Governing Equations

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Contents

Chapter 1

Particle description

Table 1.1: Various coordinates of classical mechanics.

Classical coordinates	$\mathbf{x}(t)$	$\mathbf{v}(t)$
Generalized coordinates	$\mathbf{q}$	$\dot{\mathbf{q}}$
Canonical coordinates	$\mathbf{q}$	$\mathbf{p}$
Time-dependent canonical coordinates	$\tilde{\mathbf{q}}(t)$	$\tilde{\mathbf{p}}(t)$

1.1 Lagrangian mechanics

- Define the position $\mathbf{x} = \mathbf{x}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ of a particle.
- Define the Lagrangian as $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the generalized position and generalized velocity, respectively.
- The equations of motion are obtained from the Euler-Lagrange equation, which is

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] = \left(\frac{\partial L}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}. \tag{1.1}$$

- For example, the Lagrangian of a particle in an electromagnetic field where $\phi = \phi(\mathbf{q}, t)$ is the electric potential and $\mathbf{A} = \mathbf{A}(\mathbf{q}, t)$ is the magnetic potential, is

$$L = \frac{1}{2}m\dot{q}_i\dot{q}_i + e\dot{q}_iA_i - e\phi. \tag{1.2}$$

The derivatives in the Euler-Lagrange equation are

$$\frac{\partial L}{\partial q_i} = e\dot{q}_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i} \tag{1.3}$$

$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i + eA_i \tag{1.4}$$

$$\begin{aligned}\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}}\right] &= \frac{d}{dt}\left[mv_i + eA_i(\mathbf{x}, t)\right] \\ &= m\frac{dv_i}{dt} + ev_j\left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}}. \end{aligned} \tag{1.5}$$

Thus, the Euler-Lagrange equation becomes

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.6}$$

In vector notation, this is written as

$$m\frac{d\mathbf{v}}{dt} = \left(-e\mathbf{v}\cdot\nabla_q\mathbf{A} - e\frac{\partial \mathbf{A}}{\partial t} + e\nabla_q(\mathbf{v}\cdot\mathbf{A}) - e\nabla_q\phi\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.7}$$

Using the vector identity (4) from Griffiths book, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}, \tag{1.8}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{q}, t)$.

1.2 Hamiltonian mechanics

- Define the Hamiltonian as $H = H(\mathbf{q}, \mathbf{p}, t)$, where $\mathbf{q}$ and $\mathbf{p}$ are the canonical position and momentum. For all purposes here, the canonical position is the same as the generalized position.
- The Hamiltonian is obtained from the Lagrangian using

$$H = \left(\dot{\mathbf{q}}\cdot\mathbf{p} - L\right)_{\dot{\mathbf{q}}=\mathbf{f}(\mathbf{q},\mathbf{p})}, \tag{1.9}$$

where the dependency of $\dot{\mathbf{q}}$ on $\mathbf{q}$ and $\mathbf{p}$ is obtained from evaluating

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \tag{1.10}$$

- For example, for a particle in an electromagnetic field we have

$$H = \left[\dot{q}_i p_i - \left(\frac{1}{2}m\dot{q}_i\dot{q}_i + e\dot{q}_i A_i - e\phi\right)\right]_{\dot{\mathbf{q}}=f(\mathbf{q},\mathbf{p})}. \tag{1.11}$$

Evaluating eq. (1.10) gives $p_i = m\dot{q}_i + eA_i$, which allows us to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$ and $\mathbf{p}$ as $\dot{q}_i = \frac{1}{m}(p_i - eA_i)$. Thus

$$\begin{aligned} H &= \frac{1}{m}(p_i - eA_i)p_i - \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) - e\frac{1}{m}(p_i - eA_i)A_i + e\phi \\ &= \frac{1}{2m}(p_i - eA_i)(p_i - eA_i) + e\phi. \end{aligned} \tag{1.12}$$

- We introduce the variables $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(t)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}(t)$, which are defined by

$$\tilde{\mathbf{q}} = \mathbf{x} \tag{1.13}$$

$$\tilde{\mathbf{p}} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)_{\mathbf{q}=\mathbf{x},\dot{\mathbf{q}}=\mathbf{v}} \tag{1.14}$$

