l)} dg \quad (4.39)$$

We note that $\mathcal{F}_{\alpha\beta}^{(lk)} = 4\mathcal{K}_{lk}(g_{\alpha\beta})$, where $\mathcal{K}_{lk}(g_{\alpha\beta})$ is the notation from the Stanton-Murillo paper.

Part II

Kinetic Description

Chapter 5

Kinetic equations

We denote the distribution function for a species α as $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where \mathbf{r} and \mathbf{v} are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_\alpha d\mathbf{r} d\mathbf{v} = N_\alpha, \quad (5.1)$$

where N_α is the total number of particles corresponding to species α .

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution along with Maxwell's equations

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\alpha = C_\alpha + S_\alpha \quad (5.2)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \quad (5.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (5.4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (5.5)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (5.6)$$

$$\mathbf{J} = \sum_{\alpha} Z_\alpha e \int \mathbf{v} f_\alpha d\mathbf{v} \quad (5.7)$$

$$\rho_e = \sum_{\alpha} Z_\alpha e \int f_\alpha d\mathbf{v}. \quad (5.8)$$

In the above,

- m_α is the species mass
- e is the charge
- Z_α the charge number
- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ the charge current
- $\rho_e = \rho_e(\mathbf{r}, t)$ the charge density
- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the electric field

- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ the magnetic field.

The terms C_α and S_α represent collision and source terms.

If we express the collision term in the usual way, that is $C_\alpha = \sum_\beta C_{\alpha\beta}$, then we can make the following statements:

1. Conservation of particles:

$$\int C_{\alpha\alpha} d\mathbf{v} = 0 \quad \forall \alpha \quad \int C_{\alpha\beta} d\mathbf{v} = 0 \quad \forall \alpha, \beta | \beta \neq \alpha. \quad (5.9)$$

2. Conservation of momentum:

$$\int m_\alpha \mathbf{v} C_{\alpha\alpha} d\mathbf{v} = 0 \quad \forall \alpha \quad \sum_\alpha \sum_{\beta, \beta \neq \alpha} \int m_\alpha \mathbf{v} C_{\alpha\beta} d\mathbf{v} = 0. \quad (5.10)$$

3. Conservation of energy:

$$\int \frac{1}{2} m_\alpha v^2 C_{\alpha\alpha} d\mathbf{v} = 0 \quad \forall \alpha \quad \sum_\alpha \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2} m_\alpha v^2 C_{\alpha\beta} d\mathbf{v} = 0. \quad (5.11)$$

Part III

Fluid Description

Chapter 6

Fluid equations

We now define the particle density $n_\alpha = n_\alpha(\mathbf{r}, t)$, the fluid velocity $\mathbf{u}_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$ and the fluid energy per unit mass $E_\alpha = E_\alpha(\mathbf{r}, t)$ as follows

$$n_\alpha = \int f_\alpha d\mathbf{v} \quad (6.1)$$

$$\mathbf{u}_\alpha = \frac{1}{n_\alpha} \int \mathbf{v} f_\alpha d\mathbf{v} \quad (6.2)$$

$$E_\alpha = \frac{1}{n_\alpha} \int \frac{1}{2} v^2 f_\alpha d\mathbf{v}. \quad (6.3)$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_\alpha}{\partial t} + \nabla \cdot (\mathbf{v} f_\alpha) + \nabla_v \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] = C_\alpha + S_\alpha \quad (6.4)$$

6.1 Mass

Integrating eq. (6.4) over all \mathbf{v} we obtain

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha \quad (6.5)$$

where

$$\hat{S}_\alpha = \int S_\alpha d\mathbf{v} \quad (6.6)$$

is an external source of mass.

6.2 Momentum

Multiplying eq. (6.4) by \mathbf{v} and then integrating over all \mathbf{v} leads to

$$\begin{aligned} \frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(\int \mathbf{v} \mathbf{v} f_\alpha d\mathbf{v} \right) + \int \nabla_v \cdot \left[\mathbf{v} \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] - \nabla_v \mathbf{v} \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] d\mathbf{v} = \\ \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \end{aligned} \quad (6.7)$$

We note that the third term in eq. (6.7) is zero since we are integrating over all space, and that $\nabla_v \mathbf{v}$ is the identity matrix. We thus have

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(\int \mathbf{v} \mathbf{v} f_\alpha d\mathbf{v} \right) - \frac{Z_\alpha e n_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \quad (6.8)$$

To proceed, we decompose \mathbf{v} into a mean and a fluctuation, that is, $\mathbf{v} = \mathbf{u}_\alpha + \mathbf{w}_\alpha$. Using this decomposition

$$\int \mathbf{v} \mathbf{v} f_\alpha d\mathbf{v} = \int (\mathbf{u}_\alpha \mathbf{u}_\alpha + 2\mathbf{u}_\alpha \mathbf{w}_\alpha + \mathbf{w}_\alpha \mathbf{w}_\alpha) f_\alpha d\mathbf{v} = n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v}. \quad (6.9)$$

Thus, eq. (6.8) becomes

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \quad (6.10)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \mathbf{w}_\alpha C_{\alpha\beta} d\mathbf{v} + \int \mathbf{v} S_\alpha d\mathbf{v}. \quad (6.11)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (6.12)$$

where the stress tensor is

$$\boldsymbol{\sigma}_\alpha = - \int m_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v}, \quad (6.13)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_\alpha = \sum_{\beta, \beta \neq \alpha} \int m_\alpha \mathbf{w}_\alpha C_{\alpha\beta} d\mathbf{v}, \quad (6.14)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_\alpha = \int m_\alpha \mathbf{v} S_\alpha d\mathbf{v}. \quad (6.15)$$

The stress tensor is typically decomposed into isotropic p_α and anisotropic (shear) \mathbf{t}_α tensors as follows

$$\boldsymbol{\sigma}_\alpha = -p_\alpha \mathbf{I} + \mathbf{t}_\alpha, \quad (6.16)$$

where P_α is given by

$$p_\alpha = \frac{1}{3} \int m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v}. \quad (6.17)$$

Thus, conservation of momentum becomes

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha. \quad (6.18)$$

6.3 Energy

Multiplying eq. (6.4) by $\frac{1}{2}v^2$ and then integrating over all \mathbf{v} leads to

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_\alpha d\mathbf{v} \right] + \int \nabla_v \cdot \left[\frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] \\ - \nabla_v \left[\frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \right] \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] d\mathbf{v} = \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (6.19)$$

