Ordinary differential equations

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1 Initial Value Problems (IVP's)

Denoted by conditions that are specified at a point in space.

1.1 Linear

1.1.1 First Order

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$y'' + p(t)y' + q(t)y = g(t)$ and $y(t_o) = y_o$, $y'(t_o) = y'_o$ has a unique solution in the interval where $p(t)$, $q(t)$, and $g(t)$ are continuous.

Lets form the matrix $\mathbf{X} = \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix}$, where y_1 and y_2 are solutions to the ODE.

The Wronskian is defined as $W(\mathbf{X})(t) = det(\mathbf{X}(t))$

- Homogenous ODE

  Def: one for which zero is a solution. Look like $y'' + p(t)y' + q(t)y = 0$

If $W(\mathbf{X})(t_o) \neq 0$, then c_1 and c_2 are well defined for any y_o and y'_o specified at t_o, and therefore $y = c_1 y_1 + c_2 y_2$ is the solution to any y_o and y'_o, which makes it the general solution.

If $W(\mathbf{X})(t_o) \neq 0$ then y_1 and y_2 are independent, and are the fundamental solutions, and $\mathbf{X}$ becomes the fundamental matrix.

- Constant Coefficient
  Assume e^{rt}, gives characteristic equation.
  - Real distinct roots $r = r_1, r_2$: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - Real repeated roots $r_1 = r_2$: $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$
  - Complex roots $r = \lambda \pm i\mu$: $y = c_1 e^{\lambda t} cos(\mu t) + c_2 e^{\lambda t} sin(\mu t)$
- Non-Constant Coefficient

- Nonhomogeneous ODE

  Def: not a homogeneous ODE. Look like $y'' + p(t)y' + q(t)y = g(t)$

  The general solution is $y = c_1 y_1 + c_2 y_2 + y_p$

  - Undetermined Coefficients (works for constant coefficients only)
    Assume a particular solution analogous to the forcing term up to undetermined coefficients, plug in the ODE to get the coefficient, form the general solution, and find coefficients through the initial conditions.
  - Variation of Parameters
    $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$. Compute $y^{p'}$ (assuming $u'_1(t)y_1(t) + u'_2(t)y_2(t) = 0$) and $y^{p''}$, plug them in ODE to obtain $u'_1(t)y'_1(t) + u'_2(t)y'_2(t) = g(t)$. Finally, solve for $u_1(t)$ and $u_2(t)$, which gives $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^t \mathbf{X}(\sigma)^{-1} \begin{pmatrix} 0 \\ g(\sigma) \end{pmatrix} d\sigma$.

1.1.3 Higher Order

1.1.4 System of eqs.

$\mathbf{x}' = \mathbf{P(t)x + g(t)}$
Lets form the matrix $\mathbf{X} = [\mathbf{x}^{(1)}(t)...\mathbf{x}^{(n)}(t)]$.

- Homogenous System $\mathbf{x}' = \mathbf{P(t)x}$

  As with second order equations, if $W(\mathbf{X}) \neq 0$, then $\mathbf{x} = c_1\mathbf{x}^{(1)} + ...c_n\mathbf{x}^{(n)}$ is the general solution, $\mathbf{x}^{(1)}...\mathbf{x}^{(n)}$ are linearly independent, they are the fundamental solutions, and $\mathbf{X}$ becomes the fundamental matrix.

  - Constant Coefficient $\mathbf{x}' = \mathbf{Ax}$
    Assume $\boldsymbol{\xi} e^{rt}$, gives $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = 0$
    - real distinct eigenvalues: $\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + ... + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}$
    - real repeated eigenvalues: $\mathbf{x} = c_1 \boldsymbol{\xi}^{rt} + c_2 \left[\boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}\right]$, where $\boldsymbol{\xi}$ is the eigenvector and $\boldsymbol{\eta}$ satisfies $(\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.