- The equations of motion are obtained from

$$\frac{d\tilde{q}_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.15}$$

$$\frac{d\tilde{p}_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} \tag{1.16}$$

- For example, for a particle in an electromagnetic field we have

$$\tilde{p}_i = mv_i + eA_i(\mathbf{x}, t) \tag{1.17}$$

and thus

$$\frac{d\tilde{p}_i}{dt} = m\frac{dv_i}{dt} + ev_j \left(\frac{\partial A_i}{\partial q_j}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial A_i}{\partial t}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.18}$$

$$\begin{aligned}
\frac{\partial H}{\partial q_i} &= \frac{\partial}{\partial q_i}\left[\frac{1}{2m}(p_j - eA_j)(p_j - eA_j) + e\phi\right] \\
&= \frac{1}{m}(p_j - eA_j)\frac{\partial}{\partial q_i}(p_j - eA_j) + e\frac{\partial \phi}{\partial q_i} \\
&= -\frac{e}{m}(p_j - eA_j)\frac{\partial A_j}{\partial q_i} + e\frac{\partial \phi}{\partial q_i}
\end{aligned} \tag{1.19}$$

$$\begin{aligned}
\left(\frac{\partial H}{\partial q_i}\right)_{\mathbf{q}=\tilde{\mathbf{q}},\mathbf{p}=\tilde{\mathbf{p}}} &= -\frac{e}{m}\left[mv_j + eA_j(\mathbf{x}, t) - eA_j(\mathbf{x}, t)\right]\left(\frac{\partial A_j}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} + e\left(\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}} \\
&= \left(-ev_j\frac{\partial A_j}{\partial q_i} + e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}.
\end{aligned} \tag{1.20}$$

Equation (1.16) thus leads to

$$m\frac{dv_i}{dt} = \left(-ev_j\frac{\partial A_i}{\partial q_j} - e\frac{\partial A_i}{\partial t} + ev_j\frac{\partial A_j}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.21}$$

This is the same as eq. (1.6) and thus, as shown before, the above can be expressed as

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)_{\mathbf{q}=\mathbf{x}}. \tag{1.22}$$

Chapter 2

Kinetic description

We denote the distribution function for a species α as $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where $\mathbf{r}$ and $\mathbf{v}$ are the sample space variables for position and velocity. Note that the distribution function is appropriately normalized such that

$$\int f_\alpha \, d\mathbf{r} d\mathbf{v} = N_\alpha, \tag{2.1}$$

where N_α is the total number of particles corresponding to species α.

The dynamics of a plasma can be characterized by the Boltzmann evolution equation for the distribution along with Maxwell's equations

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = C_\alpha + S_\alpha \tag{2.2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.6}$$

$$\mathbf{J} = \sum_\alpha Z_\alpha e \int \mathbf{v} f_\alpha \, d\mathbf{v} \tag{2.7}$$

$$\rho_e = \sum_\alpha Z_\alpha e \int f_\alpha \, d\mathbf{v}. \tag{2.8}$$

In the above,

- m_α is the species mass
- e is the charge
- Z_α the charge number
- $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ the charge current
- $\rho_e = \rho_e(\mathbf{r}, t)$ the charge density
- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the electric field

- $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ the magnetic field.

The terms C_α and S_α represent collision and source terms.

If we express the collision term in the usual way, that is $C_\alpha = \sum_\beta C_{\alpha\beta}$, then we can make the following statements:

1. Conservation of particles:

$$\int C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall\alpha \qquad\qquad \int C_{\alpha\beta}\, d\mathbf{v} = 0 \quad \forall\alpha, \beta|\beta \neq \alpha. \tag{2.9}$$

2. Conservation of momentum:

$$\int m_\alpha \mathbf{v} C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall\alpha \qquad\qquad \sum_\alpha \sum_{\beta,\beta\neq\alpha} \int m_\alpha \mathbf{v} C_{\alpha\beta} d\mathbf{v} = 0. \tag{2.10}$$