We note that the third term above is zero since we are integrating over all space, and that $\nabla_v[1/2(\mathbf{v} \cdot \mathbf{v})] = \mathbf{v}$. Thus, we have

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_\alpha d\mathbf{v} \right] - \frac{Z_\alpha e n_\alpha}{m_\alpha} \mathbf{E} \cdot \mathbf{u}_\alpha = \\ \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (6.20)$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} f_\alpha d\mathbf{v} = \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) (\mathbf{u}_\alpha + \mathbf{w}_\alpha) f_\alpha d\mathbf{v} = n_\alpha E_\alpha \mathbf{u}_\alpha + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \quad (6.21)$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \mathbf{w}_\alpha f_\alpha d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha + 2\mathbf{u}_\alpha \cdot \mathbf{w}_\alpha + \mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \quad (6.22)$$

$$= \mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v}. \quad (6.23)$$

Using the expressions above, eq. (6.20) becomes

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot (n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} \mathbf{E} \cdot \mathbf{u}_\alpha = -\nabla \cdot \left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \\ + \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (6.24)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\begin{aligned} \frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot (n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha} \mathbf{E} \cdot \mathbf{u}_\alpha = -\nabla \cdot \left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} \right) - \nabla \cdot \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v} \\ + \mathbf{u}_\alpha \cdot \sum_{\beta, \beta \neq \alpha} \int \mathbf{w}_\alpha C_{\alpha\beta} d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2}(\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) C_{\alpha\beta} d\mathbf{v} + \int \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \end{aligned} \quad (6.25)$$

Multiplying by mass leads to the following equation

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \nabla \cdot (\mathbf{u}_\alpha \cdot \boldsymbol{\sigma}_\alpha) - \nabla \cdot \mathbf{q}_\alpha \\ + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \end{aligned} \quad (6.26)$$

where heat flux due to random motion is

$$\mathbf{q}_\alpha = \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha d\mathbf{v}, \quad (6.27)$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_\alpha = \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) C_{\alpha\beta} d\mathbf{v}, \quad (6.28)$$

and the external source of energy is

$$\hat{Q}_\alpha = \int \frac{1}{2} m_\alpha (\mathbf{v} \cdot \mathbf{v}) S_\alpha d\mathbf{v}. \quad (6.29)$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha &= \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha \\ &+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \end{aligned} \quad (6.30)$$

We also note that the energy $m_\alpha n_\alpha E_\alpha$ can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (6.9) one obtains

$$\begin{aligned} m_\alpha n_\alpha E_\alpha &= \int \frac{1}{2} m_\alpha (\mathbf{v} \cdot \mathbf{v}) f_\alpha d\mathbf{v} \\ &= \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v} + \frac{1}{2} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \\ &= \frac{3}{2} P_\alpha + \frac{1}{2} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \\ &= \frac{3}{2} P_\alpha + m_\alpha n_\alpha K_\alpha. \end{aligned} \quad (6.31)$$

where $K_\alpha = \frac{1}{2} \mathbf{u}_\alpha \cdot \mathbf{u}_\alpha$ is the kinetic energy of species α .

6.4 Kinetic and Internal Energies

The equation for the kinetic energy is obtained by dotting eq. (6.18) with \mathbf{u}_α . For this, we first show that

$$\mathbf{u}_\alpha \cdot \left[\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) \right] \quad (6.32)$$

$$= \mathbf{u}_\alpha \cdot \left\{ \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right\} \quad (6.33)$$

$$= \mathbf{u}_\alpha \cdot \left[m_\alpha \hat{S}_\alpha \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right] \quad (6.34)$$

$$= 2m_\alpha \hat{S}_\alpha K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \quad (6.35)$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \quad (6.36)$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K_\alpha \mathbf{u}_\alpha). \quad (6.37)$$

Thus, the equation for the turbulent kinetic energy is

$$\begin{aligned} & \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \\ & - \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \end{aligned} \quad (6.38)$$

Subtracting the above equation from eq. (6.30) leads to

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_\alpha \right) + \nabla \cdot \left(\frac{3}{2} p_\alpha \mathbf{u}_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \quad (6.39)$$

6.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha, \quad (6.40)$$

- Momentum

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (6.41)$$

- Total Energy

$$\begin{aligned} \frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha &= \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha \\ &+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \end{aligned} \quad (6.42)$$

- Kinetic Energy

$$\begin{aligned} & \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \\ & - \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \end{aligned} \quad (6.43)$$

- Internal Energy

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_\alpha \right) + \nabla \cdot \left(\frac{3}{2} p_\alpha \mathbf{u}_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \quad (6.44)$$

Chapter 7

Basic concepts

7.1 Debye length

7.2 Plasma frequency

7.3 The coupling parameter

Coulomb interactions are those which occur when two charge particles head towards each other. We can define two types of Coulomb interactions: strong and weak. Strong Coulomb interactions are those for which the particle's Coulomb potential energy is larger than its kinetic energy, and viceversa for weak Coulomb interactions. Thus, we can also define two types of plasma regimes:

- Strongly-coupled plasmas: plasmas where the Coulomb interactions are mostly strong and thus drive the dynamics of its evolution. Coulomb interactions tend to be strong when the inter-particle distances are small, and thus this regime would be dominated by *short-range* interactions. These plasmas are also described as exhibiting *collisional* behavior, since a strong Coulomb interaction essentially means a collision has occurred.
- Weakly-coupled plasmas: plasmas where the Coulomb interactions are mostly weak, and as a result do not drive the dynamics of its evolution. The plasma dynamics are instead driven by *long-range* effects caused by smooth electromagnetic fields that result from integrating a large number of particles. These plasmas are also described as exhibiting *collective* behavior, since the long-range electromagnetic fields follow from the collective integration of many particles.

We describe an approximate Coulomb potential energy for particles in a plasma as

$$U = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha}. \quad (7.1)$$

The impact parameter that has been used above is a_α , the sphere radius. This provides a decent measure on the average spacing between particles in a plasma. Since the volume of a single particle is $1/n_\alpha$, and if we assume that this volume is given by $4/3\pi a_\alpha^3$, then equating these two gives the expression for the sphere radius

$$a_\alpha = \left(\frac{3}{4\pi n_\alpha} \right)^{1/3}. \quad (7.2)$$

The thermal velocity of a particle is given by

$$v_{T_\alpha} = \sqrt{\frac{k_B T_\alpha}{m_\alpha}} \quad (7.3)$$

A measure of the kinetic energy of a particle is given in terms of the thermal velocity as shown below

$$K = m_\alpha v_{T_\alpha}^2 = k_B T_\alpha \quad (7.4)$$

The ratio of the particle's Coulomb potential energy and its kinetic energy is referred to as the coupling parameter Γ_α . That is

$$\Gamma_\alpha = \frac{q_\alpha^2}{4\pi\epsilon_0 a_\alpha k_B T_\alpha}. \quad (7.5)$$

$\Gamma_\alpha > 1$ denotes a strongly coupled plasma, and $\Gamma_\alpha < 1$ denotes a weakly coupled plasma.

