Numerical Methods[1]

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[1]Disclaimer: a large chunk of the work in this document is not original; instead, parts of this document are personal notes obtained from the following books: Lomax, Pulliam, Zing; Larsson, Thomee; Hirsch; Leveque; Blazek; Laney; Moin.

Contents

Part I

Introductory Concepts

- Analytical solution u satisfies

$$D(u) = Au - f = 0 \quad \text{in } \Omega, \tag{1}$$

where A is the differential operator.

- Discrete numerical solution U_j satisfies

$$D_h(U_j^n) = 0 \quad \text{for all } n, j, \tag{2}$$

where D_h is the discrete operator for D.

- Continuous numerical solution v satisfies

$$D_h v = 0 \quad \text{in } \Omega, \tag{3}$$

Thus, $v(t_n, x_j) = U_j^n$ for all n, j.

- Local truncation error

$$l_j^{n+1} = u(t_{n+1}, x_j) - U_j^{n+1}, \tag{4}$$

given that $U_j^n = u(t_n, x_j)$ for all j.

- Global truncation error

$$e_j^{n+1} = u(t_{n+1}, x_j) - U_j^{n+1}, \tag{5}$$

given that $U_j^0 = u(t_0, x_j)$ for all j.

- Truncation error τ: difference between the discrete and analytical equations when applied to a given continuous function g

$$\tau = D_h g - Dg. \tag{6}$$

The truncation error allows us to compute global truncation errors. Also, consider the truncation error for the analytical solution, that is, $\tau = D_h u - Du$. Using equation (1), one obtains that u is also the solution to $D_h u = \tau$. This means that the **analytical solution satisfies the numerical equation but with an additional source term equal to** τ. Conversely, consider the truncation error for the function v, $\tau = D_h v - Dv$. Using equation (3), one obtains that v is also a solution to $Dv = \tau$. This means that the **numerical solution satisfies the analytical equation but with an additional source term equal to** τ.

- Order of accuracy r defined by $\tau^n = \mathcal{O}(\Delta t^r)$ as $\Delta t \to 0$ for ODE's and $\tau^n = \mathcal{O}(\Delta x^r)$ as $\Delta x \to 0$ for PDE's, where Δt is related to Δx.

Part II

Numerical Solution of ODE's

Chapter 1

List of time integrators

Consider the IVP for $u = u(t)$

$$\frac{du}{dt} = f(t,u) \quad \text{in } (0,\infty) \tag{1.1}$$

with initial condition $u(0) = u^0$. We will discretize time into a set of finite values t^n, and express the numerical solution at a given time t^n as U^n.

1.1 Explicit

- Explicit Euler method (Forward Euler method)

$$u^{n+1} = u^n + \Delta t f(t_n, u^n) \qquad 1^{\text{st}} \text{ O.A.} \tag{1.2}$$

- Explicit Midpoint (Modified Euler, 2^{nd} order RK)

$$\begin{aligned} u^{n+1/2} &= u^n + \frac{\Delta t}{2} f(t_n, u^n) \\ u^{n+1} &= u^n + \Delta t f(t_{n+1/2}, u^{n+1/2}) \end{aligned} \tag{1.3}$$

1.2 Implicit

- Implicit Euler (Backward Euler)

$$u^{n+1} = u^n + \Delta t f(t_{n+1}, u^{n+1}) \qquad 1^{\text{st}} \text{ O.A.} \tag{1.4}$$

- Trapezoidal (Crank-Nicholson)

$$u^{n+1} = u^n + \frac{\Delta t}{2}[f(t_n, u^n) + f(t_{n+1}, u^{n+1})] \qquad 2^{\text{nd}} \text{ O.A.} \tag{1.5}$$

- Implicit Midpoint

$$u^{n+1} = u^n + \Delta t f\left(t_{n+1/2}, \frac{1}{2}\left(u^n + u^{n+1}\right)\right) \tag{1.6}$$

1.3 Predictor-Corrector

- Heun's (Explicit Trapezoidal, Improved Euler)

$$\begin{aligned} \tilde{u}^{n+1} &= u^n + \Delta t f(t_n, u^n) \\ u^{n+1} &= u^n + \frac{\Delta t}{2}[f(t_n, u^n) + f(t_{n+1}, \tilde{u}^{n+1})] \quad 2^{\text{ nd}} \text{ O.A.} \end{aligned} \tag{1.7}$$

1.4 Runge-Kutta

- 4^{th} order Runge-Kutta

$$\begin{aligned} k_1 &= f(t_n, u^n) \\ k_2 &= f(t_{n+1/2}, u^n + \frac{\Delta t}{2} k_1) \\ k_3 &= f(t_{n+1/2}, u^n + \frac{\Delta t}{2} k_2) \\ k_4 &= f(t_n, u^n + \Delta t k_3) \\ u^{n+1} &= u^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \tag{1.8}$$

1.5 Multi-Step

- Leapfrog: consider the system of equations for $x = x(t)$ and $v = v(t)$

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = f(x). \tag{1.9}$$

  The Leapfrog method is a staggered scheme defined as follows

$$v^{n+1/2} = v^{n-1/2} + f(x^n)\Delta t \tag{1.10}$$

$$x^{n+1} = x^n + v^{n+1/2}\Delta t \tag{1.11}$$

- Adams-Bashforth

Chapter 2

Solving non-linear equations

The implicit schemes result in non-linear equations that require some sort of non-linear solver. For this chapter, we'll assume the governing ODE is

$$\frac{du}{dt} = f(u) \qquad \text{in } (0, \infty), \tag{2.1}$$

or, for a system of equations,

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}) \qquad \text{in } (0, \infty). \tag{2.2}$$

2.1 Newton's method

The Euler scheme requires finding u^{n+1} so that the following is satisfied

$$u^{n+1} = u^n + \Delta t f(u^{n+1}). \tag{2.3}$$

That is, we need to find the root of the nonlinear equation

$$g(x) = x - u^n - \Delta t f(x). \tag{2.4}$$

Newton's method finds the root of a non-linear equation $g(x)$ by iterating through m as follows

$$x^{m+1} = x^m - \frac{g(x^m)}{g'(x^m)}. \tag{2.5}$$

Applying Newton's method to eq. (2.4), we have

$$x^{m+1} = x^m - \frac{x^m - u^n - \Delta t f(x^m)}{1 - \Delta t f'(x^m)}, \tag{2.6}$$

The initial value for the non-linear solver is $x^0 = u^n$. Thus, using the above, u^{n+1} is approximated by x^m as $m \to \infty$. Note that this is only one example of what is referred to as a fixed-point iteration. There are other fixed-point-iteration methods that can be used to solve eq. (2.4) that are not Newton's method.