3. Conservation of energy:

$$\int \frac{1}{2} m_\alpha v^2 C_{\alpha\alpha}\, d\mathbf{v} = 0 \quad \forall\alpha \qquad\qquad \sum_\alpha \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2} m_\alpha v^2 C_{\alpha\beta}\, d\mathbf{v} = 0. \tag{2.11}$$

Chapter 3

Fluid description

We now define the particle density $n_\alpha = n_\alpha(\mathbf{r}, t)$, the fluid velocity $\mathbf{u}_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$ and the fluid energy per unit mass $E_\alpha = E_\alpha(\mathbf{r}, t)$ as follows

$$n_\alpha = \int f_\alpha \, d\mathbf{v} \tag{3.1}$$

$$\mathbf{u}_\alpha = \frac{1}{n_\alpha} \int \mathbf{v} f_\alpha \, d\mathbf{v} \tag{3.2}$$

$$E_\alpha = \frac{1}{n_\alpha} \int \frac{1}{2} v^2 f_\alpha \, d\mathbf{v}. \tag{3.3}$$

Their evolution equations are obtained by taking the appropriate moments of the Boltzmann plasma equation. Before doing so, we re-write the Boltzmann equation as

$$\frac{\partial f_\alpha}{\partial t} + \nabla \cdot (\mathbf{v} f_\alpha) + \nabla_v \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] = C_\alpha + S_\alpha \tag{3.4}$$

3.1 Mass

Integrating eq. (3.4) over all $\mathbf{v}$ we obtain

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha \tag{3.5}$$

where

$$\hat{S}_\alpha = \int S_\alpha \, d\mathbf{v} \tag{3.6}$$

is an external source of mass.

3.2 Momentum

Multiplying eq. (3.4) by $\mathbf{v}$ and then integrating over all $\mathbf{v}$ leads to

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v} \right) + \int \nabla_v \cdot \left[\mathbf{v} \frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] - \nabla_v \mathbf{v} \cdot \left[\frac{Z_\alpha e}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \right] d\mathbf{v} = \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} \, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \tag{3.7}$$

We note that the third term in eq. (3.7) is zero since we are integrating over all space, and that $\nabla_v \mathbf{v}$ is the identity matrix. We thus have

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v} \right) - \frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} \, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (3.8)$$

To proceed, we decompose $\mathbf{v}$ into a mean and a fluctuation, that is, $\mathbf{v} = \mathbf{u}_\alpha + \mathbf{w}_\alpha$. Using this decomposition

$$\int \mathbf{v}\mathbf{v} f_\alpha \, d\mathbf{v} = \int (\mathbf{u}_\alpha \mathbf{u}_\alpha + 2\mathbf{u}_\alpha \mathbf{w}_\alpha + \mathbf{w}_\alpha \mathbf{w}_\alpha) f_\alpha \, d\mathbf{v} = n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha \, d\mathbf{v}. \quad (3.9)$$

Thus, eq. (3.8) becomes

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \mathbf{v} C_{\alpha\beta} \, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (3.10)$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha d\mathbf{v} + \sum_{\beta, \beta \neq \alpha} \int \mathbf{w}_\alpha C_{\alpha\beta} \, d\mathbf{v} + \int \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (3.11)$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \nabla \cdot \boldsymbol{\sigma}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (3.12)$$

where the stress tensor is

$$\boldsymbol{\sigma}_\alpha = -\int m_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha \, d\mathbf{v}, \quad (3.13)$$

the momentum transferred between unlike particles due to friction of collisions is

$$\mathbf{R}_\alpha = \sum_{\beta, \beta \neq \alpha} \int m_\alpha \mathbf{w}_\alpha C_{\alpha\beta} \, d\mathbf{v}, \quad (3.14)$$

and the external source of momentum is

$$\hat{\mathbf{M}}_\alpha = \int m_\alpha \mathbf{v} S_\alpha \, d\mathbf{v}. \quad (3.15)$$