7.4 The plasma parameter

The standard plasma parameter Λ_α is defined as

$$\Lambda_\alpha = \frac{4}{3}\pi\lambda_{D\alpha}^3 n_\alpha. \quad (7.6)$$

There is a one to one relationship between the coupling parameter and the standard plasma parameter. Simple algebra shows that

$$\Gamma_\alpha = (1/3)\Lambda_\alpha^{-2/3}. \quad (7.7)$$

Thus, the coupling and plasma parameters are inversely proportional to each other. $\Lambda_\alpha < 1$ implies strongly-coupled plasmas, and $\Lambda_\alpha > 1$ weakly-coupled plasmas. Since Λ_α represents the number of particles per Debye sphere, it is interesting to see that a large number of particles within such a sphere is needed to be in the weakly-coupled-plasma regime. However, this does not correspond to a plasma with large density, in fact, it corresponds to the opposite. The explicit n_α term in the definition $\Lambda_\alpha = (4/3)\pi\lambda_{D\alpha}^3 n_\alpha$ is dominated by the n_α in the denominator of $\lambda_{D\alpha}$. In other words, low plasma densities lead to large Debye spheres, which in turn leads to many particles per Debye sphere, and hence a weakly-coupled plasma.

7.5 Electron degeneracy

- DeBroglie wavelength

$$\lambda_{B\alpha} = \frac{h}{\sqrt{2\pi m_\alpha v_{T_\alpha}}} \quad (7.8)$$

- Quantum plasma parameter

$$\chi_\alpha = \frac{4}{3}\pi\lambda_{B\alpha}^3 n_\alpha \quad (7.9)$$

- Fermi energy:

$$E_{f\alpha} = \frac{\hbar^2}{2m_\alpha} (3\pi^2 n_\alpha)^{2/3} \quad (7.10)$$

- The Fermi energy can be used to define the Fermi temperature $T_{f\alpha}$, Fermi velocity $v_{f\alpha}$, Fermi momentum $p_{f\alpha}$, and Fermi wavevector $k_{f\alpha}$

$$E_{f\alpha} = k_B T_{f\alpha} = \frac{1}{2}m_\alpha v_{f\alpha}^2 = \frac{p_{f\alpha}^2}{2m_\alpha} = \frac{(\hbar k_{f\alpha})^2}{2m_\alpha} \quad (7.11)$$

- Degeneracy parameter:

$$\Theta_\alpha = \frac{k_B T_\alpha}{E_{f\alpha}} = \left(\frac{2^{10}\pi}{3^4} \right)^{1/3} \chi_\alpha^{-2/3} \quad (7.12)$$

Chapter 8

Magnetohydrodynamics

8.1 Two-fluid equations

The starting point are the multi-fluid conservation laws and the Maxwell equations. We assume there are two species: electrons and ions. Additionally, we assume no sources. Thus, the starting governing equations are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (8.1)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (8.2)$$

$$\frac{\partial m_i n_i \mathbf{u}_i}{\partial t} + \nabla \cdot (m_i n_i \mathbf{u}_i \mathbf{u}_i) - Z e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = -\nabla p_i + \nabla \cdot \mathbf{t}_i + \mathbf{R}_i, \quad (8.3)$$

$$\frac{\partial m_e n_e \mathbf{u}_e}{\partial t} + \nabla \cdot (m_e n_e \mathbf{u}_e \mathbf{u}_e) + e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e, \quad (8.4)$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_i \right) + \nabla \cdot \left(\frac{3}{2} p_i \mathbf{u}_i \right) = -p_i \nabla \cdot \mathbf{u}_i + \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \quad (8.5)$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_e \right) + \nabla \cdot \left(\frac{3}{2} p_e \mathbf{u}_e \right) = -p_e \nabla \cdot \mathbf{u}_e + \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \quad (8.6)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \quad (8.7)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.8)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (8.9)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (8.10)$$

$$\mathbf{J} = e(Z n_i \mathbf{u}_i - n_e \mathbf{u}_e) \quad (8.11)$$

$$\rho_e = e(Z n_i - n_e) \quad (8.12)$$

These equations correspond to eq. (2.22) in Freidberg's Ideal MHD book, but for ions that are not singly charged. For the sections below, however, we'll assume singly-charged ions.

8.2 Low-frequency, long-wavelength, asymptotic expansions

Two assumptions:

1. Transform full Maxwell's equations to low-frequency pre-Maxwell's equations. Formally achieved with $\epsilon_0 \rightarrow 0$. This has two consequences:
 - $\epsilon_0 \partial \mathbf{E} / \partial t \rightarrow 0$
For this to be achieved it is required that $w/k \ll c$ and $V_{Ti}, V_{Te} \ll c$.
 - $\epsilon_0 \nabla \cdot \mathbf{E} \rightarrow 0$
For this to be achieved it is required that $w \ll w_{pe}$ and $a \gg \lambda_D$.
2. Neglect electron inertia in the electron momentum equations. Formally achieved with $m_e \rightarrow 0$.

Due to the first assumption, the Maxwell equations eqs. (8.7) to (8.12) are now written as

$$n_i - n_e = 0 \quad (8.13)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.14)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (8.15)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (8.16)$$

8.3 Single-fluid equations

We define single-fluid variables as

$$\rho = m_i n_i + m_e n_e = m_i n \quad (8.17)$$

$$\mathbf{v} = \frac{m_i n_i \mathbf{u}_i + m_e n_e \mathbf{u}_e}{m_i n_i + m_e n_e} = \mathbf{u}_i \quad (8.18)$$

$$p = p_i + p_e = n(T_i + T_e) \quad (8.19)$$

$$T = \frac{T_i + T_e}{2}. \quad (8.20)$$

The two conservation of mass equations eqs. (8.1) and (8.2) will lead to two single-fluid equations. The first is obtained by multiplying eq. (8.1) by m_i , and the second is obtained by multiplying the ion and electron mass equations by e and then subtracting. The results are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (8.21)$$

$$\nabla \cdot \mathbf{J} = 0. \quad (8.22)$$

Note that the second equation above is superfluous since it also follows from taking the divergence of eq. (8.16)

