Part I

Hydrodynamics

Chapter 1

Kinematics of fluid motion

1.1 Eulerian & Lagrangian reference frames

There are two types of frameworks used to described properties of fluids: Eulerian and Lagrangian. The Eulerian framework is used to describe fields, for which fluid properties are of the form $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$, where t and $\mathbf{x}$ denote time and position. Lagrangian fields are used to describe moving particles, for which fluid properties are of the form $\mathbf{f}^+ = \mathbf{f}^+(t, \mathbf{y})$, where t denotes time and $\mathbf{y}$ the location of the particle at an initial time. Eulerian and Lagrangian properties are related to each other according to

$$\mathbf{f}^+(t, \mathbf{y}) = \mathbf{f}(t, \mathbf{x}^+(t, \mathbf{y})). \tag{1.1}$$

There are two Lagrangian properties that deserve further attention. The position and velocity of a particle are given by $\mathbf{x}^+ = \mathbf{x}^+(t, \mathbf{y})$ and $\mathbf{v}^+ = \mathbf{v}^+(t, \mathbf{y})$, respectively. These two are related to each other according to

$$\frac{\partial \mathbf{x}^+}{\partial t} = \mathbf{v}^+. \tag{1.2}$$

1.2 Material derivative

We are interested in knowing how $\mathbf{f}^+(t, \mathbf{y})$ for a hypothetical particle changes as we move along the particle's path. Applying the chain rule and product rule,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{f}^+(t, \mathbf{y}) &= \frac{\partial}{\partial t}\mathbf{f}(t, \mathbf{x}^+(t, \mathbf{y})) \\
&= \frac{\partial}{\partial t}\mathbf{f}(t, x_1^+(t, \mathbf{y}), x_2^+(t, \mathbf{y}), x_3^+(t, \mathbf{y})) \\
&= \left(\frac{\partial \mathbf{f}(t, x_1, x_2, x_3)}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+(t,\mathbf{y})} + \frac{\partial x_1^+(t, \mathbf{y})}{\partial t}\left(\frac{\partial \mathbf{f}(t, x_1, x_2, x_3)}{\partial x_1}\right)_{\mathbf{x}=\mathbf{x}^+(t,\mathbf{y})} + ... \\
&= \left(\frac{\partial \mathbf{f}(t, x_1, x_2, x_3)}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+(t,\mathbf{Y})} + v_1^+(t, \mathbf{y})\left(\frac{\partial \mathbf{f}(t, x_1, x_2, x_3)}{\partial x_1}\right)_{\mathbf{x}=\mathbf{x}^+(t,\mathbf{y}),..} + ...
\end{aligned} \tag{1.3}$$

In abridged notation this becomes

$$\boxed{\frac{\partial \mathbf{f}^+}{\partial t} = \left(\frac{\partial \mathbf{f}}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+} + \mathbf{v}^+ \cdot (\nabla \mathbf{f})_{\mathbf{x}=\mathbf{x}^+}} \tag{1.4}$$

For an arbitrary particle, the Lagrangian velocity $\mathbf{v}^+$ has an Eulerian counterpart $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ such that the following is satisfied $\mathbf{v}^+ = \mathbf{v}(t, \mathbf{x}^+)$. For this same particle, we can also introduce

the Lagrangian variable $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$ and the Eulerian counterpart $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, such that the following is satisfied $\mathbf{u}^+ = \mathbf{u}(t, \mathbf{x}^+)$. We emphasize that $\mathbf{u}^+$ is not the velocity of the particle, it is instead an additional particle property since $\mathbf{v}^+$ is the actual velocity of the particle. There is a special type of particle, referred to as a fluid particle, which is one that moves with the flow. For this particle $\mathbf{v} = \mathbf{u}$. We denote properties of a fluid particle with the subscript u, e.g. $\mathbf{x}_u^+$, $\mathbf{f}_u^+$, etc.

