

Curvilinear coordinates

May 9, 2024

1 Basic definitions

1.1 Cartesian coordinates

Consider Euclidean space, which is represented by cartesian coordinates. We denote the coordinates as (x^1, x^2, x^3) and the basis unit vectors as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. The position vector is $\mathbf{x} = x^1\mathbf{x}_1 + x^2\mathbf{x}_2 + x^3\mathbf{x}_3$.

Notice that the components were denoted with upper scripts and the unit vectors with lower scripts. For cartesian coordinates it doesn't matter if we use upper or lower scripts, that is, $x^i = x_i$ and $\mathbf{x}_i = \mathbf{x}^i$. For other coordinate systems, as we'll see in the sections below, upper scripts and lower scripts have different meanings.

1.2 Curvilinear Coordinates

We will now define a new coordinate system, a curvilinear coordinate system, in relation to the standard cartesian coordinates. The coordinates of this curvilinear system are denoted as (u^1, u^2, u^3) . We introduce the following two sets of functions

$$\hat{u}^i = \hat{u}^i(x^1, x^2, x^3), \quad (1)$$

$$\hat{x}^i = \hat{x}^i(u^1, u^2, u^3), \quad (2)$$

such that they satisfy

$$\hat{x}^i(\hat{u}^1, \hat{u}^2, \hat{u}^3) = x^i \quad (3)$$

$$\hat{u}^i(\hat{x}^1, \hat{x}^2, \hat{x}^3) = u^i, \quad (4)$$

for $i = 1, 2, 3$. In other words, \hat{u}^i and \hat{x}^i are functions that map between coordinates in the cartesian and curvilinear systems.

We can now take the derivative of either eq. (3) or eq. (4). For example, the derivative d/du^j of eq. (4) gives

$$\left(\frac{\partial \hat{u}^i}{\partial x^k} \right)_{x^i = \hat{x}^i} \frac{\partial \hat{x}^k}{\partial u^j} = \delta_j^i. \quad (5)$$

We can evaluate the above at $u^i = \hat{u}^i$, so that

$$\frac{\partial \hat{u}^i}{\partial x^k} \left(\frac{\partial \hat{x}^k}{\partial u^j} \right)_{u^i = \hat{u}^i} = \delta_j^i. \quad (6)$$

We now define two basis vectors as follows

$$\mathbf{e}_i = \left(\frac{\partial \hat{x}^1}{\partial u^i} \right)_{u^i = \hat{u}^i} \mathbf{x}_1 + \left(\frac{\partial \hat{x}^2}{\partial u^i} \right)_{u^i = \hat{u}^i} \mathbf{x}_2 + \left(\frac{\partial \hat{x}^3}{\partial u^i} \right)_{u^i = \hat{u}^i} \mathbf{x}_3 \quad (7)$$

$$\mathbf{e}^i = \frac{\partial \hat{u}^i}{\partial x^1} \mathbf{x}^1 + \frac{\partial \hat{u}^i}{\partial x^2} \mathbf{x}^2 + \frac{\partial \hat{u}^i}{\partial x^3} \mathbf{x}^3 \quad (8)$$

The dot product of these two vectors is given by

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i. \quad (9)$$

The two coordinate bases are not necessarily constant, orthogonal, of unit length, or dimensionless.

At the end, one way to think about it is that for every point $[x^1, x^2, x^3]$ in the cartesian coordinate system, there is a corresponding coordinate given by $[\hat{u}^1, \hat{u}^2, \hat{u}^3]$, and that at every point of these new coordinates, there are two coordinate bases, given by eq. (7) and eq. (8). The latter basis is typically expresses as $\nabla \hat{u}^1, \nabla \hat{u}^2$, and $\nabla \hat{u}^3$.

