Flux coordinates

April 12, 2025

1 Flux coordinates

Imagine that Euclidean space is permeated by a set of surfaces, which we call flux surfaces. Each of those surfaces is labeled with a different value of the variable ψ. We also note that the flux surfaces are not stationary, they can move around as time progresses.

We now introduce the function $\hat{\psi} = \hat{\psi}(t, x^1, x^2, x^3)$. This function is defined in such a way that for all values x^1, x^2, x^3 that are part of a given flux surface at a specific time t, then $\hat{\psi}$ will evaluate to the value of ψ corresponding to that flux surface. The velocity of the flux surfaces is given by $\mathbf{V}_\psi = \mathbf{V}_\psi(t, x^1, x^2, x^3)$. Thus, by definition

$$\frac{\partial \hat{\psi}}{\partial t} + \mathbf{V}_\psi \cdot \nabla \hat{\psi} = 0. \tag{1}$$

A flux coordinate is defined as one in which $\hat{u}^1 = \hat{\psi}$.

1.1 Flux-surface averaging

To begin, we define the following. $D(\psi, t)$ is the volume enclosed at time t by the flux surface labelled by ψ. The surface of $D(\psi, t)$ is labelled as $\partial D(\psi, t)$. Additionally, $\Delta(\psi, t) = D(\psi + \Delta\psi, t) - D(\psi, t)$.

The flux surface average of a function is given by

$$\langle f \rangle_\psi = \lim_{\Delta\psi \to 0} \frac{\int_{\Delta(\psi,t)} f \, dV}{\int_{\Delta(\psi,t)} dV}. \tag{2}$$

This can be re-written as shown below

$$\langle f \rangle_\psi = \lim_{\Delta\psi \to 0} \frac{\frac{1}{\Delta\psi} \int_{\Delta(\psi,t)} f \, dV}{\frac{1}{\Delta(\psi)} \int_{\Delta(\psi,t)} dV} = \lim_{\Delta\psi \to 0} \frac{\frac{1}{\Delta\psi} \left(\int_{D(\psi+\Delta\psi,t)} f \, dV - \int_{D(\psi,t)} f \, dV \right)}{\frac{1}{\Delta\psi} \left(\int_{D(\psi+\Delta\psi,t)} dV - \int_{D(\psi,t)} dV \right)} = \frac{\frac{\partial}{\partial\psi} \int_{D(\psi,t)} f \, dV}{\frac{\partial}{\partial\psi} \int_{D(\psi,t)} dV}. \tag{3}$$

Defining $V' = V'(\psi, t)$ as

$$V' = \frac{\partial}{\partial\psi} \int_{D(\psi,t)} dV. \tag{4}$$

the second expression for the flux surface average is written as

$$\langle f \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial\psi} \int_{D(\psi,t)} f \, dV. \tag{5}$$

A third expression for $\langle g \rangle_\psi$ follows from using **??** for the above. Thus,

$$\langle f \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_0^\psi \int_{\partial D(\psi', t)} f \, \frac{dS}{|\nabla \hat{\psi}|} d\psi' = \frac{1}{V'} \int_{\partial D(\psi, t)} f \, \frac{dS}{|\nabla \hat{\psi}|}. \tag{6}$$

1.1.1 Average of spatial derivatives

We use the second definition of the flux-surface average, given by eq. (5), and then the divergence theorem to obtain

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{D(\psi, t)} \nabla \cdot \mathbf{A} \, dV = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{\partial D(\psi, t)} \mathbf{A} \cdot \frac{\nabla \hat{\psi}}{|\nabla \hat{\psi}|} \, dS. \tag{7}$$

We now use the third definition eq. (6) to obtain

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \mathbf{A} \cdot \nabla \hat{\psi} \rangle_\psi. \tag{8}$$

1.1.2 Average of time derivatives

Using the Reynolds transport theorem we show

$$\begin{aligned} \frac{\partial}{\partial t} \int_{D(\psi, t)} f \, dV &= \int_{D(\psi, t)} \frac{\partial f}{\partial t} \, dV + \int_{\partial D(\psi, t)} f \mathbf{V}_\psi \cdot \frac{\nabla \hat{\psi}}{|\nabla \hat{\psi}|} \, dS \\ &= \int_{D(\psi, t)} \frac{\partial f}{\partial t} \, dV + V' \langle f \mathbf{V}_\psi \cdot \nabla \hat{\psi} \rangle_\psi. \end{aligned} \tag{9}$$

We now take the derivative of both sides by ψ and then divide by V'.

$$\frac{1}{V'} \frac{\partial}{\partial t} V' \langle f \rangle_\psi = \left\langle \frac{\partial f}{\partial t} \right\rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle f \mathbf{V}_\psi \cdot \nabla \hat{\psi} \rangle_\psi. \tag{10}$$

Re-arranging and using eq. (1)

$$\left\langle \frac{\partial f}{\partial t} \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial t} V' \langle f \rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle f \frac{\partial \hat{\psi}}{\partial t} \right\rangle_\psi. \tag{11}$$