

Marbl

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1 Governing equations

We introduce the flow variables for density $\rho = \rho(\mathbf{x}, t)$, velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, internal energy $e = e(\mathbf{x}, t)$, and stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$. These are defined within a domain $\Omega = \Omega(t)$, which can be moving. The governing equations that dictate their evolution in the laboratory reference frame are

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}, \quad (1)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot \boldsymbol{\sigma}, \quad (2)$$

$$\rho \left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e \right) = \boldsymbol{\sigma} : \nabla \mathbf{u}. \quad (3)$$

A note on notation. The products that involve a tensor $\boldsymbol{\tau}$ can be expressed in Einstein notation as

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}, \quad (4)$$

$$\boldsymbol{\tau} \cdot \nabla \alpha = \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \quad (5)$$

$$\mathbf{f} \cdot \boldsymbol{\tau} \cdot \nabla \alpha = f_i \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \quad (6)$$

$$\boldsymbol{\tau} : \nabla \mathbf{f} = \tau_{ij} \frac{\partial f_i}{\partial x_j}. \quad (7)$$

where α is a scalar and \mathbf{f} a vector. In these notes we'll mostly be using indices i and j for FE expansions, rather than for Einstein notation.

2 Finite element expansion

We introduce the coefficients $\hat{d}_i = \hat{d}_i(t)$, $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(t)$ and $\hat{e}_i = \hat{e}_i(t)$, as well as the basis functions $\phi_i = \phi_i(\mathbf{x}, t) \in L^2$, and $w_i = w_i(\mathbf{x}, t) \in H^1$. We note that $\hat{\mathbf{u}}_i$ is a vector whose components are $\hat{u}_{i,\alpha} = \hat{u}_{i,\alpha}(t)$ for $\alpha = x, y, z$. These coefficients are used in the following expansions

$$\rho = \sum_j^{N_\phi} \hat{d}_j \phi_j, \quad (8)$$

$$\mathbf{u} = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j, \quad (9)$$

$$e = \sum_j^{N_\phi} \hat{e}_j \phi_j, \quad (10)$$

The basis functions are defined so that they are Lagrangian, that is,

$$\frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j = 0, \quad (11)$$

$$\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j = 0. \quad (12)$$

3 Semi-discrete momentum conservation

We begin by showing that

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \sum_j^{N_w} \left(\frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \frac{\partial w_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_j^{N_w} \hat{\mathbf{u}}_j \nabla w_j \right) \\ &= \sum_j^{N_w} \left[\frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \left(\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j \right) \right] \\ &= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j. \end{aligned} \quad (13)$$

The finite element formulation of the momentum equation is thus

$$\int_{\Omega} \rho \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j w_i dV = - \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i dV \quad \text{for } i = 1, \dots, N_w. \quad (14)$$

The above is re-written as

$$\sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)} = - \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i dV \quad \text{for } i = 1, \dots, N_w. \quad (15)$$

where the mass bilinear form $m_{ij}^{(w)}$ is given by

$$m_{ij}^{(w)} = \int_{\Omega} \rho w_i w_j dV. \quad (16)$$

We now introduce the vector \mathbf{U} whose components are $\hat{\mathbf{u}}_i$. We also introduce the matrix $\mathbf{M}^{(w)}$ whose components are $m_{ij}^{(w)}$. Thus, the left-hand side of eq. (15) can be written as $\mathbf{M}^{(w)} d\mathbf{U}/dt$. We also introduce the vector bilinear form

$$\mathbf{f}_{ij} = \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j dV. \quad (17)$$

This is a *vector* bilinear form since \mathbf{f}_{ij} has components $f_{ij,\alpha} = f_{ij,\alpha}(t)$, for $\alpha = x, y, z$, where α denotes the first index of $\boldsymbol{\sigma}$. We introduce the matrix \mathbf{F} , whose components are \mathbf{f}_{ij} . We also expand the field with constant value of one as follows

$$1 = \sum_i^{N_\phi} \hat{c}_i \phi_i. \quad (18)$$

If we define the vector \mathbf{C} as that with components \hat{c}_i , we can show that

$$\begin{aligned} \mathbf{FC} &= \sum_j^{N_\phi} \mathbf{f}_{ij} \hat{c}_j && \text{for } i = 1, \dots, N_w \\ &= \sum_j^{N_\phi} \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j dV \hat{c}_j && \text{for } i = 1, \dots, N_w \\ &= \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i \left(\sum_j^{N_\phi} \hat{c}_j \phi_j \right) dV && \text{for } i = 1, \dots, N_w \\ &= \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i dV && \text{for } i = 1, \dots, N_w \end{aligned} \quad (19)$$

The above is the negative of the right-hand side of eq. (15). Thus, combining all together we get

$$\mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} = -\mathbf{FC}. \quad (20)$$

4 Semi-discrete energy conservation

As with momentum conservation, we have

$$\begin{aligned} \frac{\partial \mathbf{e}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{e} &= \sum_j^{N_\phi} \left(\frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \frac{\partial \phi_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_j^{N_\phi} \hat{\mathbf{e}}_j \nabla \phi_j \right) \\ &= \sum_j^{N_\phi} \left[\frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \left(\frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j \right) \right] \\ &= \sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j. \end{aligned} \quad (21)$$

For the right-hand side of the energy conservation equation, we have

$$\boldsymbol{\sigma} : \nabla \mathbf{u} = \boldsymbol{\sigma} : \nabla \left(\sum_k^{N_w} \hat{\mathbf{u}}_k w_k \right) = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k. \quad (22)$$

The finite element formulation of the energy equation is thus

$$\begin{aligned} \int_{\Omega} \rho \sum_j^{N_{\phi}} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j \phi_i dV &= \int_{\Omega} \left(\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k \right) \phi_i dV && \text{for } i = 1, \dots, N_w \\ &= \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_k \phi_i dV && \text{for } i = 1, \dots, N_w \end{aligned} \quad (23)$$

The above is re-written as

$$\sum_j^{N_{\phi}} \frac{d\hat{\mathbf{e}}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} \quad \text{for } i = 1, \dots, N_w. \quad (24)$$

where the mass bilinear form $m_{ij}^{(\phi)}$ is given by

$$m_{ij}^{(\phi)} = \int_{\Omega} \rho \phi_i \phi_j dV. \quad (25)$$

Note that in eq. (24) there is a dot product in the right-hand side, that is, the right-hand side expanded out is

$$\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} = \sum_k^{N_w} \sum_{\alpha=x,y,z} \hat{u}_{k,\alpha} f_{ki,\alpha}. \quad (26)$$

We now introduce the vector \mathbf{E} whose components are \hat{e}_i . We also introduce the matrix $\mathbf{M}^{(\phi)}$ whose components are $m_{ij}^{(\phi)}$. Thus, eq. (24) can be succinctly written as

$$\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{U}. \quad (27)$$

Note again that on the right-hand side above there is a matrix-vector product *and* a dot product.