As a side note, let's assume u^{n+1} is approximated sufficiently well by x^1, that is, only one Newton iteration is needed. Then we have

$$u^{n+1} = u^n + \frac{\Delta t f(u^n)}{1 - \Delta t f'(u^n)}, \tag{2.7}$$

which we re-write as

$$\left(\frac{1}{\Delta t} - f'(u^n)\right)\Delta u = f(u^n), \tag{2.8}$$

where $\Delta u = u^{n+1} - u^n$. The above is the approximation that gets used for pseudo-time stepping, so that the Backward Euler scheme can converge to the steady state solution as one iterates through n.

To solve for the root of a system of non-linear equations $\mathbf{g}(\mathbf{x})$, Newton's method is as follows

$$\mathbf{x}^{m+1} = \mathbf{x}^m - \mathbf{J}^{-1}(\mathbf{x}^m)\mathbf{g}(\mathbf{x}^m). \tag{2.9}$$

In the above, $\mathbf{J}^{-1}(\mathbf{x})$ is the inverse of the Jacobian matrix $\mathbf{J}(\mathbf{x})$, which is given by

$$\mathbf{J}(\mathbf{x}) = \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial g_1(\mathbf{x})}{\partial x_1} & \frac{\partial g_1(\mathbf{x})}{\partial x_2} & \cdots \\ \frac{\partial g_2(\mathbf{x})}{\partial x_1} & \frac{\partial g_2(\mathbf{x})}{\partial x_2} & \\ \vdots & & \ddots \end{pmatrix}. \tag{2.10}$$

For the backward Euler scheme, the equivalent of eq. (2.6) would be

$$\mathbf{x}^{m+1} = \mathbf{x}^m - \left(\mathbf{I} - \Delta t \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^m}\right)^{-1} \left(\mathbf{x}^m - \mathbf{u}^n - \Delta t \mathbf{f}(\mathbf{x}^m)\right), \tag{2.11}$$

and the equivalent of eq. (2.8) would be

$$\left(\frac{\mathbf{I}}{\Delta t} - \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{u}^n}\right)\Delta \mathbf{u} = \mathbf{f}(\mathbf{u}^n), \tag{2.12}$$

where $\Delta \mathbf{u} = \mathbf{u}^{n+1} - \mathbf{u}^n$.

Part III

Finite Difference for PDE's

Chapter 3

Introduction

3.1 Finite difference formulas

Forward	Backward	Central	Central (2nd order)
$\partial U_j = \frac{U_{j+1} - U_j}{h}$	$\bar{\partial} U_j = \frac{U_j - U_{j-1}}{h}$	$\hat{\partial} U_j = \frac{U_{j+1} - U_{j-1}}{2h}$	$\partial\bar{\partial} U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$

3.2 Fourier analysis

This analysis is based on a test function that is periodic and thus has the following form

$$u(x) = \sum_{n=-\infty}^{n=\infty} \hat{u}_n e^{i\kappa_n x}. \tag{3.1}$$

We are interested in using the different numerical methods to compute the resulting approximate derivatives of the generic mode $e^{i\kappa x}$. We note that one can then combine the approximate derivatives of all the modes to obtain the approximate derivative of our test function u.

The analytically derivative of our generic mode $e^{i\kappa x}$ is $i\kappa e^{i\kappa x}$. We now assume that the approximate derivatives at a point x_j obtained using numerical schemes will be of the form $i\kappa^* e^{i\kappa x_j}$, where κ^* is a modified wave number. The closer κ^* is to κ the higher the accuracy of the numerical scheme. For example, for the central first order scheme

$$\delta U_j = \frac{U_{j+1} - U_{j-1}}{2h} \tag{3.2}$$

we have

$$i\kappa^* e^{i\kappa x_j} = \frac{e^{i\kappa(x_j+h)} - e^{i\kappa(x_j-h)}}{2h} \tag{3.3}$$

which leads to

$$\kappa^* = \frac{\sin \kappa h}{h}. \tag{3.4}$$

For the sixth-order compact scheme

$$\alpha\delta U_{j-1} + \delta U_j + \alpha\delta U_{j+1} = \frac{a}{2h}\left(U_{j+1} - U_{j-1}\right) + \frac{b}{4h}\left(U_{j+2} - U_{j-2}\right) \tag{3.5}$$

we have

$$\alpha i\kappa^* e^{i\kappa(x_j-h)} + i\kappa^* e^{i\kappa x_j} + \alpha i\kappa^* e^{i\kappa(x_j+h)} = \frac{a}{2h}\left[e^{i\kappa(x_j+h)} - e^{i\kappa(x_j-h)}\right] + \frac{b}{2h}\left[e^{i\kappa(x_j+2h)} - e^{i\kappa(x_j-2h)}\right] \tag{3.6}$$

which leads to

$$\kappa^* = \frac{\frac{a}{h}\sin\kappa h + \frac{b}{2h}\sin 2\kappa h}{1 + 2\alpha\cos\kappa h}. \tag{3.7}$$

A spectral method, on the other hand, will explicitly express the derivative of a mode as $i\kappa e^{i\kappa x_j}$, up to the last mode $\kappa = \frac{2\pi}{L}\frac{N}{2}$, and will not be able to capture higher modes. Thus,

$$\kappa^* = \begin{cases} \kappa & \text{for } \frac{2\pi}{L}\left(-\frac{N}{2}+1\right) \le \kappa \le \frac{2\pi}{L}\frac{N}{2} \\ 0 & \text{o.w.} \end{cases} \tag{3.8}$$

Chapter 4

Elliptic

Define the following norm: $|U|_S = \max_{x_j \in S} |U_j|$.
For $Au = -au'' + cu = f$ in $\Omega = (0,1)$, where $a(x) > 0$ and $c(x) \geq 0$, with boundary conditions $u(0) = U_0$ and $u(1) = U_m$, and $A_h = -a_j \partial\bar{\partial} U_j + c_j U_j$:

