Spectral Methods

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Contents

A continuous function is denoted as $f(x)$ whereas a discrete function is denoted as f_m. Vectors are denoted in bold.

1 Fourier Analysis

1.1 Fourier Series

- Definition:

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{n}=-\infty}^{\infty} \hat{\mathbf{f}}_{\mathbf{n}} e^{i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}$$

$$\hat{\mathbf{f}}_{\mathbf{n}} = \frac{1}{L^3}\int_{\mathbb{L}^3} \mathbf{f}(\mathbf{x}) e^{-i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}} d\mathbf{x}$$

where

$$\mathbf{k_n} = \frac{2\pi}{L}\mathbf{n} \qquad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

Note: $\sum_{\mathbf{n}=-\infty}^{\infty} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty}$

- Parseval's identity:

$$\frac{1}{L^3} \int_{\mathbb{L}^3} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}^*(\mathbf{x})\, d\mathbf{x} = \sum_{\mathbf{n}=-\infty}^{\infty} \hat{\mathbf{f}}_\mathbf{n} \cdot \hat{\mathbf{g}}_\mathbf{n}^*$$

1.2 Discrete Fourier Series

- Definition:

$$\mathbf{f_m} = \sum_{\mathbf{n}=-N/2}^{N/2-1} \hat{\mathbf{f}}_\mathbf{n} e^{i\mathbf{k_n}\cdot\mathbf{x_m}}$$

$$\hat{\mathbf{f}}_\mathbf{n} = \frac{1}{N^3} \sum_{\mathbf{m}=0}^{N-1} \mathbf{f_m} e^{-i\mathbf{k_n}\cdot\mathbf{x_m}}$$

where

$$\mathbf{k_n} = \frac{2\pi}{L}\mathbf{n} \qquad \mathbf{x_m} = \frac{L}{N}\mathbf{m} \qquad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \qquad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

- Parseval's identity:

$$\frac{1}{N^3} \sum_{\mathbf{m}=0}^{N-1} \mathbf{f_m} \cdot \mathbf{g_m^*} = \sum_{\mathbf{n}=-N/2}^{N/2-1} \hat{\mathbf{f}}_\mathbf{n} \cdot \hat{\mathbf{g}}_\mathbf{n}^*$$

1.3 Fourier Transform

- Definition:

$$\mathbf{f}(\mathbf{x}) = \int_{\mathbb{R}^n} \hat{\mathbf{f}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}$$

$$\hat{\mathbf{f}}(\mathbf{k}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}$$

- Common functions

$f(\mathbf{x})$	$\hat{f}(\mathbf{k})$
$e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}$	$\delta(\mathbf{k}-\boldsymbol{\lambda})$
$\delta(\mathbf{x}-\mathbf{y})$	$\frac{1}{(2\pi)^n} e^{-i\mathbf{k}\cdot\mathbf{y}}$

- Parseval's Identity:
$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathbf{f}(\mathbf{x})\cdot\mathbf{g}^*(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^3}\hat{\mathbf{f}}(\mathbf{k})\hat{\mathbf{g}}^*(\mathbf{k})\,d\mathbf{k}$$

- Convolution:
  Given
$$h(\mathbf{x}) = \int_{\mathbb{R}^3} f(\mathbf{x}-\mathbf{s})g(\mathbf{s})\,dx$$
  then
$$\hat{h}(\mathbf{k}) = (2\pi)^3\hat{f}(\mathbf{k})\hat{g}(\mathbf{k})$$

1.4 Spectral forms of common terms

We will use in this section both the hat notation and the $\mathcal{F}$ notation. That is, for the Fourier coefficient of a Fourier series, we use
$$\hat{\mathbf{f}}_{\mathbf{n}} = \mathcal{F}^{(s)}\{\mathbf{f}(\mathbf{x})\}_{\mathbf{n}},$$
and for the Fourier coefficient of a Fourier transform, we use
$$\hat{\mathbf{f}}(\mathbf{k}) = \mathcal{F}^{(t)}\{\mathbf{f}(\mathbf{x})\}(\mathbf{k}).$$