1.3 Vectors

Since there are two coordinate bases, one can define two types of vectors at every point in the domain. One is a vector in terms of contravariant components v^i

$$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + v^3 \mathbf{e}_3, \quad (10)$$

and the other a vector in terms of covariant components v_i

$$\mathbf{v} = v_1 \mathbf{e}^1 + v_2 \mathbf{e}^2 + v_3 \mathbf{e}^3. \quad (11)$$

Note that, due to eq. (9), we have $v^i = \mathbf{v} \cdot \mathbf{e}^i$ and $v_i = \mathbf{v} \cdot \mathbf{e}_i$.

We now define the metric coefficients g_{ij} and g^{ij} as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \quad (12)$$

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j \quad (13)$$

$$(14)$$

Thus, the dot product of two vectors in the curvilinear reference frame simplifies to the following

$$\mathbf{v} \cdot \mathbf{w} = v^i w_i = v_i w^i = g_{ij} v^i w^j = g^{ij} v_i w_j. \quad (15)$$

The cross product can be computed as

$$\mathbf{v} \times \mathbf{w} = \epsilon_{ijk} \sqrt{g} v^i w^j \mathbf{e}^k \quad (16)$$

$$\mathbf{v} \times \mathbf{w} = \epsilon^{ijk} \frac{1}{\sqrt{g}} v_i w_j \mathbf{e}_k, \quad (17)$$

where $g = \det(g_{ij})$. One can also define $g^{-1} = \det(g^{ij})$.

2 Calculus

2.1 Integration

- Volume integrals: Define dV_u as an infinitesimal volume in curvilinear coordinates, Ω_u as a finite volume of integration in curvilinear coordinates, and f_u as a function whose input is in curvilinear coordinates, that is, $f_u = f_u(u^1, u^2, u^3)$. Then, the volume integral can be computed using

$$\int_{\Omega_u} f_u dV_u = \int_{\Omega_u} f_u |\det(J_{ij})| du^1 du^2 du^3, \quad (18)$$

where $J_{ij} = \partial \hat{x}^i / \partial u^j$ is the Jacobian. Given that Eulerian coordinates can be thought of as an instance of curvilinear coordinates, we have

$$\int_{\Omega_u} f_u dV_u = \int_{\Omega_x} f_x dV_x = \int_{\Omega_x} f_x dx^1 dx^2 dx^3 \quad (19)$$

One thing to note is that $f_x(x^1, x^2, x^3)$ and $f_u(u^1, u^2, u^3)$ are equal when evaluated at the same point in space. In other words, these functions satisfy

$$f_u(u^1, u^2, u^3) = f_x(\hat{x}^1, \hat{x}^2, \hat{x}^3). \quad (20)$$

Equating eqs. (18) and (19) allows us to write the standard rule for integration by substitution

$$\int_{\Omega_x} f_x(x^1, x^2, x^3) dx^1 dx^2 dx^3 = \int_{\Omega_u} f_x(\hat{x}^1, \hat{x}^2, \hat{x}^3) |\det(J_{ij})| du^1 du^2 du^3. \quad (21)$$

- Surface integrals: Define dS_u as an infinitesimal surface in curvilinear coordinates, Γ_u as a finite surface of integration in curvilinear coordinates that belongs to the $u^1 = \text{constant}$ surfaces, and f_u as a function whose input is defined using curvilinear coordinates. Then, a surface integral can be computed using

$$\int_{\Gamma_u} f_u dS_u = \int_{\Gamma_u} f_u J |\nabla \hat{u}^1| du^2 du^3. \quad (22)$$

Note that now we can write

$$\begin{aligned} \int_{\Omega_u} f_u dV_u &= \int_{u_l^1}^{u_u^1} \int_{\Gamma_u} f_u J du^1 du^2 du^3 = \\ &= \int_{u_l^1}^{u_u^1} \int_{\Gamma_u} f_u \frac{J |\nabla \hat{u}^1| du^2 du^3}{|\nabla \hat{u}^1|} du^1 = \int_{u_l^1}^{u_u^1} \int_{\Gamma_u} f_u \frac{dS_u}{|\nabla \hat{u}^1|} du^1. \end{aligned} \quad (23)$$