- **Lemma 4.2**
$$|U|_{\bar{\Omega}} \leq \max\{|U_0|, |U_M|\} + C|A_h U|_\Omega$$

- **Theorem 4.1** The error bound follows,
$$|U - u|_\Omega \leq Ch^2 ||u||_C^4$$

For $Au = -\Delta u = f$ in $\Omega = (0,1) \times (0,1)$, with boundary doncitions $u = 0$ in Γ, and $A_h = -\Delta_h = -\partial_1\bar{\partial}_1 U_j - \partial_2\bar{\partial}_2 U_j$:

- **Lemma 4.4**
$$|U|_{\bar{\Omega}} \leq |U|_\Gamma + C|\Delta_h U|_\Omega$$

- **Theorem 4.2** The error bound follows,
$$|U - u|_\Omega \leq Ch^2 ||u||_C^4$$

Chapter 5

Parabolic

For $u_t = u_{xx}$ in $\mathbf{R} \times \mathbf{R}_+$, with initial condition $u(\cdot, 0) = v$ in $\mathbf{R}$.

- Each scheme can be associated with its discrete solution operator E_k, defined in either of the following two ways, where U_j^n is defined only at mesh points, and $u^n(x)$ is the corresponding numerical solution defined over all space,

$$U_j^{n+1} = (E_k U^n)_j = \sum_p a_p U_{j-p}^n$$

$$u^{n+1}(x) = (E_k u^n)(x) = \sum_p a_p u^n(x - x_p)$$

- Repeated application yields $U_j^n = (E_k^n V)_j$ or $u^n(x) = (E_k^n v)(x)$, where V_i and $v(x)$ are the initial conditions, specified at mesh points and over all space, respectively.

- The **stability** of E_k (and hence the associated scheme) is defined by $||U^n||_{l_p} = ||E_k^n V||_{l_p} \leq ||V||_{l_p}$ and $||u^n||_{L_p} = ||E_k^n v||_{L_p} \leq ||v||_{L_p}$.

- The **symbol** or characteristic polynomial of E_k is defined as follows,

$$\tilde{E}(\xi) = \sum_p a_p e^{-ip\xi}.$$

  If we make use of the following Fourier series and Fourier transform,

$$\hat{V}(\xi) = h \sum_{j=-\infty}^{\infty} V_j e^{-ij\xi}$$

$$\hat{v}(\xi) = \int_{-\infty}^{\infty} v(x) e^{-ix\xi} dx$$

  then we obtain $(E_k V)\hat{}(\xi) = \tilde{E}(\xi)\hat{V}(\xi)$ and $(E_k v)\hat{}(\xi) = \tilde{E}(h\xi)\hat{v}(\xi)$, which for repeated application leads to,

$$(E_k^n V)\hat{}(\xi) = \tilde{E}^n(\xi)\hat{V}(\xi)$$
$$(E_k^n v)\hat{}(\xi) = \tilde{E}^n(h\xi)\hat{v}(\xi).$$

- Using Parseval's theorem one arrives at the **von Neumann's stability condition**, namely, $|\tilde{E}(\xi)| \leq 1|$ for all ξ is sufficient for stability in l_2 and L_2, and necessary for stability in l_∞.

- The **order of accuracy** r is defined by $\tau^n = \mathcal{O}(h^r)$ as $h \to 0$. Since $u^{n+1}(x) = E_k u^n(x) + k\tau^n(x)$, where $u^n(x)$ is now the analytical solution and k is the time step, then the order of accuracy is also obtained from $u^{n+1}(x) - E_k u^n(x) = k\mathcal{O}(h^r)$. This expression for order of accuracy is equivalent to $\tilde{E}(\xi) = e^{-\lambda\xi^2} + \mathcal{O}(|\xi|^{r+2})$, as $\xi \to 0$.

Chapter 6

Hyperbolic

For $u_t = au_x$ in $\mathbf{R} \times \mathbf{R}_+$, with initial condition $u(\cdot, 0) = v$ in $\mathbf{R}$:

- As for parabolic equations, von Neumann's condition $|\tilde{E}(\xi)| \leq 1$ for all ξ is a necessary and sufficient condition for stability in the L_2 norm.
- Order of accuracy is defined in the same manner as for the parabolic case, except that now the symbol of E_k is given by $\tilde{E}(\xi) = e^{ia\lambda\xi} + \mathcal{O}(|\xi|^{r+1})$ as $\xi \to 0$.
- **CFL condition** for stability: a necessary condition for stability is that the domain of dependence of the finite difference scheme at (x, t) contains the domain of dependence of the continuous problem.
- The Friedrichs scheme (first order accurate) follows,

$$(E_k U^n)(x) = 1/2(1 + a\lambda)U^n(x + h) + 1/2(1 - a\lambda)U^n(x - h)$$

- The Lax-Wendroff scheme (second order accurate) follows,

$$(E_k U^n)(x) = 1/2(a^2\lambda^2 + a\lambda)U^n(x + h) + (1 - a^2\lambda^2)U^n(x) + 1/2(a^2\lambda^2 - a\lambda)U^n(x - h)$$

Part IV

Finite Volume for PDE's

Chapter 7

Elliptic

Chapter 8

Parabolic

Chapter 9

Hyperbolic

9.1 One-dimensional case

Consider the hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{9.1}$$

where $f = f(u)$ is the flux of u. The finite volume method consists on discretizing the spatial domain into volumes, which we will denote by the index i, and which are defined by $x \in [x_{i-1/2}, x_{i+1/2}]$, where $x_{i-1/2}$ and $x_{i+1/2}$ represent the boundaries of the finite volume i.

We now proceed by averaging the equation over each control volume, that is

$$\frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{1+1/2}} \frac{\partial u}{\partial t} dx + \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{1+1/2}} \frac{\partial f}{\partial x} dx = 0, \tag{9.2}$$

where $\Delta x_i = x_{i+1/2} - x_{i-1/2}$. Moving the time derivative and using the Divergence theorem, the above becomes

$$\frac{d}{dt}\left(\frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} u\, dx\right) + \frac{1}{\Delta x_i}(f_{i+1/2} - f_{i-1/2}) = 0, \tag{9.3}$$

where $f_{i\pm 1/2}$ is the flux evaluated at $x_{i\pm 1/2}$. The spatially-discrete numerical solution then satisfies

$$\frac{dU_i}{dt} + \frac{1}{\Delta x_i}(F_{i+1/2} - F_{i-1/2}) = 0, \tag{9.4}$$

where $U_i = U_i(t)$ is the discrete solution at some specific point within the finite volume, and is used to approximate the average of u. $F_{i\pm 1/2}$ are the so-called numerical fluxes.