Given the the particle property $\mathbf{f}_u^+ = \mathbf{f}(t, \mathbf{x}_u^+)$, we have

$$\begin{aligned} \frac{\partial \mathbf{f}_u^+}{\partial t} &= \left(\frac{\partial \mathbf{f}}{\partial t}\right)_{\mathbf{x}=\mathbf{x}_u^+} + \mathbf{v}_u^+ \cdot (\nabla \mathbf{f})_{\mathbf{x}=\mathbf{x}_u^+} \\ &= \left(\frac{\partial \mathbf{f}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{f}\right)_{\mathbf{x}=\mathbf{x}_u^+}. \end{aligned} \tag{1.5}$$

The term in parenthesis above is referred to as the material derivative, and it is labeled as

$$\frac{D\mathbf{f}}{Dt} = \frac{\partial \mathbf{f}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{f}. \tag{1.6}$$

An application of the above follows. Say we are given a fluid whose temperature and velocity fields are T and u, respectively. The PDE for temperature is

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2}. \tag{1.7}$$

We now introduce a fluid particle with Lagrangian temperature $T_u^+ = T_u^+(t, y)$. This Lagrangian property is related to its corresponding Eulerian field by $T_u^+(t, y) = T_u(t, x_u^+(t, y))$. Using eq. (1.5) we find the rate of change of T_u^+ to be

$$\frac{\partial T_u^+}{\partial t} = \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right)_{x=x_u^+} = \left(\frac{\partial^2 T}{\partial x^2}\right)_{x=x_u^+}. \tag{1.8}$$

Thus, we can interpret the PDE as describing the rate of change of a particle's property as the particle moves through space. This rate of change is given by the RHS term of eq. (1.7) evaluated at the particle's position.

1.3 Reynolds transport theorem

Define the one-dimensional domain $[a_0, b_0]$. Also define the one-dimensional domain $L^+ = L^+(t, a_0, b_0)$, which is given by

$$L^+ = \{x^+ : y \in [a_0, b_0]\}. \tag{1.9}$$

Note that $L^+(0, a_0, b_0) = [a_0, b_0]$. The left boundary of this volume is given by $a^+ = a^+(t, a_0, b_0)$ and the right boundary by $b^+ = b^+(t, a_0, b_0)$. These satisfy $a^+(0, a_0, b_0) = a_0$ and $b^+(0, a_0, b_0) = b_0$.

We introduce the Lagrangian function $I^+ = I^+(t, a_0, b_0)$ and its Eulerian counterpart $I = I(t, a, b)$. These two are given by

$$I(t, a, b) = \int_a^b f(t, x)\, dx, \tag{1.10}$$

$$I^+ = I(t, a^+, b^+). \tag{1.11}$$

We can then write

$$\frac{dI^+}{dt} = \left[\frac{\partial I(t,a,b)}{\partial t}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} + \left[\frac{\partial I(t,a,b)}{\partial a}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} \frac{da^+}{dt} + \left[\frac{\partial I(t,a,b)}{\partial b}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} \frac{db^+}{dt} \tag{1.12}$$

$$= \left[\frac{\partial I(t,a,b)}{\partial t}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} + \left[\frac{\partial I(t,a,b)}{\partial a}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} v(t,a^+) + \left[\frac{\partial I(t,a,b)}{\partial b}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} v(t,b^+). \tag{1.13}$$

Due to the fundamental theorem of calculus (see appendix in my PDEs notes), we can re-write the above as

$$\frac{dI^+}{dt} = \left[\frac{\partial I(t,a,b)}{\partial t}\right]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} - [f(t,a)]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} v(t,a^+) + [f(t,b)]_{\substack{a\,=\,a^+\\ b\,=\,b^+}} v(t,b^+)$$

$$= \int_{a^+}^{b^+} \frac{\partial f(t,x)}{\partial t}\, dx - f(t,a^+)v(t,a^+) + f(t,b^+)v(t,b^+). \tag{1.14}$$

