Classical Turbulence [1]

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[1]Disclaimer: a large chunk of the work in this document is not original; instead, parts of this document are personal notes obtained from books and papers I've read, principal among which are Pope (2000), etc.

Contents

Part I

Introductory Concepts

Chapter 1

Governing Equations for Fluid Dynamics

Before we derive the conservation equations for turbulent quantities, we first briefly summarize the conservation equations for the instantaneous fluid variables.

- Conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U_i}{\partial x_i} = 0. \tag{1.1}$$

- Conservation of momentum

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial \rho U_i U_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} + \rho F_i. \tag{1.2}$$

- Conservation of total energy

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho\left(E + \frac{P}{\rho}\right)U_j\right] = \frac{\partial U_i T_{ij}}{\partial x_j} + \rho F_j U_j - \frac{\partial q_j}{\partial x_j}. \tag{1.3}$$

- Conservation of scalar

$$\frac{\partial \rho Y}{\partial t} + \frac{\partial \rho Y U_j}{\partial x_j} = -\frac{\partial J_j}{\partial x_j} + W \tag{1.4}$$

In the above, ρ is the density, U_i the velocity, Y the scalar, J_j the diffusive scalar flux, W the scalar source, P the pressure, T_{ij} the shear-stress tensor, F_i a body force, E the total energy, and q the heat-conduction flux. The diffusive scalar flux, the shear-stress tensor, and the heat-conduction flux are given by

$$T_{ij} = 2\mu S_{ij}^* \tag{1.5}$$

$$q_i = -\kappa \frac{\partial T}{\partial x_i} \tag{1.6}$$

$$J_i = -\rho D \frac{\partial Y}{\partial x_i} \tag{1.7}$$

where S_{ij}^* is the deviatoric rate-of-strain tensor, and T the temperature. The transport coefficients are the diffusivity D, the viscosity μ and the thermal conductivity μ, which are obtained from

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^n \tag{1.8}$$

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$$\kappa = \frac{\mu C_p}{Pr} \tag{1.9}$$

$$D = \frac{\mu}{\rho Sc} \tag{1.10}$$

In the above, μ_0 is a reference viscosity, T_0 a reference temperature, n the power-law exponent, Sc the Schmidt number, Pr the Prandlt number, and C_p the heat capacity at constant pressure. A perfect gas is assumed, i.e.,

$$P = \rho R T, \tag{1.11}$$

$$U = C_v T, \tag{1.12}$$

$$R = \frac{R_u}{M}. \tag{1.13}$$

To obtain closure, the following is still needed

$$E = U + K, \tag{1.14}$$

$$K = \frac{1}{2} U_i U_i, \tag{1.15}$$

$$S_{ij}^* = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial U_k}{\partial x_k} \delta_{ij} \tag{1.16}$$

From the governing equations, additional transport PDEs can be derived.

- Conservation of kinetic energy

$$\frac{\partial \rho K}{\partial t} + \frac{\partial \rho K U_j}{\partial x_j} = -\frac{\partial U_j P}{\partial x_j} + \frac{\partial U_i T_{ij}}{\partial x_j} + P \frac{\partial U_i}{\partial x_i} - T_{ij} \frac{\partial U_i}{\partial x_j} + \rho F_j U_j. \tag{1.17}$$

- Conservation of internal energy

$$\frac{\partial \rho U}{\partial t} + \frac{\partial \rho U U_j}{\partial x_j} = -\frac{\partial q_j}{\partial x_j} - P \frac{\partial U_i}{\partial x_i} + T_{ij} \frac{\partial U_i}{\partial x_j}. \tag{1.18}$$

- Conservation of squared scalar

$$\frac{\partial \rho Y^2}{\partial t} + \frac{\partial \rho Y^2 U_j}{\partial x_j} = -2 \frac{\partial Y J_j}{\partial x_j} + 2 J_j \frac{\partial Y}{\partial x_j} + 2 Y W. \tag{1.19}$$

Chapter 2

Governing Equations for Turbulent Flows

2.1 Decompositions

There are multiple ways to decompose the instantaneous velocity field U_i, depending on what properties of flow variables one wants to focus on. Here we mentioned the most common ones.

$$\text{Reynolds decomposition: } U_i = \langle U_i \rangle + u_i \tag{2.1}$$

$$\text{Favre decomposition: } U_i = \widetilde{U}_i + u_i'' \tag{2.2}$$

$$\text{LES decomposition: } U_i = \overline{U}_i + u_i' \tag{2.3}$$

For the above, we have

$$\langle U_i \rangle = \int V_i f(\mathbf{V}) d\mathbf{V} \qquad \widetilde{U}_i = \frac{\langle \rho U_i \rangle}{\langle \rho \rangle} \qquad \overline{U}_i = \int G(\mathbf{r}, \mathbf{x}) U_i(\mathbf{x} - \mathbf{r}) d\mathbf{r}, \tag{2.4}$$

where V_i is the sample space variable corresponding to U_i, and $G(\mathbf{r}, \mathbf{x})$ is the LES filter.

2.2 Favre-averaged Navier-Stokes equations

A new kind of averaging is introduced, namely, the Favre average

$$\widetilde{a} = \frac{\langle \rho a \rangle}{\langle \rho \rangle} \tag{2.5}$$

This allows us to express expectations that involve density as follows

$$\langle \rho a \rangle = \langle \rho \rangle \widetilde{a} \tag{2.6}$$

$$\langle \rho ab \rangle = \langle \rho \rangle \widetilde{ab} \tag{2.7}$$

$$\langle \rho abc \rangle = \langle \rho \rangle \widetilde{abc} \tag{2.8}$$

We also note that, for the decomposition $a = \widetilde{a} + a''$, we have

$$\widetilde{a''} = \widetilde{a - \widetilde{a}} = 0 \tag{2.9}$$

Thus, the expressions that would hold for the Reynolds average also hold for the Favre average, that is

$$\widetilde{ab} = \widetilde{a}\widetilde{b} + \widetilde{a''b''} \tag{2.10}$$

$$\widetilde{abc} = \widetilde{a}\widetilde{b}\widetilde{c} + \widetilde{a}\widetilde{b''c''} + \widetilde{b}\widetilde{a''c''} + \widetilde{c}\widetilde{a''b''} + \widetilde{a''b''c''}. \tag{2.11}$$

Taking the average of the conservation of mass equation leads to

$$\frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial \langle \rho \rangle \widetilde{U}_j}{\partial x_j} = 0. \tag{2.12}$$

Before taking the average of the conservation of momentum equation, we make use of eqs. (2.7) and (2.10) to show

$$\langle \rho U_i U_j \rangle = \langle \rho \rangle \widetilde{U_i U_j} = \langle \rho \rangle \left(\widetilde{U}_i \widetilde{U}_j + \widetilde{u_i'' u_j''} \right). \tag{2.13}$$

The second term on the right-hand side is the Favre stress tensor $\tau_{ij} = \widetilde{u_i'' u_j''}$. The average of the conservation of momentum equation can now be expressed as

$$\frac{\partial \langle \rho \rangle \widetilde{U}_i}{\partial t} + \frac{\partial \langle \rho \rangle \widetilde{U}_i \widetilde{U}_j}{\partial x_j} = -\frac{\partial \langle P \rangle}{\partial x_i} + \frac{\partial \langle T_{ij} \rangle}{\partial x_j} - \frac{\partial \langle \rho \rangle \tau_{ij}}{\partial x_j} + \langle \rho \rangle \widetilde{F}_i, \tag{2.14}$$

Before taking the average of the conservation of energy equations, we use eqs. (2.7), (2.8), (2.10) and (2.11) to obtain the following expansions

$$\langle \rho U U_j \rangle = \langle \rho \rangle \widetilde{U U_j} = \langle \rho \rangle \left(\widetilde{U} \widetilde{U}_j + \widetilde{u'' u_j''} \right) \tag{2.15}$$

$$\left\langle \rho \frac{1}{2} U_i U_i \right\rangle = \langle \rho \rangle \frac{1}{2} \widetilde{U_i U_i} = \langle \rho \rangle \left(\frac{1}{2} \widetilde{U}_i \widetilde{U}_i + \frac{1}{2} \widetilde{u_i'' u_i''} \right) \tag{2.16}$$

$$\left\langle \rho \frac{1}{2} U_i U_i U_j \right\rangle = \langle \rho \rangle \frac{1}{2} \widetilde{U_i U_i U_j} = \langle \rho \rangle \left(\frac{1}{2} \widetilde{U}_i \widetilde{U}_i \widetilde{U}_j + \frac{1}{2} \widetilde{u_i'' u_i''} \widetilde{U}_j + \widetilde{U}_i \widetilde{u_i'' u_j''} + \frac{1}{2} \widetilde{u_i'' u_i'' u_j''} \right) \tag{2.17}$$

We simplify the notation above by introducing

$$\bar{K} = \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \qquad k = \frac{1}{2} \widetilde{u_i'' u_i''} \tag{2.18}$$

Thus, we have for the energy density

$$\langle \rho E \rangle = \langle \rho U \rangle + \left\langle \rho \frac{1}{2} U_i U_i \right\rangle = \langle \rho \rangle \left(\widetilde{U} + \bar{K} + k \right), \tag{2.19}$$

and for the energy flux

$$\langle \rho E U_j \rangle = \langle \rho U U_j \rangle + \left\langle \rho \frac{1}{2} U_i U_i U_j \right\rangle = \langle \rho \rangle \left(\widetilde{U} + \bar{K} + k \right) \widetilde{U}_j + \langle \rho \rangle \widetilde{u'' u_j''} + \langle \rho \rangle \widetilde{U}_i \tau_{ij} + \langle \rho \rangle \frac{1}{2} \widetilde{u_i'' u_i'' u_j''} \tag{2.20}$$

We also note that $\widetilde{E} = \widetilde{U} + \bar{K} + k$. The average of the conservation of energy equation can now be expressed as

$$\frac{\partial \langle \rho \rangle \widetilde{E}}{\partial t} + \frac{\partial \langle \rho \rangle \widetilde{E} \widetilde{U}_j}{\partial x_j} = -\frac{\partial \langle \rho \rangle \widetilde{u'' u_j''}}{\partial x_j} - \frac{\partial \langle \rho \rangle \widetilde{U}_i \tau_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\langle \rho \rangle \frac{1}{2} \widetilde{u_i'' u_i'' u_j''} \right)$$
$$- \frac{\partial \langle U_j P \rangle}{\partial x_j} + \frac{\partial \langle U_i T_{ij} \rangle}{\partial x_j} - \frac{\partial \langle q_j \rangle}{\partial x_j} + \langle \rho \rangle \widetilde{F}_i \widetilde{U}_i + \langle \rho f_i'' u_i'' \rangle \tag{2.21}$$

The term $\langle U_j P \rangle$ is treated as follows

$$\langle U_j P \rangle = \tilde{U}_j \langle P \rangle + \langle u_j'' P \rangle$$

$$= \langle \rho \rangle \frac{\tilde{U}_j \langle P \rangle}{\langle \rho \rangle} + R \langle \rho \rangle \widetilde{u_j'' T}$$

$$= \langle \rho \rangle \frac{\tilde{U}_j \langle P \rangle}{\langle \rho \rangle} + C_p \langle \rho \rangle \widetilde{u_j'' t''} - \langle \rho \rangle \widetilde{u_j'' u''}, \tag{2.22}$$

where we have used $P = \rho R T$, $R = C_p - C_v$ and $U = C_v T$. For $\langle U_j T_{ij} \rangle$ we have

$$\langle U_i T_{ij} \rangle = \tilde{U}_i \langle T_{ij} \rangle + \langle u_i'' T_{ij} \rangle. \tag{2.23}$$

These expansions are then plugged into eq. (2.21).

Before taking the average of the conservation equation for a scalar, we make use of eqs. (2.7) and (2.10) to show

$$\langle \rho Y U_j \rangle = \langle \rho \rangle \widetilde{Y U_j} = \langle \rho \rangle \left(\tilde{Y} \tilde{U}_j + \widetilde{y'' u_j''} \right). \tag{2.24}$$

The second term on the right-hand side above is the turbulent scalar flux. The average of the conservation equation for a scalar can now be expressed as

$$\frac{\partial \rho \tilde{Y}}{\partial t} + \frac{\partial \langle \rho \rangle \tilde{Y} \tilde{U}_j}{\partial x_j} = -\frac{\partial \langle J_j \rangle}{\partial x_j} - \frac{\partial \langle \rho \rangle \widetilde{y'' u_j''}}{\partial x_j} + \langle W \rangle. \tag{2.25}$$

To summarize, the Favre-averaged governing equations are

$$\frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial \langle \rho \rangle \tilde{U}_j}{\partial x_j} = 0, \tag{2.26}$$

$$\frac{\partial \langle \rho \rangle \tilde{U}_i}{\partial t} + \frac{\partial \langle \rho \rangle \tilde{U}_i \tilde{U}_j}{\partial x_j} = -\frac{\partial \langle P \rangle}{\partial x_i} + \frac{\partial \langle T_{ij} \rangle}{\partial x_j} - \frac{\partial \langle \rho \rangle \tau_{ij}}{\partial x_j} + \langle \rho \rangle \tilde{F}_i, \tag{2.27}$$

$$\frac{\partial \langle \rho \rangle \tilde{E}}{\partial t} + \frac{\partial}{\partial x_j} \left[\langle \rho \rangle \left(\tilde{E} + \frac{\langle P \rangle}{\langle \rho \rangle} \right) \tilde{U}_j \right] = -\frac{\partial}{\partial x_j} \left[\tilde{U}_i \left(\langle \rho \rangle \tau_{ij} - \langle T_{ij} \rangle \right) \right]$$
$$- \frac{\partial}{\partial x_j} \left(\langle \rho \rangle \frac{1}{2} \widetilde{u_i'' u_i'' u_j''} - \langle u_i'' T_{ij} \rangle \right) - \frac{\partial \langle q_j \rangle}{\partial x_j} - \frac{\partial}{\partial x_j} \left(C_p \langle \rho \rangle \widetilde{u_j'' t''} \right) + \langle \rho \rangle \tilde{F}_i \tilde{U}_i + \langle \rho f_i'' u_i'' \rangle. \tag{2.28}$$

$$\frac{\partial \langle \rho \rangle \tilde{Y}}{\partial t} + \frac{\partial \langle \rho \rangle \tilde{Y} \tilde{U}_j}{\partial x_j} = -\frac{\partial \langle J_j \rangle}{\partial x_j} - \frac{\partial \langle \rho \rangle \widetilde{y'' u_j''}}{\partial x_j} + \langle W \rangle. \tag{2.29}$$

The averages of the diffusive scalar flux, the shear-stress tensor, and the heat-conduction flux are approximated as

$$\langle T_{ij} \rangle \approx 2\mu \tilde{S}_{ij}^*, \tag{2.30}$$

$$\langle q_i \rangle \approx \kappa \frac{\partial \tilde{T}}{\partial x_i}. \tag{2.31}$$

$$\langle J_i \rangle \approx -\langle \rho \rangle D \frac{\partial \tilde{Y}}{\partial x_i}, \tag{2.32}$$

The transport coefficients are computed using

$$\mu = \mu_0 \left(\frac{\tilde{T}}{T_0} \right)^n \tag{2.33}$$

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$$\kappa = \frac{\mu C_p}{Pr} \tag{2.34}$$

$$D = \frac{\mu}{\langle \rho \rangle Sc} \tag{2.35}$$

The averaged equations for a perfect gas are

$$\langle P \rangle = \langle \rho \rangle R \widetilde{T} \tag{2.36}$$

$$\widetilde{U} = C_v \widetilde{T} \tag{2.37}$$

$$R = \frac{R_u}{M} \tag{2.38}$$

Additional exact relations are below

$$\widetilde{E} = \widetilde{U} + \bar{K} + k \tag{2.39}$$

$$\bar{K} = \frac{1}{2} \widetilde{U}_i \widetilde{U}_i \tag{2.40}$$

$$k = \frac{1}{2} \widetilde{u_i'' u_i''} \tag{2.41}$$

$$\widetilde{S}_{ij}^* = \frac{1}{2} \left(\frac{\partial \widetilde{U}_i}{\partial x_j} + \frac{\partial \widetilde{U}_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial \widetilde{U}_k}{\partial x_k} \delta_{ij} \tag{2.42}$$

Finally, the following models can be used to provide closure

$$\tau_{ij} = ... \tag{2.43}$$

$$-\langle \rho \rangle \frac{1}{2} \widetilde{u_i'' u_i'' u_j''} + \langle u_i'' T_{ij} \rangle \approx \left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j}, \tag{2.44}$$

$$C_p \langle \rho \rangle \widetilde{u_j'' t''} \approx -\kappa_t \frac{\partial \widetilde{T}}{\partial x_j} \qquad \text{where} \qquad \kappa_T = \frac{\mu_T C_p}{Pr_T}. \tag{2.45}$$

2.3 Additional Favre-averaged equations

2.3.1 Vorticity

2.3.2 Fluctuating velocity

To derive an equation for the favre-fluctuating velocity, we will first focus on the left-hand side of eq. (1.2) only. Plugging in the decomposition $U_i = \widetilde{U}_i + u_i''$ in the left-hand side of eq. (1.2), one obtains

$$\frac{\partial \rho \widetilde{U}_i}{\partial t} + \frac{\partial \rho u_i''}{\partial t} + \frac{\partial \rho \widetilde{U}_i U_j}{\partial x_j} + \frac{\partial \rho u_i'' U_j}{\partial x_j}. \tag{2.46}$$

We will now look at the first and third terms above. Thus,

$$\frac{\partial \rho \widetilde{U}_i}{\partial t} + \frac{\partial \rho \widetilde{U}_i U_j}{\partial x_j} = \frac{\partial \rho}{\partial t} \widetilde{U}_i + \rho \frac{\partial \widetilde{U}_i}{\partial t} + \frac{\partial \rho U_j}{\partial x_j} \widetilde{U}_i + \rho U_j \frac{\partial \widetilde{U}_i}{\partial x_j}$$

$$= \rho \frac{\partial \widetilde{U}_i}{\partial t} + \rho \widetilde{U}_j \frac{\partial \widetilde{U}_i}{\partial x_j} + \rho u_j'' \frac{\partial \widetilde{U}_i}{\partial x_j}. \tag{2.47}$$

The fourth term of eq. (2.46) is decomposed as

$$
\begin{aligned}
\frac{\partial \rho u_i'' U_j}{\partial x_j} &= \frac{\partial \rho u_i'' \widetilde{U}_j}{\partial x_j} + \frac{\partial \rho u_i'' u_j''}{\partial x_j} \\
&= \frac{\partial \rho u_i''}{\partial x_j} \widetilde{U}_j + \rho u_i'' \frac{\partial \widetilde{U}_j}{\partial x_j} + \frac{\partial \rho u_i'' u_j''}{\partial x_j}.
\end{aligned}
\tag{2.48}
$$

Combining all of the results above, one obtains

$$
\frac{\partial \rho u_i''}{\partial t} + \rho \frac{\partial \widetilde{U}_i}{\partial t} + \rho \widetilde{U}_j \frac{\partial \widetilde{U}_i}{\partial x_j} + \rho u_j'' \frac{\partial \widetilde{U}_i}{\partial x_j} + \frac{\partial \rho u_i''}{\partial x_j} \widetilde{U}_j + \rho u_i'' \frac{\partial \widetilde{U}_j}{\partial x_j} + \frac{\partial \rho u_i'' u_j''}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} + \rho F_i.
\tag{2.49}
$$

2.3.3 Kinetic energy of the mean flow

The equation for the kinetic energy of the mean flow is obtained by multiplying eq. (2.27)—the Favre-averaged equation for velocity—by $\widetilde{U}_i$, which leads to

$$
\frac{\partial \langle \rho \rangle \bar{K}}{\partial t} + \frac{\partial \langle \rho \rangle \bar{K} \widetilde{U}_j}{\partial x_j} = \langle \rho \rangle \tau_{ij} \frac{\partial \widetilde{U}_i}{\partial x_j} + \langle P \rangle \frac{\partial \widetilde{U}_i}{\partial x_i} - \langle T_{ij} \rangle \frac{\partial \widetilde{U}_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\widetilde{U}_i \langle \rho \rangle \tau_{ij} + \widetilde{U}_j \langle P \rangle - \widetilde{U}_i \langle T_{ij} \rangle \right) + \langle \rho \rangle \tilde{F}_i \widetilde{U}_i
\tag{2.50}
$$

2.3.4 Turbulent kinetic energy

To obtain the governing equation for the turbulent kinetic energy one subtracts eq. (2.50) from the Reynolds average of eq. (1.17), which is the equation for conservation of kinetic energy. We first regroup the shear stress terms of eq. (1.17)

$$
\frac{\partial U_i T_{ij}}{\partial x_j} - T_{ij} \frac{\partial U_i}{\partial x_j} = U_i \frac{\partial T_{ij}}{\partial x_j}
\tag{2.51}
$$

and eq. (2.50)

$$
-\langle T_{ij} \rangle \frac{\partial \widetilde{U}_i}{\partial x_j} + \frac{\partial \widetilde{U}_i \langle T_{ij} \rangle}{\partial x_j} = \widetilde{U}_i \frac{\partial \langle T_{ij} \rangle}{\partial x_j}.
\tag{2.52}
$$

For the average of the right-hand side of eq. (2.51), we will use the decompositions $U_i = \langle U_i \rangle + u_i$ and $T_{ij} = \langle T_{ij} \rangle + t_{ij}$, instead of the Favre decomposition and eq. (2.23). This thus allows us to write

$$
\begin{aligned}
\left\langle U_i \frac{\partial T_{ij}}{\partial x_j} \right\rangle - \widetilde{U}_i \frac{\partial \langle T_{ij} \rangle}{\partial x_j} &= \langle U_i \rangle \frac{\partial \langle T_{ij} \rangle}{\partial x_j} + \left\langle u_i \frac{\partial t_{ij}}{\partial x_j} \right\rangle - \widetilde{U}_i \frac{\partial \langle T_{ij} \rangle}{\partial x_j} \\
&= \langle u_i'' \rangle \frac{\partial \langle T_{ij} \rangle}{\partial x_j} + \left\langle u_i \frac{\partial t_{ij}}{\partial x_j} \right\rangle \\
&= \langle u_i'' \rangle \frac{\partial \langle T_{ij} \rangle}{\partial x_j} + \frac{\partial \langle u_i t_{ij} \rangle}{\partial x_j} - \left\langle t_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle.
\end{aligned}
\tag{2.53}
$$

The same procedure is followed for the pressure terms. We first regroup the pressure terms of eq. (1.17)

$$
-\frac{\partial U_i P}{\partial x_i} + P \frac{\partial U_i}{\partial x_i} = -U_i \frac{\partial P}{\partial x_i}
\tag{2.54}
$$

and eq. (2.90)

$$
\langle P \rangle \frac{\partial \widetilde{U}_i}{\partial x_i} - \frac{\partial \widetilde{U}_i \langle P \rangle}{\partial x_i} = -\widetilde{U}_i \frac{\partial \langle P \rangle}{\partial x_i}.
\tag{2.55}
$$

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Again, the Reynolds decomposition of velocity is used for the average of the right-hand side of eq. (2.54). This allows us to write

$$-\left\langle U_i \frac{\partial P}{\partial x_i} \right\rangle + \widetilde{U}_i \frac{\partial \langle P \rangle}{\partial x_i} = -\langle U_i \rangle \frac{\partial \langle P \rangle}{\partial x_i} - \left\langle u_i \frac{\partial p}{\partial x_i} \right\rangle + \widetilde{U}_i \frac{\partial \langle P \rangle}{\partial x_i}$$

$$= -\langle u_i'' \rangle \frac{\partial \langle P \rangle}{\partial x_i} - \left\langle u_i \frac{\partial p}{\partial x_i} \right\rangle$$

$$= -\langle u_i'' \rangle \frac{\partial \langle P \rangle}{\partial x_i} - \frac{\partial \langle u_i p \rangle}{\partial x_i} + \left\langle p \frac{\partial u_i}{\partial x_i} \right\rangle. \tag{2.56}$$

The overall equation for the turbulent kinetic energy is shown below

$$\frac{\partial \langle \rho \rangle k}{\partial t} + \frac{\partial \langle \rho \rangle k \widetilde{U}_j}{\partial x_j} = -\langle u_i'' \rangle \frac{\partial \langle P \rangle}{\partial x_i} + \langle u_i'' \rangle \frac{\partial \langle T_{ij} \rangle}{\partial x_j}$$

$$- \langle \rho \rangle \tau_{ij} \frac{\partial \widetilde{U}_i}{\partial x_j} + \left\langle p \frac{\partial u_i}{\partial x_i} \right\rangle - \left\langle t_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle - \frac{\partial}{\partial x_j} \left(\frac{1}{2} \langle \rho u_j'' u_i'' u_i'' \rangle + \langle u_j p \rangle - \langle u_i t_{ij} \rangle \right) + \langle \rho f_i'' u_i'' \rangle \tag{2.57}$$

We define the dissipation ϵ as

$$\langle \rho \rangle \epsilon = \left\langle t_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle, \tag{2.58}$$

Since $t_{ij} = T_{ij} - \langle T_{ij} \rangle$, the above becomes

$$\langle \rho \rangle \epsilon = \left\langle T_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle = \left\langle 2\mu \frac{\partial u_i}{\partial x_j} \right\rangle \langle S_{ij}^* \rangle + \left\langle 2\mu s_{ij}^* \frac{\partial u_i}{\partial x_j} \right\rangle \tag{2.59}$$

Neglecting the molecular viscosity fluctuations, as stated in Sagaut & Cambon (Homogeneous compressible dynamics) gives

$$\langle \rho \rangle \epsilon = 2\langle \mu \rangle \left\langle s_{ij}^* \frac{\partial u_i}{\partial x_j} \right\rangle. \tag{2.60}$$

Expanding s_{ij}^*,

$$\langle \rho \rangle \epsilon = 2\langle \mu \rangle \left\langle s_{ij} s_{ij} - \frac{1}{3} \left(\frac{\partial u_k}{\partial x_k} \right)^2 \right\rangle. \tag{2.61}$$

The above agrees with Sarkar et al, JFM, 1991. [**?**]. Using the identities

$$s_{ij} s_{ij} = \frac{1}{2} w_i w_i + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}, \tag{2.62}$$

and

$$\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} - 2 \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_i} u_j \right) + \left(\frac{\partial u_k}{\partial x_k} \right)^2, \tag{2.63}$$

and neglecting the first two terms in the equation above due to an order of magnitude estimate for high turbulence Reynolds number as suggested by Sarkar et al, we express the dissipation as

$$\langle \rho \rangle \epsilon = \langle \mu \rangle \langle w_i w_i \rangle + \frac{4}{3} \langle \mu \rangle \left\langle \left(\frac{\partial u_k}{\partial x_k} \right)^2 \right\rangle. \tag{2.64}$$

The first component above is referred to as the solenoidal dissipation, while the second component is the compressible dissipation.

