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1 Governing equations

Define $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $e = e(\mathbf{x}, t)$, and $p = p(\mathbf{x}, t)$.

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \tag{2}$$

$$\rho \left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e \right) = -p \nabla \cdot \mathbf{u} \tag{3}$$

2 Finite element expansion

We introduce the coefficients $R_i = R_i(t)$, $\mathbf{U}_i = \mathbf{U}_i(t)$, $E_i = E_i(t)$, $P_i = P_i(t)$, as well as the basis functions $\phi_i = \phi_i(\mathbf{x}, t)$, and $w_i = w_i(\mathbf{x}, t)$. We note that $\mathbf{U}_i$ is a vector whose components are $U_{i,\alpha}$ for $\alpha = x, y, z$, where $U_{i,\alpha} = U_{i,\alpha}(t)$. These coefficients are used in the following expansions

$$\rho = \sum_i^{N_\rho} R_i \phi_i, \tag{4}$$

$$\mathbf{u} = \sum_i^{N_u} \mathbf{U}_i w_i, \tag{5}$$

$$e = \sum_i^{N_e} E_i \phi_i, \tag{6}$$

$$p = \sum_i^{N_p} P_i \phi_i. \tag{7}$$

The basis functions are defined so that they satisfy

$$\frac{\partial \phi_i}{\partial t} + \mathbf{u} \cdot \nabla \phi_i = 0, \tag{8}$$

$$\frac{\partial w_i}{\partial t} + \mathbf{u} \cdot \nabla w_i = 0. \tag{9}$$

3 Semi-discrete momentum conservation

We begin by showing that

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \left(\sum_j^{N_u} \frac{d\mathbf{U}_j}{dt} w_j + \mathbf{U}_j \frac{\partial w_j}{\partial t} \right) + \mathbf{u} \cdot \left(\sum_j^{N_u} \mathbf{U}_j \nabla w_j \right) \\
&= \sum_j^{N_u} \frac{d\mathbf{U}_j}{dt} w_j + \mathbf{U}_j \left(\frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j \right) \\
&= \sum_j^{N_u} \frac{d\mathbf{U}_j}{dt} w_j. \qquad (10)
\end{aligned}$$

Define $\Omega = \Omega(t)$. The finite element formulation of the momentum equation is thus

$$\int_\Omega \rho \sum_j^{N_u} \frac{d\mathbf{U}_j}{dt} w_j w_i \, dV = \int_\Omega \sum_j^{N_p} P_j \phi_j \nabla w_i \, dV \qquad \text{for } i = 1, ..., N_u. \qquad (11)$$

The above is re-written as

$$\sum_j^{N_u} \frac{d\mathbf{U}_j}{dt} m_{ij} = \sum_j^{N_p} P_j \mathbf{d}_{ji} \qquad \text{for } i = 1, ..., N_u. \qquad (12)$$

where the mass bilinear form m_{ij} is given by

$$m_{ij} = \int_\Omega \rho w_i w_j \, dV, \qquad (13)$$

and the derivative bilinear form $\mathbf{d}_{ij}$ by

$$\mathbf{d}_{ij} = \int_\Omega \phi_i \nabla w_j \, dV. \qquad (14)$$

Note that $\mathbf{d}_{ij}$ is a vector whose components are $d_{ij,\alpha}$, for $\alpha = x, y, z$, where α determines which component of the ∇ operator is being used. Equation (12) can thus be expanded as

$$\begin{aligned}
\sum_j^{N_u} \frac{dU_{j,x}}{dt} m_{ij} &= \sum_i^{N_p} P_j d_{ji,x} \qquad \text{for } i = 1, ..., N_u, \\
\sum_j^{N_u} \frac{dU_{j,y}}{dt} m_{ij} &= \sum_i^{N_p} P_j d_{ji,y} \qquad \text{for } i = 1, ..., N_u, \\
\sum_j^{N_u} \frac{dU_{j,z}}{dt} m_{ij} &= \sum_i^{N_p} P_j d_{ji,z} \qquad \text{for } i = 1, ..., N_u. \qquad (15)
\end{aligned}$$

We'll now write the above in matrix notation. Introduce the vector $\mathbf{U}_x = \mathbf{U}_x(t)$ whose components are $U_{i,x}$ for $i = 1, ..., N_v$. The analogous holds for $\mathbf{U}_y$ and $\mathbf{U}_z$. Additionally, we introduce the matrix $\mathbf{D}_x = \mathbf{D}_x(t)$ whose components are $d_{ij,x}$. The analogous holds for D_y and D_z. Finally, the vector $\mathbf{P} = \mathbf{P}(t)$ consists of the components $P_j i$. Equation (15) can now be written as

$$
\begin{aligned}
M\frac{\mathbf{U}_x}{dt} &= \mathbf{D}_x^T\mathbf{P}, \\
M\frac{\mathbf{U}_y}{dt} &= \mathbf{D}_y^T\mathbf{P}, \\
M\frac{\mathbf{U}_z}{dt} &= \mathbf{D}_z^T\mathbf{P}.
\end{aligned} \tag{16}
$$