The analogue in 3D for the above is typically referred to as the Reynolds transport theorem. Consider the volume Ω_0 as a predefined set of $\mathbf{y}$ vectors. The control volume $\Omega^+ = \Omega^+(t,\Omega_0)$ is then given by

$$\Omega^+ = \{\mathbf{x}^+ : \mathbf{y} \in \Omega_0\}. \tag{1.15}$$

Note that $\Omega^+(0,\Omega_0) = \Omega_0$. The boundary of this volume is denoted by $\partial\Omega^+$. The velocity of the boundary at any point $\mathbf{x} \in \partial\Omega^+$ is given by the Eulerian field $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$ at such $\mathbf{x}$ value. Similarly, the unit normal to the boundary at any $\mathbf{x} \in \partial\Omega^+$ is given by the Eulerian field $\mathbf{n} = \mathbf{n}(t,\mathbf{x})$ at such an $\mathbf{x}$. For this case $I^+ = I^+(t,\Omega_0)$ and its Eulerian counterpart is $I = I(t,\Omega)$. These two are then given by

$$I = \int_\Omega f(t,\mathbf{x})\, dV, \tag{1.16}$$

$$I^+ = I(t,\Omega^+). \tag{1.17}$$

The derivation of the Reynolds transport theorem is not so simple and hence it is not reproduced in here. The result is

$$\frac{dI^+}{dt} = \int_{\Omega^+} \frac{\partial f(t,\mathbf{x})}{\partial t}\, dV + \int_{\partial\Omega^+} f(t,\mathbf{x})\mathbf{v}\cdot\mathbf{n}\, dS, \tag{1.18}$$

Using Gauss's theorem, we get

$$\boxed{\frac{dI^+}{dt} = \int_{\Omega^+} \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{v})\, dV.} \tag{1.19}$$

The above is the analogue of eq. (1.4).

If the moving volume is attached to the fluid, then we use the notation $\Omega^+ = \Omega_u^+$ and $\mathbf{v} = \mathbf{u}$. Thus, given $I_u^+ = I(t,\Omega_u^+)$, we have

$$\frac{dI_u^+}{dt} = \int_{\Omega_u^+} \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\, dV$$

$$= \left(\int_\Omega \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\, dV\right)_{\Omega=\Omega_u^+}. \tag{1.20}$$

This is the analogue of eq. (1.5). The term in parenthesis above is typically labeled as follows

$$\frac{DI}{Dt} = \int_\Omega \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\, dV. \tag{1.21}$$

This is the analogue of eq. (1.6).

1.4 The Jacobian matrix

We introduce the Jacobian matrix $\mathbf{J}^+$ with components $J^+_{ij} = J^+_{ij}(t, \mathbf{y})$ defined by

$$J^+_{ij} = \frac{\partial x^+_i}{\partial y_j}. \tag{1.22}$$

Thus, the total differential for the Lagrangian velocity can be expressed as follows

$$dx^+_i = \frac{\partial x^+_i}{\partial t} dt + \frac{\partial x^+_i}{\partial y_j} dy_j = v^+_i dt + J^+_{ij} dy_j, \tag{1.23}$$

The evolution equation for the Jacobian matrix is derived as follows

$$\frac{\partial J^+_{ij}}{\partial t} = \frac{\partial}{\partial y_j}\left(\frac{\partial x^+_i}{\partial t}\right) = \frac{\partial v^+_i}{\partial y_j} = \frac{\partial v_i(t, \mathbf{x}^+)}{\partial y_j} = \frac{\partial x^+_k}{\partial y_j}\left(\frac{\partial v_i}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}^+} = J^+_{kj}\left(\frac{\partial v_i}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}^+} \tag{1.24}$$

We now introduce $J^+ = J^+(t, \mathbf{y})$ as

$$J^+ = \det(\mathbf{J}^+). \tag{1.25}$$

Jacobi's formula for some matrix $\mathbf{a} = \mathbf{a}(t)$ is as follows

$$\frac{\partial \det(\mathbf{a})}{\partial t} = \det(\mathbf{a}) tr\left(\mathbf{a}^{-1} \cdot \frac{\partial \mathbf{a}}{\partial t}\right). \tag{1.26}$$

