

Linear Algebra

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March 20, 2025

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6.1	Basic Definitions

- Imagine a subspace W of \mathbb{R}^n . The set of all vectors \mathbf{z} that are orthogonal to all the vectors in W is called the **orthogonal complement** of W , and is denoted as W^\perp . (Note: this is an orthogonal complement, all complements need not be orthogonal).

- $(\text{Row } A)^\perp = \text{Null } A$ and $(\text{Col } A)^\perp = \text{Null } A^T$

6.2 Orthogonal Sets

- An **orthogonal set** is a set of vectors that are orthogonal with each other.
- Assume $0 = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \dots + c_p \mathbf{u}_p$. By multiplying by \mathbf{u}_i , we can show $c_i = 0$ for all i . Thus, the vectors in an orthogonal set are linearly independent, and therefore form an invertible matrix.
- Because they are linearly independent, they also form a basis for $\text{span}\{\mathbf{u}_i, \dots, \mathbf{u}_p\}$.
- Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for W . For each \mathbf{y} in W , we have: $\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p$, where

$$c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$$

- An orthogonal set composed of unit vectors is called an **orthonormal set**.
- If an $m \times n$ matrix U has orthonormal columns, then $U^T U = I$.
- If such matrix is square, then $U^{-1} = U^T$ (which is the definition of an **orthogonal matrix**).

$$U \text{ is invertible} \rightarrow (U^T U)U^{-1} = IU^{-1} \rightarrow U^T = U^{-1}$$

- For an $m \times n$ matrix with orthonormal columns:

$$\|U\mathbf{x}\| = \|\mathbf{x}\| \quad \& \quad (U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

6.3 Orthogonal Projections

- An **orthogonal projection** of \mathbf{y} onto W is the component $\hat{\mathbf{y}}$ such that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ and \mathbf{z} is orthogonal to W .
- The vector $\hat{\mathbf{y}}$ can be expressed in terms of any orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ of W as follows,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$$

- Let W be a subspace of \mathbb{R}^n , \mathbf{y} any vector in \mathbb{R}^n and $\hat{\mathbf{y}}$ the orthogonal projection of \mathbf{y} onto W . Then $\hat{\mathbf{y}}$ is the closest point in W to \mathbf{y} (or in other words, the best approximation to \mathbf{y}), because:

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad \forall \mathbf{v} \in W \text{ distinct from } \hat{\mathbf{y}}$$

- If the orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is orthonormal, then the matrix $U = [\mathbf{u}_1 \cdots \mathbf{u}_p]$ can be used to form the orthogonal projector $P = UU^T$, and

$$\hat{\mathbf{y}} = UU^T \mathbf{y}$$

7 Symmetric Matrices and Quadratic Forms