2.3.5 Internal Energy

We take the Reynolds average of the conservation form of the internal energy equation. For this, we will use

$$\left\langle T_{ij}\frac{\partial U_i}{\partial x_j}\right\rangle = \langle T_{ij}\rangle\frac{\partial\langle U_i\rangle}{\partial x_j} + \left\langle t_{ij}\frac{\partial u_i}{\partial x_j}\right\rangle \tag{2.65}$$

and

$$\left\langle -P\frac{\partial U_i}{\partial x_i}\right\rangle = -\langle P\rangle\frac{\partial\langle U_i\rangle}{\partial x_i} - \left\langle p\frac{\partial u_i}{\partial x_i}\right\rangle. \tag{2.66}$$

For the above, we expand $\langle U_i\rangle$ as $\langle U_i\rangle = \widetilde{U}_i + \langle u_i''\rangle$ to obtain

$$\left\langle T_{ij}\frac{\partial U_i}{\partial x_j}\right\rangle = \langle T_{ij}\rangle\frac{\partial\widetilde{U}_i}{\partial x_j} + \langle T_{ij}\rangle\frac{\partial\langle u_i''\rangle}{\partial x_j} + \left\langle t_{ij}\frac{\partial u_i}{\partial x_j}\right\rangle$$

$$= \langle T_{ij}\rangle\frac{\partial\widetilde{U}_i}{\partial x_j} - \langle u_i''\rangle\frac{\partial\langle T_{ij}\rangle}{\partial x_j} + \left\langle t_{ij}\frac{\partial u_i}{\partial x_j}\right\rangle + \frac{\partial\langle u_i''\rangle\langle T_{ij}\rangle}{\partial x_j} \tag{2.67}$$

and

$$\left\langle -P\frac{\partial U_i}{\partial x_i}\right\rangle = -\langle P\rangle\frac{\partial\widetilde{U}_i}{\partial x_i} - \langle P\rangle\frac{\partial\langle u_i''\rangle}{\partial x_i} - \left\langle p\frac{\partial u_i}{\partial x_i}\right\rangle$$

$$= -\langle P\rangle\frac{\partial\widetilde{U}_i}{\partial x_i} + \langle u_i''\rangle\frac{\partial\langle P\rangle}{\partial x_i} - \left\langle p\frac{\partial u_i}{\partial x_i}\right\rangle - \frac{\partial\langle u_i''\rangle\langle P\rangle}{\partial x_i}. \tag{2.68}$$

Thus, using the above, the average of the internal energy equation becomes

$$\frac{\partial\langle\rho\rangle\widetilde{U}}{\partial t} + \frac{\partial\langle\rho\rangle\widetilde{U}\widetilde{U}_j}{\partial x_j} = -\langle P\rangle\frac{\partial\widetilde{U}_i}{\partial x_i} + \langle T_{ij}\rangle\frac{\partial\widetilde{U}_i}{\partial x_j} + \langle u_i''\rangle\frac{\partial\langle P\rangle}{\partial x_i} - \langle u_i''\rangle\frac{\partial\langle T_{ij}\rangle}{\partial x_j}$$

$$- \left\langle p\frac{\partial u_i}{\partial x_i}\right\rangle + \left\langle t_{ij}\frac{\partial u_i}{\partial x_j}\right\rangle - \frac{\partial}{\partial x_j}\left(\langle\rho u''u_j''\rangle + \langle u_j''\rangle\langle P\rangle - \langle u_i''\rangle\langle T_{ij}\rangle + \langle q_j\rangle\right) \tag{2.69}$$

Note that we can write

$$\langle\rho u''u_j''\rangle + \langle u_j''\rangle\langle P\rangle = \langle\rho u''u_j''\rangle + \langle pu_j''\rangle + \langle u_j''\rangle\langle P\rangle - \langle pu_j''\rangle = \langle\rho h''u_j''\rangle - \langle pu_j''\rangle, \tag{2.70}$$

so as to be in agreement with the equations in Lele 1994.

2.3.6 Summary of conservation equations for $\bar{K}$, k, $\widetilde{U}$

$$\frac{\partial\langle\rho\rangle\bar{K}}{\partial t} + \frac{\partial\langle\rho\rangle\bar{K}\widetilde{U}_j}{\partial x_j} = \langle\rho\rangle\tau_{ij}\frac{\partial\widetilde{U}_i}{\partial x_j} + \langle P\rangle\frac{\partial\widetilde{U}_i}{\partial x_i} - \langle T_{ij}\rangle\frac{\partial\widetilde{U}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\widetilde{U}_i\langle\rho\rangle\tau_{ij} + \widetilde{U}_j\langle P\rangle - \widetilde{U}_i\langle T_{ij}\rangle\right) + \langle\rho\rangle\widetilde{F}_i\widetilde{U}_i \tag{2.71}$$

$$\frac{\partial\langle\rho\rangle k}{\partial t} + \frac{\partial\langle\rho\rangle k\widetilde{U}_j}{\partial x_j} = -\langle u_i''\rangle\frac{\partial\langle P\rangle}{\partial x_i} + \langle u_i''\rangle\frac{\partial\langle T_{ij}\rangle}{\partial x_j}$$

$$- \langle\rho\rangle\tau_{ij}\frac{\partial\widetilde{U}_i}{\partial x_j} + \left\langle p\frac{\partial u_i}{\partial x_i}\right\rangle - \left\langle t_{ij}\frac{\partial u_i}{\partial x_j}\right\rangle - \frac{\partial}{\partial x_j}\left(\frac{1}{2}\langle\rho u_j''u_i''u_i''\rangle + \langle u_j p\rangle - \langle u_i t_{ij}\rangle\right) + \langle\rho f_i''u_i''\rangle \tag{2.72}$$

$$\frac{\partial\langle\rho\rangle\widetilde{U}}{\partial t} + \frac{\partial\langle\rho\rangle\widetilde{U}\widetilde{U}_j}{\partial x_j} = -\langle P\rangle\frac{\partial\widetilde{U}_i}{\partial x_i} + \langle T_{ij}\rangle\frac{\partial\widetilde{U}_i}{\partial x_j} + \langle u_i''\rangle\frac{\partial\langle P\rangle}{\partial x_i} - \langle u_i''\rangle\frac{\partial\langle T_{ij}\rangle}{\partial x_j}$$

$$- \left\langle p\frac{\partial u_i}{\partial x_i}\right\rangle + \left\langle t_{ij}\frac{\partial u_i}{\partial x_j}\right\rangle - \frac{\partial}{\partial x_j}\left(\langle\rho u''u_j''\rangle + \langle u_j''\rangle\langle P\rangle - \langle u_i''\rangle\langle T_{ij}\rangle + \langle q_j\rangle\right) \tag{2.73}$$

Note that if you add up all three equations, one obtains a conservation equation for $\widetilde{E}$. After this addition, the only terms on the right-hand side that remain are the transport and forcing terms.

2.3.7 Reynolds stresses

2.3.8 Two-point velocity correlations

2.3.9 Squared averaged scalar

The equation for the squared averaged scalar is obtained by multiplying eq. (2.29)—the Favre-averaged equation for the scalar—by $\widetilde{Y}$, which leads to

$$\frac{\partial \langle\rho\rangle \widetilde{Y}^2}{\partial t} + \frac{\partial \langle\rho\rangle \widetilde{Y}^2 \widetilde{U}_j}{\partial x_j} = 2\langle J_j\rangle \frac{\partial \widetilde{Y}}{\partial x_j} + 2\langle\rho\rangle \widetilde{y''u_j''} \frac{\partial \widetilde{Y}}{\partial x_j} - 2\frac{\partial}{\partial x_j}\left(\widetilde{Y}\langle J_j\rangle + \widetilde{Y}\langle\rho\rangle \widetilde{y''u_j''}\right) + 2\widetilde{Y}\langle W\rangle. \quad (2.74)$$

2.3.10 Scalar variance

To obtain the governing equation for the scalar variance, one subtracts eq. (2.74) from the Reynolds average of eq. (1.19). We first regroup the diffusive scalar flux terms of eq. (1.19)

$$-\frac{\partial Y J_j}{\partial x_j} + J_j \frac{\partial Y}{\partial x_j} = -Y\frac{\partial J_j}{\partial x_j}, \quad (2.75)$$

and eq. (2.74)

$$\langle J_j\rangle \frac{\partial \widetilde{Y}}{\partial x_j} - \frac{\partial \widetilde{Y}\langle J_j\rangle}{\partial x_j} = -\widetilde{Y}\frac{\partial \langle J_j\rangle}{\partial x_j}. \quad (2.76)$$

For the average of the right-hand side of eq. (2.75), we will use the decomposition $Y = \langle Y\rangle + y$ and $J_i = \langle J_i\rangle + j_i$, instead of the traditional Favre decompositions. This thus allows us to write

$$-\left\langle Y\frac{\partial J_j}{\partial x_j}\right\rangle + \widetilde{Y}\frac{\partial \langle J_j\rangle}{\partial x_j} = -\langle Y\rangle \frac{\partial \langle J_j\rangle}{\partial x_j} - \left\langle y\frac{\partial j_j}{\partial x_j}\right\rangle + \widetilde{Y}\frac{\partial \langle J_j\rangle}{\partial x_j}$$
$$= -\langle y''\rangle \frac{\partial \langle J_j\rangle}{\partial x_j} - \left\langle y\frac{\partial j_j}{\partial x_j}\right\rangle$$
$$= -\langle y''\rangle \frac{\partial \langle J_j\rangle}{\partial x_j} - \frac{\partial \langle y j_j\rangle}{\partial x_j} + \left\langle j_j\frac{\partial y}{\partial x_j}\right\rangle. \quad (2.77)$$

Using the above, the overall equation for the scalar variance is written as

$$\frac{\partial \langle\rho\rangle \widetilde{y''y''}}{\partial t} + \frac{\partial \langle\rho\rangle \widetilde{y''y''}\widetilde{U}_j}{\partial x_j} = -2\langle y''\rangle \frac{\partial \langle J_j\rangle}{\partial x_j}$$
$$- 2\langle\rho\rangle \widetilde{y''u_j''}\frac{\partial \widetilde{Y}}{\partial x_j} + 2\left\langle j_j\frac{\partial y}{\partial x_j}\right\rangle - 2\frac{\partial}{\partial x_j}\left(\frac{1}{2}\langle\rho\rangle \widetilde{y''y''u_j''} + \langle y j_j\rangle\right) + 2\langle y''\rangle\langle W\rangle + 2\langle y''w\rangle. \quad (2.78)$$

As was done for the TKE, one can define a scalar dissipation ϵ_Y as

$$\langle\rho\rangle\epsilon_Y = -2\left\langle j_j\frac{\partial y}{\partial x_j}\right\rangle. \quad (2.79)$$

Since $j_j = J_j - \langle J_j\rangle$, the above becomes

$$\langle\rho\rangle\epsilon_Y = -2\left\langle J_j\frac{\partial y}{\partial x_j}\right\rangle = 2\left\langle \rho D\frac{\partial y}{\partial x_j}\right\rangle \frac{\partial \langle Y\rangle}{\partial x_j} + 2\left\langle \rho D\frac{\partial y}{\partial x_j}\frac{\partial y}{\partial x_j}\right\rangle. \quad (2.80)$$

2.4 Reynolds-averaged Navier-Stokes equations

Assuming a constant density, the Favre average is equivalent to the Reynolds average. Thus, for incompressible flow the continuity eq. (2.26) becomes

$$\frac{\partial \langle U_i \rangle}{\partial x_i} = 0. \tag{2.81}$$

The conservation of momentum eq. (2.27), along with the definition of the shear stress tensor in eq. (1.5), leads to

$$\frac{\partial \langle U_i \rangle}{\partial t} + \langle U_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} + \nu \frac{\partial^2 \widetilde{U}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \langle F_i \rangle, \tag{2.82}$$

where $\tau_{ij} = \langle u_i u_j \rangle$. Taking the divergence of the momentum equation, we obtain the equation for the averaged pressure

$$\frac{1}{\rho} \frac{\partial^2 \langle P \rangle}{\partial x_j \partial x_j} = -\frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial \langle U_j \rangle}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_i \partial x_j} + \frac{\partial \langle F_i \rangle}{\partial x_i}. \tag{2.83}$$

Taking the Reynolds average of the scalar transport eq. (1.4) leads to

$$\frac{\partial \rho \langle \Phi \rangle}{\partial t} + \frac{\partial \rho \langle U_j \rangle \langle \Phi \rangle}{\partial x_j} = -\frac{\partial \langle J_j \rangle}{\partial x_j} - \frac{\partial \rho \langle u_j \phi \rangle}{\partial x_j} \tag{2.84}$$

2.5 Additional Reynolds-averaged equations

2.5.1 Vorticity

The equation for the mean vorticity $\langle W_i \rangle$, also denoted by Ω_i is as follows

$$\frac{\partial \langle W_i \rangle}{\partial t} + \langle U_j \rangle \frac{\partial \langle W_i \rangle}{\partial x_j} = \langle W_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} + \nu \frac{\partial^2 \langle W_i \rangle}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j} [\langle u_i w_j \rangle - \langle u_j w_i \rangle]. \tag{2.85}$$

2.5.2 Fluctuating velocity and pressure

Equation (2.49) for incompressible flows becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial \langle U_i \rangle}{\partial t} + \langle U_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} + u_j \frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \langle U_j \rangle + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial \langle U_i \rangle}{\partial x_j \partial x_j} + F_i. \tag{2.86}$$

We now subtract eq. (2.82) from the above, to obtain

$$\frac{\partial u_i}{\partial t} + \langle U_j \rangle \frac{\partial u_i}{\partial x_j} = -u_j \frac{\partial \langle U_i \rangle}{\partial x_j} - \frac{\partial u_i u_j}{\partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial u_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i. \tag{2.87}$$

Assuming incompressible flow, and taking the divergence of eq. (1.2), one obtains the equation for pressure,

$$\frac{1}{\rho} \frac{\partial^2 P}{\partial x_j \partial x_j} = -\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} + \frac{\partial F}{\partial x_i}. \tag{2.88}$$

Subtracting eq. (2.83) from the above leads to

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_j \partial x_j} = -2 \frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial u_j}{\partial x_i} - \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} + \frac{\partial f_i}{\partial x_i}. \tag{2.89}$$

2.5.3 Kinetic energy of the mean flow

Assuming incompressible flow, eq. (2.50) becomes

$$\frac{\partial \bar{K}}{\partial t} + \langle U_j \rangle \frac{\partial \bar{K}}{\partial x_j} = \tau_{ij} \frac{\partial \langle U_i \rangle}{\partial x_j} - \frac{\langle T_{ij} \rangle}{\rho} \frac{\partial \langle U_i \rangle}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\langle U_i \rangle \tau_{ij} + \frac{\langle U_j \rangle \langle P \rangle}{\rho} - \frac{\langle U_i \rangle \langle T_{ij} \rangle}{\rho} \right) + \langle F_i \rangle \langle U_i \rangle.$$
(2.90)

2.5.4 Turbulent kinetic energy

Assuming incompressible flow, eq. (2.57) becomes

$$\frac{\partial k}{\partial t} + \langle U_j \rangle \frac{\partial k}{\partial x_j} = -\tau_{ij} \frac{\partial \langle U_i \rangle}{\partial x_j} - \left\langle \frac{t_{ij}}{\rho} \frac{\partial u_i}{\partial x_j} \right\rangle - \frac{\partial}{\partial x_j} \left(\frac{1}{2} \langle u_j u_i u_i \rangle + \frac{\langle u_j p \rangle}{\rho} - \frac{\langle u_i t_{ij} \rangle}{\rho} \right) + \langle f_i u_i \rangle.$$ (2.91)

The production term can be rewritten as $-\tau_{ij} S_{ij}$, that is, production of TKE is influenced by S_{ij} only. The second term is referred to as the dissipation ϵ, which can be rewritten as

$$\epsilon = \left\langle \frac{t_{ij}}{\rho} s_{ij} \right\rangle = 2\nu \langle s_{ij} s_{ij} \rangle.$$
(2.92)

2.5.5 Reynolds-stresses

The transport equation for the Reynolds stresses is obtained by multiplying equation (2.87) by u_j, adding to it the same product but with indices i and j switched, and then performing an average of the resulting equation. Thus, one obtains

$$\frac{\partial \tau_{ij}}{\partial t} + \langle U_k \rangle \frac{\partial \tau_{ij}}{\partial x_k} = \mathcal{P}_{ij} + \Pi_{ij} - \epsilon_{ij} - \frac{\partial}{\partial x_k} \left(\mathcal{T}_{kij}^{(u)} + \mathcal{T}_{kij}^{(\nu)} \right) + \langle u_j f_i \rangle + \langle u_i f_j \rangle.$$
(2.93)

The production is given by

$$\mathcal{P}_{ij} = -\tau_{ik} \frac{\partial \langle U_j \rangle}{\partial x_k} - \tau_{jk} \frac{\partial \langle U_i \rangle}{\partial x_k},$$
(2.94)

the velocity-pressure-gradient tensor by

$$\Pi_{ij} = -\frac{1}{\rho} \left\langle u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right\rangle,$$
(2.95)

the dissipation tensor by

$$\epsilon_{ij} = 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle,$$
(2.96)

the turbulent transport by

$$\mathcal{T}_{kij}^{(u)} = \langle u_i u_j u_k \rangle,$$
(2.97)

and the viscous diffusion by

$$\mathcal{T}_{kij}^{(\nu)} = -\nu \frac{\partial \tau_{ij}}{\partial x_k}.$$
(2.98)

The velocity-pressure-gradient tensor has two common decompositions. One of these is the following

$$\Pi_{ij} = \phi_{ij} - \frac{\partial}{\partial x_k} \mathcal{T}_{kij}^{(p)},$$
(2.99)

where ϕ_{ij} is the pressure-rate-of-strain tensor and $\mathcal{T}_{kij}^{(p)}$ is the pressure transport tensor. These are defined as follows

$$\phi_{ij} = \left\langle \frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\rangle \qquad \text{and} \qquad \mathcal{T}_{kij}^{(p)} = \frac{1}{\rho}\langle u_i p\rangle \delta_{jk} + \frac{1}{\rho}\langle u_j p\rangle \delta_{ik}. \tag{2.100}$$

An alternate decomposition is as follows

$$\Pi_{ij} = \hat{\phi}_{ij} - \frac{\partial}{\partial x_k}\mathcal{T}_{kij}^{(p')}, \tag{2.101}$$

where $\hat{\phi}_{ij}$ is the anisotropic velocity-pressure-gradient tensor and

$$-\frac{\partial}{\partial x_k}\mathcal{T}_{kij}^{(p')} = \frac{1}{3}\Pi_{kk}\delta_{ij} \qquad \mathcal{T}_{kij}^{(p')} = \frac{2}{3}\delta_{ij}\mathcal{T}_k^{(p)} \qquad \mathcal{T}_k^{(p)} = \frac{1}{\rho}\langle u_k p\rangle. \tag{2.102}$$

We label $\mathcal{T}_{kij}^{(p')}$ as the isotropic pressure transport.

Assuming no body forces, we can decompose the fluctuating pressure p into three contributions, namely

$$p = p^{(r)} + p^{(s)} + p^{(h)}, \tag{2.103}$$

where $p^{(r)}$ is the rapid pressure, $p^{(s)}$ is the slow pressure, and $p^{(h)}$ is the harmonic pressure (also referred to as Stokes pressure). Thus, $p^{(r)}$ satisfies

$$\frac{1}{\rho}\frac{\partial^2 p^{(r)}}{\partial x_j \partial x_j} = -2\frac{\partial \langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}, \tag{2.104}$$

$p^{(s)}$ satisfies

$$\frac{1}{\rho}\frac{\partial^2 p^{(s)}}{\partial x_j \partial x_j} = -\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j}, \tag{2.105}$$

and $p^{(h)}$ satisfies

$$\frac{1}{\rho}\frac{\partial^2 p^{(h)}}{\partial x_j \partial x_j} = 0. \tag{2.106}$$

The rapid pressure is the only non-zero pressure in homogeneous RDT, the slow pressure is the only non-zero contribution in decaying homogeneous turbulence. The boundary conditions are specified on $p^{(h)}$ so that the total pressure satisfies the required boundary conditions.

Using the pressure decompositions defined above we can correspondingly decompose the redistribution tensors. For example,

$$\phi_{ij} = \phi_{ij}^{(r)} + \phi_{ij}^{(s)} + \phi_{ij}^{(h)}, \tag{2.107}$$

where, instead of using p, $p^{(r)}$ is used in the definition of $\phi_{ij}^{(r)}$, $p^{(s)}$ in the definition of $\phi_{ij}^{(s)}$, and $p^{(h)}$ in the definition of $\phi_{ij}^{(h)}$. Similarly

$$\Pi_{ij} = \Pi_{ij}^{(r)} + \Pi_{ij}^{(s)} + \Pi_{ij}^{(h)}, \tag{2.108}$$

where the different Π_{ij}'s are defined using their corresponding pressures.

2.5.6 Anisotropic Reynolds-stresses

The reynolds stresses can be decomposed into an isotropic and an anisotropic component as follows

$$\tau_{ij} = \frac{2}{3}k\delta_{ij} + a_{ij}. \tag{2.109}$$

A non-dimensional anisotropy is defined as follows

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\tau_{ij}}{2k} - \frac{1}{3}\delta_{ij}. \tag{2.110}$$

Taking the material derivative of b_{ij} one obtains

$$\frac{\bar{D}b_{ij}}{\bar{D}t} = \frac{1}{2k}\frac{\bar{D}\tau_{ij}}{\bar{D}t} - \frac{\tau_{ij}}{2k^2}\frac{\bar{D}k}{\bar{D}t}$$

Thus, the transport equation for the non-dimensional anisotropy without body forces can be expressed as follows

$$2k\left(\frac{\partial b_{ij}}{\partial t} + \langle U_k\rangle \frac{\partial b_{ij}}{\partial x_k}\right) =$$
$$\mathcal{P}_{ij} + \phi_{ij} - \epsilon_{ij} - \frac{\partial}{\partial x_k}\left(\mathcal{T}_{kij}^{(u)} + \mathcal{T}_{kij}^{(p)} + \mathcal{T}_{kij}^{(\nu)}\right) - \frac{\tau_{ij}}{k}\left[\mathcal{P} - \epsilon - \frac{\partial}{\partial x_k}\left(\mathcal{T}_k^{(u)} + \mathcal{T}_k^{(p)} + \mathcal{T}_k^{(\nu)}\right)\right]. \tag{2.111}$$

2.5.7 Two-point velocity correlations

2.5.8 Squared averaged scalar

Assuming incompressible flow for eq. (2.74) leads to

$$\frac{\partial\langle Y\rangle^2}{\partial t} + \langle U_j\rangle\frac{\partial\langle Y\rangle^2}{\partial x_j} = 2\frac{\langle J_j\rangle}{\rho}\frac{\partial\langle Y\rangle}{\partial x_j} + 2\langle yu_j\rangle\frac{\partial\langle Y\rangle}{\partial x_j} - 2\frac{\partial}{\partial x_j}\left(\frac{\langle Y\rangle\langle J_j\rangle}{\rho} + \langle Y\rangle\langle yu_j\rangle\right) \tag{2.112}$$

2.5.9 Scalar variance

Assuming incompressible flow for eq. (2.78) leads to

$$\frac{\partial\langle yy\rangle}{\partial t} + \langle U_j\rangle\frac{\partial\langle yy\rangle}{\partial x_j} = -2\langle yu_j\rangle\frac{\partial\langle Y\rangle}{\partial x_j} + 2\left\langle \frac{j_j}{\rho}\frac{\partial y}{\partial x_j}\right\rangle - 2\frac{\partial}{\partial x_j}\left(\frac{1}{2}\langle yyu_j\rangle + \frac{\langle yj_j\rangle}{\rho}\right). \tag{2.113}$$

Part II

Homogeneous Turbulence

Chapter 3

Introduction

3.1 Evolution of the mean variables

In this section we derive the time evolutions of $\langle U_i \rangle$ and $\langle P \rangle$. We know that for homogeneous turbulence, gradients of turbulent statistics for homogeneous turbulence vanish. The incompressible conservation of TKE eq. (2.91) for homogeneous flows thus becomes

$$\frac{\partial k}{\partial t} = -\tau_{ij} \frac{\partial \langle U_i \rangle}{\partial x_j} - \epsilon + \langle f_i u_i \rangle. \tag{3.1}$$

This shows that the derivative of mean velocity is a function of time only, which we label as $G_{ij} = G_{ij}(t)$. Thus, we express the mean velocity as $\langle U_i \rangle = G_{ij} x_j + C_i$, or

$$\langle U_i \rangle = G_{ij} x_j, \tag{3.2}$$

after neglecting the constant of integration.

The transport equation for velocity **??** then simplifies to the following for homogeneous turbulence

$$\frac{\partial \langle U_i \rangle}{\partial t} + \langle U_k \rangle G_{ik} = -\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} + \langle F_i \rangle. \tag{3.3}$$

Differentiating the above by x_j shows that G_{ij} evolves according to

$$\frac{dG_{ij}}{dt} + G_{kj} G_{ik} = -\frac{1}{\rho} \frac{\partial^2 \langle P \rangle}{\partial x_i \partial x_j} + \frac{\partial \langle F_i \rangle}{\partial x_j}. \tag{3.4}$$

Given no body forces, eq. (3.4) shows that the second gradient of $\langle P \rangle$ is a function of time only, which we label as $C_{ij} = C_{ij}(t)$. We note that C_{ij} is symmetric. We then express the pressure as $\langle P \rangle = C + C_i x_i + \frac{1}{2} C_{ij} x_i x_j$, or

$$\langle P \rangle = \frac{1}{2} C_{ij} x_i x_j, \tag{3.5}$$

after neglecting the constants of integration.

Using the definition of S_{ij} one can show that

$$\begin{aligned}
\frac{dS_{ij}}{dt} &= \frac{1}{2} \left(\frac{dG_{ij}}{dt} + \frac{dG_{ji}}{dt} \right) \\
&= \frac{1}{2} \left(-G_{kj} G_{ik} - G_{ki} G_{jk} - \frac{2}{\rho} \frac{\partial^2 \langle P \rangle}{\partial x_i \partial x_j} + \frac{\partial \langle F_i \rangle}{\partial x_j} + \frac{\partial \langle F_j \rangle}{\partial x_i} \right) \\
&= -S_{ik} S_{jk} - W_{ik} W_{kj} - \frac{1}{\rho} \frac{\partial^2 \langle P \rangle}{\partial x_i \partial x_j} + F_{ij}^{(s)},
\end{aligned} \tag{3.6}$$

where

$$F_{ij}^{(s)} = \frac{1}{2} \left(\frac{\partial \langle F_i \rangle}{\partial x_j} + \frac{\partial \langle F_j \rangle}{\partial x_i} \right). \tag{3.7}$$

Following the same approach for W_{ij} one obtains

$$\frac{dW_{ij}}{dt} = -S_{kj}W_{ik} - W_{kj}S_{ik} + F_{ij}^{(w)}, \tag{3.8}$$

where

$$F_{ij}^{(w)} = \frac{1}{2} \left(\frac{\partial \langle F_i \rangle}{\partial x_j} - \frac{\partial \langle F_j \rangle}{\partial x_i} \right). \tag{3.9}$$

To know the evolution of the mean velocity and pressure for the case of incompressible homogeneous turbulence, one needs to know the evolution of G_{ij} and C_{ij}. To do so, one can specify an evolution of S_{ij}, and use eq. (3.8) to compute W_{ij}, to thus obtain the evolution of G_{ij}. C_{ij} then follows from either eq. (3.4) or eq. (3.6).