The stress tensor is typically decomposed into isotropic p_α and anisotropic (shear) $\mathbf{t}_\alpha$ tensors as follows

$$\boldsymbol{\sigma}_\alpha = -p_\alpha \mathbf{I} + \mathbf{t}_\alpha, \quad (3.16)$$

where P_α is given by

$$p_\alpha = \frac{1}{3} \int m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha d\mathbf{v}. \quad (3.17)$$

Thus, conservation of momentum becomes

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha. \quad (3.18)$$

3.3 Energy

Multiplying eq. (3.4) by $\frac{1}{2}v^2$ and then integrating over all $\mathbf{v}$ leads to

$$\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v}\right] + \int \nabla_v \cdot \left[\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_\alpha\right]$$
$$- \nabla_v \left[\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\right] \cdot \left[\frac{Z_\alpha e}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_\alpha\right] d\mathbf{v} = \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}. \tag{3.19}$$

We note that the third term above is zero since we are integrating over all space, and that $\nabla_v[1/2(\mathbf{v}\cdot\mathbf{v})] = \mathbf{v}$. Thus, we have

$$\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla \cdot \left[\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v}\right] - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha =$$
$$\sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}. \tag{3.20}$$

To proceed with the derivation we first note that

$$\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{v} f_\alpha \, d\mathbf{v} \;=\; \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}_\alpha + \mathbf{w}_\alpha) f_\alpha \, d\mathbf{v} \;=\; n_\alpha E_\alpha \mathbf{u}_\alpha + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \tag{3.21}$$

The last term on the right-hand side above can be re-written as

$$\int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} = \int \frac{1}{2}(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha + 2\mathbf{u}_\alpha\cdot\mathbf{w}_\alpha + \mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v} \tag{3.22}$$

$$= \mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha \mathbf{w}_\alpha f_\alpha \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}. \tag{3.23}$$

Using the expressions above, eq. (3.20) becomes

$$\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla\cdot(n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = -\nabla\cdot\left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}\right) - \nabla\cdot\int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}$$
$$+ \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) C_{\alpha\beta} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}. \tag{3.24}$$

Conservation of particles is used to modify the collisional term to thus obtain

$$\frac{\partial n_\alpha E_\alpha}{\partial t} + \nabla\cdot(n_\alpha E_\alpha \mathbf{u}_\alpha) - \frac{Z_\alpha e n_\alpha}{m_\alpha}\mathbf{E}\cdot\mathbf{u}_\alpha = -\nabla\cdot\left(\mathbf{u}_\alpha \cdot \int \mathbf{w}_\alpha\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}\right) - \nabla\cdot\int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha)\mathbf{w}_\alpha f_\alpha \, d\mathbf{v}$$
$$+ \mathbf{u}_\alpha \cdot \sum_{\beta,\beta\neq\alpha} \int \mathbf{w}_\alpha C_{\alpha\beta} \, d\mathbf{v} + \sum_{\beta,\beta\neq\alpha} \int \frac{1}{2}(\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha) C_{\alpha\beta} \, d\mathbf{v} + \int \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) S_\alpha \, d\mathbf{v}. \tag{3.25}$$

Multiplying by mass leads to the following equation

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla\cdot(m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E}\cdot\mathbf{u}_\alpha = \nabla\cdot(\mathbf{u}_\alpha\cdot\boldsymbol{\sigma}_\alpha) - \nabla\cdot\mathbf{q}_\alpha$$
$$+ \mathbf{u}_\alpha\cdot\mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \tag{3.26}$$

where heat flux due to random motion is

$$\mathbf{q}_\alpha = \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) \mathbf{w}_\alpha f_\alpha \, d\mathbf{v}, \tag{3.27}$$

the heat generated and transferred between unlike particles due to collisional dissipation is

$$Q_\alpha = \sum_{\beta, \beta \neq \alpha} \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) C_{\alpha\beta} \, d\mathbf{v}, \tag{3.28}$$

and the external source of energy is

$$\hat{Q}_\alpha = \int \frac{1}{2} m_\alpha (\mathbf{v} \cdot \mathbf{v}) S_\alpha \, d\mathbf{v}. \tag{3.29}$$

