Partial Differential Equations

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Chapter 1

Fundamental PDEs

1.1 Laplace and Poisson equations

1.1.1 Fundamental solutions

Assuming solutions to the Laplace eq. depend only on the distance from origin ($|x|$), we obtain, up to a constant, the "fundamental solutions" of the Laplace eq.

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

1.1.2 Solving Poisson's Eq.

Let $f \in C_c^2(\mathbb{R}^n)$, $n > 2$, and Φ be the fundamental solution to the Laplace equation. Define $u(x)$ as follows,

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy.$$

Then,

- $u \in C^2(\mathbb{R}^n)$
- $-\Delta u = f(x) \quad (x \in \mathbb{R}^n)$.

1.1.3 Mean value property

Let $U \subset \mathbb{R}^n$ be an open set, and let $B(x,r)$ be a ball centered at $x \in \mathbb{R}^n$ contained in U. Assume $u(x)$ is harmonic in U and that $u \in C^2(U)$. Then,

$$u(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} u dy = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u dS.$$

- **Max and Min principle**: let $u(x)$ be a harmonic function in a connected domain U and assume $u \in C^2(U) \cap C(\bar{U})$. Then,

$$\max_{x \in \bar{U}} u(x) = \max_{y \in \partial U} u(y)$$
$$\min_{x \in \bar{U}} u(x) = \min_{y \in \partial U} u(y)$$

  Moreover, if $u(x)$ achieves its max or min in the interior of U, then $u(x)$ is constant in U.

- Strict positivity: assume U is a connected domain, g is continuous on ∂U, $g \geq 0$ and $\neq 0$, and that u solves,

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

  Then $u > 0$ at all $x \in U$.
- Uniqueness: let g be continuous on ∂U and f be continuous in U. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ to the boundary value problem,

$$\begin{cases} \Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

- **Regularity**: Let $u \in C^2(U)$ be a harmonic function in a domain U. Then $u \in C^\infty(U)$.
- **Estimates on derivatives**: Let $u(x)$ be a harmonic function in a domain U and let $B(y_0, r)$ be a ball contained in U centered at a point $y_0 \in U$. Then there exist universal constants C_n and D_n that depend only on the dimension n so that,

$$|u(y_0)| \leq \frac{C_n}{r^n} \int_{B(y_0,r)} |u(y)| \, dy$$

$$|\nabla u(y_0)| \leq \frac{D_n}{r^{n+1}} \int_{B(y_0,r)} |u(y)| \, dy$$

- **Liouville theorem**: Let $u(x)$ be a harmonic bounded function in $\mathbb{R}^n$. Then $u(x)$ is equal identically to a constant.
- **Harnack's inequality**: Let U be an open set and let V be strictly contained in U. Then there exists a constant C that depends on U and V but nothing else so that for any non-negative harmonic function u in U we have,

$$\sup_{x \in V} u(x) \leq C \inf_{x \in V} u(x)$$

1.1.4 Green's function

- Dirichlet problem: Suppose $u \in C^2(\bar{U})$ solves

$$\begin{cases} \Delta u = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

  then u would be of the following form,

$$u(x) = \int_U G(x,y) f(y) \, dy - \int_{\partial U} g(y) \frac{\partial G}{\partial n}(x,y) \, dS(y) \qquad (x \in U),$$

  where $G(x,y)$ is the Green's function for the region U, and is defined as

$$G(x,y) = \Phi(y - x) - \phi(y;x),$$

  where $\phi(y;x)$ is a corrector function that, for a fixed x, solves the following

$$\begin{cases} \Delta \phi(y;x) = 0 & \text{in } U \\ \phi(y;x) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

  Finally, reciprocity holds for $G(x,y)$, that is, for all $x, y \in U, x \neq y$, we have $G(x;y) = G(y;x)$.

- Neuman problem: Suppose $u \in C^2(\bar{U})$ solves

$$\begin{cases} \Delta u = f & \text{in } U \\ \frac{\partial u}{\partial n} = g & \text{on } \partial U, \end{cases}$$

then u would be of the following form,

$$u(x) = \int_U N(x,y) f(y)\, dy + \int_{\partial U} N(x,y) g(y)\, dS(y) \qquad (x \in U),$$

where $N(x,y)$ is the Green's function for the region U, and is defined as

$$G(x,y) = \Phi(x-y) - h(y;x),$$

where $h(y;x)$ is a corrector function that, for a fixed x, solves the following

$$\begin{cases} \Delta h(y;x) = 0 & \text{in } U \\ \frac{\partial h(y;x)}{\partial n} = \frac{\partial \Phi(x-y)}{\partial n} + \frac{1}{|\partial U|} & \text{on } \partial U. \end{cases}$$

Finally, if there were to be a solution we would need $\int_U f(x)\, dx = -\int_{\partial U} g(y) dS(y)$.