3.2   Evolution of the fluctuating variables

3.2.1   Governing Equations

Just as in section 3.1 our goal was to find evolution equations for the mean variables $\langle U_i \rangle$ and $\langle P \rangle$, in this section we now seek to find the equations for the fluctuations $u_i = u_i(t, \mathbf{x})$ and $p = p(t, \mathbf{x})$. Under the assumption of homogeneous turbulence, the PDEs for these fluctuating variables become

$$\frac{\partial u_i}{\partial t} + G_{jk}x_k \frac{\partial u_i}{\partial x_j} = -u_j G_{ij} - \frac{\partial u_i u_j}{\partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{3.10}$$

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_j \partial x_j} = -2G_{ij} \frac{\partial u_j}{\partial x_i} - \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}. \tag{3.11}$$

3.2.2   Rogallo Transformation

It is convenient to use the method of change of variables to transform the PDEs above to eliminate the convective term on the equation for the fluctuating velocity. Given functions $\mathring{u}_i = \mathring{u}_i(t, \mathring{\mathbf{x}})$ and $\mathring{p} = \mathring{p}(t, \mathring{\mathbf{x}})$ suppose the velocity and pressure fields can be re-expressed as follows

$$u_i = \mathring{u}_i(t, \mathbf{f}) \tag{3.12}$$

$$p = \mathring{p}(t, \mathbf{f}), \tag{3.13}$$

for some function $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$. Thus, derivatives in the original frame convert to the following in the transformed frame

$$\frac{\partial}{\partial t} = \left[\frac{\partial}{\partial t} \right]_{\mathring{\mathbf{x}}=\mathbf{f}} + \left[\frac{\partial}{\partial \mathring{x}_k} \right]_{\mathring{\mathbf{x}}=\mathbf{f}} \frac{\partial f_k}{\partial t} \tag{3.14}$$

$$\frac{\partial}{\partial x_i} = \left[\frac{\partial}{\partial \mathring{x}_k} \right]_{\mathring{\mathbf{x}}=\mathbf{f}} \frac{\partial f_k}{\partial x_i} \tag{3.15}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} = \left[\frac{\partial^2}{\partial \mathring{x}_p \partial \mathring{x}_k} \right]_{\mathring{\mathbf{x}}=\mathbf{f}} \frac{\partial f_p}{\partial x_i} \frac{\partial f_k}{\partial x_j} + \left[\frac{\partial}{\partial \mathring{x}_k} \right]_{\mathring{\mathbf{x}}=\mathbf{f}} \frac{\partial^2 f_k}{\partial x_i \partial x_j}. \tag{3.16}$$

Lets now pick a specific form for the function $\mathbf{f}$, namely $\mathbf{f} = \mathbf{Ax}$, where $\mathbf{A} = \mathbf{A}(t)$ is a square matrix. In tensor notation this is equivalent to

$$f_i = A_{in}x_n. \tag{3.17}$$

The first two terms in the PDE for the fluctuating velocity thus become

$$\frac{\partial u_i}{\partial t} + G_{jk}x_k\frac{\partial u_i}{\partial x_j} = \left[\frac{\partial \mathring{u}_i}{\partial t}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}} + \left[\frac{\partial \mathring{u}_i}{\partial \mathring{x}_k}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}}\frac{dA_{kn}}{dt}x_n + G_{jn}x_n\left[\frac{\partial \mathring{u}_i}{\partial \mathring{x}_k}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}}A_{kj} \tag{3.18}$$

If we choose A_{kn} to satisfy

$$\frac{dA_{kn}}{dt} + G_{jn}A_{kj} = 0, \tag{3.19}$$

with initial condition $A_{ij} = \delta_{ij}$ at $t = 0$, we then obtain

$$\frac{\partial u_i}{\partial t} + G_{jk}x_k\frac{\partial u_i}{\partial x_j} = \left[\frac{\partial \mathring{u}_i}{\partial t}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}}. \tag{3.20}$$

Using the above we can re-write the equation for the fluctuating velocity (3.10) as follows

$$\left[\frac{\partial \mathring{u}_i}{\partial t}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}} = -[\mathring{u}_j]_{\mathring{\mathbf{x}}=\mathbf{Ax}}G_{ij} - \left[\frac{\partial \mathring{u}_i\mathring{u}_j}{\partial \mathring{x}_k}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}}A_{kj} - \left[\frac{1}{\rho}\frac{\partial \mathring{p}}{\partial \mathring{x}_k}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}}A_{ki} + \nu\left[\frac{\partial^2\mathring{u}_i}{\partial \mathring{x}_p\partial \mathring{x}_k}\right]_{\mathring{\mathbf{x}}=\mathbf{Ax}}A_{pj}A_{kj},$$
$$\tag{3.21}$$

which is equivalent to

$$\frac{\partial \mathring{u}_i}{\partial t} = -\mathring{u}_jG_{ij} - \frac{\partial \mathring{u}_i\mathring{u}_j}{\partial \mathring{x}_k}A_{kj} - \frac{1}{\rho}\frac{\partial \mathring{p}}{\partial \mathring{x}_k}A_{ki} + \nu\frac{\partial^2\mathring{u}_i}{\partial \mathring{x}_p\partial \mathring{x}_k}A_{pj}A_{kj}. \tag{3.22}$$

A similar analysis for the fluctuating pressure equation leads to

$$\frac{1}{\rho}\frac{\partial^2\mathring{p}}{\partial \mathring{x}_m\partial \mathring{x}_l}A_{mk}A_{lk} = -2G_{ij}\frac{\partial \mathring{u}_j}{\partial \mathring{x}_t}A_{ti} - \frac{\partial^2\mathring{u}_i\mathring{u}_j}{\partial \mathring{x}_p\partial \mathring{x}_k}A_{pi}A_{kj}. \tag{3.23}$$

3.3 The Reynolds-stress evolution equation

The transport equation for the Reynolds stresses simplifies to the following evolution equation

$$\frac{d\tau_{ij}}{dt} = \mathcal{P}_{ij} + \phi_{ij} - \epsilon_{ij}, \tag{3.24}$$

where $\phi_{ij} = \phi_{ij}^{(r)} + \phi_{ij}^{(s)}$, that is, $\phi^{(h)} = 0$. The fluctuating pressure satisfies the following Poisson equation

$$\frac{\partial^2 p}{\partial x_j\partial x_j} = -2\rho\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \rho\frac{\partial^2 u_iu_j}{\partial x_i\partial x_j}, \tag{3.25}$$

where the first term on the right hand side corresponds to the rapid pressure whereas the second corresponds to the slow pressure.

The solution to the PDE above is obtained using Green's function, that is, for

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_k\partial x_k} = S(\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^3) \tag{3.26}$$

the solution is of the following form,

$$f(\mathbf{x}) = \int_{\mathbb{R}^3} S(\mathbf{y})G(\mathbf{x} - \mathbf{y})d\mathbf{y} \tag{3.27}$$

where

$$G(\mathbf{r}) = \frac{-1}{4\pi ||\mathbf{r}||_2} \quad \text{for } \mathbb{R}^3. \tag{3.28}$$

Since the deformation $\partial \langle U_i \rangle / \partial x_j$ is constant and $\partial u_j(\mathbf{x}) / \partial x_i$ as a function of $\mathbf{y}$ becomes $\partial u_j(\mathbf{y}) / \partial y_i$, we obtain,

$$p(\mathbf{x}) = -2\rho \frac{\partial \langle U_i \rangle}{\partial x_j} \int_{\mathbb{R}^3} \frac{\partial u_j(\mathbf{y})}{\partial y_i} G(\mathbf{x} - \mathbf{y}) d\mathbf{y} - \rho \int_{\mathbb{R}^3} \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} G(\mathbf{x} - \mathbf{y}) d\mathbf{y} \tag{3.29}$$

3.3.1   The rapid redistribution

Using equation (3.29) we have

$$\left\langle \frac{p^{(r)}(\mathbf{x})}{\rho} \frac{\partial u_i(\mathbf{x})}{\partial x_j} \right\rangle = -2 \frac{\partial \langle U_k \rangle}{\partial x_l} \int_{\mathbb{R}^3} \left\langle \frac{\partial u_i(\mathbf{x})}{\partial x_j} \frac{\partial u_l(\mathbf{y})}{\partial y_k} \right\rangle G(\mathbf{x} - \mathbf{y}) d\mathbf{y}. \tag{3.30}$$

Since the two-point correlation is a function of $\mathbf{x} - \mathbf{y}$ and not of $\mathbf{x}$ and $\mathbf{y}$ independently, we can rewrite the expectation inside the integral as follows,

$$\left\langle \frac{\partial u_i(\mathbf{x})}{\partial x_j} \frac{\partial u_l(\mathbf{y})}{\partial y_k} \right\rangle = \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_k} \langle u_i(\mathbf{x}) u_l(\mathbf{y}) \rangle = \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_k} \mathcal{R}_{il}(\mathbf{x} - \mathbf{y}) = \left[-\frac{\partial}{\partial r_j} \frac{\partial}{\partial r_k} \mathcal{R}_{il}(\mathbf{r}) \right]_{\mathbf{r} = \mathbf{x} - \mathbf{y}} \tag{3.31}$$

where we have used the chain rule to obtain the last equality. Hence the expectation of the pressure-velocity gradient has the following form,

$$\left\langle \frac{p^{(r)}(\mathbf{x})}{\rho} \frac{\partial u_i(\mathbf{x})}{\partial x_j} \right\rangle = 2 \frac{\partial \langle U_k \rangle}{\partial x_l} \int_{\mathbb{R}^3} \left[\frac{\partial^2 \mathcal{R}_{il}(\mathbf{r})}{\partial r_j \partial r_k} \right]_{\mathbf{r} = \mathbf{x} - \mathbf{y}} G(\mathbf{x} - \mathbf{y}) d\mathbf{y}. \tag{3.32}$$

We then proceed by integration by substitution, with the substitution $\mathbf{y} = \mathbf{x} + \mathbf{r}$.

$$\left\langle \frac{p^{(r)}(\mathbf{x})}{\rho} \frac{\partial u_i(\mathbf{x})}{\partial x_j} \right\rangle = 2 \frac{\partial \langle U_k \rangle}{\partial x_l} \int_{\mathbb{R}^3} \frac{\partial^2 \mathcal{R}_{il}(\mathbf{r})}{\partial r_j \partial r_k} G(\mathbf{r}) d\mathbf{r}. \tag{3.33}$$

Using the definition of the Green's function, we rewrite the above as,

$$\left\langle \frac{p^{(r)}(\mathbf{x})}{\rho} \frac{\partial u_i(\mathbf{x})}{\partial x_j} \right\rangle = 2 \frac{\partial \langle U_k \rangle}{\partial x_l} \int_{\mathbb{R}^3} \frac{-1}{4\pi ||\mathbf{r}||_2} \frac{\partial^2 \mathcal{R}_{il}(\mathbf{r})}{\partial r_j \partial r_k} d\mathbf{r} = 2 \frac{\partial \langle U_k \rangle}{\partial x_l} M_{iljk} \tag{3.34}$$

where we have defined the tensor

$$M_{iljk} = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{||\mathbf{r}||_2} \frac{\partial^2 \mathcal{R}_{il}(\mathbf{r})}{\partial r_j \partial r_k} d\mathbf{r} \tag{3.35}$$

The rapid pressure-rate-of-strain term can now be written in the succinct notation as follows,

$$\phi_{ij}^{(r)} = 2 \frac{\partial \langle U_k \rangle}{\partial x_l} (M_{iljk} + M_{jlik}). \tag{3.36}$$

Chapter 4

Spectral Analysis

4.1 Definitions

One can use spectral tools such as the Fourier transform to analyze the fluid equations from a different perspective. In this section we provide definitions for various spectral quantities required for this spectral approach.

4.1.1 Three-dimensional spectral analysis

We begin with the more general three-dimensional case.

Velocity

The fluctuating velocity is expressed either as a Fourier series (denoted here as the discrete approach) or as an inverse Fourier transform (denoted as the continuous approach). Thus, for fluctuating velocity we have

- Discrete

$$u_j(\mathbf{x}, t) = \sum_{\boldsymbol{\kappa}} \hat{u}_j(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa} \cdot \mathbf{x}}, \tag{4.1}$$

$$\hat{u}_j(\boldsymbol{\kappa}, t) = \frac{1}{L^3} \int_{\mathbb{L}^3} u_j(\mathbf{x}, t) e^{-i\boldsymbol{\kappa} \cdot \mathbf{x}} d\mathbf{x}. \tag{4.2}$$

The above represents the entire flow as a juxtaposition of cubic domains

$$\mathbb{L}^3 = \{(x, y, z) : 0 < x < L, 0 < y < L, 0 < z < L\} \tag{4.3}$$

The velocity field is periodic across the cubes, i.e. $u_i(\mathbf{x} + NL, t) = u_i(\mathbf{x}, t)$ for all integer vectors N. Note that $u_i(\mathbf{x}, t)$ corresponds to the velocity of the original homogeneous flow within a cube of equal dimensions. This representation maintains spatial homogeneity over $\mathbb{R}^3$. The effects of the artificially imposed periodicity decreases as the ratio of L over the integral length scale approaches infinity.

- Continuous

$$u_j(\mathbf{x}, t) = \int_{\mathbb{R}^3} \hat{u}_j(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa} \cdot \mathbf{x}} d\boldsymbol{\kappa}, \tag{4.4}$$

$$\hat{u}_j(\boldsymbol{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} u_j(\mathbf{x}, t) e^{-i\boldsymbol{\kappa} \cdot \mathbf{x}} d\mathbf{x}. \tag{4.5}$$

Write something here about why we can use the Fourier transform...generalized functions.

Spectra

Using the above representations, the two-point velocity correlation can be used to define the discrete ($\hat{\mathcal{R}}_{ij}$) and continuous (E_{ij}) velocity spectrum tensors.

- Discrete

$$\mathcal{R}_{ij}(\mathbf{r}, t) = \sum_{\boldsymbol{\kappa}} \hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}, \tag{4.6}$$

$$\hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}, t) = \frac{1}{L^3} \int_{\mathbb{L}^3} \mathcal{R}_{ij}(\mathbf{r}, t) e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} d\mathbf{r}. \tag{4.7}$$

- Continuous

$$\mathcal{R}_{ij}(\mathbf{r}, t) = \int_{\mathbb{R}^3} E_{ij}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} d\boldsymbol{\kappa}, \tag{4.8}$$

$$E_{ij}(\boldsymbol{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{R}_{ij}(\mathbf{r}, t) e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} d\mathbf{r}. \tag{4.9}$$

Additional identities

- Discrete

$$\langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}', t) \rangle = \hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}', t) \delta_{\boldsymbol{\kappa}', \boldsymbol{\kappa}}. \tag{4.10}$$

- Continuous

$$\langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}', t) \rangle = E_{ij}(\boldsymbol{\kappa}', t) \delta(\boldsymbol{\kappa}' - \boldsymbol{\kappa}). \tag{4.11}$$

4.1.2 One-dimensional spectral analysis

All of the relations above apply for transforms along one direction only. These are shown below.

Velocity

- Discrete

$$u_j(\mathbf{e}_1 x_1, t) = \sum_{\kappa_1} \hat{u}_j^{(\mathbf{e}_1)}(\kappa_1, t) e^{i\kappa_1 x_1} \tag{4.12}$$

$$\hat{u}_j^{(\mathbf{e}_1)}(\kappa_1, t) = \frac{1}{L} \int_0^L u_j(\mathbf{e}_1 x_1, t) e^{-i\kappa_1 x_1} \, dx_1, \tag{4.13}$$

- Continuous

$$u_j(\mathbf{e}_1 x_1, t) = \int_{-\infty}^{\infty} \hat{u}_j^{(\mathbf{e}_1)}(\kappa_1, t) e^{i\kappa_1 x_1} \, d\kappa_1 \tag{4.14}$$

$$\hat{u}_j^{(\mathbf{e}_1)}(\kappa_1, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_j(\mathbf{e}_1 x_1, t) e^{-i\kappa_1 x_1} \, dx_1. \tag{4.15}$$

Spectra

- Discrete

$$\mathcal{R}_{ij}(\mathbf{e}_1 r_1, t) = \sum_{\kappa_1} \hat{\mathcal{R}}_{ij}^{(\mathbf{e}_1)}(\kappa_1, t) e^{i\kappa_1 x_1} \tag{4.16}$$

$$\hat{\mathcal{R}}_{ij}^{(\mathbf{e}_1)}(\kappa_1, t) = \frac{1}{L} \int_0^L \mathcal{R}_{ij}(\mathbf{e}_1 r_1, t) e^{-i\kappa_1 r_1} \, dr_1, \tag{4.17}$$

- Continuous

$$\mathcal{R}_{ij}(\mathbf{e}_1 r_1, t) = \int_{-\infty}^{\infty} E_{ij}^{(\mathbf{e}_1)}(\kappa_1, t) e^{i\kappa_1 x_1} \, d\kappa_1 \tag{4.18}$$

$$E_{ij}^{(\mathbf{e}_1)}(\kappa_1, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{R}_{ij}(\mathbf{e}_1 r_1, t) e^{-i\kappa_1 r_1} \, dr_1 \tag{4.19}$$

Additional identities

- Discrete

$$\langle \hat{u}_i^{(\mathbf{e}_1)*}(\kappa_1, t) \hat{u}_j^{(\mathbf{e}_1)}(\kappa_1', t) \rangle = \hat{\mathcal{R}}_{ij}^{(\mathbf{e}_1)}(\kappa_1', t) \delta_{\kappa_1', \kappa_1}. \tag{4.20}$$

- Continuous

$$\langle \hat{u}_i^{(\mathbf{e}_1)*}(\kappa_1, t) \hat{u}_j^{(\mathbf{e}_1)}(\kappa_1', t) \rangle = E_{ij}^{(\mathbf{e}_1)}(\kappa_1', t) \delta(\kappa_1' - \kappa_1). \tag{4.21}$$

The relationship between one- and three-dimensional spectra are given by

- Discrete

$$\hat{\mathcal{R}}_{ij}^{(\mathbf{e}_1)}(\kappa_1, t) = \sum_{\kappa_2} \sum_{\kappa_3} \hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}) \tag{4.22}$$

- Continuous

$$E_{ij}^{(1d)}(\kappa_1, t) = \iint_{-\infty}^{\infty} E_{ij}(\boldsymbol{\kappa}, t) \, d\kappa_2 \, d\kappa_3 \tag{4.23}$$

4.1.3 Kinetic Energy spectrum

The energy spectrum is defined as

$$E(\kappa) = \oint \frac{1}{2} E_{ii}(\boldsymbol{\kappa}) \, dS(\kappa) = \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} E_{ii}(\boldsymbol{\kappa}) \kappa^2 \sin(\theta) \, d\theta d\phi \tag{4.24}$$

A relationship can be obtained between the discrete and continuous approaches. If we denote $\boldsymbol{\kappa}_D$ and $\boldsymbol{\kappa}_C$ as the wave vectors of the discrete and continuous approach, this relationship is as follows

$$E_{ij}(\boldsymbol{\kappa}_C, t) = \sum_{\boldsymbol{\kappa}_D} \delta(\boldsymbol{\kappa}_C - \boldsymbol{\kappa}_D) \hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}_D, t). \tag{4.25}$$

4.1.4 Spectral representation of statistics

Using expressions from the previous section we obtain

- Discrete

$$\langle u_i u_j \rangle = \sum_{\boldsymbol{\kappa}} \hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}, t) \tag{4.26}$$

$$k = \sum_{\boldsymbol{\kappa}} \frac{1}{2} \hat{\mathcal{R}}_{ii}(\boldsymbol{\kappa}, t) \tag{4.27}$$

$$\left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} \right\rangle = \sum_{\boldsymbol{\kappa}} \kappa_k \kappa_l \hat{\mathcal{R}}_{ij}(\boldsymbol{\kappa}, t) \tag{4.28}$$

$$\langle s^2 \rangle = \langle 2 s_{ij} s_{ij} \rangle = \sum_{\boldsymbol{\kappa}} \kappa_i \kappa_i \hat{\mathcal{R}}_{jj}(\boldsymbol{\kappa}, t) \tag{4.29}$$

- Continuous

$$\langle u_i u_j \rangle = \int_{\mathbb{R}^3} E_{ij}(\boldsymbol{\kappa}, t) d\boldsymbol{\kappa} \tag{4.30}$$

$$k = \int_0^\infty E(\kappa, t)\, d\kappa \tag{4.31}$$

$$\left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} \right\rangle = \int_{\mathbb{R}^3} \kappa_k \kappa_l E_{ij}(\boldsymbol{\kappa}, t)\, d\boldsymbol{\kappa} \tag{4.32}$$

$$\langle s^2 \rangle = \langle 2 s_{ij} s_{ij} \rangle = 2 \int_0^\infty \kappa^2 E(\kappa, t)\, d\kappa \tag{4.33}$$

4.2   The fluctuating velocity

We rewrite below eq. (3.22) and **??** but with the $^\circ$ superscript neglected for convenience.

$$\frac{\partial u_i}{\partial t} = -u_j G_{ij} - \frac{\partial u_i u_j}{\partial x_k} A_{kj} - \frac{1}{\rho} \frac{\partial p}{\partial x_k} A_{ki} + \nu \frac{\partial^2 u_i}{\partial x_p \partial x_k} A_{pj} A_{kj}. \tag{4.34}$$

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_m \partial x_l} A_{mk} A_{lk} = -2 G_{ij} \frac{\partial u_j}{\partial x_t} A_{ti} - \frac{\partial^2 u_i u_j}{\partial x_p \partial x_k} A_{pi} A_{kj}. \tag{4.35}$$

The Fourier transform of the above two leads to

$$\frac{\partial \hat{u}_i}{\partial t} = -\hat{u}_j G_{ij} - i A_{kj} \kappa_k \hat{N}_{ij} - i \frac{1}{\rho} A_{ki} \kappa_k \hat{p} - \nu A_{pj} \kappa_p A_{kj} \kappa_k \hat{u}_i. \tag{4.36}$$

$$-\frac{1}{\rho} A_{mk} \kappa_m A_{lk} \kappa_l \hat{p} = -2 G_{ij} i A_{ti} \kappa_t \hat{u}_j + A_{pi} \kappa_p A_{kj} \kappa_k \hat{N}_{ij}. \tag{4.37}$$

where $\hat{u}_i = \hat{u}_i(t, \boldsymbol{\kappa})$, $\hat{p} = \hat{p}(t, \boldsymbol{\kappa})$, and $\hat{N}_{ij} = \hat{N}_{ij}(t, \boldsymbol{\kappa})$ are the Fourier transforms of u_i, p, and $u_i u_j$, respectively.

Define the function $\hat{\boldsymbol{\kappa}} = \hat{\boldsymbol{\kappa}}(t, \boldsymbol{\kappa})$, which is determined by the following ODE

$$\frac{\partial \hat{\kappa}_i}{\partial t} = -G_{ki} \hat{\kappa}_k, \tag{4.38}$$

with initial condition $\hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa}$ at $t = 0$. Since the ODE is linear we can assume solutions of the form $\hat{\kappa}_i = B_{ji} \kappa_j$, where $B_{ij} = B_{ij}(t)$. Plugging this form into the ODE and the initial condition above, we require that B_{ij} satisfy

$$\frac{dB_{ji}}{dt} + G_{ki} B_{jk} = 0, \tag{4.39}$$

along with $B_{ij} = \delta_{ij}$ at $t = 0$. Since these requirements are identical to those satisfied by A_{ij}, we conclude that

$$\hat{\kappa}_i = A_{ji} \kappa_j, \tag{4.40}$$

which in vector notation becomes $\hat{\boldsymbol{\kappa}} = \mathbf{A}^T \boldsymbol{\kappa}$. Using this definition for $\hat{\boldsymbol{\kappa}}$ the evolution equation for $\hat{u}_i$ and $\hat{p}$ become

$$\frac{\partial \hat{u}_i}{\partial t} = -\hat{u}_j G_{ij} - i \hat{\kappa}_j \hat{N}_{ij} - i \frac{1}{\rho} \hat{\kappa}_i \hat{p} - \nu \hat{\kappa}^2 \hat{u}_i, \tag{4.41}$$

$$-\frac{1}{\rho} \hat{\kappa}^2 \hat{p} = -2 G_{ij} i \hat{\kappa}_i \hat{u}_j + \hat{\kappa}_i \hat{\kappa}_j \hat{N}_{ij}. \tag{4.42}$$

Combining the above two we have

$$\frac{\partial \hat{u}_i}{\partial t} = -M_{ij}\hat{u}_j + s_i - \nu\hat{\kappa}^2\hat{u}_i, \tag{4.43}$$

where $M_{ij} = M_{ij}(t, \boldsymbol{\kappa})$ is given by

$$M_{ij} = G_{ij} - 2\frac{\hat{\kappa}_i\hat{\kappa}_k}{\hat{\kappa}^2}G_{kj}, \tag{4.44}$$

$s_i = s_i(t, \boldsymbol{\kappa})$ by

$$s_i = -i\hat{\kappa}_j L_{ik}\hat{N}_{kj}, \tag{4.45}$$

and $L_{ij} = L_{ij}(t, \boldsymbol{\kappa})$ by

$$L_{ij} = \delta_{ij} - \frac{\hat{\kappa}_i\hat{\kappa}_j}{\hat{\kappa}^2}. \tag{4.46}$$

This last tensor is termed the projection operator since $L_{ij}y_j$ is the component of some generic vector $\mathbf{y}$ orthogonal to $\hat{\boldsymbol{\kappa}}$. The term $-M_{ij}\hat{v}_j$, which is linear, arises from the combination of the first terms on the RHS of the velocity and pressure equations, whereas the term s_i, which is non-linear, arises from combination of the second terms on the RHS of the velocity and pressure equations.