Using the decomposition for the stress tensor, the conservation of energy equation becomes

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \tag{3.30}$$

We also note that the energy $m_\alpha n_\alpha E_\alpha$ can be decomposed into internal and kinetic energies. Using the trace of the decomposition shown in eq. (3.9) one obtains

$$\begin{aligned} m_\alpha n_\alpha E_\alpha &= \int \frac{1}{2} m_\alpha (\mathbf{v} \cdot \mathbf{v}) f_\alpha \, d\mathbf{v} \\ &= \int \frac{1}{2} m_\alpha (\mathbf{w}_\alpha \cdot \mathbf{w}_\alpha) f_\alpha \, d\mathbf{v} + \frac{1}{2} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \\ &= \frac{3}{2} P_\alpha + \frac{1}{2} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \\ &= \frac{3}{2} P_\alpha + m_\alpha n_\alpha K_\alpha. \end{aligned} \tag{3.31}$$

where $K_\alpha = \frac{1}{2} \mathbf{u}_\alpha \cdot \mathbf{u}_\alpha$ is the kinetic energy of species α.

3.4 Kinetic and Internal Energies

The equation for the kinetic energy is obtained by dotting eq. (3.18) with $\mathbf{u}_\alpha$. For this, we first show that

$$\mathbf{u}_\alpha \cdot \left[\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) \right] \tag{3.32}$$

$$= \mathbf{u}_\alpha \cdot \left\{ \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right\} \tag{3.33}$$

$$= \mathbf{u}_\alpha \cdot \left[m_\alpha \hat{S}_\alpha \mathbf{u}_\alpha + m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) \right] \tag{3.34}$$

$$= 2 m_\alpha \hat{S}_\alpha K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \tag{3.35}$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \left[\frac{\partial m_\alpha n_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) \right] K_\alpha + m_\alpha n_\alpha \left(\frac{\partial K_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla K_\alpha \right) \tag{3.36}$$

$$= m_\alpha \hat{S}_\alpha K_\alpha + \frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha). \tag{3.37}$$

Thus, the equation for the turbulent kinetic energy is

$$\frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha =$$
$$- \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha. \quad (3.38)$$

Subtracting the above equation from eq. (3.30) leads to

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha\right) + \nabla \cdot \left(\frac{3}{2}p_\alpha \mathbf{u}_\alpha\right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha. \quad (3.39)$$

3.5 Summary

To summarize, we have,

- Particle density

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = \hat{S}_\alpha, \quad (3.40)$$

- Momentum

$$\frac{\partial m_\alpha n_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = -\nabla p_\alpha + \nabla \cdot \mathbf{t}_\alpha + \mathbf{R}_\alpha + \hat{\mathbf{M}}_\alpha, \quad (3.41)$$

- Total Energy

$$\frac{\partial m_\alpha n_\alpha E_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha E_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) - \nabla \cdot \mathbf{q}_\alpha$$
$$+ \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + Q_\alpha + \hat{Q}_\alpha, \quad (3.42)$$

- Kinetic Energy

$$\frac{\partial m_\alpha n_\alpha K_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha K \mathbf{u}_\alpha) - Z_\alpha e n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha =$$
$$- \nabla \cdot (\mathbf{u}_\alpha p_\alpha) + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{t}_\alpha) + p_\alpha \nabla \cdot \mathbf{u}_\alpha - \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha - m_\alpha K_\alpha \hat{S}_\alpha.$$
$$(3.43)$$

- Internal Energy

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha\right) + \nabla \cdot \left(\frac{3}{2}p_\alpha \mathbf{u}_\alpha\right) = -p_\alpha \nabla \cdot \mathbf{u}_\alpha + \mathbf{t}_\alpha : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{q}_\alpha + Q_\alpha + \hat{Q}_\alpha - \mathbf{u}_\alpha \cdot \hat{\mathbf{M}}_\alpha + m_\alpha K_\alpha \hat{S}_\alpha.$$
$$(3.44)$$