1.2 Heat Equation

1.2.1 Fundamental solution

The function

$$\Phi(t,x) = \frac{1}{(4\alpha\pi t)^{n/2}} e^{\frac{-|x|^2}{4\alpha t}} \quad (x \in \mathbb{R}^n, t > 0)$$

is called the fundamental solution, or heat Kernel, of the heat equation

$$\frac{\partial u}{\partial t} = \alpha \Delta u,$$

where α is called the thermal diffusivity. Moreover, $\int_{\mathbb{R}^n} \Phi(t,x) dx = 1$ for $\alpha = 1$. We will from now on assume $\alpha = 1$.

1.2.2 Classification

- The homogeneous **initial-value problem** follows,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

- The homogeneous **initial/boundary-value problem** follows,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } U_T \\ u = g & \text{on } \Gamma_T, \end{cases}$$

  where U_T and Γ_T are as defined in the book.

- For either problem we could add $f(t,x)$ to the PDE to obtain the **inhomogeneous** case. When referring to the initial-value or the initial/boundary-value problem, we are referring to both the homogeneous and inhomogeneous cases.

1.2.3 Homogeneous initial-value problem

Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and Φ be the fundamental solution to the heat equation. Define $u(t,x)$ as follows,

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) g(y) dy \qquad (x \in \mathbb{R}^n, t > 0).$$

Then,

- $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$
- u satisfies the PDE for the homogeneous initial-value problem.
- $\lim_{(t,x)\to(0,x^0)} u(t,x) = g(x^0)$ for each point $x^0 \in \mathbb{R}^n$, and for any $x \in \mathbb{R}^n, t > 0$.

1.2.4 Inhomogeneous initial-value problem

Define $v(t,x)$ as follows

$$u(t,x) = \int_0^t v(t,x;s)\, ds,$$

where $v(t,x;s) = \int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y)\, dy$, parameterized by s, is the solution to the initial-value problem

$$\begin{cases} \frac{\partial v(t,x;s)}{\partial t} - \Delta v(t,x;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ v(t=s,x;s) = f(s,x) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases}$$

Then,

- $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$
- u satisfies the PDE for the inhomogeneous initial-value problem.
- $\lim_{(t,x)\to(0,x^0)} u(t,x) = 0$ for each point $x^0 \in \mathbb{R}^n$, and for any $x \in \mathbb{R}^n, t > 0$.

1.2.5 Maximum principle

- Assume $u \in C_1^2(U_\Gamma) \cap C(\bar{U}_T)$ solves the homogeneous initial/boundary-value problem. Then,

  $$\max_{\bar{U}_T} u = \max_{\Gamma_T} u$$

  - Uniqueness: Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_\Gamma) \cap C(\bar{U}_T)$ of the initial/boundary-value problem.

- Assume $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves the homogeneous initial-value problem and satisfies the estimate

  $$u(t,x) \le A e^{a|x|^2} \qquad (x \in \mathbb{R}^n, 0 \le t \le T),$$

  for constants $A, a > 0$. Then,

  $$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$$

  - Uniqueness: Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of the initial-value problem, that also satisfies the growth estimate

    $$u(t,x) \le A e^{a|x|^2} \qquad (x \in \mathbb{R}^n, 0 \le t \le T),$$

    for constants $A, a > 0$.

1.2.6 Estimates on derivatives

For the homogeneous initial-value problem, we have

$$|u(t,x)| \le \frac{C_n}{t^{n/2}} \int_{\mathbb{R}^n} |g(y)|\, dy \qquad (x \in \mathbb{R}^n, t > 0)$$

and

$$|\nabla u(t,x)| \le \frac{D_n}{t^{(n+1)/2}} \int_{\mathbb{R}^n} |g(y)|\, dy \qquad (x \in \mathbb{R}^n, t > 0).$$

1.3 Wave Equation

1.3.1 The Cauchy Problem

Consider the Cauchy problem

$$\begin{cases} \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = p,\ u_t = q & \text{on } \mathbb{R} \times (t = 0). \end{cases}$$

Since the wave equation has solution of the form $\phi(t,x) = f(x - ct) + g(x + ct)$, a solution to the Cauchy problem follows,

$$u(t,x) = \frac{1}{2}[p(x - ct) + p(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} q(y)\, dy.$$

1.3.2 Energy Methods

- **Definition**

  $$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{c^2(x)} u_t(t,x)^2 + \nabla u(t,x)^2\, dx$$

  It is easy to show that $E(t) = E(0)$ for $t \ge 0$.

- **Uniqueness**: Using conservation of energy for $w(t,x) = u(t,x) - v(t,x)$, it can be easily shown that the Cauchy problem has a unique solution.