The Fourier coefficient $\hat{N}_{ij}$ can be expressed as follows

$$\begin{aligned}
\hat{N}_{ij}(t, \boldsymbol{\kappa}) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} u_i(t, \mathbf{x})u_j(t, \mathbf{x})e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}d\mathbf{x} \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\hat{u}_i(t, \mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}d\mathbf{p}\right)\left(\int_{\mathbb{R}^3}\hat{u}_j(t, \mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}d\mathbf{q}\right)e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}d\mathbf{x} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\hat{u}_i(t, \mathbf{p})\hat{u}_j(t, \mathbf{q})\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}d\mathbf{x}d\mathbf{p}d\mathbf{q} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\hat{u}_i(t, \mathbf{p})\hat{u}_j(t, \mathbf{q})\boldsymbol{\delta}(\mathbf{p}+\mathbf{q}-\boldsymbol{\kappa})d\mathbf{p}d\mathbf{q} \\
&= \int_{\mathbb{R}^3}\hat{u}_i(t, \mathbf{p})\hat{u}_j(t, \boldsymbol{\kappa}-\mathbf{p})d\mathbf{p}. \tag{4.47}
\end{aligned}$$

4.3 The velocity-spectrum tensor

We proceed in deriving the evolution equation for the velocity-spectrum tensor as follows,

$$\begin{aligned}
\frac{\partial}{\partial t}[\hat{u}_i^*(t, \boldsymbol{\kappa})\hat{u}_j(t, \boldsymbol{\kappa}')] =& \frac{\partial \hat{u}_i^*(t, \boldsymbol{\kappa})}{\partial t}\hat{u}_j(t, \boldsymbol{\kappa}') + \hat{u}_i^*(t, \boldsymbol{\kappa})\frac{\partial \hat{u}_j(t, \boldsymbol{\kappa}')}{\partial t} \\
=& \left[-M_{ik}(t, \boldsymbol{\kappa})\hat{u}_k^*(t, \boldsymbol{\kappa}) + s_i^*(t, \boldsymbol{\kappa}) - \nu\hat{\kappa}^2(t, \boldsymbol{\kappa})\hat{u}_i^*(t, \boldsymbol{\kappa})\right]\hat{u}_j(t, \boldsymbol{\kappa}') \\
&+ \hat{u}_i^*(t, \boldsymbol{\kappa})\left[-M_{jk}(t, \boldsymbol{\kappa}')\hat{u}_k(t, \boldsymbol{\kappa}') + s_j(t, \boldsymbol{\kappa}') - \nu\hat{\kappa}^2(t, \boldsymbol{\kappa}')\hat{u}_j(t, \boldsymbol{\kappa}')\right] \\
=& -M_{ik}(t, \boldsymbol{\kappa})\hat{u}_k^*(t, \boldsymbol{\kappa})\hat{u}_j(t, \boldsymbol{\kappa}') - M_{jk}(t, \boldsymbol{\kappa}')\hat{u}_i^*(t, \boldsymbol{\kappa})\hat{u}_k(t, \boldsymbol{\kappa}') \\
&+ \hat{u}_i^*(t, \boldsymbol{\kappa})s_j(t, \boldsymbol{\kappa}') + s_i^*(t, \boldsymbol{\kappa})\hat{u}_j(t, \boldsymbol{\kappa}') \\
&- \nu\hat{\kappa}^2(t, \boldsymbol{\kappa})\hat{u}_i^*(t, \boldsymbol{\kappa})\hat{u}_j(t, \boldsymbol{\kappa}') - \nu\hat{\kappa}^2(t, \boldsymbol{\kappa}')\hat{u}_i^*(t, \boldsymbol{\kappa})\hat{u}_j(t, \boldsymbol{\kappa}') \tag{4.48}
\end{aligned}$$

We now define the tensor $T_{ij} = T_{ij}(t, \boldsymbol{\kappa})$ using the following equation

$$\langle\hat{u}_i^*(t, \boldsymbol{\kappa})s_j(t, \boldsymbol{\kappa}') + s_i^*(t, \boldsymbol{\kappa})\hat{u}_j(t, \boldsymbol{\kappa}')\rangle = T_{ij}(t, \boldsymbol{\kappa}')\boldsymbol{\delta}(\boldsymbol{\kappa}' - \boldsymbol{\kappa}). \tag{4.49}$$

Taking the expectation of both sides of equation (4.48), and using equations (4.11) and (4.49) one obtains

$$\frac{\partial E_{ij}(t, \boldsymbol{\kappa}')}{\partial t} \boldsymbol{\delta}(\boldsymbol{\kappa}' - \boldsymbol{\kappa}) = \left[-M_{ik}(t, \boldsymbol{\kappa}) E_{kj}(t, \boldsymbol{\kappa}') - M_{jk}(t, \boldsymbol{\kappa}') E_{ik}(t, \boldsymbol{\kappa}') + T_{ij}(t, \boldsymbol{\kappa}') \right.$$
$$\left. -\nu \hat{\kappa}^2(t, \boldsymbol{\kappa}) E_{ij}(t, \boldsymbol{\kappa}') - \nu \hat{\kappa}^2(t, \boldsymbol{\kappa}') E_{ij}(t, \boldsymbol{\kappa}') \right] \boldsymbol{\delta}(\boldsymbol{\kappa}' - \boldsymbol{\kappa}). \tag{4.50}$$

Integrating over $\boldsymbol{\kappa}'$ the evolution equation for the velocity-spectrum tensor $E_{ij} = E_{ij}(t, \boldsymbol{\kappa})$ becomes

$$\frac{\partial E_{ij}}{\partial t} = P_{ij} + T_{ij} - 2\nu \hat{\kappa}^2 E_{ij}, \tag{4.51}$$

where $P_{ij} = -M_{ik} E_{kj} - M_{jk} E_{ik}$.

A clarification note   At this point it is important to note that we've been working with Fourier transforms of variables that were defined in the Rogallo reference frame, that is, those that have the superscript $^\circ$ (this superscript was neglected for convenience). The relationship between statistics of $u_i(t, \mathbf{x})$ and $\mathring{u}_i(t, \mathring{\mathbf{x}})$ is now discussed. For the two-point correlations, both quantities are related as

$$\langle u_i(t, \mathbf{x}) u_j(t, \mathbf{x}') \rangle = \left[\langle \mathring{u}_i(t, \mathring{\mathbf{x}}) \mathring{u}_j(t, \mathring{\mathbf{x}}') \rangle \right]_{\mathring{\mathbf{x}} = \mathbf{Ax}, \, \mathring{\mathbf{x}}' = \mathbf{Ax}'}. \tag{4.52}$$

Using equation (4.8) we can express $\langle \mathring{u}_i(t, \mathring{\mathbf{x}}) \mathring{u}_j(t, \mathring{\mathbf{x}}') \rangle$ as the following integral

$$\langle \mathring{u}_i(t, \mathring{\mathbf{x}}) \mathring{u}_j(t, \mathring{\mathbf{x}}') \rangle = \int_{\mathbb{R}^3} E_{ij}(t, \boldsymbol{\kappa}) e^{i\boldsymbol{\kappa} \cdot (\mathring{\mathbf{x}}' - \mathring{\mathbf{x}})} d\boldsymbol{\kappa}. \tag{4.53}$$

Since

$$\boldsymbol{\kappa} \cdot (\mathbf{Ax}' - \mathbf{Ax}) = \kappa_i A_{ij}(x_j' - x_j) = \hat{\kappa}_j(x_j' - x_j) \tag{4.54}$$

the two-point velocity correlation for u_i becomes

$$\langle u_i(t, \mathbf{x}) u_j(t, \mathbf{x}') \rangle = \int_{\mathbb{R}^3} E_{ij}(t, \boldsymbol{\kappa}) e^{i\hat{\boldsymbol{\kappa}}(t, \boldsymbol{\kappa}) \cdot (\mathbf{x}' - \mathbf{x})} d\boldsymbol{\kappa}. \tag{4.55}$$

The Reynolds stresses are therefore obtained from the following

$$\tau_{ij} = \int_{\mathbb{R}^3} E_{ij} \, d\boldsymbol{\kappa}. \tag{4.56}$$

We now show a summary of the correspondence between the different terms in the evolution equations for the fluctuating velocity (2.87), the energy spectrum (4.51), and the Reynolds stresses (2.93).

u_i:	$\dfrac{\partial u_i}{\partial t}$	$\langle U_j \rangle \dfrac{\partial u_i}{\partial x_j}$	$-u_j \dfrac{\partial \langle U_i \rangle}{\partial x_j}$	$-\dfrac{1}{\rho} \dfrac{\partial p^{(r)}}{\partial x_i}$	$-\dfrac{1}{\rho} \dfrac{\partial p^{(s)}}{\partial x_i}$	$-\dfrac{\partial u_i u_j}{\partial x_j}$	$\nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j}$	$\dfrac{\partial \tau_{ij}}{\partial x_j}$
E_{ij}:	$\dfrac{\partial E_{ij}}{\partial t}$	makes $\hat{\boldsymbol{\kappa}}$	P_{ij}		T_{ij}		$-2\nu \hat{\kappa}^2 E_{ij}$	Null
τ_{ij}:	$\dfrac{\partial \tau_{ij}}{\partial t}$	$\langle U_k \rangle \dfrac{\partial \tau_{ij}}{\partial x_k}$	$\mathcal{P}_{ij}$	Π_{ij}		$\mathcal{T}_{kij}^{(u)}$	ϵ_{ij}	$\mathcal{T}_{kij}^{(\nu)}$	Null

Chapter 5

Rapid-distortion theory

- Homogeneous RDT used much more (origin. Taylor, Prandt'l, 1930's)

- Viscous effects can be included but often neglected

- Inhomogeneous RDT developed by Hunt (1973, JFM) and collaborators. See Hunt & Carruthers (1990, JFM), Goldsten, Atassi, Peake.

5.1   When are mean distortions rapid enough?

Start with the homogeneous equations for the turbulent fluctuations,

$$\frac{\partial u_i}{\partial t} + G_{jk}x_k\frac{\partial u_i}{\partial x_j} = -u_j G_{ij} - \frac{\partial u_i u_j}{\partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{5.1}$$

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j \partial x_j} = -2G_{ij}\frac{\partial u_j}{\partial x_i} - \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}. \tag{5.2}$$

We will now assume orders of magnitude for the variables above. This ordering is as follow

$$u_i \sim u_0 \qquad p \sim p_0 \qquad G_{ij} \sim S \qquad t \sim S^{-1} \qquad x_i \sim l_0. \tag{5.3}$$

In the above, u_0 is the characteristic velocity of large-scale eddies, l_0 the length scale of large-scale eddies, p_0 the characteristic scale for pressure, and S^{-1} the time scale of the deformation. Using the above, the terms in the pressure equation have the following order of magnitude

$$\underbrace{\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j \partial x_j}}_{\frac{p_0}{\rho l_0^2}} = -\underbrace{2G_{ij}\frac{\partial u_j}{\partial x_i}}_{\frac{Su_0}{l_0}} - \underbrace{\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}}_{\frac{u_0^2}{l_0^2}}. \tag{5.4}$$

The linear term is of much larger magnitude when

$$\frac{\frac{Su_0}{l_0}}{\frac{u_0^2}{l_0^2}} \gg 1 \rightarrow \frac{Sl_0}{u_0} \gg 1. \tag{5.5}$$

If this is satisfied, then

$$\frac{p_0}{\rho l_0^2} \sim \frac{Su_0}{l_0} \rightarrow p_0 \sim \rho S u_0 l_0. \tag{5.6}$$

Using this ordering for pressure, we obtain the following orders of magnitude for the terms in the fluctuating velocity equation

$$\underbrace{\frac{\partial u_i}{\partial t}}_{Su_0} + \underbrace{G_{jk}x_k\frac{\partial u_i}{\partial x_j}}_{Su_0} = -\underbrace{u_j G_{ij}}_{Su_0} - \underbrace{\frac{\partial u_i u_j}{\partial x_j}}_{\frac{u_0^2}{l_0}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{Su_0} + \underbrace{\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{\frac{\nu u_0}{l_0^2}}. \tag{5.7}$$

Again, the linear terms are of much larger magnitudes when

$$\frac{Su_0}{\frac{u_0^2}{l_0}} \gg 1 \rightarrow \frac{Sl_0}{u_0} \gg 1, \tag{5.8}$$

The viscous terms can be neglected when

$$\frac{Su_0}{\frac{u_0 \nu}{l_0^2}} \gg 1 \rightarrow \frac{Sl_0^2}{\nu} \gg 1 \rightarrow \frac{Sl_0}{u_0}\frac{u_0 l_0}{\nu} \gg 1 \rightarrow \frac{Sl_0}{u_0}Re_t \gg 1. \tag{5.9}$$

Thus, one could conclude that the equations for the fluctuating velocities and pressure can be approximated by linear equations when $Sl_0/u_0 \gg 1$, that is, when the time scale of the deformation is much faster than the time scale of the energetic eddies. The linearized equations are shown below

$$\frac{\partial u_i}{\partial t} + G_{jk}x_k\frac{\partial u_i}{\partial x_j} = -u_j G_{ij} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{5.10}$$

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j \partial x_j} = -2G_{ij}\frac{\partial u_j}{\partial x_i}. \tag{5.11}$$

However, it is important to note the following points:

- For rapid distortions S^{-1} is a very small number. Thus, the time is of the order of S^{-1} only for the very first initial period. For longer times, scaling t by S^{-1} and still assuming that τ is of $O(1)$ does not hold.

- Having u_i and x_i to be of the order of u_o and l_o respectively implies that all velocities and lengths are of the order of u_o and l_o. However, turbulence is a multi scale problem, and thus some velocities and lengths will scale as u_η and η, respectively. Hence, for RDT to apply to all scales we also need to have $S\eta/u_\eta \gg 1$.

5.2 Evolution of a Fourier mode

The evolution of a velocity Fourier mode for generic homogeneous deformations is given by equation (4.43). Since the governing equations for rapid distortions do not include the non-linear and viscous terms, the evolution equation for the Fourier modes simplifies to

$$\frac{\partial \hat{u}_i}{\partial t} = -M_{ij}\hat{u}_j \tag{5.12}$$

Since the evolution equation for $\hat{u}_i$ is linear, we can assume solutions of the form

$$\hat{u}_i = H_{ik}\hat{u}_k^{(0)} \tag{5.13}$$

where $\hat{u}_i^{(0)} = \hat{u}_i(0, \boldsymbol{\kappa})$, and $H_{ij} = H_{ij}(t, \boldsymbol{\kappa})$ is a real non-random function that satisfies

$$\frac{\partial H_{ik}}{\partial t} = -M_{ij}H_{jk} \tag{5.14}$$

with initial condition $H_{ik} = \delta_{ik}$ for $t = 0$.

5.3 Evolution of the spectrum

The evolution equation for the velocity-spectrum tensor for generic homogeneous deformations is given by equation (4.51). Again, neglecting the non-linear and viscous terms we obtain

$$\frac{\partial E_{ij}}{\partial t} = -M_{ik}E_{kj} - M_{jk}E_{ik}, \tag{5.15}$$

which, after plugging in for M_{ij}, can be re-expressed as follows

$$\frac{\partial E_{ij}}{\partial t} = -G_{ik}E_{kj} - G_{jk}E_{ik} + 2G_{pk}\left(\frac{\hat{\kappa}_i\hat{\kappa}_p}{\hat{\kappa}^2}E_{kj} + \frac{\hat{\kappa}_j\hat{\kappa}_p}{\hat{\kappa}^2}E_{ik}\right). \tag{5.16}$$

We could, however, compute the velocity-spectrum tensor using the transfer functions H_{ij}, due to the linearity of the governing equations. Using equation (4.11) we obtain

$$\begin{aligned}
E_{ij}(t,\boldsymbol{\kappa}')\boldsymbol{\delta}(\boldsymbol{\kappa}' - \boldsymbol{\kappa}) &= \langle \hat{v}_i^*(t,\boldsymbol{\kappa})\hat{v}_j(t,\boldsymbol{\kappa}')\rangle \\
&= \langle H_{ik}(t,\boldsymbol{\kappa})\hat{v}_k^*(0,\boldsymbol{\kappa})H_{jp}(t,\boldsymbol{\kappa}')\hat{v}_p(0,\boldsymbol{\kappa}')\rangle \\
&= H_{ik}(t,\boldsymbol{\kappa})H_{jp}(t,\boldsymbol{\kappa}')\langle \hat{v}_k^*(0,\boldsymbol{\kappa})\hat{v}_p(0,\boldsymbol{\kappa}')\rangle \\
&= H_{ik}(t,\boldsymbol{\kappa})H_{jp}(t,\boldsymbol{\kappa}')E_{kp}(0,\boldsymbol{\kappa}')\boldsymbol{\delta}(\boldsymbol{\kappa}' - \boldsymbol{\kappa}).
\end{aligned} \tag{5.17}$$

Integrating both sides over $\boldsymbol{\kappa}'$ we obtain

$$E_{ij} = H_{ik}H_{jp}E_{kp}^{(0)}, \tag{5.18}$$

where $E_{ij}^{(0)} = E_{ij}(0,\boldsymbol{\kappa})$.

The Reynolds stresses can be obtained from an alternate velocity-spectrum tensor. Define this tensor as $\tilde{E}_{ij} = \tilde{E}_{ij}(t,\boldsymbol{\chi})$, such that

$$E_{ij} = \left[\tilde{E}_{ij}\right]_{\boldsymbol{\chi}=\hat{\boldsymbol{\kappa}}}. \tag{5.19}$$

We thus obtain the following

$$\begin{aligned}
\frac{\partial E_{ij}}{\partial t} &= \left[\frac{\partial \tilde{E}_{ij}}{\partial t}\right]_{\boldsymbol{\chi}=\hat{\boldsymbol{\kappa}}} + \left[\frac{\partial \tilde{E}_{ij}}{\partial \chi_k}\right]_{\boldsymbol{\chi}=\hat{\boldsymbol{\kappa}}}\frac{\partial \hat{\kappa}_k}{\partial t} \\
&= \left[\frac{\partial \tilde{E}_{ij}}{\partial t}\right]_{\boldsymbol{\chi}=\hat{\boldsymbol{\kappa}}} - \left[\frac{\partial \tilde{E}_{ij}}{\partial \chi_k}\right]_{\boldsymbol{\chi}=\hat{\boldsymbol{\kappa}}}G_{rk}\hat{\kappa}_r \\
&= \left[\frac{\partial \tilde{E}_{ij}}{\partial t} - \frac{\partial \tilde{E}_{ij}}{\partial \chi_k}G_{rk}\chi_r\right]_{\boldsymbol{\chi}=\hat{\boldsymbol{\kappa}}}.
\end{aligned} \tag{5.20}$$

Using the above in equation (5.16) we obtain the following evolution equation for $\tilde{E}_{ij}$

$$\frac{\partial \tilde{E}_{ij}}{\partial t} - \frac{\partial \tilde{E}_{ij}}{\partial \chi_k}G_{rk}\chi_r = -G_{ik}\tilde{E}_{kj} - G_{jk}\tilde{E}_{ik} + 2G_{pk}\left(\frac{\chi_i\chi_p}{\chi^2}\tilde{E}_{kj} + \frac{\chi_j\chi_p}{\chi^2}\tilde{E}_{ik}\right). \tag{5.21}$$

Jacobi's formula allows us to compute the evolution of the determinant of A_{ij}. Thus

$$\begin{aligned}
\frac{d}{dt}\det(A_{ij}) &= \det(A_{ij})\text{tr}\left(A_{ik}^{-1}\frac{dA_{kj}}{dt}\right) \\
&= \det(A_{ij})\text{tr}(-A_{ik}^{-1}G_{qj}A_{kq}) \\
&= -\det(A_{ij})\text{tr}(G_{qj}\delta_{iq}) \\
&= -\det(A_{ij})G_{kk}.
\end{aligned} \tag{5.22}$$

Since $\det(A_{ij}) = 1$ for $t = 0$, the determinant of A_{ij} remains equal to one for all times t when the flow is incompressible. Thus the Reynolds stresses can be computed from $\tilde{E}_{ij}$ as well, since using integration by substitution one can show that

$$
\begin{aligned}
\tau_{ij} &= \int E_{ij} d\boldsymbol{\kappa} \\
&= \int \left[\tilde{E}_{ij} \right]_{\boldsymbol{\chi} = \hat{\boldsymbol{\kappa}}} d\boldsymbol{\kappa} \\
&= \int \left[\tilde{E}_{ij} \right]_{\boldsymbol{\chi} = \hat{\boldsymbol{\kappa}}} |\det(A_{pq})| d\boldsymbol{\kappa} \\
&= \int \left[\tilde{E}_{ij} \right]_{\boldsymbol{\chi} = \hat{\boldsymbol{\kappa}}} \left| \det\left(\frac{\partial \hat{\kappa}_q}{\partial \kappa_p} \right) \right| d\boldsymbol{\kappa} \\
&= \int \tilde{E}_{ij} d\boldsymbol{\chi}.
\end{aligned}
\tag{5.23}
$$

Part III

Inhomogeneous turbulence

Chapter 6

Free shear flows

Chapter 7

Wall-bounded flows

7.1 Fully-developed flows

7.1.1 Channels

Balance of forces

For fully developed channel flow, with channel half height δ, the streamwise and wall normal momentum equations simplify to the following

$$0 = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial x} + \nu \frac{d^2 \langle U \rangle}{dy^2} - \frac{d\langle uv \rangle}{dy} \tag{7.1}$$

$$0 = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial y} - \frac{d\langle vv \rangle}{dy} \tag{7.2}$$

Differentiating the wall normal equation with respect to x leads to

$$\frac{\partial}{\partial y}\frac{\partial \langle P \rangle}{\partial x} = 0, \tag{7.3}$$

that is, the pressure drop is independent of y. Integrating the streamwise equation with respect to y, from zero to some location y, leads to

$$\rho\nu\frac{d\langle U \rangle}{dy} - \rho\langle uv \rangle = \rho\nu\frac{d\langle U \rangle}{dy}\bigg|_{wall} + \frac{\partial \langle P \rangle}{\partial x}y$$

$$= \tau_{wall} + \frac{\partial \langle P \rangle}{\partial x}y. \tag{7.4}$$

Evaluating at $y = \delta$ shows that

$$\frac{\partial \langle P \rangle}{\partial x} = -\frac{\tau_{wall}}{\delta}. \tag{7.5}$$

Thus, the integrated momentum equation becomes

$$\rho\nu\frac{d\langle U \rangle}{dy} - \rho\langle uv \rangle = \tau_{wall}\left(1 - \frac{y}{\delta}\right). \tag{7.6}$$

Using the following wall scales

$$u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}} \qquad \delta_\nu = \frac{\nu}{u_\tau} \tag{7.7}$$

and the friction Reynolds number $Re_\tau = u_\tau\delta/\nu$ the integrated momentum equation can be re-expressed in wall units as follows

$$\frac{du^+}{dy^+} - \langle uv \rangle^+ = 1 - \frac{y^+}{Re_\tau}. \tag{7.8}$$

Channel laws

We choose the set of physical variables associated with the channel to consist of the following: ρ, ν, δ, u_τ (corresponds to the pressure gradient), y, and $d\langle U\rangle/dy$. Buckingham Π theorem tells us that there are three non-dimensional groups formed from the above variables, and they are related by a single equation. This is shown below

$$\frac{d\langle U\rangle}{dy} = \frac{u_\tau}{y}\Phi\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right). \tag{7.9}$$

It is postulated that very close to the wall one encounters the inner layer ($y/\delta < 0.1$) throughout which the velocity is independent of channel scales such as δ. Thus, equation (7.9) becomes

$$\frac{d\langle U\rangle}{dy} = \frac{u_\tau}{y}\Phi_I\left(\frac{y}{\delta_\nu}\right). \tag{7.10}$$

It is also postulated that far from the wall one encounters the outer layer ($y/\delta_\nu > 50$) throughout which the velocity is independent of the wall scales such as δ_ν. Thus equation (7.9) becomes

$$\frac{d\langle U\rangle}{dy} = \frac{u_\tau}{y}\Phi_O\left(\frac{y}{\delta}\right). \tag{7.11}$$

Rewriting the inner layer in wall units one obtains

$$\frac{du^+}{dy^+} = \frac{1}{y^+}\Phi_I(y^+). \tag{7.12}$$

and thus

$$u^+ = \int_0^{y^+} \frac{1}{y'}\Phi_I(y')dy'. \tag{7.13}$$

The integral above is referred to as the **law of the wall**, and experimental data shows this integral to be universal. At the wall $u^+ = 0$ and $du^+/dy^+ = 1$ (since $\tau_{wall} = \rho\nu d\langle U\rangle/dy$). Thus a Taylor series expansion about $y^+ = 0$ shows that

$$u^+ = y^+ + H.O.T. \tag{7.14}$$

Indeed, simulations show $u^+ = y^+$ up to around $y^+ < 5$, and this region is referred to as the viscous sublayer.

The overlap between the inner layer and the outer layer is described by the **log law**, though the limits of the log law may expand beyond those of the inner and outer layer (i.e $y/\delta_\nu > 30$, $y/\delta < 0.3$). Within this layer the effect of both wall and channel scales is negligible and hence Φ_I becomes a constant equal to κ^{-1}. Thus

$$\frac{du^+}{dy^+} = \frac{1}{\kappa y^+}. \tag{7.15}$$

Evaluating the law of the wall we obtain

$$u^+ = \int_0^a \frac{1}{y'}\Phi_I(y')dy' + \int_a^{y^+} \frac{1}{\kappa y'}dy' = C + \frac{1}{\kappa}[\ln(y^+) - \ln(a)] = \frac{1}{\kappa}\ln(y^+) + B, \tag{7.16}$$

where $a < y^+$ is a location within the log law and C and B are constants.

Layer	Limits	Law
Inner	$y/\delta < 0.1$	Law of the wall
Overlap	$y/\delta < 0.1$, $y^+ > 50$	Log law (expands to $y/\delta < 0.3$, $y^+ > 30$)
Outer	$y^+ > 50$	Velocity-defect law

Table 7.1: Summary of wall layers and corresponding laws.

Integrating equation (7.11) between y and δ leads to the **velocity-defect law**. That is

$$\frac{U_0 - \langle U \rangle}{u_\tau} = \int_y^\delta \frac{1}{y'} \Phi_O(\frac{y'}{\delta}) dy' = \int_{y/\delta}^1 \frac{1}{y''} \Phi_O(y'') dy''. \tag{7.17}$$

The integral above, as opposed to the law of the wall, is not expected to be universal. Within the log law Φ_O becomes a constant equal to κ^{-1}. Hence evaluating the velocity-defect law leads to

$$\frac{U_0 - \langle U \rangle}{u_\tau} = \int_{y/\delta}^b \frac{1}{\kappa y''} dy'' + \int_b^1 \frac{1}{y''} \Phi_O(y'') dy'' = \frac{1}{\kappa}[\ln(b) - \ln(y/\delta)] + C_1 = -\frac{1}{\kappa}\ln(y/\delta) + B_1 \tag{7.18}$$

where $b > y/\delta$ is a location within the log law and C_1 and B_1 are constants.

7.1.2 Pipes

For fully developed pipe flow the streamwise and radial momentum equations simplify to the following

$$0 = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial z} + \nu\left(\frac{\partial^2 \langle u_z \rangle}{\partial r^2} + \frac{1}{r}\frac{\partial \langle u_z \rangle}{\partial r}\right) - \frac{\partial \langle u_z u_r \rangle}{\partial r} - \frac{\langle u_z u_r \rangle}{r}. \tag{7.19}$$

$$0 = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial r} - \frac{\partial \langle u_r u_r \rangle}{\partial r} - \frac{\langle u_r u_r \rangle}{r} + \frac{\langle u_\theta u_\theta \rangle}{r} \tag{7.20}$$

Differentiating the radial equation with respect to z leads to

$$\frac{\partial}{\partial r}\frac{\partial \langle P \rangle}{\partial z} = 0, \tag{7.21}$$

that is, the pressure drop is independent of r. We now rewrite the streamwise equation as follows

$$0 = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial z} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \langle u_z \rangle}{\partial r}\right) - \frac{1}{r}\frac{\partial r\langle u_z u_r \rangle}{\partial r}. \tag{7.22}$$

Multiplying the above by r, and then integrating with respect to r, from some location r to the radius of the pipe R leads to

$$\rho\nu R\frac{\partial \langle u_z \rangle}{\partial r} - \left(\rho\nu r\frac{\partial \langle u_z \rangle}{\partial r} - r\rho\langle u_z u_r \rangle\right) = \frac{\partial \langle P \rangle}{\partial z}\frac{R^2}{2} - \frac{\partial \langle P \rangle}{\partial z}\frac{r^2}{2} \tag{7.23}$$

which is rewritten as

$$-R\tau_{wall} - \left(\rho\nu r\frac{\partial \langle u_z \rangle}{\partial r} - r\rho\langle u_z u_r \rangle\right) = \frac{\partial \langle P \rangle}{\partial z}\frac{R^2}{2} - \frac{\partial \langle P \rangle}{\partial z}\frac{r^2}{2}. \tag{7.24}$$

Evaluation at $r = 0$ shows that

$$\frac{\partial \langle P \rangle}{\partial z} = -\frac{2\tau_{wall}}{R} \tag{7.25}$$

Thus, the integrated momentum equation becomes

$$\rho\nu\frac{\partial\langle u_z\rangle}{\partial r} - \rho\langle u_z u_r\rangle = -\tau_{wall}\frac{r}{R} \qquad (7.26)$$

which is of equivalent form to that of the channel if one uses the substitution $r = R - r'$ since then each of the first and second terms of the left-hand side would change signs.