Godunov Scheme:

$$F^n_{j+1/2} = \begin{cases} \min_{U_j \le u \le U_{j+1}} f(u) & \text{for} \quad U_j < U_{j+1} \\ \min_{U_{j+1} \le u \le U_j} f(u) & \text{for} \quad U_j > U_{j+1} \end{cases}$$

Lax-Friedrichs:

$$F^n_{j+1/2} = \frac{h}{2k}(U_j - U_{j+1} + \frac{1}{2}(f(U_j) + f(U_{j+1}))$$

9.2 Multi-dimensional case

We follow a similar approach for the multidimensional case. Consider a generic conservation equation

$$\frac{\partial w_i}{\partial t} + \frac{\partial f^{(c)}_{ij}}{\partial x_j} = \frac{\partial f^{(v)}_{ij}}{\partial x_j} + q_i \tag{9.5}$$

where w_i is the vector of conservative variables, $f_{ij}^{(c)}$ the convective flux tensor, $f_{ij}^{(v)}$ the viscous flux tensor, and q_i some heat source.

Averaging the equation over a generic finite volume denoted by the index I, one obtains

$$\frac{d}{dt}\left(\frac{1}{\Omega_I}\int_{\Omega_I} w_i \, dV\right) + \frac{1}{\Omega_I}\int_{\delta\Omega_I} f_i^{(c)} \, dS = \frac{1}{\Omega_I}\int_{\delta\Omega_I} f_i^{(v)} \, dS + \frac{1}{\Omega_I}\int_{\Omega_I} q_i \, dV, \tag{9.6}$$

where $f_i^{(c)} = f_{ij}^{(c)} n_j$ and $f_i^{(v)} = f_{ij}^{(v)} n_j$ are the vectors of convective and viscous fluxes, respectively. In vector notation, this is written as

$$\frac{d}{dt}\left(\frac{1}{\Omega_I}\int_{\Omega_I} \mathbf{w} \, dV\right) + \frac{1}{\Omega_I}\int_{\delta\Omega_I} \mathbf{f}^{(c)} \, dS = \frac{1}{\Omega_I}\int_{\delta\Omega_I} \mathbf{f}^{(v)} \, dS + \frac{1}{\Omega_I}\int_{\Omega_I} \mathbf{q} \, dV. \tag{9.7}$$

A specific example is the Navier-Stokes equations, for which we have

$$\mathbf{w} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix} \qquad \mathbf{f}^{(c)} = \begin{bmatrix} \rho(u_j n_j) \\ \rho u(u_j n_j) + p n_1 \\ \rho v(u_j n_j) + p n_2 \\ \rho w(u_j n_j) + p n_3 \\ \rho\left(E + \frac{p}{\rho}\right)(u_j n_j) \end{bmatrix} \qquad \mathbf{f}^{(v)} = \begin{bmatrix} 0 \\ \tau_{1j} n_j \\ \tau_{2j} n_j \\ \tau_{3j} n_j \\ u_i \tau_{ij} n_j + \kappa \frac{\partial T}{\partial x_j} n_j \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} 0 \\ \rho g_1 \\ \rho g_2 \\ \rho g_3 \\ \rho u_i g_i \end{bmatrix}. \tag{9.8}$$

The spatially-discrete numerical solution then satisfies

$$\frac{d\mathbf{W}_I}{dt} + \frac{1}{\Omega_I}\sum_{K\in N(I)} \mathbf{F}_K^{(c)} \, \Delta S_K = \frac{1}{\Omega_I}\sum_{K\in N(I)} \mathbf{F}_K^{(v)} \, \Delta S_K + \mathbf{Q}_I \tag{9.9}$$

where $\mathbf{W}_I = \mathbf{W}_I(t)$ is the numerical solution at some specific point within the finite volume, and is used to approximate the average of the vector $\mathbf{w}$. The faces of the finite volumes are indexed, and the set $N(I)$ consists of the indices of the faces of the finite volume I. The variables $\mathbf{F}_K^{(c)} = \mathbf{F}_K^{(c)}(\mathbf{W}_1, \mathbf{W}_2, \dots)$ and $\mathbf{F}_K^{(v)} = \mathbf{F}_K^{(v)}(\mathbf{W}_1, \mathbf{W}_2, \dots)$ are the numerical fluxes, and ΔS_K the surface area, for face K. The averaged source term $\mathbf{Q}_I = \mathbf{Q}_I(\mathbf{W}_I)$ is approximated as the source vector $\mathbf{q}$ evaluated using $\mathbf{W}_I$. The above is typically rewritten as

$$\frac{d\mathbf{W}_I}{dt} = -\frac{1}{\Omega_I}\mathbf{R}_I, \tag{9.10}$$

where the residual $\mathbf{R}_I = \mathbf{R}_I(\mathbf{W}_1, \mathbf{W}_2, \dots)$ is given by

$$\mathbf{R}_I = \sum_{K\in N(I)} \mathbf{F}_K^{(c)} \, \Delta S_K - \sum_{K\in N(I)} \mathbf{F}_K^{(v)} \, \Delta S_K - \mathbf{Q}_I \, \Omega_I \tag{9.11}$$

The Jacobian is then

$$\frac{\partial \mathbf{R}_I}{\partial \mathbf{W}_J} = \sum_{K\in N(I)} \frac{\partial \mathbf{F}_K^{(c)}}{\partial \mathbf{W}_J} \, \Delta S_K - \sum_{K\in N(I)} \frac{\partial \mathbf{F}_K^{(v)}}{\partial \mathbf{W}_J} \, \Delta S_K - \frac{\partial \mathbf{Q}_I}{\partial \mathbf{W}_J} \, \Omega_I \tag{9.12}$$

Note the the Jacobian is only non zero when $J \in M(I)$, where $M(I)$ represents the set of all finite volumes that the numerical fluxes of finite volume I depend on.