The two conservation of momentum equations will also lead to two single-fluid equations. The first is obtained by adding the ion and electron conservation of momentum equations to obtain

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u} \right) - \mathbf{J} \times \mathbf{B} + \nabla p = \nabla \cdot \mathbf{t}_i + \nabla \cdot \mathbf{t}_e. \quad (8.23)$$

For the second equation, we use $m_e \rightarrow 0$ and quasineutrality in the electron momentum equation to obtain

$$en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = -\nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e, \quad (8.24)$$

Assuming quasi-neutrality, the definition of the current given in eq. (8.11) is now

$$\mathbf{J} = en(\mathbf{u}_i - \mathbf{u}_e), \quad (8.25)$$

which is also written as

$$\mathbf{J} = en(\mathbf{v} - \mathbf{u}_e). \quad (8.26)$$

Using the above in the electron continuity equation gives

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en}(\mathbf{J} \times \mathbf{B} - \nabla p_e + \nabla \cdot \mathbf{t}_e + \mathbf{R}_e). \quad (8.27)$$

The two conservation of energy equations will also lead to two single-fluid equations. Each is evaluated using the single-fluid variables. As part of this derivation, we first rewrite the ion and electron internal energy eqs. (8.5) and (8.6) as

$$\frac{1}{\gamma - 1} \left(\frac{\partial p_i}{\partial t} + \mathbf{u}_i \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{u}_i \right) = \mathbf{t}_i : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{q}_i + Q_i, \quad (8.28)$$

$$\frac{1}{\gamma - 1} \left(\frac{\partial p_e}{\partial t} + \mathbf{u}_e \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u}_e \right) = \mathbf{t}_e : \nabla \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_e, \quad (8.29)$$

where we have used $\gamma = 5/3$ (the ratio of specific heats for monoatomic systems). We then note that

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho} \frac{\partial \rho}{\partial t} - \frac{1}{\rho} \nabla \rho \cdot \mathbf{v} = -\frac{\partial \ln \rho}{\partial t} - \nabla \ln \rho \cdot \mathbf{v}, \quad (8.30)$$

and thus

$$\gamma \nabla \cdot \mathbf{v} = -\frac{1}{\rho^\gamma} \frac{\partial \rho^\gamma}{\partial t} - \frac{1}{\rho^\gamma} \nabla \rho^\gamma \cdot \mathbf{v}. \quad (8.31)$$

The result above allows us to write

$$\begin{aligned} \frac{\partial p_\alpha}{\partial t} + \mathbf{v} \cdot \nabla p_\alpha + \gamma p_\alpha \nabla \cdot \mathbf{v} &= \frac{\partial p_\alpha}{\partial t} - p_\alpha \frac{1}{\rho^\gamma} \frac{\partial \rho^\gamma}{\partial t} + \mathbf{v} \cdot \nabla p_\alpha - p_\alpha \frac{1}{\rho^\gamma} \nabla \rho^\gamma \cdot \mathbf{v} \\ &= \rho^\gamma \left[\frac{\partial}{\partial t} \left(\frac{p_\alpha}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p_\alpha}{\rho^\gamma} \right) \right]. \end{aligned} \quad (8.32)$$

Thus, the ion energy equation becomes

$$\frac{\partial}{\partial t} \left(\frac{p_i}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p_i}{\rho^\gamma} \right) = \frac{\gamma - 1}{\rho^\gamma} (\mathbf{t}_i : \nabla \mathbf{v} - \nabla \cdot \mathbf{q}_i + Q_i), \quad (8.33)$$

and the electron energy equation becomes

$$\frac{\partial}{\partial t} \left(\frac{p_e}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p_e}{\rho^\gamma} \right) = \frac{\gamma - 1}{\rho^\gamma} \left[\mathbf{t}_e : \nabla \left(\mathbf{v} - \frac{\mathbf{J}}{en} \right) - \nabla \cdot \mathbf{q}_e + Q_e \right] + \frac{1}{en} \mathbf{J} \cdot \nabla \left(\frac{p_e}{\rho^\gamma} \right). \quad (8.34)$$

8.4 Resistive MHD

The electron collision term is modeled as

$$\mathbf{R}_e = m_e n_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e), \quad (8.35)$$

where ν_{ei} is the momentum exchange collision frequency. Assuming quasi-neutrality, the expression for current in eq. (8.25) can be used to obtain

$$\mathbf{R}_e = \frac{m_e \nu_{ei}}{e} \mathbf{J}. \quad (8.36)$$

Neglecting all terms on the right-hand side of eq. (8.27) except for the electron collision term, we have

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en} \mathbf{R}_e. \quad (8.37)$$

Using eq. (8.36) in the above, we have

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{m_e \nu_{ei}}{e^2 n} \mathbf{J}, \quad (8.38)$$

which we re-write as

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}, \quad (8.39)$$

where

$$\eta = \frac{m_e \nu_{ei}}{e^2 n} \quad (8.40)$$

is the resistivity.

8.5 Ideal MHD

The ideal MHD equations are obtained by neglecting the right-hand sides of eqs. (8.23), (8.27), (8.33) and (8.34). Summing the two pressure equations, the resulting equations would be

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (8.41)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \quad (8.42)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \quad (8.43)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = 0, \quad (8.44)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.45)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (8.46)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (8.47)$$

Given the vector identity

$$\frac{1}{2} \nabla (B^2) = \mathbf{B} \times (\nabla \times \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (8.48)$$

we can use Ampere's law to re-write the $\mathbf{J} \times \mathbf{B}$ term in the velocity equation as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} \left[(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (B^2) \right]. \quad (8.49)$$

Similarly, given the vector identity

$$\nabla \times (\mathbf{B} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{v} + \mathbf{B} (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \mathbf{B}), \quad (8.50)$$

we can use Ohm's law to re-write the $\nabla \times \mathbf{E}$ term in Faraday's law as

$$\nabla \times \mathbf{E} = \nabla \times (-\mathbf{v} \times \mathbf{B}) = (\mathbf{v} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{v} + \mathbf{B} (\nabla \cdot \mathbf{v}). \quad (8.51)$$

Thus, the ideal MHD equations can be summarized as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (8.52)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.53)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \frac{1}{\mu_0} \left[(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (B^2) \right] \quad (8.54)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} - \mathbf{B} (\nabla \cdot \mathbf{v}) \quad (8.55)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = 0, \quad (8.56)$$