7.1.3 Ducts

Part IV

Modeling and Simulation

Chapter 8

RANS Modeling

This chapter focuses on models for the Reynolds stresses themselves, without use of the Boussinesq approximation such as in Eddy-Viscosity Models (EVMs). Nonetheless a few comments are first mentioned regarding EVMs. The $k - \epsilon$ model, which was quite popular some long time ago, has the following characteristics:

- One of the first two-equation models that showed moderate success across in a variety of engineering problems.

- Lack of sensitivity to adverse pressure gradients, where the model over predicts the shear stress and thus delays or completely prevents separation (Menter 1993).

- Numerical stiffness of the low-Reynolds number variants (Menter 1993).

- There are no natural or direct boundary conditions for the ϵ equation, and the boundary conditions that have actually been proposed introduce additional non-linearities that can upset the numerical procedure (Menter 1993).

- The exact equation for the dissipation, evaluated at the wall, includes unclosed higher order correlations such as the dissipation of dissipation (Speziale 1992).

The $k - \omega$ model, another popular two-equation closure, has the following characteristics:

- Significantly better performance under adverse pressure gradient conditions than the $k - \epsilon$ model.

- Increased computational robustness than $k - \epsilon$ for integration up to a solid boundary (Speziale 1992).

- The leading order terms of the exact ω equations at the wall are of closed form (Speziale 1992).

- Solutions of TKE are asymptotically inconsistent near a solid wall.

- There is a strong sensitivity to free stream boundary values for w specified outside the shear layer (Menter 1993).

8.1 The Theory of Matrix Polynomials Applied to Reynolds-Stress Modleing

Assume Π_{ij} is a model for the physical quantity Σ_{ij}. It is quite common for differential and algebraic Reynolds-stress models to define Π_{ij} as an algebraic functional relationship whose inputs are tensors $\mathbf{C}^{(1)}, ..., \mathbf{C}^{(n)}$ and scalars $s^{(1)}, ..., s^{(m)}$. Thus,

$$\Pi_{ij} = F_{ij}(s^{(1)}, ..., s^{(m)}, \mathbf{C}^{(1)}, ..., \mathbf{C}^{(n)}). \tag{8.1}$$

An assumption that can be used to narrow down the functional form of F_{ij} is to have both the exact physical tensor and its model satisfy the same transformation properties. For example, if Σ_{ij} is objective we could enforce Π_{ij} to be objective as well. An objective tensor is one that satisfies $\tilde{\Sigma}_{ij} = O_{ip}O_{jq}\Sigma_{pq}$ under Euclidean transformations, where $\tilde{\Sigma}_{ij}$ is such tensor in the transformed reference frame and O_{ij} is a time-dependent rotation matrix.

Enforcing such assumption would require F_{ij} to consist of a sum of terms as follows

$$F_{ij} = \sum_k f^{(k)} \mathcal{F}_{ij}^{(k)}. \tag{8.2}$$

The functions $f^{(k)}$ depend on the scalars $s^{(1)}, ..., s^{(m)}$ only, and each $\mathcal{F}_{ij}^{(k)}$ is of the form

$$\mathcal{F}_{ij}^{(k)} = c_{ip}^{(a_k)} c_{pq}^{(b_k)} c_{qr}^{(c_k)} ... \tag{8.3}$$

where $a_k, b_k, c_k, ...$ can each take any value between one and n. Additionally, the scalars $s^{(1)}, ..., s^{(m)}$ and the tensors $\mathbf{C}^{(1)}, ..., \mathbf{C}^{(n)}$ would need to be objective, where an objective scalar is one that satisfies $\tilde{s} = s$. If all of these requirements are satisfied, then we can show that

$$\begin{aligned}
\tilde{\Pi}_{ij} &= F_{ij}(\tilde{s}^{(1)}, ..., \tilde{s}^{(m)}, \tilde{\mathbf{C}}^{(1)}, ..., \tilde{\mathbf{C}}^{(n)}) \\
&= F_{ij}(s^{(1)}, ..., s^{(m)}, \mathbf{Q}\mathbf{C}^{(1)}\mathbf{Q}^T, ..., \mathbf{Q}\mathbf{C}^{(n)}\mathbf{Q}^T) \\
&= Q_{ip}Q_{jq}F_{pq}(s^{(1)}, ..., s^{(m)}, \mathbf{C}^{(1)}, ..., \mathbf{C}^{(n)}) \\
&= Q_{ip}Q_{jq}\Pi_{pq},
\end{aligned} \tag{8.4}$$

which shows Π_{ij} is indeed objective. It is relevant to note that $\Pi_{ij} = F_{ji}$ is also a valid form.

8.1.1 Models as a function of one tensor

8.1.2 Models as a function of two tensors

We now focus on a model for the non-dimensional anisotropic Reynolds stresses b_{ij}. We further assume that b_{ij} depends only on the non-dimensionalized anisotropic rate-of-strain tensor S_{ij}^*, and the non-dimensionalized rate-of-rotation tensor W_{ij}^*.

As shown in Spencer & Rivlin (The theory of matrix polynomials and its applications to the mechanics of isotropic continua, 1959) one can use the Cayley-Hamilton theorem to show that any product of the form shown in (8.3) that consists of the two tensors $\mathbf{S}^*$ and $\mathbf{W}^*$ can be expanded as a linear combination of a finite number of tensors. For this case such tensors are

$$\begin{gathered}
\mathbf{I}, \\
\mathbf{S}^*, \quad \mathbf{W}^*, \\
\mathbf{S}^{*2}, \quad \mathbf{W}^{*2}, \quad \mathbf{S}^*\mathbf{W}^*, \quad \mathbf{W}^*\mathbf{S}^*, \\
\mathbf{W}^*\mathbf{S}^{*2}, \quad \mathbf{S}^{*2}\mathbf{W}^*, \quad \mathbf{W}^{*2}\mathbf{S}^*, \quad \mathbf{S}^*\mathbf{W}^{*2}, \\
\mathbf{W}^{*2}\mathbf{S}^{*2}, \quad \mathbf{S}^{*2}\mathbf{W}^{*2}, \quad \mathbf{W}^*\mathbf{S}^*\mathbf{W}^{*2}, \quad \mathbf{S}^*\mathbf{W}^*\mathbf{S}^{*2}, \\
\mathbf{S}^*\mathbf{W}^{*2}\mathbf{S}^{*2}, \quad \mathbf{W}^*\mathbf{S}^{*2}\mathbf{W}^{*2}.
\end{gathered} \tag{8.5}$$

Thus, the infinite sum for F_{ij} shown in (8.2) becomes a finite linear combination as follows

$$F_{ij} = \sum_{k=1}^{17} g^{(k)} \mathcal{G}_{ij}^{(k)}, \tag{8.6}$$

where the tensors $\mathcal{G}_{ij}^{(k)}$ are shown above.

Since the Reynolds stresses are symmetric, its model must be symmetric as well. The symmetric form is given by $F_{ij} + F_{ji}$. Combining terms in the sums for F_{ij} and F_{ji} that have the same coefficient $g^{(k)}$ we obtain

$$F_{ij} + F_{ji} = \sum_{k=1}^{12} h^{(k)} \mathcal{H}_{ij}^{(k)} \tag{8.7}$$

where each of the $\mathcal{H}_{ij}^{(k)}$'s is shown below.

$$\mathbf{I},$$
$$\mathbf{S}^*,$$
$$\mathbf{S}^{*2}, \quad \mathbf{W}^{*2}, \quad \mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*,$$
$$\mathbf{W}^*\mathbf{S}^{*2} - \mathbf{S}^{*2}\mathbf{W}^*, \quad \mathbf{W}^{*2}\mathbf{S}^* + \mathbf{S}^*\mathbf{W}^{*2}$$
$$\mathbf{W}^{*2}\mathbf{S}^{*2} + \mathbf{S}^{*2}\mathbf{W}^{*2}, \quad \mathbf{W}^*\mathbf{S}^*\mathbf{W}^{*2} - \mathbf{W}^{*2}\mathbf{S}^*\mathbf{W}^*, \quad \mathbf{S}^*\mathbf{W}^*\mathbf{S}^{*2} - \mathbf{S}^{*2}\mathbf{W}^*\mathbf{S}^*,$$
$$\mathbf{S}^*\mathbf{W}^{*2}\mathbf{S}^{*2} + \mathbf{S}^{*2}\mathbf{W}^{*2}\mathbf{S}^*, \quad \mathbf{W}^*\mathbf{S}^{*2}\mathbf{W}^{*2} - \mathbf{W}^{*2}\mathbf{S}^{*2}\mathbf{W}^*. \tag{8.8}$$

As shown in Spencer & Rivlin (1959) the sum $\mathbf{S}^*\mathbf{W}^{*2}\mathbf{S}^{*2} + \mathbf{S}^{*2}\mathbf{W}^{*2}\mathbf{S}^*$ can be expresses as a linear combination of the other terms, and hence can be dropped from the set above. Since b_{ij} is anisotropic, a model for this tensor needs to be anisotropic as well. This is given by

$$F_{ij} + F_{ji} - \frac{2}{3} F_{kk} \delta_{ij} = \sum_{k=1}^{10} p^{(k)} \hat{\mathcal{P}}_{ij}^{(k)} \tag{8.9}$$

where the deviatoric tensors $\hat{\mathcal{P}}_{ij}^{(k)}$ are shown in table blah. The above sum is thus used as the model for b_{ij}.

$\hat{\boldsymbol{\mathcal{P}}}^{(1)} = \mathbf{S}^*$	$\hat{\boldsymbol{\mathcal{P}}}^{(6)} = \mathbf{W}^{*2}\mathbf{S}^* + \mathbf{S}^*\mathbf{W}^{*2} - \frac{2}{3}\{\mathbf{W}^{*2}\mathbf{S}^*\}\mathbf{I}$
$\hat{\boldsymbol{\mathcal{P}}}^{(2)} = \mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*$	$\hat{\boldsymbol{\mathcal{P}}}^{(7)} = \mathbf{W}^*\mathbf{S}^*\mathbf{W}^{*2} - \mathbf{W}^{*2}\mathbf{S}^*\mathbf{W}^*$
$\hat{\boldsymbol{\mathcal{P}}}^{(3)} = \mathbf{S}^{*2} - \frac{1}{3}\{\mathbf{S}^{*2}\}\mathbf{I}$	$\hat{\boldsymbol{\mathcal{P}}}^{(8)} = \mathbf{S}^*\mathbf{W}^*\mathbf{S}^{*2} - \mathbf{S}^{*2}\mathbf{W}^*\mathbf{S}^*$
$\hat{\boldsymbol{\mathcal{P}}}^{(4)} = \mathbf{W}^{*2} - \frac{1}{3}\{\mathbf{W}^{*2}\}\mathbf{I}$	$\hat{\boldsymbol{\mathcal{P}}}^{(9)} = \mathbf{W}^{*2}\mathbf{S}^{*2} + \mathbf{S}^{*2}\mathbf{W}^{*2} - \frac{2}{3}\{\mathbf{W}^{*2}\mathbf{S}^{*2}\}\mathbf{I}$
$\hat{\boldsymbol{\mathcal{P}}}^{(5)} = \mathbf{W}^*\mathbf{S}^{*2} - \mathbf{S}^{*2}\mathbf{W}^*$	$\hat{\boldsymbol{\mathcal{P}}}^{(10)} = \mathbf{W}^*\mathbf{S}^{*2}\mathbf{W}^{*2} - \mathbf{W}^{*2}\mathbf{S}^{*2}\mathbf{W}^*$

The ten tensors above constitute the so-called integrity basis for Reynolds stresses modeled as a function of S_{ij}^* and W_{ij}^*.

8.2 Differential Reynolds-stress models

There are three main components that need to be modeled to obtain closure for the Reynolds-stress differential transport equation. These are the velocity-pressure-gradient tensor Π_{ij}, the dissipation tensor ϵ_{ij}, and the turbulent transport $\mathcal{T}_{kij}^{(u)}$.

8.2.1 Velocity-pressure-gradient tensor

A model for the pressure-rate-of-strain tensor needs to be:

1. Symmetric,

2. Traceless,

3. Objective,

4. Linear in G_{ij} (the rapid term is linear in G_{ij} and the slow term depends only on fluctuating quantities)

5. Capable of guaranteeing realizability for τ,

6. Linear in τ_{ij} for the RDT limit.

As described in section (8.1), if we require a model to satisfy a minimum of tensor properties, then there is a finite number of linearly independent tensors, termed the integrity basis, that can be used to define the model. Thus, a model for the non-dimensional pressure-rate-of-strain tensor ϕ_{ij}/ϵ that satisfies requirements 1through 4 from above, and depends on the non-dimensional variables $S_{ij}^* = \tau \hat{S}_{ij}$, $W_{ij}^* = \tau W_{ij}$, and b_{ij}, would be a polynomial expansion of the following form (Speziale 1991)

$$\frac{\phi_{ij}}{\epsilon} = \sum_{k=1}^{7} h^{(k)} \mathcal{H}_{ij}^{(k)}, \tag{8.10}$$

where the functions $h^{(k)}$ depend on model coefficients and invariants of the input tensors, and $\mathcal{H}_{ij}^{(1)}...\mathcal{H}_{ij}^{(7)}$ are the eight members of the integrity basis. These members are shown below

$\mathcal{H}^{(1)} = \mathbf{b}$	$\mathcal{H}^{(5)} = \mathbf{W}^*\mathbf{b} - \mathbf{b}\mathbf{W}^*$
$\mathcal{H}^{(2)} = \mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I}$	$\mathcal{H}^{(6)} = \mathbf{S}^*\mathbf{b}^2 + \mathbf{b}^2\mathbf{S}^* - \frac{2}{3}\{\mathbf{S}^*\mathbf{b}^2\}\mathbf{I}$
$\mathcal{H}^{(3)} = \mathbf{S}^*$	$\mathcal{H}^{(7)} = \mathbf{W}^*\mathbf{b}^2 - \mathbf{b}^2\mathbf{W}^*$
$\mathcal{H}^{(4)} = \mathbf{S}^*\mathbf{b} + \mathbf{b}\mathbf{S}^* - \frac{2}{3}\{\mathbf{S}^*\mathbf{b}\}\mathbf{I}$	

The most generic linear-in-$\mathbf{b}$ model would have $h^{(2)} = h^{(6)} = h^{(7)} = 0$. The LRR model by Lumley (1975) has the following $h^{(k)}$ coefficients

$$h^{(1)} = -2C_R \qquad h^{(3)} = \frac{4}{5} \qquad h^{(4)} = \frac{18C_2 + 12}{11} \qquad h^{(5)} = \frac{20 - 14C_2}{11}. \tag{8.11}$$

The SSG model by Speziale (1991) has the following coefficients

$$h^{(1)} = -C_1 - C_1^*\frac{\mathcal{P}}{\epsilon} \qquad h^{(2)} = C_2 \qquad h^{(3)} = C_3 - C_3^*\{\mathbf{b}^2\} \qquad h^{(4)} = C_4 \qquad h^{(5)} = C_5, \tag{8.12}$$

and is thus a non-linear model that does not satisfy the fifth requirement shown above.

8.2.2 Dissipation

The simplest model for the dissipation is to assume an isotropic form due to high Reynolds numbers. That is

$$\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}. \tag{8.13}$$

An alternate model is that of Rotta (1951), which is

$$\epsilon_{ij} = \frac{\tau_{ij}}{k}\epsilon. \tag{8.14}$$

A third approach is to express the dissipation as follows

$$\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij} + \hat{\epsilon}_{ij}, \tag{8.15}$$

where $\hat{\epsilon}_{ij}$ is the anisotropic dissipation. The anisotropic dissipation can thus be combined with the pressure-rate of strain so that a model is formulated for the sum $\phi_{ij} + \hat{\epsilon}_{ij}$.

8.2.3 Turbulent transport

Common models for the turbulent transport $\mathcal{T}_{ijk}^{(u)}$ are those of Mellor and Herring (1973) and Hanjalic and Launder (1972). The term $\mathcal{T}_i^{(p)}$ that appears in the pressure transport tensor is modeled as $c\mathcal{T}_{ijj}^{(u)}$, where c is some constant. However, if both $\mathcal{T}_{ijk}^{(u)}$ and $\mathcal{T}_{ijk}^{(p')}$ are to be treated together, then common models for such sum are those of Shir (1973) and Daly and Harlow (1970),

8.3 Algebraic Reynolds-stress models

A common method for obtaining an algebraic Reynolds stresses model is to use the quasi-equilibrium assumption

$$\frac{Db_{ij}}{Dt} = 0, \tag{8.16}$$

and to model the Reynolds stress transport terms using the transport terms of the TKE, as shown below

$$\frac{\partial}{\partial x_k}\left(\mathcal{T}_{kij}^{(u)} + \mathcal{T}_{kij}^{(p)} + \mathcal{T}_{kij}^{(\nu)}\right) = \frac{\tau_{ij}}{k}\frac{\partial}{\partial x_k}\left(\mathcal{T}_k^{(u)} + \mathcal{T}_k^{(p)} + \mathcal{T}_k^{(\nu)}\right). \tag{8.17}$$

The above states that the inhomogeneous contribution due to the transport terms of the Reynolds stresses can be modeled in terms of the inhomogeneous contribution due to the transport terms of the turbulent kinetic energy. Thus equation (2.111) simplifies to the following

$$0 = \mathcal{P}_{ij} + \phi_{ij} - \epsilon_{ij} - \frac{\tau_{ij}}{k}\left(\mathcal{P} - \epsilon\right). \tag{8.18}$$

For incompressible flows, the production over dissipation can be expressed in terms of S_{ij}^* and W_{ij}^* as follows

$$\begin{aligned}
\frac{\mathcal{P}_{ij}}{\epsilon} &= \frac{1}{\epsilon}\left(-\frac{2}{3}k\frac{\partial\langle U_j\rangle}{\partial x_i} - 2kb_{ik}(S_{jk} + W_{jk}) - \frac{2}{3}k\frac{\partial\langle U_i\rangle}{\partial x_j} - 2kb_{jk}(S_{ik} + W_{ik})\right) \\
&= -\frac{4}{3}\mathbf{S}^* - 2(\mathbf{S}^*\mathbf{b} + \mathbf{b}\mathbf{S}^*) - 2(\mathbf{W}^*\mathbf{b} - \mathbf{b}\mathbf{W}^*).
\end{aligned} \tag{8.19}$$

Assuming the most generic linear model for the pressure-rate-rate-of-strain, equation (8.18) becomes

$$a_1\mathbf{b} + a_2\mathbf{S}^* + a_3\left(\mathbf{S}^*\mathbf{b} + \mathbf{b}\mathbf{S}^* - \frac{2}{3}\{\mathbf{S}^*\mathbf{b}\}\mathbf{I}\right) + a_4(\mathbf{W}^*\mathbf{b} - \mathbf{b}\mathbf{W}^*) = 0, \qquad (8.20)$$

where

$$a_1 = \left(\frac{h^{(1)}}{2} + 1 - \frac{\mathcal{P}}{\epsilon}\right) \qquad a_2 = \frac{1}{2}\left(-\frac{4}{3} + h^{(3)}\right) \qquad a_3 = \frac{1}{2}\left(-2 + h^{(4)}\right) \qquad a_4 = \frac{1}{2}\left(-2 + h^{(5)}\right). \qquad (8.21)$$

Linearity in $\mathbf{b}$ is required so that when the polynomial expansion (**??**) is used to replace $\mathbf{b}$ a system of linear equations is obtained for the coefficients of such expansion.

The EASM of Rumsey & Gatski (cite) uses the linearize SSG model to compute the pressure-rate-of-strain tensor. Thus, we have

$$a_1 = -\left(\frac{C_1^*}{2} + 1\right)\frac{\mathcal{P}}{\epsilon} + 1 - \frac{C_1}{2} \qquad a_2 = \frac{1}{2}\left(-\frac{4}{3} + C_3\right) \qquad a_3 = \frac{1}{2}\left(-2 + c_4\right) \qquad a_4 = \frac{1}{2}\left(-2 + C_5\right). \qquad (8.22)$$

On the other hand, the EARSM of Wallin & Johansson (cite) uses Rotta's model to compute the slow-pressure-rate-of-strain tensor and the LRR model for the rapid part. Thus, the a_i coefficients become

$$a_1 = \left(-C_R + 1 - \frac{\mathcal{P}}{\epsilon}\right) \qquad a_2 = -\frac{4}{15} \qquad a_3 = -\frac{5 - 9C_2}{11} \qquad a_4 = -\frac{7C_2 + 1}{11} \qquad (8.23)$$

For such model, a value for C_2 of 5/9 can be chosen, which simplifies the implicit algebraic equation to

$$N\mathbf{b} = -\frac{3}{5}\mathbf{S}^* - (\mathbf{W}^*\mathbf{b} - \mathbf{b}\mathbf{W}^*), \qquad (8.24)$$

where

$$N = \frac{9}{4}\left(C_R - 1 + \frac{\mathcal{P}}{\epsilon}\right). \qquad (8.25)$$

Equation (8.24) is an implicit algebraic equation for the Reynolds stresses since the factor N depends on $\mathcal{P}/\epsilon$, which itself depends on b_{ij}.

The method for obtaining and explicit expression for b_{ij} from equation (8.24) is the following. First, the polynomial expansion (**??**) is used to replace b_{ij} in the implicit equation, except for the stresses that define N since this variable is left as a yet undetermined function. All higher order tensors in this new equation are replaced by combinations of elements of the integrity basis using the Cayley-Hamilton theorem. By matching coefficients corresponding to the same element of the integrity basis one obtains expressions for the weights $p^{(k)}$ as a function of N. Finally, the polynomial expansion for b_{ij} with the new expressions for $p^{(k)}$ is used in the definition of N (8.25). This leads to a non-linear algebraic equation for N and its solution is used in the expressions for $p^{(k)}$.

Chapter 9

Probability Density Function Methods

9.1 The Eulerian Modeling Approach

The transport equation for the one-point, one-time, Eulerian PDF $f = f(\mathbf{V}; \mathbf{x}, t)$ is derived in Appendix H of Pope (2000) and is shown below

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = -\frac{\partial}{\partial V_i}\left(f \left\langle \frac{DU_i}{Dt} \middle| \mathbf{V} \right\rangle \right). \tag{9.1}$$

This is a mathematical identity that contains no physics. The physics comes in by replacing the material derivative of U_i using the Navier-Stokes equations, which leads to

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} - \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right) \frac{\partial f}{\partial V_i} - \frac{\partial}{\partial V_i}\left(f \left\langle -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \middle| \mathbf{V} \right\rangle \right). \tag{9.2}$$

The equation above can be multiplied by V_j and then averaged over the velocity sample space to obtain the RANS momentum equation. For the first term on the left hand side one would obtain

$$\int V_j \frac{\partial f}{\partial t} d\mathbf{V} = \frac{\partial}{\partial t}\int V_j f d\mathbf{V} = \frac{\partial \langle U_j \rangle}{\partial t}, \tag{9.3}$$

whereas for the second term one obtains

$$\int V_j V_i \frac{\partial f}{\partial x_i} d\mathbf{V} = \frac{\partial}{\partial x_i}\int V_j V_i f d\mathbf{V} = \frac{\partial \langle U_j U_i \rangle}{\partial x_i}. \tag{9.4}$$

For the terms on the right hand side we make use of the identity

$$\int B_k \frac{\partial}{\partial V_i}(f A_j) d\mathbf{V} = -\int \frac{\partial B_k}{\partial V_i} A_j f d\mathbf{V}, \tag{9.5}$$

where $B_k = B_k(\mathbf{V})$ and $A_j = A_j(\mathbf{V})$. Thus, the first term on the right hand side becomes

$$\int \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} - \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k x_k} \right) V_j \frac{\partial f}{\partial V_i} d\mathbf{V} = \left(-\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} + \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right) \delta_{ij} \int f d\mathbf{V}$$

$$= -\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_j} + \nu \frac{\partial^2 \langle U_j \rangle}{\partial x_k x_k}. \tag{9.6}$$

Given $f_{\mathbf{U}\phi} = f_{\mathbf{U}\phi}(\mathbf{V}, \psi; \mathbf{x}, t)$ the joint PDF of $\mathbf{U}$ and ϕ, and $f_{\phi|\mathbf{U}} = f_{\phi|\mathbf{U}}(\psi|\mathbf{V}; \mathbf{x}, t)$ the conditional PDF for ϕ conditioned on $\mathbf{V}$, we write the following

$$\langle \phi \rangle = \int \int \psi f_{\mathbf{U}\phi} d\psi d\mathbf{V} = \int \int \psi f_{\phi|\mathbf{U}} f d\psi d\mathbf{V} = \int \langle \phi|\mathbf{V} \rangle f d\mathbf{V}, \qquad (9.7)$$

The derivation for the second term on the right hand side thus proceeds as follows

$$-\int V_j \frac{\partial}{\partial V_i} \left(f \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \middle| \mathbf{V} \right\rangle \right) d\mathbf{V} = \delta_{ij} \int \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \middle| \mathbf{V} \right\rangle f d\mathbf{V}$$
$$= \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} \right\rangle$$
$$= 0. \qquad (9.8)$$

Given the PDF one can compute $\langle U_i \rangle$ and τ_{ij}. The averaged pressure $\langle P \rangle$ can then be obtained from a poisson equation. Thus, the last term in the PDF equation (9.2) is the only unclosed term, and it is typically modeled using the Generalized Langevin Model (GLM), as shown below

$$-\frac{\partial}{\partial V_i} \left(f \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \middle| \mathbf{V} \right\rangle \right) = -\frac{\partial}{\partial V_i} [f G_{ij}(V_j - \langle U_j \rangle)] + \frac{1}{2} C_0 \epsilon \frac{\partial^2 f}{\partial V_j \partial V_j}. \qquad (9.9)$$

The coefficients $G_{ij} = G_{ij}(\mathbf{x}, t)$ and $C_0 = C_0(\mathbf{x}, t)$ are specific to each of the different versions of the GLM. As with the exact unclosed term, the modeled terms vanish when the RANS momentum equations are formed.

Multiplying equation (9.2) by $v_j v_k$ and integrating over the velocity space one obtains the corresponding transport equation for the Reynolds stresses

$$\frac{\partial \tau_{ij}}{\partial t} + \langle U_k \rangle \frac{\partial \tau_{ij}}{\partial x_k} = \mathcal{P}_{ij} - \frac{\partial \mathcal{T}_{kij}^{(u)}}{\partial x_k} + G_{ik} \tau_{kj} + G_{jk} \tau_{ki} + C_0 \epsilon \delta_{ij}. \qquad (9.10)$$

In other words, the velocity-pressure-gradient tensor, the dissipation tensor, and viscous diffusion are modeled by the GLM as follows

$$\Pi_{ij} - \epsilon_{ij} - \frac{\partial \mathcal{T}_{kij}^{(\nu)}}{\partial x_k} = G_{ik} \tau_{kj} + G_{jk} \tau_{ki} + C_0 \epsilon \delta_{ij} \qquad (9.11)$$

This shows that to each GLM there is a corresponding DRSM. The accuracy of the above framework ultimately depends on how accurately the modeled PDF represents the exact PDF.