9.2.1 An example from turbulence modeling

Consider the SST turbulence model. The two transport equations solved by the model are

$$\frac{\partial \rho k}{\partial t} + \frac{\partial \rho k u_j}{\partial x_j} = P - \beta^* \rho w k + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_k \mu_t)\frac{\partial k}{\partial x_j}\right] \tag{9.13}$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho w u_j}{\partial x_j} = \frac{\gamma}{\nu_t} P - \beta \rho w^2 + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_w \mu_t)\frac{\partial w}{\partial x_j}\right] + 2(1 - F_1)\frac{\rho \sigma_{w2}}{w}\frac{\partial k}{\partial x_j}\frac{\partial w}{\partial x_j} \tag{9.14}$$

The vector of conservative variables is $\mathbf{w} = [\rho k, \rho w]^T$. The convective and viscous flux vectors are

$$\mathbf{f}^{(c)} = \begin{bmatrix} \rho k (u_j n_j) \\ \rho w (u_j n_j) \end{bmatrix} \qquad \mathbf{f}^{(v)} = \begin{bmatrix} (\mu + \sigma_k \mu_t)\frac{\partial k}{\partial x_j} n_j \\ (\mu + \sigma_w \mu_t)\frac{\partial w}{\partial x_j} n_j \end{bmatrix}. \tag{9.15}$$

The source $\mathbf{q}$ is

$$\mathbf{q} = \begin{bmatrix} P - \beta^* \rho w k \\ \frac{\gamma}{\nu_t} P - \beta \rho w^2 + 2(1 - F_1)\frac{\rho \sigma_{w2}}{w}\frac{\partial k}{\partial x_j}\frac{\partial w}{\partial x_j} \end{bmatrix} \tag{9.16}$$

Convective numerical fluxes: A simple upwinding scheme is as follows

$$\mathbf{F}_K^{(c)} = \begin{bmatrix} (\rho k)_l a_0 + (\rho k)_r a_1 \\ (\rho w)_l a_0 + (\rho w)_r a_1 \end{bmatrix} \tag{9.17}$$

where subscripts l and r denote values at the center nodes of the control volumes to the left and right of the K^{th} control surface, respectively. The variables a_0 and a_1 are given by

$$a_0 = \begin{cases} q_{lr} & \text{if } q_{lr} > 0 \\ 0 & \text{o.w.} \end{cases} \qquad a_1 = \begin{cases} 0 & \text{if } q_{lr} > 0 \\ q_{lr} & \text{o.w.} \end{cases} \tag{9.18}$$

where

$$q_{lr} = \frac{(\mathbf{u}_l \cdot \mathbf{n}) + (\mathbf{u}_r \cdot \mathbf{n})}{2}. \tag{9.19}$$

The corresponding Jacobian is

$$\frac{\partial \mathbf{F}_K^{(c)}}{\partial \mathbf{W}_J} = \begin{bmatrix} \frac{\partial \mathbf{F}_K^{(c)}(1)}{\partial \mathbf{W}_J(1)} & \frac{\partial \mathbf{F}_K^{(c)}(1)}{\partial \mathbf{W}_J(2)} \\ \frac{\partial \mathbf{F}_K^{(c)}(2)}{\partial \mathbf{W}_J(1)} & \frac{\partial \mathbf{F}_K^{(c)}(2)}{\partial \mathbf{W}_J(2)} \end{bmatrix} = \begin{bmatrix} a_0 \delta_{lJ} + a_1 \delta_{rJ} & 0 \\ 0 & a_0 \delta_{lJ} + a_1 \delta_{rJ} \end{bmatrix}. \tag{9.20}$$

Viscous numerical fluxes: the viscous fluxes can be discretized as follows

$$\mathbf{F}_K^{(v)} = \begin{bmatrix} (\mu + \sigma_k \mu_t)_{avg} \left[\frac{k_r - k_l}{l_{lr}} \mathbf{t}_{lr} + \nabla k_{avg} - (\nabla k_{avg} \cdot \mathbf{t}_{lr})\, \mathbf{t}_{lr}\right] \cdot \mathbf{n} \\ (\mu + \sigma_\omega \mu_t)_{avg} \left[\frac{\omega_r - \omega_l}{l_{lr}} \mathbf{t}_{lr} + \nabla \omega_{avg} - (\nabla \omega_{avg} \cdot \mathbf{t}_{lr})\, \mathbf{t}_{lr}\right] \cdot \mathbf{n} \end{bmatrix}. \tag{9.21}$$

where the subscript $_{avg}$ means the left and right values have been averaged, $\mathbf{t}_{lr}$ is the unit vector that points from the center of the l^{th} control volume (the one to the left of the face) to the center of the r^{th} control volume (the one to the right of the face), and l_{lr} is the length between these two centers. The corresponding Jacobian is approximated as follows

$$\frac{\partial \mathbf{F}_K^{(v)}}{\partial \mathbf{W}_J} = \begin{bmatrix} \frac{\partial \mathbf{F}_K^{(v)}(1)}{\partial \mathbf{W}_J(1)} & \frac{\partial \mathbf{F}_K^{(v)}(1)}{\partial \mathbf{W}_J(2)} \\ \frac{\partial \mathbf{F}_K^{(v)}(2)}{\partial \mathbf{W}_J(1)} & \frac{\partial \mathbf{F}_K^{(v)}(2)}{\partial \mathbf{W}_J(2)} \end{bmatrix}, \tag{9.22}$$

where

$$\frac{\partial \mathbf{F}_K^{(v)}(1)}{\partial \mathbf{W}_J(1)} = (\mu + \sigma_k \mu_t)_{avg} \frac{1}{l_{lr}} \mathbf{t}_{lr} \cdot \mathbf{n} \frac{\delta_{rJ}}{\rho_r} - (\mu + \sigma_k \mu_t)_{avg} \frac{1}{l_{lr}} \mathbf{t}_{lr} \cdot \mathbf{n} \frac{\delta_{lJ}}{\rho_l}, \tag{9.23}$$

$$\frac{\partial \mathbf{F}_K^{(v)}(2)}{\partial \mathbf{W}_J(2)} = (\mu + \sigma_\omega \mu_t)_{avg} \frac{1}{l_{lr}} \mathbf{t}_{lr} \cdot \mathbf{n} \frac{\delta_{rJ}}{\rho_r} - (\mu + \sigma_\omega \mu_t)_{avg} \frac{1}{l_{lr}} \mathbf{t}_{lr} \cdot \mathbf{n} \frac{\delta_{lJ}}{\rho_l} \tag{9.24}$$

and $\frac{\partial \mathbf{F}_K^{(v)}(1)}{\partial \mathbf{W}_J(2)} = \frac{\partial \mathbf{F}_K^{(v)}(2)}{\partial \mathbf{W}_J(1)} = 0$.