If we assume incompressibility, then the above simplifies to

$$\nabla \cdot \mathbf{v} = 0, \quad (8.57)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (8.58)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \frac{1}{\mu_0} \left[(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (B^2) \right] \quad (8.59)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} \quad (8.60)$$

Chapter 9

Transport and coupling coefficients

9.1 Plasma diffusion

9.2 Plasma viscosity

9.3 Plasma heat conduction

9.4 Momentum exchange collision frequency ν_{ei}

9.5 Electron-ion thermal coupling w_{ei}

Appendix A

Lagrangian and Eulerian PDFs

A.1 Eulerian PDF

Consider an Eulerian velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. The Eulerian PDF $f = f(\mathbf{V}; \mathbf{x}, t)$ gives the probability that the velocity field will have a value of \mathbf{V} at location \mathbf{x} and at time t . We'll also introduce the fine-grained Eulerian PDF $f' = f'(\mathbf{V}; \mathbf{x}, t)$, which is defined as

$$f'(\mathbf{V}; \mathbf{x}, t) = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}). \quad (\text{A.1})$$

Note: a delta function of a 3D argument means the following $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$. The Eulerian PDF can be obtained from the fine-grained Eulerian PDF using

$$f(\mathbf{V}; \mathbf{x}, t) = \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle. \quad (\text{A.2})$$

The proof is as follows,

$$\begin{aligned} \langle f'(\mathbf{V}; \mathbf{x}, t) \rangle &= \langle \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) f(\mathbf{V}'; \mathbf{x}, t) d\mathbf{V}' \\ &= f(\mathbf{V}; \mathbf{x}, t). \end{aligned} \quad (\text{A.3})$$

A.2 Lagrangian PDF

Consider a Lagrangian particle with velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$ and position $\mathbf{x}^+(t, \mathbf{y})$. The Lagrangian PDF $f_L = f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$ gives the probability that the particle that started at location \mathbf{y} at the reference time t_0 will have a velocity \mathbf{V} and position \mathbf{x} at time t . We'll also introduce the fine-grained Eulerian PDF $f'_L = f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$, which is defined as

$$f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}). \quad (\text{A.4})$$

Note: a delta function of a 3D argument means the following $\delta(\mathbf{a}) = \delta(a_1)\delta(a_2)\delta(a_3)$. The Lagrangian PDF can be obtained from the fine-grained Lagrangian PDF using

$$f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) = \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle. \quad (\text{A.5})$$

The proof is as follows,

$$\begin{aligned} \langle f'_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) \rangle &= \langle \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) \rangle \\ &= \int \delta(\mathbf{V}' - \mathbf{V}) \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{V}', \mathbf{x}'; t|\mathbf{y}) d\mathbf{V}' d\mathbf{x}' \\ &= f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}). \end{aligned} \quad (\text{A.6})$$

A.3 Relation between Lagrangian and Eulerian PDFs

As a quick side note, we mention that the inverse of \mathbf{x}^+ is $\mathbf{y}^+ = \mathbf{y}^+(t, \mathbf{z})$, which gives the initial location of a fluid particle that at time t is located at position \mathbf{z} . Thus, $\mathbf{x}^+(t, \mathbf{y}^+(t, \mathbf{z})) = \mathbf{z}$.

We begin as follows

$$\begin{aligned} \int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} &= \int \delta(\mathbf{u}^+(t, \mathbf{y}) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y} \\ &= \int \delta(\mathbf{u}(\mathbf{x}^+(t, \mathbf{y}), t) - \mathbf{V}) \delta(\mathbf{x}^+(t, \mathbf{y}) - \mathbf{x}) |\det D\mathbf{x}^+| d\mathbf{y}, \end{aligned} \quad (\text{A.7})$$

where we have introduced $|\det D\mathbf{x}^+|$, which is the absolute value of the determinant of the Jacobean $\partial\mathbf{x}^+/\partial\mathbf{y}$, and is equal to one for incompressible flows. Using integration by substitution we obtain

$$\int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = \int \delta(\mathbf{u}(\mathbf{z}, t) - \mathbf{V}) \delta(\mathbf{z} - \mathbf{x}) d\mathbf{z} = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}) \quad (\text{A.8})$$

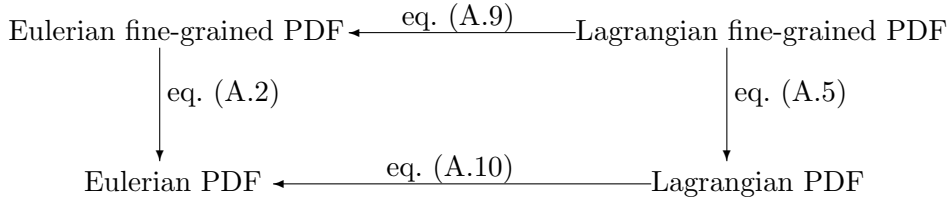
Given the definition of $f'(\mathbf{V}; \mathbf{x}, t)$, we have

$$\int f'_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = f'(\mathbf{V}; \mathbf{x}, t). \quad (\text{A.9})$$

Taking the expectation of the above we obtain

$$\int f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) d\mathbf{y} = f(\mathbf{V}; \mathbf{x}, t). \quad (\text{A.10})$$

A summary of all of the relations derived thus far is given by the following graph



A.4 Evolution equation for fine-grained Eulerian PDF

A.5 Evolution equation for fine-grained Lagrangian PDF

Appendix B

Electromagnetism

This chapter first focuses on electrostatics and magnetostatics, which can be understood as follows

$$\begin{aligned} \text{stationary charges} &\rightarrow \text{constant electric fields} = \text{electrostatics} \\ \text{stationary currents} &\rightarrow \text{constant magnetic fields} = \text{magnetostatics}. \end{aligned} \quad (\text{B.1})$$

B.1 Electrostatics

- Coulomb's Law

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}} \quad (\text{B.2})$$

- Electric Field \mathbf{E} derived from $\mathbf{F} = Q\mathbf{E}$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \quad (\text{B.3})$$

- If there are multiple point charges

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \quad (\text{B.4})$$

- **Charge distributions and fields:** if the charges are so small and so numerous that they can be described using a continuous distribution (i.e. $q_i \rightarrow dq = \rho d\tau$, where ρ is a charge density and $d\tau$ an infinitesimal volume)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau' \quad (\text{B.5})$$

If the charge distribution is localized to a surface or a line, then the analogous of the above is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r^2} \hat{\mathbf{r}} da' \quad \text{or} \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{r^2} \hat{\mathbf{r}} dl' \quad (\text{B.6})$$

Taking the divergence and curl of eq. (B.5):

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad (\text{B.7})$$

$$\nabla \times \mathbf{E} = 0 \quad (\text{B.8})$$

- **Fields and potentials**

Since $\nabla \times \mathbf{E} = 0$ we have

$$\mathbf{E} = -\nabla V. \quad (\text{B.9})$$

where V is the electric potential. Fundamental theorem of calculus can be used to express the potential $V(\mathbf{r})$ as

$$V(\mathbf{r}) - V(\mathcal{O}) = - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \quad (\text{B.10})$$

where \mathcal{O} is the reference point, at which one usually defines $V(\mathcal{O}) = 0$ (e.g. sea-level as the altitude at which height is equal to zero).