- General derivative:
$$\mathcal{F}^{(s)}\left\{\frac{\partial\mathbf{f}(\mathbf{x})}{\partial x_j}\right\}_{\mathbf{n}} = i\kappa_{\mathbf{n},j}\hat{\mathbf{f}}_{\mathbf{n}}$$
$$\mathcal{F}^{(t)}\left\{\frac{\partial\mathbf{f}(\mathbf{x})}{\partial x_j}\right\}(\mathbf{k}) = i\kappa_j\hat{\mathbf{f}}(\mathbf{k})$$

- Derivative in spectral space:
$$\frac{\partial\hat{\mathbf{f}}(\mathbf{k})}{\partial\kappa_j} = -i\mathcal{F}^{(t)}\left\{x_j\mathbf{f}(\mathbf{x})\right\}(\mathbf{k})$$

- Divergence:
$$\mathcal{F}^{(s)}\left\{\nabla\cdot\mathbf{f}(\mathbf{x})\right\}_{\mathbf{n}} = i\mathbf{k}_{\mathbf{n}}\cdot\hat{\mathbf{f}}_{\mathbf{n}}$$
$$\mathcal{F}^{(t)}\left\{\nabla\cdot\mathbf{f}(\mathbf{x})\right\}(\mathbf{k}) = i\mathbf{k}\cdot\hat{\mathbf{f}}(\mathbf{k})$$

- Curl:
$$\mathcal{F}^{(s)}\left\{\nabla\times\mathbf{f}(\mathbf{x})\right\}_{\mathbf{n}} = i\mathbf{k}_{\mathbf{n}}\times\hat{\mathbf{f}}_{\mathbf{n}}$$
$$\mathcal{F}^{(t)}\left\{\nabla\times\mathbf{f}(\mathbf{x})\right\}(\mathbf{k}) = i\mathbf{k}\times\hat{\mathbf{f}}(\mathbf{k})$$

- Laplacian:
$$\mathcal{F}^{(s)}\left\{\nabla^2\mathbf{f}(\mathbf{x})\right\}_{\mathbf{n}} = -\kappa_{\mathbf{n}}^2\hat{\mathbf{f}}(\mathbf{k})$$
$$\mathcal{F}^{(t)}\left\{\nabla^2\mathbf{f}(\mathbf{x})\right\}(\mathbf{k}) = -\kappa^2\hat{\mathbf{f}}(\mathbf{k})$$

2 Chebyshev Analysis

2.1 Chebyshev Series

- Definition:

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

$$a_n = \frac{2}{\pi C_n} \int_{-1}^{1} f(x) T_n(x) \frac{dx}{\sqrt{1-x^2}}$$

where

$$C_n = \begin{cases} 2 & n = 0 \\ 1 & O.W. \end{cases}$$

2.2 Discrete Chebyshev Series

- Definition:

$$f_j = \sum_{n=0}^{\infty} a_n T_n(x_j)$$

$$a_n = \frac{2}{NC_n} \sum_{j=0}^{N} \frac{1}{C_j} f_j T_n(x_j)$$

where

$$C_n = \begin{cases} 2 & n = 0, N \\ 1 & O.W. \end{cases}$$

3 Classical Orthogonal Polynimials

Orthogonal polynomials are the members of the set $\{P_n(x)\}_{n=1}^{\infty}$, where $P_n(x)$ is a polynomial of degree n.

They satisfy the orthogonality relation:

$$\langle P_n, P_m \rangle = \int_a^b P_n(x) P_m(x) w(x) dx = \langle P_n, P_n \rangle \delta_{nm}$$

These orthogonal polynomials satisfy the following ODE,

$$g_2(x) P_n'' + g_1(x) P_n' + a_n P_n = 0$$

and are generated from the Rodrigues' formula:

$$P_n(x) = \frac{1}{e_n w(x)} \frac{d^n}{dx^n} \{w(x)[g(x)]^n\}$$

The polynomials considered in this file are also solutions of the Sturm-Liouville BVP, that is, they satisfy the following ODE and appropriate boundary conditions.

$$\frac{1}{r(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]P_n=-\lambda_n^2 P_n$$

The polynomials are also orthogonal with respect to $r(x)$.

