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1 Governing equations

We introduce the flow variables for density $\rho = \rho(\mathbf{x}, t)$, velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, internal energy $e = e(\mathbf{x}, t)$, and stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$. These are defined within a domain $\Omega = \Omega(t)$, which can be moving. The governing equations that dictate their evolution in the laboratory reference frame are

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot \boldsymbol{\sigma}, \tag{2}$$

$$\rho \left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e \right) = \boldsymbol{\sigma} : \nabla \mathbf{u}. \tag{3}$$

A note on notation. The products that involve a tensor $\boldsymbol{\tau}$ can be expressed in Einstein notation as

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}, \tag{4}$$

$$\boldsymbol{\tau} \cdot \nabla \alpha = \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \tag{5}$$

$$\mathbf{f} \cdot \boldsymbol{\tau} \cdot \nabla \alpha = f_i \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \tag{6}$$

$$\boldsymbol{\tau} : \nabla \mathbf{f} = \tau_{ij} \frac{\partial f_i}{\partial x_j}. \tag{7}$$

where α is a scalar and $\mathbf{f}$ a vector. In these notes we'll mostly be using indices i and j for FE expansions, rather than for Einstein notation.

2 Finite element expansion

We introduce the coefficients $\hat{d}_i = \hat{d}_i(t)$, $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(t)$ and $\hat{e}_i = \hat{e}_i(t)$, as well as the basis functions $\phi_i = \phi_i(\mathbf{x}, t) \in L^2$, and $w_i = w_i(\mathbf{x}, t) \in H^1$. We note that $\hat{\mathbf{u}}_i$ is a vector whose components are $\hat{u}_{i,\alpha} = \hat{u}_{i,\alpha}(t)$ for $\alpha = x, y, z$. These coefficients are used in the following expansions

$$\rho = \sum_{j}^{N_\phi} \hat{d}_j \phi_j, \tag{8}$$

$$\mathbf{u} = \sum_{j}^{N_w} \hat{\mathbf{u}}_j w_j, \tag{9}$$

$$e = \sum_{j}^{N_\phi} \hat{e}_j \phi_j, \tag{10}$$

The basis functions are defined so that they are Lagrangian, that is,

$$\frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j = 0, \tag{11}$$

$$\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j = 0. \tag{12}$$

3 Semi-discrete momentum conservation

We begin by showing that

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \sum_{j}^{N_w} \left(\frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \frac{\partial w_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_{j}^{N_w} \hat{\mathbf{u}}_j \nabla w_j \right) \\
&= \sum_{j}^{N_w} \left[\frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \left(\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j \right) \right] \\
&= \sum_{j}^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j.
\end{aligned} \tag{13}$$

The finite element formulation of the momentum equation is thus

$$\int_\Omega \rho \sum_{j}^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j w_i \, dV = - \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \, dV \qquad \text{for } i = 1, ..., N_w. \tag{14}$$

The above is re-written as

$$\sum_{j}^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)} = - \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \, dV \qquad \text{for } i = 1, ..., N_w. \tag{15}$$

where the mass bilinear form $m_{ij}^{(w)}$ is given by

$$m_{ij}^{(w)} = \int_\Omega \rho w_i w_j \, dV. \tag{16}$$

We now introduce the vector $\mathbf{U}$ whose components are $\hat{\mathbf{u}}_i$. We also introduce the matrix $\mathbf{M}^{(w)}$ whose components are $m_{ij}^{(w)}$. Thus, the left-hand side of eq. (15) can be written as $\mathbf{M}^{(w)} \, d\mathbf{U}/dt$. We also introduce the vector bilinear form

$$\mathbf{f}_{ij} = \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \phi_j dV. \tag{17}$$

This is a *vector* bilinear form since $\mathbf{f}_{ij}$ has components $f_{ij,\alpha} = f_{ij,\alpha}(t)$, for $\alpha = x, y, z$, where α denotes the first index of $\boldsymbol{\sigma}$. We introduce the matrix $\mathbf{F}$, whose components are $\mathbf{f}_{ij}$. We also expand the field with constant value of one as follows

$$1 = \sum_{i}^{N_\phi} \hat{c}_i \phi_i. \tag{18}$$

If we define the vector $\mathbf{C}$ as that with components $\hat{c}_i$, we can show that

$$\begin{aligned}
\mathbf{FC} &= \sum_{j}^{N_\phi} \mathbf{f}_{ij} \hat{c}_j && \text{for } i = 1, ..., N_w \\
&= \sum_{j}^{N_\phi} \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV \, \hat{c}_j && \text{for } i = 1, ..., N_w \\
&= \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \left(\sum_{j}^{N_\phi} \hat{c}_j \phi_j \right) \, dV && \text{for } i = 1, ..., N_w \\
&= \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_i \, dV && \text{for } i = 1, ..., N_w
\end{aligned} \tag{19}$$

The above is the negative of the right-hand side of eq. (15). Thus, combining all together we get

$$\mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} = -\mathbf{FC}. \tag{20}$$

4 Semi-discrete energy conservation

As with momentum conservation, we have

$$\begin{aligned}
\frac{\partial \mathbf{e}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{e} &= \sum_{j}^{N_\phi} \left(\frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \frac{\partial \phi_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_{j}^{N_\phi} \hat{\mathbf{e}}_j \nabla \phi_j \right) \\
&= \sum_{j}^{N_\phi} \left[\frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \left(\frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j \right) \right] \\
&= \sum_{j}^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j.
\end{aligned} \tag{21}$$

For the right-hand side of the energy conservation equation, we have

$$\boldsymbol{\sigma} : \nabla \mathbf{u} = \boldsymbol{\sigma} : \nabla \left(\sum_{k}^{N_w} \hat{\mathbf{u}}_k w_k \right) = \sum_{k}^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k. \tag{22}$$

The finite element formulation of the energy equation is thus

$$\int_\Omega \rho \sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j \phi_i \, dV = \int_\Omega \left(\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k \right) \phi_i \, dV \qquad \text{for } i = 1, ..., N_w$$
$$= \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \int_\Omega \boldsymbol{\sigma} \cdot \nabla w_k \phi_i \, dV \qquad \text{for } i = 1, ..., N_w \tag{23}$$

The above is re-written as

$$\sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} \qquad \text{for } i = 1, ..., N_w. \tag{24}$$

where the mass bilinear form m_{ij}^{ϕ} is given by

$$m_{ij}^{(\phi)} = \int_\Omega \rho \phi_i \phi_j \, dV. \tag{25}$$

Note that in eq. (24) there is a dot product in the right-hand side, that is, the right-hand side expanded out is

$$\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} = \sum_k^{N_w} \sum_{\alpha=x,y,z} \hat{u}_{k,\alpha} f_{ki,\alpha}. \tag{26}$$

We now introduce the vector $\mathbf{E}$ whose components are $\hat{e}_i$. We also introduce the matrix $\mathbf{M}^{(\phi)}$ whose components are $m_{ij}^{(\phi)}$. Thus, eq. (24) can be succinctly written as

$$\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{U}. \tag{27}$$

Note again that on the right-hand side above there is a matrix-vector product *and* a dot product.