- **Domain of dependence**: $\phi(t^*,x^*)$ depends on the values of $\phi(t,x)$ for t that satisfies $t^* > t \ge 0$ and all x that lie inside the ball $B[x^*, c(t^* - t)]$. To prove show that two solutions with the same ICs inside the ball centered at x^* have the same value at some later time t^* and position x^*. Do this by showing $e(t)$, evaluated over the ball $B[x^*, c(t^* - t)]$ for the difference of both solutions, remains zero.

Chapter 2

Solution Methods for PDEs

2.1 Characteristics

2.2 Self-similarity

2.3 Separation of Variables

2.4 Eigenfunction Expansions

2.5 Transform Methods

Appendix A

Calculus

A.1 Fundamental theorem of Calculus

Given two functions $f(x)$ and $F(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \Leftrightarrow \quad f(x) = \frac{dF(x)}{dx}$$

where part 1 is the forward direction and part 2 is the backward direction. From such one can derive in a trivial fashion

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t) \qquad \text{and} \qquad \frac{d}{dt}\int_t^b f(x)\,dx = -f(t)$$

For multiple dimensions, where $\mathbf{f}(\mathbf{x})$ is a vector and $F(\mathbf{x})$ a scalar, we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}(\mathbf{x}) \cdot d\mathbf{l} = F(\mathbf{b}) - F(\mathbf{a}) \quad \Leftrightarrow \quad \mathbf{f}(\mathbf{x}) = \boldsymbol{\nabla} F(\mathbf{x})$$

which shows the implication that follows when the integral is independent of the path taken.

A.2 Stokes' theorem

Stokes' theorem pertains to integrals over an area:

- for a scalar $f \in C^1(\bar{\Omega})$

$$\int_{\partial\Omega} (\mathbf{n} \times \boldsymbol{\nabla} f)\,dS = \oint f\,d\mathbf{l}$$

- for a vector $\mathbf{f} \in C^1(\bar{\Omega})$

$$\int_{\partial\Omega} (\boldsymbol{\nabla} \times \mathbf{f}) \cdot \mathbf{n}\,dS = \oint \mathbf{f} \cdot d\mathbf{l}$$

A.3 Gauss's theorem

Gauss's theorem pertains to integrals over a volume:

- for a scalar $f \in C^1(\bar{\Omega})$

$$\int_{\Omega} \boldsymbol{\nabla} f\,dV = \int_{\partial\Omega} f\mathbf{n}\,dS$$

- for a vector $\mathbf{f} \in C^1(\bar{\Omega})$

$$\int_\Omega \boldsymbol{\nabla} \cdot \mathbf{f} \, dV = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} \, dS$$

$$\int_\Omega \boldsymbol{\nabla} \times \mathbf{f} \, dV = \int_{\partial\Omega} \mathbf{n} \times \mathbf{f} \, dS$$

- for a tensor $\mathbf{f} \in C^1(\bar{\Omega})$

$$\int_\Omega \frac{\partial f_{ij}}{\partial x_j} \, dV = \int_{\partial\Omega} f_{ij} n_j \, dS.$$

A.4 Integration by parts

For f and g scalars $\in C^1(\bar{\Omega})$

$$\int_\Omega (\boldsymbol{\nabla} f) g \, dV = -\int_\Omega f (\boldsymbol{\nabla} g) \, dV + \int_{\partial\Omega} f g \mathbf{n} \, dS,$$

for f a scalar and $\mathbf{g}$ a vector $\in C^1(\bar{\Omega})$

$$\int_\Omega \boldsymbol{\nabla} f \cdot \mathbf{g} \, dV = -\int_\Omega f \boldsymbol{\nabla} \cdot \mathbf{g} \, dV + \int_{\partial\Omega} f \mathbf{g} \cdot \mathbf{n} \, dS.$$

A.5 Green's first and second identities $(f, g \in C^2(\bar{\Omega}))$

$$\int_\Omega \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \, dV = -\int_\Omega f \Delta g \, dV + \int_{\partial\Omega} f \boldsymbol{\nabla} g \cdot \mathbf{n} \, dS$$

$$\int_\Omega f \Delta g - g \Delta f \, dV = \int_{\partial\Omega} f \boldsymbol{\nabla} g \cdot \mathbf{n} - g \boldsymbol{\nabla} f \cdot \mathbf{n} \, dS$$

A.6 Integration by substitution

Given the continuously differentiable function $\boldsymbol{\phi} : \mathbf{y} \to \mathbf{x}$,

$$\int_{\boldsymbol{\phi}(\Omega)} f(\mathbf{x}) \, dV_x = \int_\Omega f(\boldsymbol{\phi}(\mathbf{y})) J \, dV_y$$

where $J = |\det(D\boldsymbol{\phi})(\mathbf{y})|$, $dV_x = dx_1 ... dx_n$, and $dV_y = dy_1 ... dy_n$.