The Eulerian modeling approach thus proceeds as follows. From the exact equations for the instantaneous fluid properties, such as the Navier-Stokes equations for velocity, unclosed statistical equations are obtained, such as the RANS equation for $\langle U_i \rangle$ or τ_{ij}, or the transport equation for the Eulerian PDF f. The unclosed terms in these equations are then modeled in terms of the known variables.

9.2 The Stochastic Lagrangian Approach

An alternate modeling approach, termed the stochastic Lagrangian approach, can be followed to derive a closed modeled conservation equation for the Eulerian PDF. As for the Eulerian modeling approach, mean quantities such as $\langle U_i \rangle$ or τ_{ij} are known variables since they can be obtained from the Eulerian PDF. The mean pressure can be obtained by solving its corresponding Poisson equation, and it is thus also referred to as a known variable.

The starting point is the Lagrangian evolution equations for the position $X_i^+ = X_i^+(t, \mathbf{y})$ and velocity $U_i^+ = U_i^+(t, \mathbf{y})$ of a fluid particle that at time t_0 was located at $\mathbf{y}$. These evolution equations are shown below

$$\frac{\partial X_i^+}{\partial t} = U_i^+ \tag{9.12}$$

$$\frac{\partial U_i^+}{\partial t} = \left(-\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial x_i} + \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right)_{\mathbf{x} = \mathbf{X}^+} + \left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \right)_{\mathbf{x} = \mathbf{X}^+} . \tag{9.13}$$

The first term in parenthesis of the right-hand side of the velocity evolution equation consists of known variables, whereas the second term consists of unknown terms and thus requires modeling. The above equations can be used to derive a transport equation for a Lagrangian PDF, from which an Eulerian PDF follows. These derivations will be shown in this section.

9.2.1 Lagrangian probablity density functions

The inverse of $\mathbf{X}^+$ is $\mathbf{Y}^+ = \mathbf{Y}^+(t, \mathbf{z})$, which gives the initial location of a fluid particle that at time t is located at position $\mathbf{z}$. Thus,

$$\mathbf{X}^+(t, \mathbf{Y}^+(t, \mathbf{z})) = \mathbf{z}, \tag{9.14}$$

which represents the location of a particle at time t, whose original location corresponds to such particle being located at $\mathbf{z}$ for time t.

The Lagrangian PDF is given by $f_L = f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$, which is conditioned on the fluid particle being located at $\mathbf{y}$ at some initial time t_0. Here $\mathbf{V}$ and $\mathbf{x}$ are the sample space variables for $\mathbf{U}^+$ and $\mathbf{X}^+$. f_L can be obtained as follows

$$f_L = \langle f_L' \rangle, \tag{9.15}$$

where $f_L' = f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y})$ is the fine-grained Lagrangian PDF, which is given by

$$f_L' = \delta(\mathbf{U}^+ - \mathbf{V})\delta(\mathbf{X}^+ - \mathbf{x}). \tag{9.16}$$

We will now derive the relationship between the Lagrangian and Eulerian PDFs. We start with

$$\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) d\mathbf{y} = \int \delta(\mathbf{U}^+(t, \mathbf{y}) - \mathbf{V})\delta(\mathbf{X}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y}$$

$$= \int \delta(\mathbf{U}(t, \mathbf{X}^+(t, \mathbf{y})) - \mathbf{V})\delta(\mathbf{X}^+(t, \mathbf{y}) - \mathbf{x}) d\mathbf{y}$$

$$= \int \delta(\mathbf{U}(t, \mathbf{X}^+(t, \mathbf{y})) - \mathbf{V})\delta(\mathbf{X}^+(t, \mathbf{y}) - \mathbf{x})|\det D\mathbf{X}^+| d\mathbf{y}, \tag{9.17}$$

where we have introduced $|\det D\mathbf{X}^+|$, which is the absolute value of the determinant of the Jacobean $\partial \mathbf{X}^+/\partial \mathbf{y}$, and is equal to one for incompressible flows. Using integration by substitution we obtain

$$\int f_L'(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) d\mathbf{y} = \int \delta(\mathbf{U}(t, \mathbf{z}) - \mathbf{V})\delta(\mathbf{z} - \mathbf{x}) d\mathbf{z} = \delta(\mathbf{U}(t, \mathbf{x}) - \mathbf{V}) \tag{9.18}$$

Since $\delta(\mathbf{U}(t, \mathbf{x}) - \mathbf{V})$ is the fine-grained Eulerian PDF, taking the expectation of the above we finally obtain

$$\int f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) d\mathbf{y} = f(\mathbf{V}; \mathbf{x}, t). \tag{9.19}$$

9.2.2 Diffusion processes

A Markov process is a stochastic process whose joint PDF satisfies the following

$$f_{N-1}(V_N; t_N | V_{N-1}, t_{N-1}, V_{N-2}, t_{N-2}, ..., V_1, t_1) = f_1(V_N; t_N | V_{N-1}, t_{N-1}). \tag{9.20}$$

That is, the probability distribution for the random variable at time t depends only on one single previous realization at time $t_{N-1} < t$, and is independent of all the realizations at previous times less than t_{N-1}.

Consider the operator Δ_h which operates on $U = U(t)$, such that $\Delta_h U(t) = U(t+h) - U(t)$. Using such operator define the set of infinitesimal parameters $B_n = B_n(V, t)$

$$B_n = \lim_{h \to 0} \frac{1}{h} \langle (\Delta_h U)^n | U = V \rangle. \tag{9.21}$$

A diffusion process is a Markov process with drift coefficient $a = a(V, t)$ and diffusion coefficient $b = b(V, t)$ given by

$$a = B_1 \qquad b^2 = B_2, \tag{9.22}$$

and $B_i = 0$ for all $i \geq 3$. Diffusion processes are governed by the following stochastic differential equation

$$dU(t) = a[U(t), t]dt + b[U(t), t]dW(t) \tag{9.23}$$

where $dW(t)$ is the infinitesimal increment for a Wiener process. The corresponding conditional PDF $f_1 = f_1(V; t | V_1, t_1)$ satisfies the Fokker-Plank equation

$$\frac{\partial f_1}{\partial t} = -\frac{\partial a f_1}{\partial V} + \frac{1}{2} \frac{\partial^2 b^2 f_1}{\partial V^2}. \tag{9.24}$$

Multiplying the above by $f(V_1; t_1)$ and integrating over V_1 shows that the marginal PDF $f = f(V; t)$ satisfies the same Fokker-Plank equation.

For a vector-valued diffusion process the sample paths are given by $\mathbf{U} = \mathbf{U}(t)$. The drift coefficient is given by $a_i = a_i(\mathbf{V}, t)$, where

$$a_i = \lim_{h \to 0} \frac{1}{h} \langle (\Delta_h U_i)^n | \mathbf{U} = \mathbf{V} \rangle. \tag{9.25}$$

The coefficient $b_{ij} = b_{ij}(\mathbf{V}, t)$ is defined such that $b_{ik} b_{jk} = B_{ij}$, where $B_{ij} = B_{ij}(\mathbf{V}, t)$ is the diffusion coefficient and is given by

$$B_{ij} = \lim_{h \to 0} \frac{1}{h} \langle \Delta_h U_i \Delta_h U_j | \mathbf{U} = \mathbf{V} \rangle. \tag{9.26}$$

The associated stochastic differential equation is

$$dU_i(t) = a_i[\mathbf{U}(t), t]dt + b_{ij}[\mathbf{U}(t), t]dW_j(t), \tag{9.27}$$

and the Fokker-Plank equation for the conditional PDF $f_1 = f_1(\mathbf{V}; t | \mathbf{V}^{(1)}, t^{(1)})$ is

$$\frac{\partial f_1}{\partial t} = -\frac{\partial a_i f_1}{\partial V_i} + \frac{1}{2} \frac{\partial^2 B_{ij} f_1}{\partial V_i V_j}. \tag{9.28}$$

As with the scalar case, the marginal PDF is obtained by multiplying the above by $f(\mathbf{V}^{(1)}; t^{(1)})$ and integrating over $\mathbf{V}^{(1)}$. Such PDF solves the exact same evolution equation as the conditional PDF.

9.2.3 Evolution equations for the Lagrangian PDF

Consider the six-dimensional diffusion process $\mathbf{Z}^* = \mathbf{Z}^*(t)$ that describes the position $\mathbf{X}^* = \mathbf{X}^*(t)$ and velocity $\mathbf{U}^* = \mathbf{U}^*(t)$ of a fluid particle. Thus

$$\mathbf{Z}^* = \begin{pmatrix} Z_1^* \\ Z_2^* \\ Z_3^* \\ Z_4^* \\ Z_5^* \\ Z_6^* \end{pmatrix} = \begin{pmatrix} X_1^* \\ X_2^* \\ X_3^* \\ U_1^* \\ U_2^* \\ U_3^* \end{pmatrix} \tag{9.29}$$

The corresponding sample space is given by the vector $\{x_1, x_2, x_3, V_1, V_2, V_3\}$. The vectors $\mathbf{X}^*$ and $\mathbf{U}^*$ evolve according to

$$\frac{dX_i^*}{dt} = U_i^*, \tag{9.30}$$

$$dU_i^* = -\left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} + \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right)_{\mathbf{x}=\mathbf{X}^*} dt + G_{ij}(U_j^* - \langle U_j \rangle)dt + (C_0\epsilon)^{1/2}dW_i, \tag{9.31}$$

where the coefficient G_{ij} and the expectation $\langle U_j \rangle$ are evaluated at $\mathbf{X}^*$. The drift term involving the pressure gradient and the laplacian of $\langle U_i \rangle$ comes from the Navier-Stokes equations, whereas the remaining terms model the fluctuating pressure gradient and the Laplacian of the fluctuating velocity.

The stochastic process $\mathbf{Z}^*$ evolves according the general diffusion equation

$$dZ_i^*(t) = a_i[\mathbf{Z}^*(t), t]dt + b_{ij}[\mathbf{Z}^*(t), t]dW_j(t), \tag{9.32}$$

where

$$\mathbf{a}[\mathbf{Z}^*(t), t] = \begin{pmatrix} Z_4^*(t) \\ Z_5^*(t) \\ Z_6^*(t) \\ -\left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_1} + \nu \frac{\partial^2 \langle U_1 \rangle}{\partial x_k \partial x_k} \right) + G_{1j}(Z_{(j+3)}^*(t) - \langle U_j \rangle) \\ -\left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_2} + \nu \frac{\partial^2 \langle U_2 \rangle}{\partial x_k \partial x_k} \right) + G_{2j}(Z_{(j+3)}^*(t) - \langle U_j \rangle) \\ -\left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_3} + \nu \frac{\partial^2 \langle U_3 \rangle}{\partial x_k \partial x_k} \right) + G_{3j}(Z_{(j+3)}^*(t) - \langle U_j \rangle) \end{pmatrix} \tag{9.33}$$

and

$$\mathbf{b}[\mathbf{Z}^*(t), t] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_0\epsilon)^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_0\epsilon)^{1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_0\epsilon)^{1/2} \end{pmatrix}. \tag{9.34}$$

The Eulerian fields for the pressure gradient, viscous diffusion, and G_{ij} that appear in $\mathbf{a}[\mathbf{Z}^*(t), t]$ are evaluated at $\mathbf{x} = \mathbf{X}^*$. From the expression for $\mathbf{b}$ we obtain

$$B_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_0\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & C_0\epsilon \end{pmatrix}. \tag{9.35}$$

Using equation (9.28) it is straightforward to derive the Fokker-Plank equation for the conditional PDF $f_1(\mathbf{x}, \mathbf{V}; t | \mathbf{x}^{(1)}, \mathbf{V}^{(1)}, t^{(1)})$. Assuming a fixed value of $t^{(1)} = t_0$, and labeling $\mathbf{x}^{(1)}$ by $\mathbf{y}$ and $\mathbf{V}^{(1)}$ by $\mathbf{W}$, we re-label this PDF as $f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y})$. Its Fokker-Plank equation follows

$$
\frac{\partial}{\partial t} f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y}) = - V_i \frac{\partial}{\partial x_i} f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y}) + \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} - \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right) \frac{\partial}{\partial V_i} f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y})
$$

$$
- G_{ij} \frac{\partial}{\partial V_i} [(V_j - \langle U_j \rangle) f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y})] + \frac{1}{2} C_0 \epsilon \frac{\partial^2}{\partial V_i V_i} f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y}).
$$

(9.36)

The pressure gradient, viscous diffusion, and G_{ij} are now functions of time and the sample space variable $\mathbf{x}$. The Lagrangian PDF f_L can be obtained from $f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y})$ as shown below

$$
\begin{aligned}
f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{y}) &= \frac{f_L(\mathbf{V}, \mathbf{x}; t, \mathbf{y})}{f_L(\mathbf{y})} \\
&= \frac{\int f_L(\mathbf{V}, \mathbf{x}; t, \mathbf{W}, \mathbf{y}) d\mathbf{W}}{f_L(\mathbf{y})} \\
&= \int f_L(\mathbf{V}, \mathbf{x}; t | \mathbf{W}, \mathbf{y}) \frac{f_L(\mathbf{W}, \mathbf{y})}{f_L(\mathbf{y})} d\mathbf{W}
\end{aligned}
$$

(9.37)

Multiplying equation (9.36) by $f_L(\mathbf{W}, \mathbf{y}) / f_L(\mathbf{y})$ and integrating over $\mathbf{W}$ shows that f_L still satisfies the given Fokker-Plank, that is

$$
\frac{\partial f_L}{\partial t} = -V_i \frac{\partial f_L}{\partial x_i} + \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} - \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right) \frac{\partial f_L}{\partial V_i} - G_{ij} \frac{\partial}{\partial V_i} [(V_j - \langle U_j \rangle) f_L] + \frac{1}{2} C_0 \epsilon \frac{\partial^2 f_L}{\partial V_i V_i}.
$$

(9.38)

Using equation (9.19) we finally obtain the associated Eulerian PDF

$$
\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_i} - \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_k} \right) \frac{\partial f}{\partial V_i} - G_{ij} \frac{\partial}{\partial V_i} [(V_j - \langle U_j \rangle) f] + \frac{1}{2} C_0 \epsilon \frac{\partial^2 f}{\partial V_i V_i}.
$$

(9.39)

This is equation is identical to the evolution equation (9.9) for the Eulerian PDF using the GLM.

9.2.4 Particle methods

We consider a large number of statistically identical particles whose PDFs are denoted by f_L, as previously described. They are originally distributed uniformly over a domain $\mathcal{V}$, and thus the PDF of initial position becomes

$$
f_L(\mathbf{y}) = \frac{1}{\mathcal{V}}
$$

(9.40)

so as to satisfy the normalization condition.

Ultimately, we are interested in Eulerian quantities such as the mean velocity, which is obtained as follows

$$
\langle U_i \rangle = \int V_i f d\mathbf{V}.
$$

(9.41)

Instead of computing the corresponding Eulerian PDF for the system of particles we make use

of the following

$$f(\mathbf{V}; \mathbf{x}, t) = \int f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) d\mathbf{y}$$

$$= \int f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) d\mathbf{y} \frac{f_L(\mathbf{y})}{f_L(\mathbf{y})}$$

$$= \int f_L(\mathbf{V}, \mathbf{x}; t|\mathbf{y}) f_L(\mathbf{y}) d\mathbf{y} \frac{1}{f_L(\mathbf{y})}$$

$$= \int f_L(\mathbf{V}, \mathbf{x}; t, \mathbf{y}) d\mathbf{y} \frac{1}{f_L(\mathbf{y})}$$

$$= f_L(\mathbf{V}, \mathbf{x}; t) \frac{1}{f_L(\mathbf{y})}$$

$$= f_{L_x}(\mathbf{V}|\mathbf{x}; t) \frac{f_L(\mathbf{x}; t)}{f_L(\mathbf{y})}. \tag{9.42}$$

We note that $f_L(\mathbf{V}, \mathbf{x}; t)$ is the probability of seeing a particle with velocity $\mathbf{V}$ and position $\mathbf{x}$ at time t, irrespective of where that particle originated. The PDF $f_{L_x} = f_{L_x}(\mathbf{V}|\mathbf{x}; t)$ refers to the probability of seeing a particle with velocity $\mathbf{V}$ given that the particle, irrespective of where it originated, is located at position $\mathbf{x}$. We now require that $f_L(\mathbf{x}; t) = f_L(\mathbf{y})$ (the distribution of particles over space remains uniform) and thus we can compute statistical quantities, such as the mean velocity, as follows

$$\langle U_i \rangle = \int V_i f_{L_x} d\mathbf{V}. \tag{9.43}$$

That is, we have stablished an equality between f and f_{L_x}. If we were to use the GLM to model the particle evolution equations, the PDF f_{L_x} would also satisfy eq. (9.39).

9.3 PDFs for the fluctuating velocity

The Eulerian PDF for $\mathbf{u}$ is given by $g = g(\mathbf{v}; \mathbf{x}, t)$, where $\mathbf{v}$ is the sample space for $\mathbf{u}$. The PDF g contains no information on $\langle \mathbf{U} \rangle$, but the information provided by both g and $\langle \mathbf{U} \rangle$ is identical to that obtained from f. The relationship between f and g is as follows

$$g(\mathbf{v}; \mathbf{x}, t) = f(\langle \mathbf{U} \rangle + \mathbf{v}; \mathbf{x}, t). \tag{9.44}$$

It is important to note that similar relationships follow for multi-variable PDFs. For example, consider the joint PDFs $g(\mathbf{q}, \mathbf{v}; \mathbf{x}, t)$ and $f(\mathbf{q}, \mathbf{V}; \mathbf{x}, t)$, where $\mathbf{q}$ is the sample space variable for $-\frac{1}{\rho} \nabla \mathbf{p} + \nu \Delta \mathbf{u}$. Thus, these two PDFs are related by

$$g(\mathbf{q}, \mathbf{v}; \mathbf{x}, t) = f(\mathbf{q}, \langle \mathbf{U} \rangle + \mathbf{v}; \mathbf{x}, t). \tag{9.45}$$

We begin the derivation of the transport equation for the fluctuating Eulerian PDF as follows

$$\frac{\partial g}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial g}{\partial x_i} = \left(\frac{\partial f}{\partial V_j} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} \frac{\partial \langle U_j \rangle}{\partial t} + \left(\frac{\partial f}{\partial t} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}}$$
$$+ (\langle U_i \rangle + v_i) \left[\left(\frac{\partial f}{\partial V_j} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} \frac{\partial \langle U_j \rangle}{\partial x_i} + \left(\frac{\partial f}{\partial x_i} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} \right]. \tag{9.46}$$

Regrouping the above we obtain

$$\frac{\partial g}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial g}{\partial x_i} = \left(\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} + \left(\frac{\partial f}{\partial V_j} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} \left[\frac{\partial \langle U_j \rangle}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial \langle U_j \rangle}{\partial x_i} \right] \tag{9.47}$$

To proceed further we need the following identity. For any function $H(\mathbf{V}; \mathbf{x}, t)$ we have

$$\frac{\partial H(\langle \mathbf{U} \rangle + \mathbf{v}; \mathbf{x}, t)}{\partial v_i} = \left(\frac{\partial H(\mathbf{V}; \mathbf{x}, t)}{\partial V_j} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} \frac{\partial(\langle U_j \rangle + v_j)}{\partial v_i} = \left(\frac{\partial H(\mathbf{V}; \mathbf{x}, t)}{\partial V_i} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}}. \quad (9.48)$$

The transport equation for g then becomes

$$\frac{\partial g}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial g}{\partial x_i} = \left(\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} + \frac{\partial g}{\partial v_j} \left[\frac{\partial \langle U_j \rangle}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial \langle U_j \rangle}{\partial x_i} \right] \quad (9.49)$$

Using equation (9.2) the first term on the right-hand side becomes

$$\left(\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} = \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_j} - \nu \frac{\partial^2 \langle U_j \rangle}{\partial x_k \partial x_k} \right) \left(\frac{\partial f}{\partial V_j} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}}$$
$$- \left[\frac{\partial}{\partial V_j} \left(f \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} \middle| \mathbf{V} \right\rangle \right) \right]_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} \quad (9.50)$$

Using the identity shown in (9.48) the above becomes

$$\left(\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} \right)_{\mathbf{V} = \langle \mathbf{U} \rangle + \mathbf{v}} = \left(\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_j} - \nu \frac{\partial^2 \langle U_j \rangle}{\partial x_k \partial x_k} \right) \frac{\partial g}{\partial v_j}$$
$$- \frac{\partial}{\partial v_j} \left(g \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} \middle| \langle \mathbf{U} \rangle + \mathbf{v} \right\rangle \right). \quad (9.51)$$

Using equation (9.45) the conditional expectation in the last term can be re-expressed as

$$\int \mathbf{q} \, f(\mathbf{q} | \langle \mathbf{U} \rangle + \mathbf{v}; \mathbf{x}, t) d\mathbf{q} = \int \mathbf{q} \, g(\mathbf{q} | \mathbf{v}; \mathbf{x}, t) d\mathbf{q} = \left\langle -\frac{1}{\rho} \nabla \mathbf{p} + \nu \Delta \mathbf{u} \middle| \mathbf{v} \right\rangle. \quad (9.52)$$

Using the RANS momentum equation one finally obtains the transport equation for g

$$\frac{\partial g}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial g}{\partial x_i} = v_i \frac{\partial \langle U_j \rangle}{\partial x_i} \frac{\partial g}{\partial v_j} - \frac{\partial \tau_{jk}}{\partial x_k} \frac{\partial g}{\partial v_j} - \frac{\partial}{\partial v_j} \left(g \left\langle -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} \middle| \mathbf{v} \right\rangle \right). \quad (9.53)$$

As before, the term that needs modeling is the last one on the right-hand side. If, during the derivation of the transport equation for g, the modeled equation for f based on the GLM is used, then the modeled equation for g corresponding to the GLM becomes

$$\frac{\partial g}{\partial t} + (\langle U_i \rangle + v_i) \frac{\partial g}{\partial x_i} = v_i \frac{\partial \langle U_j \rangle}{\partial x_i} \frac{\partial g}{\partial v_j} - \frac{\partial \tau_{jk}}{\partial x_k} \frac{\partial g}{\partial v_j} - \frac{\partial g G_{jk} v_k}{\partial v_j} + \frac{1}{2} C_0 \epsilon \frac{\partial^2 g}{\partial v_k \partial v_k}. \quad (9.54)$$

We can also set up a relationship between the fluctuating Eulerian and fluctuating Lagrangian PDFs. The fluctuating Lagrangian PDF $g_L = g_L(\mathbf{v}, \mathbf{x}; t | \mathbf{y})$ is the joint PDF of $\mathbf{X}^+(t, \mathbf{y})$ and the fluctuating velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$, where $\mathbf{u}^+ = \mathbf{U}^+ - \langle \mathbf{U}^+ \rangle$. The evolution equations for these variables are

$$\frac{\partial X_i^+}{\partial t} = \langle U_i \rangle_{\mathbf{x} = \mathbf{X}^+} + u_i^+, \quad (9.55)$$

$$\frac{\partial u_i^+}{\partial t} = -u_j^+ \left(\frac{\partial \langle U_i \rangle}{\partial x_j} \right)_{\mathbf{x} = \mathbf{X}^+} + \left(\frac{\partial \tau_{ij}}{\partial x_j} \right)_{\mathbf{x} = \mathbf{X}^+} + \left(-\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \right)_{\mathbf{x} = \mathbf{X}^+}. \quad (9.56)$$

As with the Eulerian PDFs, we can relate the Lagrangian PDF and the fluctuating Lagrangian PDF as follows

$$g_L(\mathbf{v}, \mathbf{x}; t | \mathbf{y}) = f_L(\langle \mathbf{U}(\mathbf{x}, t) \rangle + \mathbf{v}, \mathbf{x}; t | \mathbf{y}). \quad (9.57)$$

Thus, plugging in equations (9.44) and (9.57) into equation (9.19), we finally obtain

$$\int g_L(\mathbf{v}, \mathbf{x}; t|\mathbf{y})d\mathbf{y} = g(\mathbf{v}; \mathbf{x}, t). \tag{9.58}$$

For homogeneous turbulence the fluctuating Eulerian PDF becomes independent of $\mathbf{x}$, that is, $g = g^{(h)}$, where $g^{(h)} = g^{(h)}(\mathbf{v}; t)$. As with the two-point velocity correlation $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{y}, t)$, which for homogeneous turbulence does not depend on $\mathbf{x}$ and $\mathbf{y}$ separately but on $\mathbf{x} - \mathbf{y}$, the fluctuating Lagrangian PDF g_L also depends on $\mathbf{x} - \mathbf{y}$ only. Thus, we write

$$g_L(\mathbf{v}, \mathbf{x}; t|\mathbf{y}) = h_L(\mathbf{v}, \mathbf{x} - \mathbf{y}; t), \tag{9.59}$$

where $h_L(\mathbf{v}, \mathbf{x} - \mathbf{y}; t)$ is the joint PDF for the fluctuating velocity $\mathbf{u}^+(t, \mathbf{y})$ and the separation distance $\mathbf{X}^+(t, \mathbf{y}) - \mathbf{y}$. Using equation (9.58) we can show that

$$g^{(h)}(\mathbf{v}; t) = \int g_L(\mathbf{v}, \mathbf{x}; t|\mathbf{y})dy = \int h_L(\mathbf{v}, \mathbf{x} - \mathbf{y}; t)d\mathbf{y} = \int h_L(\mathbf{v}, \mathbf{r}; t)d\mathbf{r}, \tag{9.60}$$

where we used integration by substitution for the variable $\mathbf{y}(\mathbf{r}) = \mathbf{x} - \mathbf{r}$. The last integral above constitutes the fluctuating Lagrangian PDF for $\mathbf{u}^+(t, \mathbf{y})$, which we label as $g_L^{(h)} = g_L^{(h)}(\mathbf{v}; t)$. Thus, for homogeneous turbulence, the PDFs for $\mathbf{u}(t, \mathbf{x})$ and $\mathbf{u}^+(t, \mathbf{y})$ are equal, that is $g^{(h)} = g_L^{(h)}$.

Chapter 10

Large-eddy simulation

10.1 Filtering

The filtering operation for a homogeneous filter is as follows

$$\overline{U}_i(\mathbf{x}) = \int_{\mathbb{R}^3} G(\mathbf{r}) U_i(\mathbf{x} - \mathbf{r}) d\mathbf{r} \tag{10.1}$$

Taking the fourier transform of the above leads to

$$\mathcal{F}\{\overline{U}_i(\mathbf{x})\} = (2\pi)^3 \mathcal{F}\{G(\mathbf{x})\} \mathcal{F}\{U_i(\mathbf{x})\} \tag{10.2}$$

due to the convolution theorem. If we define $\hat{\overline{U}}_i(\boldsymbol{\kappa}) = \mathcal{F}\{\overline{U}_i(\mathbf{x})\}$, $\hat{G}(\boldsymbol{\kappa}) = (2\pi)^3 \mathcal{F}\{G(\mathbf{x})\}$, and $\hat{U}_i(\boldsymbol{\kappa}) = \mathcal{F}\{U_i(\mathbf{x})\}$, then the above becomes

$$\hat{\overline{U}}_i(\boldsymbol{\kappa}) = \hat{G}(\boldsymbol{\kappa}) \hat{U}_i(\boldsymbol{\kappa}). \tag{10.3}$$

That is, filtering in physical space is equivalent to multiplication by $\hat{G}(\boldsymbol{\kappa})$ in spectral space.