- **Charge distributions and potentials**

Divergence of eq. (B.9) gives

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho \quad (\text{B.11})$$

whose solution is

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'. \quad (\text{B.12})$$

- Define potential energy U as the negative of the work required to move charge Q from \mathbf{a} to \mathbf{b} .

$$U = - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})] \quad (\text{B.13})$$

If the reference point is infinity, then $U(\mathbf{r}) = QV(\mathbf{r})$.

- Potential energy of a set of charges q_i

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \sum_{j=i+1}^n \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^n q_i \left(\sum_{j=1, j \neq i}^n \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i) \quad (\text{B.14})$$

where $V(\mathbf{r}_i)$ is the potential due to all charges except the one at \mathbf{r}_i . The continuous form is

$$U = \frac{1}{2} \int \rho V d\tau = \frac{\epsilon_0}{2} \int E^2 d\tau \quad (\text{B.15})$$

where now V represents the potential due to all charges. Thus, if ρ is such that it defines a set of point charges (e.g. $\delta(\mathbf{r})$), then eq. (B.15) would be equal to eq. (B.14) plus the additional terms corresponding to $i = j$. Those additional terms correspond to the energy required to create point charges, which is infinity.

- **Electrostatic conductors:** materials whose charges are free to move but are in a state of electrostatic equilibrium. $\mathbf{E} = 0$ inside, since if it were not, then charges would move and the material would not be in electrostatic equilibrium. As a consequence, $\rho = 0$ inside, all the charge is on the surface, and \mathbf{E} is perpendicular to the outer surface.
- If there is a cavity within the conductor, and within the cavity a charge q , an amount $-q$ of charge will reside in the inner surface, and an amount q on the outer surface, and that configuration will lead to $\mathbf{E} = 0$ inside the conductor.
- **Faraday cage:** if there are no charges within such cavity, then $\mathbf{E} = 0$ within the cavity as well, regardless of how many charges are outside the conductor. If \mathbf{E} was not zero inside the cavity, then its field lines would start and end on the cavity walls. Letting the field lines be part of a closed loop, the rest of which is inside the conductor, then the line integral along the closed loop would be positive, in violation of $\nabla \times \mathbf{E} = 0$.

- A capacitor consists of two conductors, one with charge Q and the other with charge $-Q$. The constant of proportionality between Q and the voltage difference between the two conductors is the capacitance $C = Q/V$. The energy stored in a capacitor is $W = \frac{1}{2}CV^2$.

B.2 Magnetostatics

- Lorentz force law: $\mathbf{F} = Q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$
- Given the charge densities λ , σ , and ρ
 - Current [Amperes]: the amount of charge that passes a point in a small amount of time.

$$\mathbf{I} = \lambda \mathbf{v} \quad (\text{B.16})$$

- Surface current density: the amount of charge that passes a line in a small amount of time.

$$\mathbf{K} = \sigma \mathbf{v} \quad (\text{B.17})$$

- Volume current density: the amount of charge that passes an area in a small amount of time.

$$\mathbf{J} = \rho \mathbf{v} \quad (\text{B.18})$$

- Magnetic component of Lorentz force

$$\mathbf{F}_{\text{mag}} = \int \mathbf{I} \times \mathbf{B} dl = \int \mathbf{K} \times \mathbf{B} da = \int \mathbf{J} \times \mathbf{B} d\tau \quad (\text{B.19})$$

- Conservation of current

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (\text{B.20})$$

- Charge currents and fields

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} dl' \quad (\text{B.21})$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} da' \quad (\text{B.22})$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau' \quad (\text{B.23})$$

Taking the divergence and curl of eq. (B.23):

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{B.24})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (\text{B.25})$$

- A steady straight-line current leads to a circular magnetic field around it. A steady circular current leads to a straight magnetic field line along the axis of the circle.

- **Fields and potentials**

Since $\nabla \cdot \mathbf{B} = 0$ we have

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{B.26})$$

where \mathbf{A} is the magnetic vector potential.

- **Charge currents and potentials**

The magnetic field is not altered if a function whose curl vanishes (that is $\nabla\lambda$) is added to \mathbf{A} . Thus, λ can be picked to make \mathbf{A} divergence-less. Taking the curl of \mathbf{B} then leads to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad (\text{B.27})$$

whose solution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau'. \quad (\text{B.28})$$

B.3 Electric Fields in Matter

B.3.1 Dipole moments

- The dipole moment of a continuous distribution of charge is define as

$$\mathbf{p}(\mathbf{r}) = \int_V \rho(\mathbf{r}')(\mathbf{r}' - \mathbf{r}) d\tau. \quad (\text{B.29})$$

- For a discrete, rather than continuous, distribution of charges, the charge density is given by $\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i)$, and thus the dipole moment becomes

$$\mathbf{p}(\mathbf{r}) = \int_V \sum_i q_i \delta(\mathbf{r}' - \mathbf{r}_i)(\mathbf{r}' - \mathbf{r}) d\tau = \sum_i q_i(\mathbf{r}_i - \mathbf{r}). \quad (\text{B.30})$$

- If there are only two opposite charges, then the dipole moment simplifies to

$$\mathbf{p}(\mathbf{r}) = q(\mathbf{r}_1 - \mathbf{r}) - q(\mathbf{r}_2 - \mathbf{r}) = q(\mathbf{r}_1 - \mathbf{r}_2) = q\mathbf{d}. \quad (\text{B.31})$$

where \mathbf{d} points from the negative to the positive charge.

- When an electric field is applied to an atom, its nucleus shifts slightly in the direction of the field, and the center of the surrounding sphere of electrons shift slightly against the field. A tiny dipole moment is induced. This dipole moment is often proportional to the electric field, i.e.

$$\mathbf{p} = \alpha \mathbf{E}, \quad (\text{B.32})$$

where α is the atomic polarizability.