3.1 Jacobi Polynomials

This is the family of polynomials for which:

$$\begin{aligned}
p(x) &= (1-x)^{\alpha+1}(1+x)^{\beta+1} \\
q(x) &= 0 \\
r(x) &= (1-x)^{\alpha}(1+x)^{\beta} \\
\lambda_n^2 &= n(n+\alpha+\beta+1)
\end{aligned}$$

3.1.1 Chebyshev

Orthogonal basis of $L_w^2[-1,1]$, with

$$w(x)=\frac{1}{\sqrt{1-x^2}}$$

$$\langle P_n, P_n\rangle = \begin{cases} \pi/2 & \text{if } n=m\neq 0 \\ \pi & \text{if } n=m=0 \end{cases}$$

Coefficients of common form ODE

$$\begin{aligned}
g_1(x) &= -x \\
g_2(x) &= 1-x^2 \\
a_n &= n^2
\end{aligned}$$

Coefficients of Rodrigues' formula

$$\begin{aligned}
g(x) &= 1-x^2 \\
e_n &= (-1)^n(2n-1)(2n-3)...1
\end{aligned}$$

Coefficients of Sturm-Liouville ODE

$$\begin{aligned}
p(x) &= \sqrt{1-x^2} \\
q(x) &= 0 \\
r(x) &= \frac{1}{\sqrt{1-x^2}} \\
\lambda_n^2 &= n^2
\end{aligned}$$

That is, $\alpha=\beta=-1/2$.

3.1.2 Legendre

Orthogonal basis of $L^2_w[-1,1]$, with

$$w(x) = \frac{1}{2}$$

$$\langle P_n, P_n \rangle = \frac{1}{2n+1}$$

Coefficients of common form ODE

$$\begin{aligned} g_1(x) &= -2x \\ g_2(x) &= 1 - x^2 \\ a_n &= n(n+1) \end{aligned}$$

Coefficients of Rodrigues' formula

$$\begin{aligned} g(x) &= 1 - x^2 \\ e_n &= (-1)^n 2^n n! \end{aligned}$$

Coefficients of Sturm-Liouville ODE

$$\begin{aligned} p(x) &= 1 - x^2 \\ q(x) &= 0 \\ r(x) &= 1 \\ \lambda_n^2 &= n(n+1) \end{aligned}$$

That is, $\alpha = \beta = 0$.

3.2 Hermite

Orthogonal basis of $L^2_w[-\infty,\infty]$, with

$$w(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\langle P_n, P_n \rangle = n!$$

Coefficients of common form ODE

$$\begin{aligned} g_1(x) &= -x \\ g_2(x) &= 1 \\ a_n &= n \end{aligned}$$

Coefficients of Rodrigues' formula

$$\begin{aligned} g(x) &= 1 \\ e_n &= (-1)^n \end{aligned}$$

Coefficients of Sturm-Liouville ODE

$$\begin{aligned} p(x) &= e^{-x^2/2} \\ q(x) &= 0 \\ r(x) &= e^{-x^2/2} \\ \lambda_n^2 &= n \end{aligned}$$

3.3 Laguerre

Orthogonal basis of $L^2_w[0, \infty]$, with

$$w(x) = \frac{1}{\sqrt{2\pi}} e^{-x}$$

$$\langle P_n, P_n \rangle = 1$$

Coefficients of common form ODE

$$\begin{aligned} g_1(x) &= 1 - x \\ g_2(x) &= x \\ a_n &= n \end{aligned}$$

Coefficients of Rodrigues' formula

$$\begin{aligned} g(x) &= x \\ e_n &= n! \end{aligned}$$

Coefficients of Sturm-Liouville ODE

$$\begin{aligned} p(x) &= xe^{-x} \\ q(x) &= 0 \\ r(x) &= e^{-x} \\ \lambda_n^2 &= n \end{aligned}$$