2.2 Differentiation

2.2.1 The grad operator

Consider the function $f_x = f_x(x^1, x^2, x^3)$ and the grad operator, which is

$$\nabla f_x = \frac{\partial f_x}{\partial x^1} \mathbf{x}_1 + \frac{\partial f_x}{\partial x^2} \mathbf{x}_2 + \frac{\partial f_x}{\partial x^3} \mathbf{x}_3. \quad (24)$$

We now introduce the function $f_u = f_u(u^1, u^2, u^3)$ and note that $f_x = f_u(\hat{u}^1, \hat{u}^2, \hat{u}^3)$. Thus

$$\frac{\partial f_x}{\partial x^1} = \left(\frac{\partial f_u}{\partial u^1} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^1}{\partial x^1} + \left(\frac{\partial f_u}{\partial u^2} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^2}{\partial x^1} + \left(\frac{\partial f_u}{\partial u^3} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^3}{\partial x^1}, \quad (25)$$

$$\frac{\partial f_x}{\partial x^2} = \left(\frac{\partial f_u}{\partial u^1} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^1}{\partial x^2} + \left(\frac{\partial f_u}{\partial u^2} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^2}{\partial x^2} + \left(\frac{\partial f_u}{\partial u^3} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^3}{\partial x^2}, \quad (26)$$

$$\frac{\partial f_x}{\partial x^3} = \left(\frac{\partial f_u}{\partial u^1} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^1}{\partial x^3} + \left(\frac{\partial f_u}{\partial u^2} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^2}{\partial x^3} + \left(\frac{\partial f_u}{\partial u^3} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \frac{\partial \hat{u}^3}{\partial x^3}. \quad (27)$$

Using the definition of \mathbf{e}^i , the grad operator can be written as

$$\nabla f_x = \left(\frac{\partial f_u}{\partial u^1} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \mathbf{e}^1 + \left(\frac{\partial f_u}{\partial u^2} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \mathbf{e}^2 + \left(\frac{\partial f_u}{\partial u^3} \right)_{\mathbf{u}=\hat{\mathbf{u}}} \mathbf{e}^3. \quad (28)$$

This shows the equivalence between the grad operator in Eulerian coordinates and curvilinear coordinates.

2.2.2 The divergence operator

2.2.3 The curl operator

The curl is given by

$$\nabla \times A = \epsilon^{ijk} \frac{1}{\sqrt{g}} \frac{\partial A_j}{\partial u^i} \mathbf{e}_k \quad (29)$$

3 Flux coordinates

Imagine that Euclidean space is permeated by a set of surfaces, which we call flux surfaces. Each of those surfaces is labeled with a different value of the variable ψ . We also note that the flux surfaces are not stationary, they can move around as time progresses.

We now introduce the function $\hat{\psi} = \hat{\psi}(t, x^1, x^2, x^3)$. This function is defined in such a way that for all values x^1, x^2, x^3 that are part of a given flux surface at a specific time t , then $\hat{\psi}$ will evaluate to the value of ψ corresponding to that flux surface. The velocity of the flux surfaces is given by $\mathbf{V}_\psi = \mathbf{V}_\psi(t, x^1, x^2, x^3)$. Thus, by definition

$$\frac{\partial \hat{\psi}}{\partial t} + \mathbf{V}_\psi \cdot \nabla \hat{\psi} = 0. \quad (30)$$

A flux coordinate is defined as one in which $\hat{u}^1 = \hat{\psi}$.

3.1 Flux-surface averaging

To begin, we define the following. $D(\psi, t)$ is the volume enclosed at time t by the flux surface labelled by ψ . The surface of $D(\psi, t)$ is labelled as $\partial D(\psi, t)$. Additionally, $\Delta(\psi, t) = D(\psi + \Delta\psi, t) - D(\psi, t)$.