- For molecules, the induced dipole moment might not necessarily be aligned with the electric field. Additionally, you get different dipole moments depending on which direction the electric field comes from. A general expression of the dipole moment is then

$$\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z, \end{aligned} \quad (\text{B.33})$$

where the polarizability is now a tensor.

- Polar molecules: molecules have an inherent dipole moment, i.e. there is no need to apply an electric field to induce the dipole moment. Applying an electric field to those molecules would create a torque so that their inherent dipole moment becomes aligned with the electric field.

- Polarized matter: matter whose constituents have a dipole moment aligned with the field (either because the dipole moments were induced or because the inherent dipole moments were rotated to be aligned with the field).
- Polarization \mathbf{P} : dipole moment per unit volume.

B.3.2 Electric displacement

- What is the electric field caused by all of the tiny little dipoles in polarized matter? The field of such a polarized object is identical to the field that would be produced by a distribution of bound charges with volume charge density ρ_b and surface charge density σ_b . These two are given by

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad (\text{B.34})$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}. \quad (\text{B.35})$$

- There are now two charge distributions, one characterized by ρ_b , which we refer to as bound charges and which follows from the dipole moments in a material, and another that consists of any extra charges that may be present and that are not bound to the material. The latter charge distribution is labeled as ρ_f .
- Combining the contributions of both, Gauss's law reads

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0}(\rho_b + \rho_f) = \frac{1}{\epsilon_0}(-\nabla \cdot \mathbf{P} + \rho_f) \quad (\text{B.36})$$

- Defining the electric displacement \mathbf{D} as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (\text{B.37})$$

allows us to write Gauss's Law as

$$\nabla \cdot \mathbf{D} = \rho_f. \quad (\text{B.38})$$

B.3.3 Linear dielectric

- A linear dielectric is one whose polarization \mathbf{P} is directly proportional to \mathbf{E} , that is,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \quad (\text{B.39})$$

In the above, χ_e is the constant of proportionality and is referred to as the electric susceptibility.

- The electric displacement for a linear dielectric then takes the form

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}. \quad (\text{B.40})$$

- Relative permittivity or dielectric constant: $\epsilon_r = 1 + \chi_e$.
- Permittivity: $\epsilon = \epsilon_0 (1 + \chi_e)$.
- The energy stored in a volume that includes a linear dielectric is given by

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau. \quad (\text{B.41})$$

B.4 Magnetic Fields in Matter

B.5 Electrodynamics

B.5.1 Ohm's Law

- Ohm's law refers to the proportionality between the force per unit charge applied to charged elements and the resulting volume current that occurs. That is,

$$\mathbf{J} = \sigma \mathbf{f}, \quad (\text{B.42})$$

where \mathbf{f} is the force per unit charge, and the proportionality σ is the conductivity. If one neglects the magnetic contribution to \mathbf{f} , which is typically done for non-plasmas, then

$$\mathbf{J} = \sigma \mathbf{E}. \quad (\text{B.43})$$

For steady currents ($\partial\rho/\partial t = 0$) and uniform conductivity

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = 0 \quad (\text{B.44})$$

and thus, the charge density is zero. This is similar to a conductor, but now we have charges moving.

- Similarly, given an applied voltage, a current will result. The constant of proportionality R , known as the resistance, is given by

$$V = IR. \quad (\text{B.45})$$

B.5.2 Electromagnetic induction

- Defined the electromotive force (emf) as

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \quad (\text{B.46})$$

- The universal flux rule states: whenever the magnetic flux through a loop

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \quad (\text{B.47})$$

changes, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad (\text{B.48})$$

will appear in the loop. This can occur in two ways:

1. Magnetic field doesn't change, loop changes:
For example, a loop of wire is pulled to the right through a constant magnetic field. In this case the emf is magnetic.
2. Magnetic field changes, loop doesn't change:
There is a stationary loop (any loop, not necessarily a physical loop of wire), and the magnetic field through it changes. In this case, the **changing magnetic field induces an electric field** and thus the emf is electric. Using eq. (B.48) we get **Faraday's law**

$$\oint \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}, \quad (\text{B.49})$$

which, in differential form is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.50})$$

- Lenz's law: Nature abhors a change in flux. Thus, as the magnetic flux changes and it induces an electric field over a loop, the resulting current goes in a direction such that it would create an opposing flux that tries to cancel the original change in magnetic flux.

- Mutual inductance:

If there is a steady current going through a wire loop, this will create a magnetic field and thus a magnetic flux through another wire loop close by. The constant of proportionality between the flux through the second loop and the current in the first is the mutual inductance M . That is

$$\Phi_2 = M_{21}I \quad (\text{B.51})$$

Note: If I ran the same current on loop two, then the flux in loop one would be $\Phi_1 = M_{12}I$. However, it can be shown that $M_{21} = M_{12}$ and thus $\Phi_1 = \Phi_2$.

Now, imagine the current in loop one changes in time. The magnetic field associated with that current changes in time, and thus the magnetic flux through loop two changes as well. That is,

$$\Phi_2(t) = MI_1(t). \quad (\text{B.52})$$

Due to Faraday's law an induced emf would be created in the second loop,

$$\mathcal{E}_2(t) = -M \frac{dI_1(t)}{dt}. \quad (\text{B.53})$$

This emf creates a current $I_2(t)$ in the second loop.

- Self inductance:

The changing magnetic field associated with the changing current in loop one also creates a changing flux within this loop. This is given by

$$\Phi_1(t) = LI_1(t), \quad (\text{B.54})$$

where L is the self-inductance. Again, the changing flux leads to an emf within loop one, called the back emf

$$\mathcal{E}(t) = -L \frac{dI(t)}{dt}. \quad (\text{B.55})$$

This emf drives a new current in loop one that opposes the original current change.

- The energy stored in magnetic fields is given by

$$W = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{J} d\tau = \frac{1}{2\mu_0} \int B^2 d\tau. \quad (\text{B.56})$$

- Ampere's law eq. (B.25) was derived using assumptions of magnetostatics. Maxwell extended Ampere's law to work for magnetodynamics, so that the divergence of eq. (B.25) would actually give zero on both sides. Thus, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{B.57})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{B.58})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{B.59})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (\text{B.60})$$

- As shown earlier, Faraday's law indicates that a changing magnetic field induces an electric field. Maxwell's correction to Ampere's law then indicates that a changing electric field induces a magnetic field.