    - complex conjugate eigenvalues (r_1, r_2) with corresponding complex conjugate eigenvectors $(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)})$:
      $\mathbf{x} = c_1 \mathrm{Re}[\boldsymbol{\xi}^{(1)} e^{r_1 t}] + c_2 \mathrm{Im}[\boldsymbol{\xi}^{(1)} e^{r_1 t}] + c_3 \boldsymbol{\xi}^{(3)} e^{r_3 t} + ... + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}$
  - Non-Constant Coefficient

- Nonhomogenous System

  The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)}(t) + ... + c_n \mathbf{x}^{(n)} + \mathbf{v}(t)$

  - Diagonalization

    If $\mathbf{A}$ can be diagonalize, perform diagonalization, form uncoupled system, solve and recombine to obtain general solution.
  - Undetermined Coefficients

    Assume a particular solution analogous to the forcing term up to undetermined vectors of coefficients, plug in the ODE to get these undetermined vectors. The result is the particular solution.
  - Variation of Parameters

    $\mathbf{v}(t) = \mathbf{X}(t)\mathbf{u}(t)$. Follow same approach as in second order case, the end result is $\mathbf{u}(t) = \int_0^t \mathbf{X}(\sigma)^{-1} \mathbf{g}(\sigma) d\sigma$.

1.2 Nonlinear

2 Boundary Value Problems (BVP's)

Denoted by conditions that are specified at more than one point in space.

2.1 Linear

2.1.1 First Order

2.1.2 Second Order

- Sturm-Liouville BVP

  One that satisfies the S-L homogeneous or nonhomogenous ODE, has bc's $a_1 y(a) + a_2 y'(a) = 0$ and $b_1 y(b) + b_2 y'(b) = 0$, and has $p(x) > 0$, $r(x) > 0$. These are also known as regular Sturm Liouville BVP.

  The goal is to find λ's (known as eigenvalues) that give non-zero solutions (known as eigenfunctions) to the ODE. This is analogous to finding λ's that give non-zero solutions (known as eigenvectors) to the matrix equation $\mathbf{A}\xi = \lambda\xi$.

  Common S-L solutions : Fourrier, Bessel, Chebyshev, Legendre, Hermite, Laguerre.

  - Homogenous Sturm-Liouville
    $My = -\lambda^2 y$ where $M = \frac{1}{r(x)} \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right]$
    - To solve, find general solution, apply boundary conditions to figure out eigenvalues that will give non zero solutions, obtain the eigenfunctions up to a multiplicative constant.

    * Infinite series of real, non-negative, distinct eigenvalues $\lambda_1 < \lambda_2 < ...$
    * One to one correspondence between eigenvalue and LI eigenfunction.
    * All eigenfunctions of S-L problem are orthogonal $\int_a^b r(x)\phi_m(x)\phi_n(x)dx = N_m\delta_{nm}$, where $N_m = \int_a^b r(x)\phi_m^2(x)dx$
    * A square integrable function f can be expanded as $f = \sum_{n=1}^{\infty} c_n\phi_n(x)$, where $c_m = \frac{\int_a^b r(x)f(x)\phi_m(x)dx}{\int_a^b r(x)\phi_m^2(x)dx}$
  - Nonhomogenous Sturm-Liouville
    $My = -\mu^2 y + f(x)$
    * Solution is of the form: $\phi = \sum_{n=1}^{\infty} b_n\phi_n(x)$
    * If $\mu \neq \lambda_n$ for all n, then $b_n = \frac{c_n}{\mu^2-\lambda_n^2}$, where $c_n = \int_a^b r(x)f(x)\phi_n(x)dx$.
    * If $\mu = \lambda_m$
      · If $c_m \neq 0$, then no solution
      · If $c_m = 0$, b_n is arbitrary, and there are infinite solutions
  - Singular Sturm-Liouville
    * A SL problem can be rewritten as $y'' + \frac{p'(x)}{p(x)}y' + \frac{q(x)+\lambda^2 r(x)}{p(x)}y = 0$.
    * A singular point of an ODE is one where the coefficients blow up. For this case, $p(x) = 0$.
    * A Singular SL problem is one where we allow a singularity at either or both of the boundaries (i.e. p=0), and the bc at the singular point is one that ensures the following is satisfied $\lim_{x\to aorb} p(x)(y_n' y_m - y_m' y_n) \to 0$
    * Since we now allow $p = 0$ at boundaries, this is an extension of the regular S-L BVP.
    * A boundary condition that ensures $\lim_{x\to aorb} p(x)(y_n' y_m - y_m' y_n) \to 0$ is satisfied basically forces the general solution to exclude the singular fundamental solution, that is, the one that blows up.

- Other BVP's (i.e. do not satisfy Sturm-Liouville ODE or bc's, are higher order, etc.)

2.1.3 Higher Order

2.2 Nonlinear