The flux surface average of a function is given by

$$\langle f \rangle_\psi = \lim_{\Delta\psi \rightarrow 0} \frac{\int_{\Delta(\psi, t)} f dV}{\int_{\Delta(\psi, t)} dV}. \quad (31)$$

This can be re-written as shown below

$$\langle f \rangle_\psi = \lim_{\Delta\psi \rightarrow 0} \frac{\frac{1}{\Delta\psi} \int_{\Delta(\psi,t)} f dV}{\frac{1}{\Delta(\psi)} \int_{\Delta(\psi,t)} dV} = \lim_{\Delta\psi \rightarrow 0} \frac{\frac{1}{\Delta\psi} \left(\int_{D(\psi+\Delta\psi,t)} f dV - \int_{D(\psi,t)} f dV \right)}{\frac{1}{\Delta\psi} \left(\int_{D(\psi+\Delta\psi,t)} dV - \int_{D(\psi,t)} dV \right)} = \frac{\frac{\partial}{\partial\psi} \int_{D(\psi,t)} f dV}{\frac{\partial}{\partial\psi} \int_{D(\psi,t)} dV}. \quad (32)$$

Defining $V' = V'(\psi, t)$ as

$$V' = \frac{\partial}{\partial\psi} \int_{D(\psi,t)} dV. \quad (33)$$

the second expression for the flux surface average is written as

$$\langle f \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial\psi} \int_{D(\psi,t)} f dV. \quad (34)$$

A third expression for $\langle g \rangle_\psi$ follows from using eq. (23) for the above. Thus,

$$\langle f \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial\psi} \int_0^\psi \int_{\partial D(\psi',t)} f \frac{dS}{|\nabla \hat{\psi}|} d\psi' = \frac{1}{V'} \int_{\partial D(\psi,t)} f \frac{dS}{|\nabla \hat{\psi}|}. \quad (35)$$

3.1.1 Average of spatial derivatives

We use the second definition of the flux-surface average, given by eq. (34), and then the divergence theorem to obtain

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial\psi} \int_{D(\psi,t)} \nabla \cdot \mathbf{A} dV = \frac{1}{V'} \frac{\partial}{\partial\psi} \int_{\partial D(\psi,t)} \mathbf{A} \cdot \frac{\nabla \hat{\psi}}{|\nabla \hat{\psi}|} dS. \quad (36)$$

We now use the third definition eq. (35) to obtain

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial\psi} V' \langle \mathbf{A} \cdot \nabla \hat{\psi} \rangle_\psi. \quad (37)$$

3.1.2 Average of time derivatives

Using the Reynolds transport theorem we show

$$\begin{aligned} \frac{\partial}{\partial t} \int_{D(\psi,t)} f dV &= \int_{D(\psi,t)} \frac{\partial f}{\partial t} dV + \int_{\partial D(\psi,t)} f \mathbf{V}_\psi \cdot \frac{\nabla \hat{\psi}}{|\nabla \hat{\psi}|} dS \\ &= \int_{D(\psi,t)} \frac{\partial f}{\partial t} dV + V' \langle f \mathbf{V}_\psi \cdot \nabla \hat{\psi} \rangle_\psi. \end{aligned} \quad (38)$$

We now take the derivative of both sides by ψ and then divide by V' .

$$\frac{1}{V'} \frac{\partial}{\partial t} V' \langle f \rangle_\psi = \left\langle \frac{\partial f}{\partial t} \right\rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial\psi} V' \langle f \mathbf{V}_\psi \cdot \nabla \hat{\psi} \rangle_\psi. \quad (39)$$

Re-arranging and using eq. (30)

$$\left\langle \frac{\partial f}{\partial t} \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial t} V' \langle f \rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial\psi} V' \left\langle f \frac{\partial \hat{\psi}}{\partial t} \right\rangle_\psi. \quad (40)$$