Sources: The discretized source terms are computed following eq. (9.16). The Jacobian is expressed as

$$\frac{\partial \mathbf{Q}_I}{\partial \mathbf{W}_J} = \begin{bmatrix} \frac{\partial \mathbf{Q}_I(1)}{\partial \mathbf{W}_J(1)} & \frac{\partial \mathbf{Q}_I(1)}{\partial \mathbf{W}_J(2)} \\ \frac{\partial \mathbf{Q}_I(2)}{\partial \mathbf{W}_J(1)} & \frac{\partial \mathbf{Q}_I(2)}{\partial \mathbf{W}_J(2)} \end{bmatrix}. \tag{9.25}$$

According to Wilcox p. 413, the elements of the matrix above are approximated as follows

$$\frac{\partial \mathbf{Q}_I(1)}{\partial \mathbf{W}_J(1)} \approx -\beta^* w_I \delta_{IJ} \qquad \frac{\partial \mathbf{Q}_I(2)}{\partial \mathbf{W}_J(2)} \approx -2\beta w_I \delta_{IJ} \qquad \frac{\partial \mathbf{Q}_I(1)}{\partial \mathbf{W}_J(2)} = \frac{\partial \mathbf{Q}_I(2)}{\partial \mathbf{W}_J(1)} \approx 0. \tag{9.26}$$

9.3 Implicit time integration

We combine eq. (9.10) for all I into a single equation as follows

$$\frac{d}{dt} \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} -\frac{1}{\Omega_1} \mathbf{R}_1 \\ -\frac{1}{\Omega_2} \mathbf{R}_2 \\ \vdots \end{pmatrix}. \tag{9.27}$$

Note that the above is of the same form as eq. (2.2). Thus, the corresponding form of eq. (2.12) would be

$$\left[\frac{1}{\Delta t} \begin{pmatrix} \mathbf{I} & 0 & \dots \\ 0 & \mathbf{I} & \\ \vdots & & \ddots \end{pmatrix} - \begin{pmatrix} -\frac{1}{\Omega_1} \frac{\partial \mathbf{R}_1(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_1} & -\frac{1}{\Omega_1} \frac{\partial \mathbf{R}_1(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_2} & \dots \\ -\frac{1}{\Omega_2} \frac{\partial \mathbf{R}_2(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_1} & -\frac{1}{\Omega_2} \frac{\partial \mathbf{R}_2(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_1} & \\ \vdots & & \ddots \end{pmatrix}_{\mathbf{x}_\alpha = \mathbf{W}_\alpha^n} \right] \begin{pmatrix} \Delta \mathbf{W}_1 \\ \Delta \mathbf{W}_2 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\Omega_1} \mathbf{R}_1(\mathbf{W}_1^n, \mathbf{W}_2^n, \dots) \\ -\frac{1}{\Omega_2} \mathbf{R}_2(\mathbf{W}_1^n, \mathbf{W}_2^n, \dots) \\ \vdots \end{pmatrix}. \tag{9.28}$$

Multiplying each of the major rows above one obtains

$$\left[\frac{1}{\Delta t} \begin{pmatrix} \Omega_1 \mathbf{I} & 0 & \dots \\ 0 & \Omega_2 \mathbf{I} & \\ \vdots & & \ddots \end{pmatrix} + \begin{pmatrix} \frac{\partial \mathbf{R}_1(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{R}_1(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_2} & \dots \\ \frac{\partial \mathbf{R}_2(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{R}_2(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_2} & \\ \vdots & & \ddots \end{pmatrix}_{\mathbf{x}_\alpha = \mathbf{W}_\alpha^n} \right] \begin{pmatrix} \Delta \mathbf{W}_1 \\ \Delta \mathbf{W}_2 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} -\mathbf{R}_1(\mathbf{W}_1^n, \mathbf{W}_2^n, \dots) \\ -\mathbf{R}_2(\mathbf{W}_1^n, \mathbf{W}_2^n, \dots) \\ \vdots \end{pmatrix}. \tag{9.29}$$

As previously noted, the only Jacobians $\partial \mathbf{R}_I / \partial \mathbf{W}_J$ that are non-zero are those for which $J \in M(I)$. Thus, the second submatrix of the above is mostly sparse. The Ith major row can be written as

$$\sum_{J=M(I)} \left[\frac{\Omega_I}{\Delta t} \delta_{IJ} + \left. \frac{\partial \mathbf{R}_I(\mathbf{x}_1, \mathbf{x}_2, \dots)}{\partial \mathbf{x}_J} \right|_{\mathbf{x}_\alpha = \mathbf{W}_\alpha^n} \right] \Delta \mathbf{W}_J = -\mathbf{R}_I(\mathbf{W}_1^n, \mathbf{W}_2^n, \dots). \tag{9.30}$$

Part V

Finite Element for PDE's

Chapter 10

Introduction

10.1 The Galerkin method

10.1.1 Weak form or variational formulation

Consider the problem in which a scalar $u = u(\mathbf{x}, t)$ is sought as the solution to the following

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au = f && \text{in } \Omega \times \mathbf{R}_+ \\ &u = 0 && \text{on } \Gamma \times \mathbf{R}_+ \\ &u(\cdot, 0) = u_0 && \text{in } \Omega \end{aligned} \tag{10.1}$$

where A is any given linear operator, Ω is the spatial domain, Γ its boundary, and $\mathbf{R}_+$ the set of positive real numbers. The weak formulation of the above is to find the weak solution $u \in U$ such that

$$\left(\frac{\partial u}{\partial t}, v\right) + (Au, v) = (f, v) \qquad \forall v \in V, \tag{10.2}$$

where $(a, b) = \int_\Omega av \, d\mathbf{x}$ is the L_2 inner product, U is some functional space, and V is either the same as U or a different functional space. Using Calculus identities, some of the derivatives in (Au, v) can be moved from operating on u to operating on v. We label the end result as $a(u, v)$, which is a bilinear form. We also label (f, v) as $L(v)$, which is a linear form. Thus, the above is rewritten as

$$\left(\frac{\partial u}{\partial t}, v\right) + a(u, v) = L(v) \qquad \forall v \in V. \tag{10.3}$$