B.6 Conservation Laws

B.6.1 Conservation of energy

- Suppose you assemble a distribution of charges and currents, which at time t produce fields \mathbf{E} and \mathbf{B} .
- The potential energy of the system, as shown in previous sections, would be

$$U_{em} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right). \quad (\text{B.61})$$

- The question then arises, how much energy would be transferred to the charges as these charges are allowed to move?
- Label U_{mech} as the energy density gained by these charges as they are allowed to move.
- The evolution of U_{mech} is then

$$\frac{\partial U_{mech}}{\partial t} = -\frac{\partial U_{em}}{\partial t} - \nabla \cdot \mathbf{S}. \quad (\text{B.62})$$

In the above \mathbf{S} is the pointing vector and is defined as

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}). \quad (\text{B.63})$$

It represents the flux of energy across space.

- As the evolution equation above shows, a decreasing potential energy in the electromagnetic field constitutes a transfer of this lost energy to that of the charges.
- In integral form, we write the above as

$$\frac{d}{dt} \int_V U_{mech} d\tau = -\frac{d}{dt} \int_V U_{em} d\tau - \oint \mathbf{S} \cdot d\mathbf{a}. \quad (\text{B.64})$$

B.6.2 Conservation of momentum

- We now ask, how much momentum would be transferred to the charges as these are allowed to move?
- Label P_{mech} as the momentum density gained by the charges as they move around.
- Label P_{em} as the momentum density stored in the electromagnetic fields themselves. This is defined as

$$P_{em} = \mu_0 \epsilon_0 \mathbf{S}. \quad (\text{B.65})$$

- The evolution of P_{mech} is then

$$\frac{\partial P_{mech}}{\partial t} = -\frac{\partial P_{em}}{\partial t} + \nabla \cdot \mathbf{T}. \quad (\text{B.66})$$

In the above \mathbf{T} is the Maxwell stress tensor and is defined as

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right). \quad (\text{B.67})$$

It represents the flux of momentum across space.

- As the evolution equation above shows, a decreasing momentum in the electromagnetic field constitutes a transfer of this lost momentum to that of the charges.
- In integral form, we write the above as

$$\frac{d}{dt} \int_V P_{mech} d\tau = -\frac{d}{dt} \int_V P_{em} d\tau + \oint \mathbf{T} \cdot d\mathbf{a}. \quad (\text{B.68})$$

B.7 Electromagnetic waves

B.7.1 Simple waves

- The simplest kind of waves can be written as

$$u(x, t) = A \sin \left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t + \phi \right) \quad (\text{B.69})$$

where

A : magnitude
 λ : wavelength
 T : period
 ϕ : phase constant

Thus, as x goes from zero to λ , for example, an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in space. Similarly, as t goes from zero to T , an additional 2π value is added to the argument of the sin, and thus a whole wave is traversed in time.

- Defining the wavevector and angular frequency as

$$k = \frac{2\pi}{\lambda} \quad w = \frac{2\pi}{T}, \quad (\text{B.70})$$

then

$$u(x, t) = A \sin(kx - wt + \phi), \quad (\text{B.71})$$

- The frequency ν is the inverse of the period, $\nu = 1/T$.
- By inspecting the form of the simple sinusoidal wave above, it is clear that the velocity of the wave is

$$v = \frac{w}{k} = \frac{\lambda}{T} = \lambda\nu. \quad (\text{B.72})$$

- A general wave can be Fourier decomposed as follows

$$u(x, t) = \sum_n \hat{u}_n e^{i(k_n z - wt + \phi)}, \quad (\text{B.73})$$

where $k_n = 2\pi n/L$. The above is often re-written as

$$u(x, t) = \sum_n \tilde{u}_n e^{i(k_n z - wt)}, \quad (\text{B.74})$$

where $\tilde{u}_n = \hat{u}_n e^{i\phi}$.

- For the more general three-dimensional case, a wave is decomposed as follows

$$\mathbf{u}(x, t) = \sum_{\mathbf{n}} \tilde{\mathbf{u}}_{\mathbf{n}} e^{i(\mathbf{k}_{\mathbf{n}} \cdot \mathbf{x} - \omega t)}, \quad (\text{B.75})$$

where $\mathbf{k}_{\mathbf{n}} = 2\pi\mathbf{n}/L$ and $\mathbf{n} = [n_1, n_2, n_3]$.

- A plane wave is one for which the only existing $\mathbf{k}_{\mathbf{n}}$'s point along a single direction. Without loss of generality, we can assume this direction is the z direction and thus write

$$\mathbf{u}(x, t) = \sum_{n_3} \tilde{\mathbf{u}}_{n_3} e^{i(k_{n_3} \cdot z - \omega t)}, \quad (\text{B.76})$$

B.7.2 Electromagnetic waves in vacuum

- The application of eq. (B.75) to electric and magnetic fields gives

$$\mathbf{E} = \sum_{\mathbf{n}} \tilde{\mathbf{E}}_{\mathbf{n}} e^{i(\mathbf{k}_{\mathbf{n}} \cdot \mathbf{x} - \omega t)}, \quad (\text{B.77})$$

and

$$\mathbf{B} = \sum_{\mathbf{n}} \tilde{\mathbf{B}}_{\mathbf{n}} e^{i(\mathbf{k}_{\mathbf{n}} \cdot \mathbf{x} - \omega t)}. \quad (\text{B.78})$$

- For $\rho = \mathbf{J} = 0$, Maxwell's equations can be combined to give the wave equations for \mathbf{E} and \mathbf{B} , that is,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \nabla^2 \mathbf{E} = 0, \quad (\text{B.79})$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \nabla^2 \mathbf{B} = 0. \quad (\text{B.80})$$

The speed of electromagnetic waves is thus $c = 1/\sqrt{\epsilon_0 \mu_0}$.

- Using eq. (B.77) in $\nabla \cdot \mathbf{E} = 0$ gives $\mathbf{k}_{\mathbf{n}} \cdot \tilde{\mathbf{E}}_{\mathbf{n}} = 0$. That is, the \mathbf{E} field is orthogonal to the direction of propagation of the mode.
- Using eq. (B.78) in $\nabla \cdot \mathbf{B} = 0$ gives $\mathbf{k}_{\mathbf{n}} \cdot \tilde{\mathbf{B}}_{\mathbf{n}} = 0$. That is, the \mathbf{B} field is orthogonal to the direction of propagation of the mode.
- Using eqs. (B.77) and (B.78) in $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ gives $\mathbf{k}_{\mathbf{n}} \times \tilde{\mathbf{E}}_{\mathbf{n}} = \omega \tilde{\mathbf{B}}_{\mathbf{n}}$. That is, the \mathbf{B} field is orthogonal to the \mathbf{E} field.