The filtered two-point velocity correlation is given by $\overline{\mathcal{R}}_{ij}(\mathbf{x}, \mathbf{r}) = \langle \overline{u}_i(\mathbf{x}) \overline{u}_j(\mathbf{x} + \mathbf{r}) \rangle$. For homogeneous turbulence this can be expresses as

$$\overline{\mathcal{R}}_{ij}(\mathbf{r}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(\mathbf{a}) G(\mathbf{b}) \mathcal{R}_{ij}(\mathbf{r} + \mathbf{a} - \mathbf{b}) \, d\mathbf{a} d\mathbf{b}. \tag{10.4}$$

The filtered spectrum is then defined as

$$\overline{E}_{ij}(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \overline{\mathcal{R}}_{ij}(\mathbf{r}) e^{-i\boldsymbol{\kappa} \cdot \mathbf{r}} \, d\mathbf{r}. \tag{10.5}$$

Thus, by plugging in eq. (10.4) into eq. (10.5) one can show that the filtered spectrum is related to the velocity-spectrum tensor as follows

$$\overline{E}_{ij}(\boldsymbol{\kappa}) = \hat{G}(\boldsymbol{\kappa})^2 E_{ij}(\boldsymbol{\kappa}). \tag{10.6}$$

If we assume the filter is isotropic in real space, then it'll also be isotropic in spectral space. Thus the above is rewritten as $\overline{E}_{ij}(\boldsymbol{\kappa}) = \hat{G}(\kappa)^2 E_{ij}(\boldsymbol{\kappa})$, which allows us to obtain

$$\overline{E}(\kappa) = \hat{G}(\kappa)^2 E(\kappa). \tag{10.7}$$

The most common filters are the box filter

$$G(r) = \frac{6}{\pi \Delta^3} H(\frac{\Delta}{2} - |r|), \tag{10.8}$$

the Gaussian filter

$$G(r) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{r^2}{2\sigma^2}\right), \tag{10.9}$$

with $\sigma^2 = \Delta^2/12$, and the Sharp spectral filter

$$\hat{G}(\kappa) = H(\frac{\pi}{\Delta} - |\kappa|). \tag{10.10}$$

A few important aspects need to be noted. All of the filters above are parameterized by the filter width Δ. Also, the box filter represents an average over a sphere instead of a box. Finally, the spectral filter annihilitates all Fourier modes with wave-vector magnitude greater than $\kappa_c = \pi/\Delta$, where κ_c is labeled the cut-off wavenumber.

10.2   Dynamics of filtered variables

10.2.1   Rate-of-strain tensor

The filtered rate-of-strain tensor is given by

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right), \tag{10.11}$$

and the characteristic filtered rate of strain is given by $\overline{S} = \left(2\overline{S}_{ij}\overline{S}_{ij}\right)^{1/2}$.

Using eq. (4.33) for a filtered isotropic turbulent field, we obtain

$$\langle \overline{S}^2 \rangle = 2 \int_0^\infty \kappa^2 \overline{E}(\kappa)\, d\kappa = 2 \int_0^\infty \kappa^2 \hat{G}(\kappa)^2 E(\kappa)\, d\kappa. \tag{10.12}$$

The factor κ^2 attenuates the integrand for small wave numbers, whereas the term $\hat{G}^2(\kappa)$ suppresses the integrand for wave numbers higher than the cuttoff wave number κ_c. Thus, most of the contributions towards $\langle \overline{S}^2 \rangle$ come from wave numbers around κ_c. Assuming κ_c is in the inertial subrange, once can use the Kolmogorov spectrum $E = C\epsilon^{2/3}\kappa^{-5/3}$ to obtain

$$\langle \overline{S}^2 \rangle = \left(2 \int_0^\infty \kappa^{1/3} \hat{G}(\kappa)^2\, d\kappa\right) C\epsilon^{2/3} = a_f C\epsilon^{2/3}\Delta^{-4/3}, \tag{10.13}$$

where

$$a_f = 2 \int_0^\infty (\kappa\Delta)^{1/3} \hat{G}(\kappa)^2 \Delta\, d\kappa \tag{10.14}$$

A transformation of variables reveals that

$$a_f = 2 \int_0^\infty s^{1/3} \hat{G}(s/\Delta)^2\, ds. \tag{10.15}$$

Since κ is always multiplied by Δ in the definitions of $\hat{G}$, a_f becomes independent of Δ, and dependent on the type of filter only.

10.2.2   Filtered Navier-Stokes equations

Assuming a homogeneous filter, spatial differentiation and filtering commute. Thus, the continuity equation follows

$$\frac{\partial \langle U_i \rangle}{\partial x_i} = 0, \tag{10.16}$$

and the momentum equation in non-conservative form is

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{U}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}. \tag{10.17}$$

In the above, $\tau_{ij} = \overline{U_i U_j} - \overline{U}_i \overline{U}_j$ is the residual-stress or subgrid-scale (SGS) tensor.

10.2.3 Residual stresses

As states in section blah blah, the residual or subgrid-scale stress is given by $\tau_{ij} = \overline{U_i U_j} - \overline{U}_i \overline{U}_j$. The residual kinetic energy is $k_r = \frac{1}{2}\tau_{ii}$ and the anisotropic residual-stress tensor is defined using $\tau_{ij} = \frac{2}{3} k_r \delta_{ij} + a_{ij}$.

Two common decompositions of the residual stresses are those of Leonard (1974)

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij}, \tag{10.18}$$

and Germano (1986)

$$\tau_{ij} = \mathcal{L}^\circ_{ij} + \mathcal{C}^\circ_{ij} + \mathcal{R}^\circ_{ij}. \tag{10.19}$$

The tensors

$$L_{ij} = \qquad \text{and} \qquad \mathcal{L}^\circ_{ij} = \tag{10.20}$$

are the Leonard stresses,

$$C_{ij} = \qquad \text{and} \qquad \mathcal{C}^\circ_{ij} = \tag{10.21}$$

the cross tresses, and

$$R_{ij} = \qquad \text{and} \qquad \mathcal{R}^\circ_{ij} = \tag{10.22}$$

the SGS Reynolds stresses. In general L_{ij} and C_{ij} are not Galilean invariant, and thus the Germano decomposition is preferred.

10.2.4 Kinetic energy

A decomposition of the filtered kinetic energy is obtained in analogy to the RANS case. That is

$$\text{RANS:} \quad \langle E \rangle = \bar{E} + k \qquad \text{LES:} \quad \overline{E} = E_f + k_r, \tag{10.23}$$

where $\overline{E}$ is the filtered kinetic energy $\frac{1}{2}\overline{U_k U_k}$, E_f is the kinetic energy of the filtered velocity field $\frac{1}{2}\overline{U}_k \overline{U}_k$ and k_r is the residual kinetic energy.

Below we compare the transport equations for $\bar{E}$ and E_f

$$\text{RANS:} \quad \frac{\bar{D}\bar{E}}{\bar{D}t} = -\mathcal{P} - \bar{\epsilon} - \frac{\partial}{\partial x_i}\left(\langle U_j \rangle \tau_{ij} + \langle U_i \rangle \frac{\langle P \rangle}{\rho} - 2\nu \langle U_j \rangle \bar{S}_{ij}\right), \tag{10.24}$$

$$\text{LES:} \quad \frac{\overline{D}E_f}{\overline{D}t} = -\mathcal{P}_r - \epsilon_f - \frac{\partial}{\partial x_i}\left(\overline{U}_j \tau_{ij} + \overline{U}_i \frac{\overline{P}}{\rho} - 2\nu \overline{U}_j \overline{S}_{ij}\right), \tag{10.25}$$

where

$$\mathcal{P} = -\tau_{ij}\bar{S}_{ij} \qquad \bar{\epsilon} = 2\nu \bar{S}_{ij}\bar{S}_{ij} \tag{10.26}$$

and

$$\mathcal{P}_r = -\tau_{ij}\overline{S}_{ij} \qquad \epsilon_f = 2\nu \overline{S}_{ij}\overline{S}_{ij}. \tag{10.27}$$

10.3 The Smagorinsky model

In an analogy to the Boussinesq approximation of RANS models, the Smagorinsky model follows

$$a_{ij} = -2\nu_r \overline{S}_{ij} \tag{10.28}$$

where ν_r is the eddy viscosity of the residual motions. In analogy now to the mixing length hypothesis, the eddy viscosity is expressed as follows

$$\nu_r = l_S^2 \overline{S} \tag{10.29}$$

where $l_S = C_S \Delta$ is the length scale and C_s is the Smagorinsky coefficient. Thus, the expected rate of production of residual kinetic energy is given by $\langle P_r \rangle = l_S^2 \langle \overline{S}^3 \rangle$.

The scaling $l_S \sim \Delta$ that results from the definition can be proved by assuming that $\langle P_r \rangle = \epsilon$. In this case

$$l_S = \left(\frac{\epsilon}{\langle \overline{S}^3 \rangle} \right)^{1/2} = \left(\frac{\left(\frac{\langle \overline{S}^2 \rangle \Delta^{4/3}}{a_f C} \right)^{3/2}}{\langle \overline{S}^3 \rangle} \right)^{1/2} = \frac{\Delta}{(a_f C)^{3/4}} \left(\frac{\langle \overline{S}^2 \rangle^{3/2}}{\langle \overline{S}^3 \rangle} \right)^{1/2}, \tag{10.30}$$

where we have used eq. (10.13) to replace ϵ. Assuming $\langle \overline{S} \rangle^{3/2} = \langle \overline{S}^3 \rangle$ and using the value of a_f corresponding to the sharp spectral filter one obtains

$$l_S \approx 0.17\Delta, \tag{10.31}$$

which allows us to identify the value of 0.17 for C_S.

10.4 Numerical aspects LES

For explicit LES, the physical modeling and numerics are separate issues, whereas for implicit LES these two aspects are entwined. Regarding the numerical aspects of explicit LES, an appropriate grid resolution and numerical scheme is required to accurately capture the time evolution of $\mathbf{W}$. One of the first questions to ask is then how refined should the mesh be? The answer depends, among other factors, on the filter width Δ. For example, the larger the value of Δ, the smoother the filtered velocity is, and therefore the coarser the mesh can be. Thus, before one can answer the question of how refined the mesh should be, one first needs to address the question of how to determine Δ.

A user could employ different methods to specify the parameter Δ of the simulation of interest. One method would be to pick Δ so that it falls in the inertial range. This is in agreement with the basis of LES, that is, to resolve the large scales in the energy-containing range, while modeling the remaining smaller scales in the universal equilibrium range. However, it is difficult to determine a priori the bounds of the inertial range, and thus a Δ within those bounds. An alternate method is the one briefly described in Pope (Ten questions concerning the large-eddy simulation of turbulent flows, New Journal of Physics, 2004). In this case, Δ is adaptively computed so that a specified percentage of turbulent kinetic energy is resolved at any given point. For example, if we want to resolve 80% of the turbulent kinetic energy, then Δ would be adaptively changed until the ratio

$$M(\mathbf{x}, t) = \frac{k_r(\mathbf{x}, t)}{K(\mathbf{x}, t) + k_r(\mathbf{x}, t)} \tag{10.32}$$

reaches a value just below 0.2, where k_r is the turbulent kinetic energy of the residual motions, and K the turbulent kinetic energy of the filtered velocity ($K = \frac{1}{2}w_i w_i$). The challenge with this approach is that a good method to compute K and estimate k_r are needed.

Once Δ is specified, and we thus have a way to gauge how much the resolved scales fluctuate, one can then tackle the issue of how refined the mesh should be so that the numerical error introduced by the numerical scheme is minimized. This of course depends on the numerical method being used. When using low-order numerical schemes, the grid would need to be finer to provide a similar amount of numerical error than when using a high-order numerical scheme. We will first consider spectral schemes, which, for the appropriate problems, provide the lowest numerical error.

Assume we have used a sharp spectral filter with filter width Δ on a one-dimensional flow field over a domain of length L. Thus, modes with wavevector up to $\kappa_c = \frac{\pi}{\Delta}$ are present in the filtered velocity field, and need to be captured by the numerical scheme. If we use a grid with spacing h, then the largest mode that the spectral scheme can capture is $\kappa_{N/2} = \frac{\pi}{h}$. Thus, we want to pick a value of h so that $\kappa_{N/2} = \kappa_c$, which means h needs to be equal to Δ.

If we instead use a non-spectral scheme, a finer mesh spacing would be needed ...

Chapter 11

Direct Numerical Simulation

11.1 Resolution requirements

11.1.1 All flows

- Compute two point correlations, which give an indication of the spatial extent of the structures. Make sure the distance from the peak of the correlation to where it levels off to zero is at most half of the domain, since the size of the structures is roughly twice this distance.

11.1.2 Homogeneous flows

- For isotropic turbulence $\mathcal{L} > 8L_{11}$, where $\mathcal{L}$ is the size of the box and L_{11} is the integral length scale. This leads to

$$\kappa_0 L_{11} = \frac{2\pi}{\mathcal{L}} L_{11} < \frac{\pi}{4}. \tag{11.1}$$

- To account for the smallest scales $\kappa_{\max}\eta > 1.5$ This leads to

$$\Delta x = \frac{\mathcal{L}}{N} = \frac{\mathcal{L}2}{2\pi N}\pi = \frac{\pi}{\kappa_{\max}} < \frac{\pi}{1.5}\eta. \tag{11.2}$$

11.1.3 Inhomogeneous flows

- For wall bounded flows check Δx^+, Δy^+, and Δz^+ to make sure they are within a "recommended" range.

For the interior of the flow check that $h/\eta < 12$ where $h = (h_x h_y h_z)^{1/3}$ so that you have at least two points to resolve scales of size 24η. Those scales correspond to a wave number $0.26/\eta$, which, given isotropic turbulence and a carefully devised model spectrum, Pope (2001) shows is the value at which the maximum dissipation takes place.

Appendix A

Vectors in rotating reference frames

A.1   Basic properties

A.1.1   Position

Consider the Eucledian transformation

$$\tilde{x}_i^+(\tilde{t}) = Q_{ij}(\tilde{t} - c)x_j^+(\tilde{t} - c) + b_i(\tilde{t} - c) \tag{A.1}$$

where $x^+(t)$ is the Lagrangian position of a fluid particle, and $\tilde{x}^+(\tilde{t})$ is the Lagrangian position of the same particle but in a rotating reference frame. The transformation above amounts to a rotation of the reference frame determined by the orthogonal matrix $\mathbf{Q}(t)$, a translation of the reference frame determined by the vector $\mathbf{b}(t)$, and a translation in time given by $\tilde{t} = t + c$.

A.1.2   Velocity

Since the velocity of a fluid particle is given by

$$u_i^+(t) = \frac{dx_i^+(t)}{dt}, \tag{A.2}$$

the components of velocity in the non-inertial reference frame are defined as

$$\tilde{u}_i^+(\tilde{t}) = \frac{d\tilde{x}_i^+(\tilde{t})}{d\tilde{t}}. \tag{A.3}$$

Thus, the relationship between the velocity components in the different reference frames is given by

$$\tilde{u}_i^+(\tilde{t}) = \frac{d}{d\tilde{t}}[Q_{ij}(\tilde{t} - c)x_j^+(\tilde{t} - c) + b_i(\tilde{t} - c)]$$
$$= Q_{ij}(\tilde{t} - c)U_j^+(\tilde{t} - c) + \dot{Q}_{ij}(\tilde{t} - c)x_j^+(\tilde{t} - c) + \dot{b}_i(\tilde{t} - c). \tag{A.4}$$

If we assume there is no translation in space or time, the above reduces to

$$\tilde{u}_i^+ = Q_{ij}u_j^+ + \dot{Q}_{ij}x_j^+, \tag{A.5}$$

where each quantity above depends on time t. Multiplying by Q_{ki}^{-1} we obtain

$$Q_{ik}\tilde{u}_i^+ = u_k^+ + Q_{ik}\dot{Q}_{ij}x_j^+. \tag{A.6}$$

Noting that the tensor $Q_{ik}\dot{Q}_{ij}$ is antisymmetric, we obtain

$$u_k^+ = Q_{ik}\tilde{u}_i^+ + Q_{ij}\dot{Q}_{ik}x_j^+. \tag{A.7}$$

We now define the angular velocity tensor Ω_{ij} and the angular velocity vector Ω_i using the following relationship

$$\Omega_{ij} = \epsilon_{jik}\Omega_k = Q_{kj}\dot{Q}_{ki}. \tag{A.8}$$

Thus (A.7) can be expressed in terms of the angular velocity tensor as

$$u_k^+ = Q_{ik}\tilde{u}_i^+ + \Omega_{kj}x_j^+, \tag{A.9}$$

or in terms of the angular velocity vector as

$$u_k^+ = Q_{ik}\tilde{u}_i^+ + \epsilon_{jki}\Omega_i x_j^+. \tag{A.10}$$

A.1.3 Acceleration

We assume again no translation in space or time. Taking the derivative of both sides of eq. (A.7)

$$\frac{du_k^+}{dt} = \dot{Q}_{ik}\tilde{u}_i^+ + Q_{ik}\frac{d\tilde{u}_i^+}{dt} + \dot{Q}_{ik}\frac{dQ_{ij}x_j^+}{dt} + Q_{ij}\ddot{Q}_{ik}x_j^+. \tag{A.11}$$

Using eq. (A.1) the above is re-written as

$$\frac{du_k^+}{dt} = Q_{ik}\frac{d\tilde{u}_i^+}{dt} + 2\dot{Q}_{ik}\tilde{u}_i^+ + Q_{ij}\ddot{Q}_{ik}x_j^+. \tag{A.12}$$

Re-writing the last term on the right-hand-side above leads to

$$\frac{du_k^+}{dt} = Q_{ik}\frac{d\tilde{u}_i^+}{dt} + 2\dot{Q}_{ik}\tilde{u}_i^+ + \frac{dQ_{ij}\dot{Q}_{ik}}{dt}x_j^+ - \dot{Q}_{ij}\dot{Q}_{ik}x_j^+. \tag{A.13}$$

We now show that

$$\begin{aligned}
\dot{Q}_{ij}\dot{Q}_{ik} &= \delta_{pi}\dot{Q}_{pj}\dot{Q}_{ik} \\
&= Q_{pq}Q_{qi}^{-1}\dot{Q}_{pj}\dot{Q}_{ik} \\
&= Q_{pq}\dot{Q}_{pj}Q_{iq}\dot{Q}_{ik}.
\end{aligned} \tag{A.14}$$

Thus, in terms of the angular velocity tensor, we have

$$\frac{du_k^+}{dt} = Q_{ik}\frac{d\tilde{u}_i^+}{dt} + 2\dot{Q}_{ik}\tilde{u}_i^+ + \frac{d\Omega_{kj}}{dt}x_j^+ - \Omega_{kq}\Omega_{jq}x_j^+. \tag{A.15}$$

In terms of the angular velocity vector, the above becomes

$$\frac{du_k^+}{dt} = Q_{ik}\frac{d\tilde{u}_i^+}{dt} + 2\dot{Q}_{ik}\tilde{u}_i^+ + \epsilon_{jki}\frac{d\Omega_i}{dt}x_j^+ - \epsilon_{qki}\Omega_i\epsilon_{qjp}\Omega_p x_j^+, \tag{A.16}$$

which, upon a simple re-arrangement of indices, gives

$$\frac{du_k^+}{dt} = Q_{ik}\frac{d\tilde{u}_i^+}{dt} + 2\dot{Q}_{ik}\tilde{u}_i^+ + \epsilon_{kij}\frac{d\Omega_i}{dt}x_j^+ + \epsilon_{kiq}\Omega_i\epsilon_{qpj}\Omega_p x_j^+. \tag{A.17}$$

A.2 Unit vectors

Let the orthogonal basis for the inertial reference frame be denoted by $\hat{a}^{(1)}$, $\hat{a}^{(2)}$, and $\hat{a}^{(3)}$, and the orthogonal basis for the rotating reference frame by $\hat{b}^{(1)}$, $\hat{b}^{(2)}$, and $\hat{b}^{(3)}$. We know that the unit vector $\hat{b}^{(i)}$ has components $\tilde{b}_j^{(i)}$ in the rotating reference frame, and components $b_j^{(i)}$ in the inertial reference frame. The vector is the same whether expressed in the rotating or inertial reference frames, that is

$$\tilde{b}_k^{(i)}\hat{b}^{(k)} = b_k^{(i)}\hat{a}^{(k)} \tag{A.18}$$

Since $\tilde{b}_k^{(i)} = \delta_{ik}$, we rewrite the above as

$$\hat{b}^{(i)} = b_k^{(i)}\hat{a}^{(k)} = Q_{jk}\tilde{b}_j^{(i)}\hat{a}^{(k)} = Q_{ik}\hat{a}^{(k)}. \tag{A.19}$$

Doting both sides by $\hat{a}^{(j)}$ shows that

$$Q_{ij} = \hat{a}^{(j)} \cdot \hat{b}^{(i)}. \tag{A.20}$$

One can also multiply both sides of eq. (A.10) by $\hat{a}^{(k)}$ and use eq. (A.19) to obtain

$$u_k^+\hat{a}^{(k)} = \tilde{u}_i^+\hat{b}^{(i)} + \epsilon_{jki}\Omega_i x_j^+\hat{a}^{(k)}. \tag{A.21}$$

If we denote $\vec{u}_{ine}^+$ as the vector $u_k^+\hat{a}^{(k)}$ and $\vec{u}_{rot}^+$ as the vector $\tilde{u}_i^+\hat{b}^{(i)}$, then the above is re-expressed as follows

$$\vec{u}_{ine}^+ = \vec{u}_{rot}^+ + \vec{\Omega} \times \vec{x}^+. \tag{A.22}$$

Similarly, multiplying both sides of eq. (A.17) gives

$$\frac{du_k^+}{dt}\hat{a}^{(k)} = \frac{d\tilde{u}_i^+}{dt}\hat{b}^{(i)} + 2\dot{Q}_{ik}\tilde{u}_i^+\hat{a}^{(k)} + \epsilon_{kij}\frac{d\Omega_i}{dt}x_j^+\hat{a}^{(k)} + \epsilon_{kiq}\Omega_i\epsilon_{qpj}\Omega_p x_j^+\hat{a}^{(k)}. \tag{A.23}$$

Using eq. (A.8) one can show that $\dot{Q}_{ik} = \epsilon_{pjk}\Omega_p Q_{ij}$. Thus, the second term on the left-hand-side above can be written as

$$\begin{aligned}
2\dot{Q}_{ik}\tilde{u}_i^+\hat{a}^{(k)} &= 2\epsilon_{pjk}\Omega_p Q_{ij}\tilde{u}_i^+\hat{a}^{(k)} \\
&= 2\Omega_p Q_{ij}\tilde{u}_i^+\hat{a}^{(p)} \times \hat{a}^{(j)} \\
&= 2\Omega_p\hat{a}^{(p)} \times \tilde{u}_i^+\hat{b}^{(i)}.
\end{aligned} \tag{A.24}$$

Therefore, in vector notation eq. (A.23) becomes

$$\vec{a}_{ine}^+ = \vec{a}_{rot}^+ + 2\vec{\Omega} \times \vec{u}_{rot}^+ + \dot{\vec{\Omega}} \times \vec{x}^+ + \vec{\Omega} \times (\vec{\Omega} \times \vec{x}^+), \tag{A.25}$$

where $\vec{a}_{ine}^+ = \frac{du_k^+}{dt}\hat{a}^{(k)}$ and $\vec{a}_{rot}^+ = \frac{d\tilde{u}_i^+}{dt}\hat{b}^{(i)}$.