Thus, for the Jacobian matrix, we have

$$\frac{\partial \det(\mathbf{J}^+)}{\partial t} = \det(\mathbf{J}^+) tr\left[(\mathbf{J}^+)^{-1} \cdot (\mathbf{G})_{\mathbf{x}=\mathbf{x}^+} \cdot \mathbf{J}^+\right], \tag{1.27}$$

where $G_{ij} = \partial v_i / \partial x_j$. The trace on the right-hand is a product of three matrices of the following form $a^{-1}_{ij} b_{jk} a_{ki}$. This is equivalent to b_{kk}. Thus, we finally have

$$\frac{\partial J^+}{\partial t} = J^+\left(\frac{\partial v_k}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}^+}. \tag{1.28}$$

Consider now a fluid particle, so that

$$J^+_{u,ij} = \frac{\partial x^+_{u,i}}{\partial y_j}, \tag{1.29}$$

and

$$J^+_u = \det(\mathbf{J}^+_u). \tag{1.30}$$

Given eq. (1.28), its evolution equation is

$$\frac{\partial J^+_u}{\partial t} = J^+_u\left(\frac{\partial u_k}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}^+_u}. \tag{1.31}$$

1.5 Material line

A material line $l_i^+ = l_i^+(t, \mathbf{y})$ is defined as the segment that joins two fluid particles that are infinitesimally close to each other, that is

$$l_i^+(t, \mathbf{y}) = x_{u,i}^+(t, \mathbf{y} + \mathbf{f}(\mathbf{y})) - x_{u,i}^+(t, \mathbf{y}), \tag{1.32}$$

where $\mathbf{f}(\mathbf{y})$ is the initial infinitesimal displacement. The rate of change of a material line can be evaluated as follows

$$\begin{aligned}\frac{\partial l_i^+}{\partial t} &= \frac{\partial}{\partial t} x_{u,i}^+(t, \mathbf{y} + \mathbf{f}(\mathbf{y})) - \frac{\partial}{\partial t} x_{u,i}^+(t, \mathbf{y}) \\ &= u_i(t, \mathbf{x}_u^+(t, \mathbf{y} + \mathbf{f}(\mathbf{y}))) - u_i(t, \mathbf{x}_u^+(t, \mathbf{y})) \\ &= u_i(t, \mathbf{x}_u^+(t, \mathbf{y}) + \mathbf{l}^+(t, \mathbf{y})) - u_i(t, \mathbf{x}_u^+(t, \mathbf{y})),\end{aligned} \tag{1.33}$$

We now make use of Taylor's theorem to obtain

$$u_i(t, \mathbf{x}_u^+ + \mathbf{l}^+) = u_i(t, \mathbf{x}_u^+) + l_j^+ \left(\frac{\partial u_i}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}_u^+} + \frac{1}{2} l_j^+ l_k^+ R_{ijk}(\mathbf{x}^+) \tag{1.34}$$

where

$$|R_{ijk}(\mathbf{x})| \leq \max_{j,k} \max_{\mathbf{y} \in B} \left| \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)_{\mathbf{x}=\mathbf{y}} \right|, \quad \mathbf{x} \in B \tag{1.35}$$

where B is the closed ball where u_i is twice continuously spatially differentiable. Define $f(\mathbf{l}^+) = l_j^+ l_k^+ R_{ijk}$, where R_{ijk} has been evaluated at a specific $\mathbf{x}^+$. Since $|l_j^+| = |\mathbf{l}^+ \cdot \mathbf{e}_j| = ||\mathbf{l}^+||\mathbf{e}_j| \cos(\theta)| < |\mathbf{l}^+|$, one can easily show that

$$|f(\mathbf{l}^+)| = |l_j^+ l_k^+ R_{ijk}| < |l_j^+||l_k^+||R_{ijk}| < C|\mathbf{l}^+|^2 \tag{1.36}$$