Different linear operators A lead to different bilinear forms. A few examples are shown in the table below

Operator	Definition	Bilinear Form
Mass	λu	$(\lambda u, v)$
Diffusion	$-\nabla \cdot (\lambda \nabla u)$	$(\lambda \nabla u, \nabla v)$
Convection	$\boldsymbol{\lambda} \cdot \nabla u$	$(\boldsymbol{\lambda} \cdot \nabla u, v)$

10.1.2 Discretization into finite spaces

Introduce $U_h \subset U$ and $V_h \subset V$ as finite dimensional spaces, that is, spaces with a finite number of basis elements. We now aim to find $u_h \in U_h$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h. \tag{10.4}$$

Label the finite basis of U_h as $\{\Phi_i\}_{i=1}^n$, where n is the total number of basis elements (also referred to as degrees of freedom), and $\Phi_i = \Phi_i(\mathbf{x})$. For the sake of simplicity, lets use $V_h = U_h$ in this description.

Any $v_h \in V_h$ can be written as $v_h = \sum_{i=1}^n V_i \Phi_i$, where $V_i = V_i(t)$. Equation (10.4) is then automatically satisfied if the following is satisfied

$$\left(\frac{\partial u_h}{\partial t}, \Phi_i\right) + a(u_h, \Phi_i) = (f, \Phi_i) \qquad \text{for } i = 1, ..., n. \tag{10.5}$$

Since $u_h \in U_h$, we can write $u_h = \sum_{j=1}^n U_j \Phi_j$, where $U_j = U_j(t)$. Thus, the above becomes

$$\sum_{j=1}^n (\Phi_j, \Phi_i) \frac{dU_j}{dt} + \sum_{j=1}^n U_j a(\Phi_j, \Phi_i) = (f, \Phi_i) \qquad \text{for } i = 1, ..., n. \tag{10.6}$$

Define the vectors $\mathbf{U}$ and $\mathbf{b}$, as those whose components are U_j and (f, Φ_i), respectively. Define the matrix $\mathbf{A}$ as that whose components are $a_{ij} = a(\Phi_j, \Phi_i)$. Similarily, define the mass matrix $\mathbf{M}$ as that whose components are $m_{i,j} = (\Phi_j, \Phi_i)$. Then, the equation above can be written as

$$\mathbf{M} \frac{d\mathbf{U}}{dt} + \mathbf{A}\mathbf{U} = \mathbf{b}. \tag{10.7}$$

10.1.3 The discontinuous Galerkin method

10.1.4 The Petrov Galerkin method

10.2 The collocation method

For the collocation method, we again aim to solve eq. (10.1) by assuming the solution is $u_h \in U_h$. This assumption allows us to write $u_h = \sum_{j=1}^n U_j \Phi_j$ and thus the governing PDE can be written as

$$\sum_{j=1}^n \frac{dU_j}{dt} \Phi_j + \sum_{j=1}^n U_j A \Phi_j = f. \tag{10.8}$$

The above equation is evaluated at a discrete set of n points, termed the collocation points. Thus we have

$$\sum_{j=1}^n \frac{dU_j}{dt} \Phi_j(x_i) + \sum_{j=1}^n U_j (A\Phi_j)(x_i) = f(x_i) \qquad \text{for } i = 1, ..., n. \tag{10.9}$$

The above can be written as

$$\tilde{\mathbf{M}} \frac{d\mathbf{U}}{dt} + \tilde{\mathbf{A}}\mathbf{U} = \tilde{\mathbf{b}}, \tag{10.10}$$

where $\tilde{\mathbf{M}}$ is a matrix whose components are $\tilde{m}_{ij} = \Phi_j(x_i)$, $\tilde{\mathbf{A}}$ is a matrix whose components are $a_{ij} = (A\Phi_j)(x_i)$, and $\tilde{\mathbf{b}}$ is a vector whose components are $f(x_i)$.

10.3 The spectral method

Spectral methods can be based on the Galerkin, spectral, or other methods, but have the defining property that U_h is the space of functions that are global within Ω; that is, they span all of Ω.

10.4 The finite element method

The finite element method is typically based on the Galerkin method, and has as its defining property the use of a functional space U_h that consists of functions that are local; that is, functions that are non-zero only among specific zones commonly referred to as elements.

Chapter 11

Elliptic

Chapter 12

Parabolic

Chapter 13

Hyperbolic

Appendix A

Notes on functional analysis

A.1 Classification of methods

- Galerkin method: introduce trial functions, and reformulate the PDE in the weak form using those trial functions. Then express the solution in terms of test functions
- Collocation method: trial functions not equal to test functions (test functions are delta functions)
- Finite element method: local trial and test functions
- Spectral method: global trial functions

A.2 Some terminology

- Cauchy sequence: a sequence v_1, v_2, v_3, ... is a Cauchy sequence if for every possitive real number ϵ, there is a possitive integer N such that for all possitive integers $m, n > N$, $||v_m - v_n|| < \epsilon$.
- Complete inner produce space: an inner product space V is complete if every Cauchy sequence $\{v_i\}_{i=1}^{\infty}$ in V has a limit $v = \lim v_i \in V$.
- Compact set: a set is compact if it is bounded and closed.
- Coercive bilinear form: a bilinear form $a(:,:)$ is coercive in a Hilbert space V if

$$a(v, v) \geq \alpha ||v||_V^2, \qquad \forall v \in V, \qquad \text{with } \alpha > 0. \tag{A.1}$$

- Bounded linear form: a linear form is bounded in the normed vector space V if there exists an $M > 0$ such that $|L(v)| \leq M||v||$, for every $v \in V$.
- Bounded bilinear form: a bilinear form is bounded in the normed vector space V if there exists and $M > 0$ such that $|a(w, v)| \leq M||w||\ ||v||$, for every $w, v \in V$.