A.3 Eulerian variables

We now introduce the Eulerian counterpart to the Lagrangian variables. In the inertial reference frame these are $\rho(t, \mathbf{x})$, $u_i(t, \mathbf{x})$, and $p(t, \mathbf{x})$. They are defined by the following expressions

$$\rho^+ = \rho(t, \mathbf{x}^+) \tag{A.26}$$

$$u_i^+ = u_i(t, \mathbf{x}^+) \tag{A.27}$$

$$p^+ = p(t, \mathbf{x}^+). \tag{A.28}$$

Similarly for the rotating reference frame, we have $\tilde{\rho}(t, \tilde{\mathbf{x}})$, $\tilde{u}_i(t, \tilde{\mathbf{x}})$, and $\tilde{p}(t, \tilde{\mathbf{x}})$. These are defined by

$$\tilde{\rho}^+ = \tilde{\rho}(t, \tilde{\mathbf{x}}^+) \tag{A.29}$$

$$\tilde{u}_i^+ = \tilde{u}_i(t, \tilde{\mathbf{x}}^+) \tag{A.30}$$

$$\tilde{p}^+ = \tilde{p}(t, \tilde{\mathbf{x}}^+). \tag{A.31}$$

We now use the transformation rules for the Lagrangian variables to derive the transformation rules for the Eulerian variables. The transformation rules for the Lagrangian variables are

$$\rho^+ = \tilde{\rho}^+ \tag{A.32}$$

$$u_k^+ = Q_{ik}\tilde{u}_i^+ + Q_{ij}\dot{Q}_{ik}x_j^+ \tag{A.33}$$

$$p^+ = \tilde{p}^+. \tag{A.34}$$

The second equation above is the previously derived eq. (A.7). Using the definition of the Eulerian variables, the above is re-written as

$$\rho(t, \mathbf{x}^+) = \tilde{\rho}(t, \tilde{\mathbf{x}}^+) \tag{A.35}$$

$$u_i(t, \mathbf{x}^+) = Q_{ik}\tilde{u}_i(t, \tilde{\mathbf{x}}^+) + Q_{ij}\dot{Q}_{ik}x_j^+ \tag{A.36}$$

$$p(t, \mathbf{x}^+) = \tilde{p}(t, \tilde{\mathbf{x}}^+). \tag{A.37}$$

Using eq. (A.1), we get

$$\rho(t, \mathbf{x}^+) = \tilde{\rho}(t, \mathbf{Q}\mathbf{x}^+) \tag{A.38}$$

$$u_i(t, \mathbf{x}^+) = Q_{ik}\tilde{u}_i(t, \mathbf{Q}\mathbf{x}^+) + Q_{ij}\dot{Q}_{ik}x_j^+ \tag{A.39}$$

$$p(t, \mathbf{x}^+) = \tilde{p}(t, \mathbf{Q}\mathbf{x}^+) + . \tag{A.40}$$

Since the above holds for any $\mathbf{x}^+$, we finally re-write it as

$$\rho(t, \mathbf{x}) = \tilde{\rho}(t, \mathbf{Q}\mathbf{x}) \tag{A.41}$$

$$u_i(t, \mathbf{x}) = Q_{ik}\tilde{u}_i(t, \mathbf{Q}\mathbf{x}) + Q_{ij}\dot{Q}_{ik}x_j \tag{A.42}$$

$$p(t, \mathbf{x}) = \tilde{p}(t, \mathbf{Q}\mathbf{x}). \tag{A.43}$$

A.4   Rotating Navier-Stokes equations

We first obtain a few set of relationships that will later be used in deriving the rotating Navier-Stokes equations. Taking the derivative of eq. (A.43) gives

$$\begin{aligned}
\frac{\partial p(t, \mathbf{x})}{\partial x_i} &= \frac{\partial \tilde{p}(t, \mathbf{Q}\mathbf{x})}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}\tilde{p}(t, Q_{1j}x_j, Q_{2j}x_j, Q_{3j}x_3) \\
&= \frac{\partial Q_{1j}x_j}{\partial x_i}\left[\frac{\partial \tilde{p}(t, \tilde{\mathbf{x}})}{\partial \tilde{x}_1}\right]_{\tilde{\mathbf{x}}=\mathbf{Q}\mathbf{x}} + \frac{\partial Q_{2j}x_j}{\partial x_i}\left[\frac{\partial \tilde{p}(t, \tilde{\mathbf{x}})}{\partial \tilde{x}_2}\right]_{\tilde{\mathbf{x}}=\mathbf{Q}\mathbf{x}} + \frac{\partial Q_{3j}x_j}{\partial x_i}\left[\frac{\partial \tilde{p}(t, \tilde{\mathbf{x}})}{\partial \tilde{x}_3}\right]_{\tilde{\mathbf{x}}=\mathbf{Q}\mathbf{x}} \\
&= Q_{ji}\left[\frac{\partial \tilde{p}(t, \tilde{\mathbf{x}})}{\partial \tilde{x}_j}\right]_{\tilde{\mathbf{x}}=\mathbf{Q}\mathbf{x}} \tag{A.44}
\end{aligned}$$

If we evaluate the above at $\mathbf{x} = \mathbf{x}^+$, we get

$$\left[\frac{\partial p(t,\mathbf{x})}{\partial x_i}\right]_{\mathbf{x}=\mathbf{x}^+} = Q_{ji}\left[\frac{\partial \tilde{p}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_j}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}. \tag{A.45}$$

Multiplying both sides by $\hat{a}^{(i)}$ gives

$$\left[\frac{\partial p(t,\mathbf{x})}{\partial x_i}\right]_{\mathbf{x}=\mathbf{x}^+}\hat{a}^{(i)} = \left[\frac{\partial \tilde{p}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_j}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(j)}. \tag{A.46}$$

We also use eq. (A.25) to write

$$\vec{a}_{ine}^+ = ... \tag{A.47}$$

The conservation of momentum equation can be written as

$$\rho(t,\mathbf{x})\left[\frac{\partial u_i(t,\mathbf{x})}{\partial t} + u_j(t,\mathbf{x})\frac{\partial u_i(t,\mathbf{x})}{\partial x_j}\right] = -\frac{\partial p(t,\mathbf{x})}{\partial x_i} + \frac{\partial \tau_{ij}(t,\mathbf{x})}{\partial x_j}. \tag{A.48}$$

Evaluating the above at $\mathbf{x} = \mathbf{x}^+$ leads to

$$\rho(t,\mathbf{x}^+)\frac{du_i^+}{dt} = -\left[\frac{\partial p(t,\mathbf{x})}{\partial x_i}\right]_{\mathbf{x}=\mathbf{x}^+} + \left[\frac{\partial \tau_{ij}(t,\mathbf{x})}{\partial x_j}\right]_{\mathbf{x}=\mathbf{x}^+}. \tag{A.49}$$

Multiplying both sides by $\hat{a}^{(i)}$ gives

$$\rho(t,\mathbf{x}^+)\vec{a}_{ine}^+ = -\left[\frac{\partial p(t,\mathbf{x})}{\partial x_i}\right]_{\mathbf{x}=\mathbf{x}^+}\hat{a}^{(i)} + \left[\frac{\partial \tau_{ij}(t,\mathbf{x})}{\partial x_j}\right]_{\mathbf{x}=\mathbf{x}^+}\hat{a}^{(i)}. \tag{A.50}$$

Using eq. (A.46) and blah blah gives

$$\rho(t,\mathbf{x}^+)\vec{a}_{ine}^+ = -\left[\frac{\partial \tilde{p}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_i}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(i)} + \left[\frac{\partial \tilde{\tau}_{ij}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_j}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(i)}. \tag{A.51}$$

Using eq. (A.35) we write the above as

$$[\tilde{\rho}(t,\tilde{\mathbf{x}})]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\,\vec{a}_{ine}^+ = -\left[\frac{\partial \tilde{p}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_i}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(i)} + \left[\frac{\partial \tilde{\tau}_{ij}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_j}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(i)}. \tag{A.52}$$

Finally, using eq. (A.25), we obtain

$$[\tilde{\rho}(t,\tilde{\mathbf{x}})]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\left[\vec{a}_{rot}^+ + 2\vec{\Omega}\times\vec{u}_{rot}^+ + \dot{\vec{\Omega}}\times\vec{x}^+ + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{x}^+\right)\right]$$
$$= -\left[\frac{\partial \tilde{p}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_i}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(i)} + \left[\frac{\partial \tilde{\tau}_{ij}(t,\tilde{\mathbf{x}})}{\partial \tilde{x}_j}\right]_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}^+}\hat{b}^{(i)}. \tag{A.53}$$

A.5   Eulerian variables (OLD!!!!!)

We now show the transformation rules for Eulerian rather than Lagrangian variables. The Eulerian velocity in the inertial reference frame $u_i(t,\mathbf{x})$ and the Eulerian velocity in the rotating reference frame $\tilde{u}_i(\tilde{t},\tilde{\mathbf{x}})$ are defined by the following expressions

$$u_i^+(t) = u_i(t,\mathbf{x}^+(t)), \tag{A.54}$$

$$\tilde{u}_i^+(\tilde{t}) = \tilde{u}_i(\tilde{t},\tilde{\mathbf{x}}^+(\tilde{t})). \tag{A.55}$$

Given the definition of the Eulerian velocities, eq. (A.4) for the velocity transformation becomes

$$\tilde{u}_i(\tilde{t}, \tilde{\mathbf{x}}^+(\tilde{t})) = Q_{ij}(\tilde{t} - c)u_j(\tilde{t} - c, \mathbf{x}^+(\tilde{t} - c)) + \dot{Q}_{ij}(\tilde{t} - c)x_j^+(\tilde{t} - c) + \dot{b}_i(\tilde{t} - c). \qquad (A.56)$$

We now use eq. (A.1), but in the form $x_j^+(\tilde{t} - c) = Q_{kj}(\tilde{t} - c)\tilde{x}_k^+(\tilde{t}) - Q_{kj}(\tilde{t} - c)b_k(\tilde{t} - c)$. Plugging this into the above gives

$$\tilde{u}_i(\tilde{t}, \tilde{\mathbf{x}}^+(\tilde{t})) = Q_{ij}(\tilde{t} - c)u_j(\tilde{t} - c, \mathbf{Q}^T(\tilde{t} - c)\tilde{\mathbf{x}}^+(\tilde{t}) - \mathbf{Q}^T(\tilde{t} - c)\mathbf{b}(\tilde{t} - c))$$
$$+ \dot{Q}_{ij}(\tilde{t} - c)Q_{kj}(\tilde{t} - c)\tilde{x}_k^+(\tilde{t}) - \dot{Q}_{ij}(\tilde{t} - c)Q_{kj}(\tilde{t} - c)b_k(\tilde{t} - c) + \dot{b}_i(\tilde{t} - c). \quad (A.57)$$

Since the above holds for any $\tilde{\mathbf{x}}^+(\tilde{t})$ we obtain

$$\tilde{u}_i(\tilde{t}, \tilde{\mathbf{x}}) = Q_{ij}(\tilde{t} - c)u_j(\tilde{t} - c, \mathbf{Q}^T(\tilde{t} - c)\tilde{\mathbf{x}} - \mathbf{Q}^T(\tilde{t} - c)\mathbf{b}(\tilde{t} - c))$$
$$+ \dot{Q}_{ij}(\tilde{t} - c)Q_{kj}(\tilde{t} - c)\tilde{x}_k - \dot{Q}_{ij}(\tilde{t} - c)Q_{kj}(\tilde{t} - c)b_k(\tilde{t} - c) + \dot{b}_i(\tilde{t} - c). \qquad (A.58)$$

This is re-written in the simpler notation

$$\tilde{u}_i = Q_{ij}u_j + \dot{Q}_{ij}Q_{kj}\tilde{x}_k - \dot{Q}_{ij}Q_{kj}b_k + \dot{b}_i, \qquad (A.59)$$

with the assumption that variables in the inertial reference frame are evaluated as follows

$$u_i = u_i(\tilde{t} - c, \mathbf{Q}^T\tilde{\mathbf{x}} - \mathbf{Q}^T\mathbf{b})$$
$$Q_{ij} = Q_{ij}(\tilde{t} - c)$$
$$\dot{Q}_{ij} = \left[\frac{dQ_{ij}(t)}{dt}\right]_{(\tilde{t}-c)}$$
$$b_i = b_i(\tilde{t} - c)$$
$$\dot{b}_i = \left[\frac{db_i(t)}{dt}\right]_{(\tilde{t}-c)}. \qquad (A.60)$$

A.6   Rotating turbulence (OLD!!!!!!)

Given eq. (A.59), the mean velocity in the transforming reference frame is expressed in terms of the mean velocity in the inertial reference as follows

$$\langle \tilde{U}_i \rangle = Q_{ij}\langle U_j \rangle + \dot{Q}_{ij}Q_{kj}\tilde{x}_k - \dot{Q}_{ij}Q_{kj}b_k + \dot{b}_i. \qquad (A.61)$$

The mean velocity in the inertial reference frame in terms of the mean velocity in the transforming reference frame is given by

$$\langle U_q \rangle = Q_{iq}\langle \tilde{U}_i \rangle - Q_{iq}\dot{Q}_{ij}Q_{kj}\tilde{x}_k + Q_{iq}\dot{Q}_{ij}Q_{kj}b_k - Q_{iq}\dot{b}_i. \qquad (A.62)$$

Since $Q_{iq}\dot{Q}_{ij}$ is an anti-symmetric tensor, we have $Q_{iq}\dot{Q}_{ij} = -Q_{ij}\dot{Q}_{iq}$. Using the fact that $Q_{ij}Q_{kj} = \delta_{ki}$ the above becomes

$$\langle U_q \rangle = Q_{iq}\langle \tilde{U}_i \rangle + \dot{Q}_{iq}\tilde{x}_i - \dot{Q}_{iq}b_i - Q_{iq}\dot{b}_i. \qquad (A.63)$$

The fluctuating velocity, which in the inertial reference frame is given by $u_i = U_i - \langle U_i \rangle$, and in the transformed reference frame by $\tilde{u}_i = \tilde{U}_i - \langle \tilde{U}_i \rangle$, then transforms as

$$\tilde{u}_i = Q_{ij}u_j. \qquad (A.64)$$

Thus the reynolds stresses in the new reference frame $\tilde{\tau}_{ij} = \langle \tilde{u}_i \tilde{u}_j \rangle$ are given by

$$\tilde{\tau}_{ij} = Q_{ip} Q_{jq} \tau_{pq}. \tag{A.65}$$

The material derivative of the Reynolds stresses in the rotating reference frame thus becomes

$$\begin{aligned}
\frac{D\tilde{\tau}_{ij}}{D\tilde{t}} &= \frac{\partial \tilde{\tau}_{ij}}{\partial \tilde{t}} + \langle \tilde{U}_k \rangle \frac{\partial \tilde{\tau}_{ij}}{\partial \tilde{x}_k} \\
&= \dot{Q}_{ip} Q_{jq} \tau_{pq} + Q_{ip} \dot{Q}_{jq} \tau_{pq} + Q_{ip} Q_{jq} \left(\frac{\partial \tau_{pq}}{\partial \tilde{t}} + \langle \tilde{U}_k \rangle \frac{\partial \tau_{pq}}{\partial \tilde{x}_k} \right)
\end{aligned} \tag{A.66}$$

Since the Reynolds stresses τ_{pq} depend on $(\tilde{t} - c, \mathbf{Q}^T \tilde{\mathbf{x}} - \mathbf{Q}^T \mathbf{b})$, we have

$$\begin{aligned}
\frac{\partial \tau_{pq}}{\partial \tilde{t}} &= \frac{\partial \tau_{pq}}{\partial t} + (\dot{Q}_{ki} \tilde{x}_k - \dot{Q}_{ki} b_k - Q_{ki} \dot{b}_k) \frac{\partial \tau_{pq}}{\partial x_i}, \\
\langle \tilde{U}_k \rangle \frac{\partial \tau_{pq}}{\partial \tilde{x}_k} &= Q_{ki} \langle \tilde{U}_k \rangle \frac{\partial \tau_{pq}}{\partial x_i}.
\end{aligned} \tag{A.67}$$

Thus, we obtain

$$\begin{aligned}
\frac{D\tilde{\tau}_{ij}}{D\tilde{t}} &= \dot{Q}_{ip} Q_{jq} \tau_{pq} + Q_{ip} \dot{Q}_{jq} \tau_{pq} + Q_{ip} Q_{jq} \left(\frac{\partial \tau_{pq}}{\partial t} + \langle U_k \rangle \frac{\partial \tau_{pq}}{\partial x_k} \right) \\
&= \dot{Q}_{ip} Q_{jq} \tau_{pq} + Q_{ip} \dot{Q}_{jq} \tau_{pq} + Q_{ip} Q_{jq} \frac{D\tau_{pq}}{Dt}
\end{aligned} \tag{A.68}$$

The first two terms on the right hand side can be expressed in terms of the Reynolds stresses in the rotating reference frame as follows

$$\begin{aligned}
\dot{Q}_{ip} Q_{jq} \tau_{pq} + Q_{ip} \dot{Q}_{jq} \tau_{pq} &= \dot{Q}_{ip} Q_{jq} Q_{kp} \tilde{\tau}_{ks} Q_{sq} + Q_{ip} \dot{Q}_{jq} Q_{sp} \tilde{\tau}_{sk} Q_{kq} \\
&= \dot{Q}_{ip} Q_{kp} \tilde{\tau}_{kj} + \dot{Q}_{jq} Q_{kq} \tilde{\tau}_{ik} \\
&= -\tilde{\Lambda}_{ik} \tilde{\tau}_{kj} + \tilde{\tau}_{ik} \tilde{\Lambda}_{kj},
\end{aligned} \tag{A.69}$$

where $\tilde{\Lambda}_{ij} = Q_{ik} \dot{Q}_{jk}$. Note the similarity between $\tilde{\Lambda}_{ij} = Q_{ik} \dot{Q}_{jk}$ and $\Omega_{ij} = Q_{kj} \dot{Q}_{ki}$. In fact, these two are related as follows $\tilde{\Lambda}_{ij} = Q_{ik} \Omega_{kl} Q_{jl}$. The transport equation for the Reynolds stresses in the rotating reference frame thus becomes

$$\frac{D\tilde{\tau}_{ij}}{D\tilde{t}} + \tilde{\Lambda}_{ik} \tilde{\tau}_{kj} - \tilde{\tau}_{ik} \tilde{\Lambda}_{kj} = Q_{ip} Q_{jq} \frac{D\tau_{pq}}{Dt}. \tag{A.70}$$

An objective scalars s, vector b_i, or tensor c_{ij} is one which transforms as follows

$$\tilde{s} = s \qquad \tilde{b}_i = Q_{ij} b_j \qquad \tilde{c}_{ij} = Q_{ip} c_{pq} Q_{jq}. \tag{A.71}$$

Thus, the Reynolds stresses are objective, but the velocity vector is not. Using equation (A.61) it is easy to show that the mean-flow gradients transform as follow

$$\frac{\partial \langle \tilde{U}_i \rangle}{\partial \tilde{x}_k} = Q_{ip} \frac{\partial \langle U_p \rangle}{\partial \tilde{x}_k} + \dot{Q}_{ip} Q_{kp} = Q_{ip} Q_{kt} \frac{\partial \langle U_p \rangle}{\partial x_t} + \dot{Q}_{ip} Q_{kp}. \tag{A.72}$$

Thus, one can easily show that the rate-of-strain tensor is objective, whereas the rate-of-rotation tensor transforms as follows

$$\tilde{W}_{ij} = Q_{ip} W_{pq} Q_{jq} + \dot{Q}_{ik} Q_{jk}, \tag{A.73}$$

which is equivalent to

$$\tilde{W}_{ij} + \tilde{\Lambda}_{ij} = Q_{ip} W_{pq} Q_{jq}. \tag{A.74}$$

We now define $\Lambda_{ij} = 0$, which implies Λ_{ij} is not objective. This is a sensible decision given that Λ_{ij} is not a typical tensor such as τ_{ij}. Thus we can rewrite the above as follows

$$\tilde{W}_{ij} + \tilde{\Lambda}_{ij} = Q_{ip}(W_{pq} + \Lambda_{pq})Q_{jq}, \tag{A.75}$$

which shows $W_{ij}^* = W_{ij} + \Lambda_{ij}$ is an objective tensor. Similarly, equation (A.70) shows that

$$\frac{D\tau_{ij}}{Dt} + \Lambda_{ik}\tau_{kj} - \tau_{ik}\Lambda_{kj} \tag{A.76}$$

is an objective tensor.

We will now show how the Reynolds stress production, redistribution, dissipation and transport terms transform under a coordinate transformation. We start with production, which involves gradients of the mean flow. Using equation (A.72) we obtain

$$-\tilde{\tau}_{jk}\frac{\partial \langle \tilde{U}_i \rangle}{\partial \tilde{x}_k} = -Q_{jq}Q_{ks}\tau_{qs}\left(Q_{ip}Q_{kt}\frac{\partial \langle U_p \rangle}{\partial x_t} + \dot{Q}_{ip}Q_{kp}\right)$$

$$= -Q_{ip}Q_{jq}\tau_{qt}\frac{\partial \langle U_p \rangle}{\partial x_t} - \dot{Q}_{ip}Q_{jq}\tau_{pq} \tag{A.77}$$

We thus have

$$\tilde{\mathcal{P}}_{ij} = -\tilde{\tau}_{ik}\frac{\partial \langle \tilde{U}_j \rangle}{\partial \tilde{x}_k} - \tilde{\tau}_{jk}\frac{\partial \langle \tilde{U}_i \rangle}{\partial \tilde{x}_k} = Q_{ip}Q_{jq}\mathcal{P}_{pq} - Q_{ip}\dot{Q}_{jq}\tau_{pq} - \dot{Q}_{ip}Q_{jq}\tau_{pq}$$

$$= Q_{ip}Q_{jq}\mathcal{P}_{pq} - \tilde{\tau}_{ik}\tilde{\Lambda}_{kj} + \tilde{\Lambda}_{ik}\tilde{\tau}_{kj}. \tag{A.78}$$

The last step was obtain using (A.69). The above is rewritten as follows

$$-\tilde{\tau}_{ik}(\tilde{S}_{kj} - \tilde{W}_{kj}^*) - (\tilde{S}_{ik} + \tilde{W}_{ik}^*)\tilde{\tau}_{kj} = Q_{ip}Q_{jq}\mathcal{P}_{pq} \tag{A.79}$$

which shows $-\tau_{ik}(S_{kj} - W_{kj}^*) - (S_{ik} + W_{ik}^*)\tau_{kj}$ is objective.

The next term to consider is the velocity-pressure-gradient correlation, which involves gradients of pressure. We note that pressure, as a scalar, transforms as

$$\tilde{P} = P, \tag{A.80}$$

from which it follows that $\langle \tilde{P} \rangle = \langle P \rangle$ and $\tilde{p} = p$. The same holds for the density ρ and kinematic viscosity ν. The expression below transforms as follow

$$\tilde{u}_i\frac{\partial \tilde{p}}{\partial \tilde{x}_j} = Q_{ip}Q_{jq}u_p\frac{\partial p}{\partial x_q}, \tag{A.81}$$

which leads to the following transformation for the velocity-pressure-gradient correlation

$$\tilde{\Pi}_{ij} = -\frac{1}{\tilde{\rho}}\left\langle \tilde{u}_i\frac{\partial \tilde{p}}{\partial \tilde{x}_j} + \tilde{u}_j\frac{\partial \tilde{p}}{\partial \tilde{x}_i} \right\rangle = -Q_{ip}Q_{jq}\frac{1}{\rho}\left\langle u_p\frac{\partial p}{\partial x_q} + u_q\frac{\partial p}{\partial x_p} \right\rangle = Q_{ip}Q_{jq}\Pi_{pq}. \tag{A.82}$$

Similarly, the dissipation transforms as follows

$$\tilde{\epsilon}_{ij} = 2\tilde{\nu}\left\langle \frac{\partial \tilde{u}_i}{\partial \tilde{x}_k}\frac{\partial \tilde{u}_j}{\partial \tilde{x}_k} \right\rangle = Q_{ip}Q_{jq}Q_{kt}Q_{ks}2\nu\left\langle \frac{\partial u_p}{\partial x_t}\frac{\partial u_q}{\partial x_s} \right\rangle = Q_{ip}Q_{jq}\epsilon_{pq}. \tag{A.83}$$

Finally, the transport terms transform as follows

$$\frac{\partial}{\partial \tilde{x}_k}\left(\langle \tilde{u}_i\tilde{u}_j\tilde{u}_k \rangle - \nu\frac{\partial \tilde{\tau}_{ij}}{\partial \tilde{x}_k}\right) = Q_{ip}Q_{jq}Q_{kr}Q_{ks}\frac{\partial}{\partial x_s}\left(\langle u_p u_q u_r \rangle - \nu\frac{\partial \tau_{pq}}{\partial x_r}\right)$$

$$= Q_{ip}Q_{jq}\frac{\partial}{\partial x_k}\left(\langle u_p u_q u_k \rangle - \nu\frac{\partial \tau_{pq}}{\partial x_k}\right). \tag{A.84}$$

We can thus conclude that

$$Q_{iq}Q_{jq}\frac{D\tau_{pq}}{Dt} = Q_{iq}Q_{jq}\mathcal{P}_{pq} + Q_{iq}Q_{jq}\Pi_{pq} - Q_{iq}Q_{jq}\epsilon_{pq} - Q_{iq}Q_{jq}\frac{\partial}{\partial x_k}\left(\mathcal{T}^{(u)}_{kpq} + \mathcal{T}^{(\nu)}_{kpq}\right)$$

$$= -\tilde{\tau}_{ik}(\tilde{S}_{kj} - \tilde{W}^*_{kj}) - (\tilde{S}_{ik} + \tilde{W}^*_{ik})\tilde{\tau}_{kj} + \tilde{\Pi}_{ij} - \tilde{\epsilon}_{ij} - \frac{\partial}{\partial \tilde{x}_k}\left(\tilde{\mathcal{T}}^{(u)}_{kij} + \tilde{\mathcal{T}}^{(\nu)}_{kij}\right). \quad \text{(A.85)}$$

Using the above in equation (A.70) leads to the transport equation for the Reynolds stresses in a rotating reference frame where every single term in the equation contains quantities evaluated in the rotating reference frame

$$\frac{D\tilde{\tau}_{ij}}{D\tilde{t}} + \tilde{\Lambda}_{ik}\tilde{\tau}_{kj} - \tilde{\tau}_{ik}\tilde{\Lambda}_{kj}$$

$$= -\tilde{\tau}_{ik}(\tilde{S}_{kj} - \tilde{W}^*_{kj}) - (\tilde{S}_{ik} + \tilde{W}^*_{ik})\tilde{\tau}_{kj} + \tilde{\Pi}_{ij} - \tilde{\epsilon}_{ij} - \frac{\partial}{\partial \tilde{x}_k}\left(\tilde{\mathcal{T}}^{(u)}_{kij} + \tilde{\mathcal{T}}^{(\nu)}_{kij}\right). \quad \text{(A.86)}$$

Appendix B

RANS equations in cylindrical coordinates

The continuity equation follows

$$\frac{1}{r}\frac{\partial\left(r\langle u_r\rangle\right)}{\partial r} + \frac{1}{r}\frac{\partial\langle u_\theta\rangle}{\partial\theta} + \frac{\partial\langle u_z\rangle}{\partial z} = 0. \tag{B.1}$$

The momentum equations follow

$$\frac{\partial\langle u_r\rangle}{\partial t} + \langle u_r\rangle\frac{\partial\langle u_r\rangle}{\partial r} + \frac{\langle u_\theta\rangle}{r}\frac{\partial\langle u_r\rangle}{\partial\theta} + \langle u_z\rangle\frac{\partial\langle u_r\rangle}{\partial z} - \frac{\langle u_\theta\rangle^2}{r} =$$
$$-\frac{1}{\rho}\frac{\partial\langle P\rangle}{\partial r} + \nu\left(\frac{\partial^2\langle u_r\rangle}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\langle u_r\rangle}{\partial\theta^2} + \frac{\partial^2\langle u_r\rangle}{\partial z^2} + \frac{1}{r}\frac{\partial\langle u_r\rangle}{\partial r} - \frac{2}{r^2}\frac{\partial\langle u_\theta\rangle}{\partial\theta} - \frac{\langle u_r\rangle}{r^2}\right)$$
$$-\frac{\partial\langle u_r u_r\rangle}{\partial r} - \frac{1}{r}\frac{\partial\langle u_r u_\theta\rangle}{\partial\theta} - \frac{\partial\langle u_r u_z\rangle}{\partial z} - \frac{\langle u_r u_r\rangle}{r} + \frac{\langle u_\theta u_\theta\rangle}{r} \tag{B.2}$$

$$\frac{\partial\langle u_\theta\rangle}{\partial t} + \langle u_r\rangle\frac{\partial\langle u_\theta\rangle}{\partial r} + \frac{\langle u_\theta\rangle}{r}\frac{\partial\langle u_\theta\rangle}{\partial\theta} + \langle u_z\rangle\frac{\partial\langle u_\theta\rangle}{\partial z} + \frac{\langle u_r\rangle\langle u_\theta\rangle}{r} =$$
$$-\frac{1}{\rho r}\frac{\partial\langle P\rangle}{\partial\theta} + \nu\left(\frac{\partial^2\langle u_\theta\rangle}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\langle u_\theta\rangle}{\partial\theta^2} + \frac{\partial^2\langle u_\theta\rangle}{\partial z^2} + \frac{1}{r}\frac{\partial\langle u_\theta\rangle}{\partial r} + \frac{2}{r^2}\frac{\partial\langle u_r\rangle}{\partial\theta} - \frac{\langle u_\theta\rangle}{r^2}\right)$$
$$-\frac{\partial\langle u_\theta u_r\rangle}{\partial r} - \frac{1}{r}\frac{\partial\langle u_\theta u_\theta\rangle}{\partial\theta} - \frac{\partial\langle u_\theta u_z\rangle}{\partial z} - 2\frac{\langle u_r u_\theta\rangle}{r} \tag{B.3}$$

$$\frac{\partial\langle u_z\rangle}{\partial t} + \langle u_r\rangle\frac{\partial\langle u_z\rangle}{\partial r} + \frac{\langle u_\theta\rangle}{r}\frac{\partial\langle u_z\rangle}{\partial\theta} + \langle u_z\rangle\frac{\partial\langle u_z\rangle}{\partial z} =$$
$$-\frac{1}{\rho}\frac{\partial\langle P\rangle}{\partial z} + \nu\left(\frac{\partial^2\langle u_z\rangle}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\langle u_z\rangle}{\partial\theta^2} + \frac{\partial^2\langle u_z\rangle}{\partial z^2} + \frac{1}{r}\frac{\partial\langle u_z\rangle}{\partial r}\right)$$
$$-\frac{\partial\langle u_z u_r\rangle}{\partial r} - \frac{1}{r}\frac{\partial\langle u_z u_\theta\rangle}{\partial\theta} - \frac{\partial\langle u_z u_z\rangle}{\partial z} - \frac{\langle u_z u_r\rangle}{r}. \tag{B.4}$$