A.3 Useful equalities and inequalities

$$|ab| = |a||b| \qquad \forall a, b \in \mathbb{C} \tag{A.2}$$

$$|a + b| \leq |a| + |b| \qquad \forall a, b \in \mathbb{C} \tag{A.3}$$

$$|(w, v)| \leq ||w||\, ||v|| \qquad \forall v, w \in \text{Inner product space (Cauchy-Schwarz inequality)} \tag{A.4}$$

$$||w + v|| \leq ||w|| + ||v|| \qquad \forall v, w \in \text{Normed space (triangle inequality)} \tag{A.5}$$

$$\left| \int_a^b v(x)\, dx \right| \leq \int_a^b |v(x)|\, dx \tag{A.6}$$

$$\left\| \int_a^b v(x, y)\, dx \right\| \leq \int_a^b ||v(x, y)||\, dx \text{ where the norm is over the y-domain.} \tag{A.7}$$

A.4 The weak derivative

If $v \in C^1(\bar{\Omega})$, then through integration by parts

$$\int_\Omega \frac{\partial v}{\partial x_i} \phi \, dx = -\int_\Omega v \frac{\partial \phi}{\partial x_i} \, dx \qquad \forall \phi \in C_0^1(\Omega). \tag{A.8}$$

However, if $v \in L_2(\Omega)$ but not necessarily in $C^1(\bar{\Omega})$, we cannot write the equation above. Instead, we ask, is there a function w such that the following holds?

$$\int_\Omega w\phi \, dx = -\int_\Omega v \frac{\partial \phi}{\partial x_i} \, dx \qquad \forall \phi \in C_0^1(\Omega). \tag{A.9}$$

This can be rewritten as $(w, \phi) = L(\phi)$, where $L(\phi) = -\int_\Omega v \frac{\partial \phi}{\partial x_i} dx$. If $L(\phi)$ is bounded in L_2, Riesz' representation theorem then states a unique solution $w \in L_2(\Omega)$ exists. This w is the weak derivative.

More generally, if $v \in C^k(\bar{\Omega})$, then through integration by parts

$$\int_\Omega D^\alpha v \phi \, dx = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, dx \qquad \forall |\alpha| \leq k, \, \forall \phi \in C_0^{|\alpha|}(\Omega). \tag{A.10}$$

However, if $v \in L_2(\Omega)$ but not necessarily in $C^k(\bar{\Omega})$, we cannot write the equation above. Instead, we ask, is there a function w such that the following holds?

$$\int_\Omega w\phi \, dx = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, dx \qquad \forall \phi \in C_0^{|\alpha|}(\Omega). \tag{A.11}$$

As before, if the left-hand-side operator is bounded, then we have a unique solution $w \in L_2(\Omega)$. This w is the weak derivative. Often, weak derivatives are referred to as "$D^\alpha v$ in the weak sense."

A.5 Function spaces

- C^0: the set of all continuous functions
- C^k the set of all functions whose derivatives up to order k all exist and are continuous. These are called continuously differentiable functions of order k.
- L_2: the set of all functions that are square integrable.

- H^k: the set of all L_2 functions whose weak partial derivatives up to order k also belong to L_2.
- Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth or polygonal boundary. Then part of the Sobolev embedding theorem can be written as

$$H^k(\Omega) \subset C^l(\bar{\Omega}) \text{ if } k > l + d/2. \tag{A.12}$$

  Thus, we have $H^m(\Omega) \subset C^{m-1}(\bar{\Omega})$ for $\Omega \in \mathbb{R}$ and $H^m(\Omega) \subset C^{m-2}(\bar{\Omega})$ for $\Omega \in \mathbb{R}^2$ or $\mathbb{R}^3$.

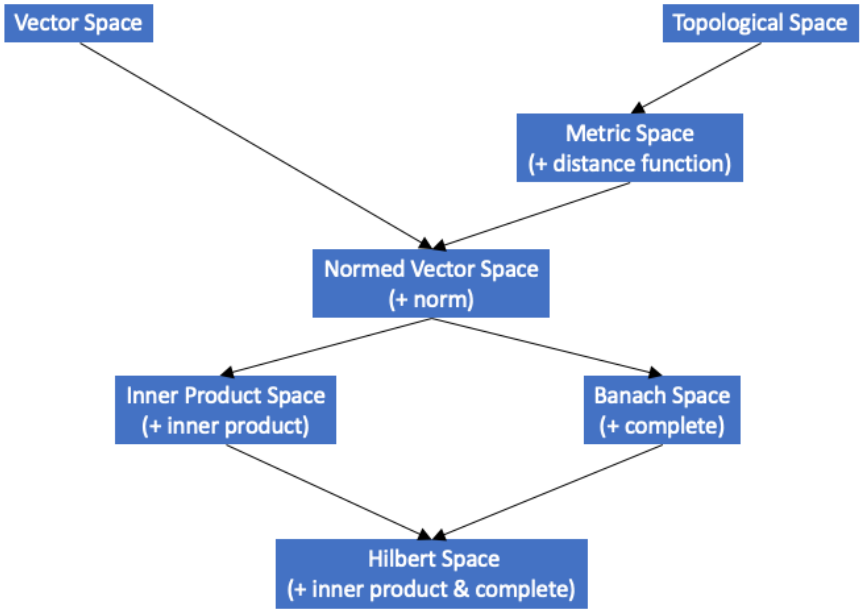

Figure A.1: Map of function spaces.

Space	Norm	Inner product
$C(M)$	$\|\|v\|\|_C = \sup_{x \in M} \|v(x)\|$	X
$C^k(M)$	$\|\|v\|\|_{C^k} = \max_{\|\alpha\| \leq k} \|\|D^\alpha v\|\|_C$ $\|v\|_{C^k} = \max_{\|\alpha\|=k} \|\|D^\alpha v\|\|_C$	X
$L_p(\Omega)$	$\|\|v\|\|_{L_p} = \left(\int_\Omega \|v\|^p \, dx \right)^{1/p}$	X
$L_2(\Omega)$	$\|\|v\|\|_{L_2} = \left(\int_\Omega \|v\|^2 \, dx \right)^{1/2}$	$(v, w) = \int_\Omega v w^* \, dx$
$H^k(\Omega)$	$\|\|v\|\|_k = \left(\sum_{\|\alpha\| \leq k} \|\|D^\alpha v\|\|^2 \right)^{1/2}$ $\|v\|_k = \left(\sum_{\|\alpha\|=k} \|\|D^\alpha v\|\|^2 \right)^{1/2}$	$(v, w)_k = \sum_{\|\alpha \leq k\|} (D^\alpha v, D^\alpha w)$