Define $g(\mathbf{l}^+) = |\mathbf{l}^+|$. For any $\epsilon > 0$, there exists a neighborhood of $\mathbf{l}^+ = 0$ such that $C|\mathbf{l}^+|^2 < \epsilon|\mathbf{l}^+|$, which can be re-written as $|f(\mathbf{l}^+)| < \epsilon|g(\mathbf{l}^+)|$. Thus $f(\mathbf{l}^+) = o[g(\mathbf{l}^+)] = o[|\mathbf{l}^+|]$ as $\mathbf{l}^+ \to 0$ and we express Taylor's theorem as follows

$$u_i(t, \mathbf{x}_u^+ + \mathbf{l}^+) = u_i(t, \mathbf{x}_u^+) + l_j^+ \left(\frac{\partial u_i}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}_u^+} + o[|\mathbf{l}^+|]. \tag{1.37}$$

(Note: In the equation above I'm always adding $f(\mathbf{l}^+)$ after the second + sign, regardless of the size of $\mathbf{l}^+$. By stating that $f(\mathbf{l}^+) = o(g(|\mathbf{l}^+|))$ as $\mathbf{l}^+ \to 0$, or by writing the Taylor's Theorem as above, I only mean that $f(\mathbf{l}^+)$ is a function with a very specific behavior as $\mathbf{l}^+ \to 0$.) The equation above leads to the evolution equation for a material line, namely

$$\frac{\partial l_i^+}{\partial t} = l_j^+ \left(\frac{\partial u_i(t, \mathbf{x})}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}_u^+} + o[|\mathbf{l}^+|]. \tag{1.38}$$

We apply the same reasoning used to obtain equation (1.37) but for the variable $x_i^+(t, \mathbf{y} + \mathbf{f}(\mathbf{y}))$, which gives

$$x_{u,i}^+(t, \mathbf{y} + \mathbf{f}(\mathbf{y})) = x_{u,i}^+(t, \mathbf{y}) + f_j(\mathbf{y}) \left(\frac{\partial x_{u,i}^+(t, \mathbf{z})}{\partial z_j}\right)_{\mathbf{z}=\mathbf{y}} + o[|\mathbf{f}(\mathbf{y})|]. \tag{1.39}$$

Using (eq. (1.32)), the above becomes

$$l_i^+(t, \mathbf{y}) = f_j(\mathbf{y}) \left(\frac{\partial x_i^+(t, \mathbf{z})}{\partial z_j}\right)_{\mathbf{z}=\mathbf{y}} + o[|\mathbf{f}(\mathbf{y})|]. \tag{1.40}$$

In terms of the Jacobian matrix, the above is rewritten as

$$l_i^+(t,\mathbf{y}) = f_j(\mathbf{y})J_{ij}^+(t,\mathbf{y}) + o[|\mathbf{f}(\mathbf{y})|]. \tag{1.41}$$

Neglecting higher order terms, we have the following ODE and its proposed solution

$$\frac{\partial l_i^+}{\partial t} = l_j^+ \left(\frac{\partial u_i}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}_u^+} \qquad l_i^+ = f_j J_{ij}^+. \tag{1.42}$$

Plugging the proposed solution for l_i^+ into its ODE, and using equation (1.24) for a fluid particle, shows that this is indeed a solution, as shown below

$$\frac{\partial l_i^+}{\partial t} = \frac{\partial}{\partial t}(f_j J_{ij}^+) = f_j \frac{\partial J_{ij}^+}{\partial t} = f_j J_{kj}^+ \left(\frac{\partial u_i}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}_u^+} = l_k^+ \left(\frac{\partial u_i}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}_u^+}. \tag{1.43}$$

Chapter 2

Conservation laws

Conservation laws describe changes in mass, momentum, energy and passive scalars. The densities of these four variables are shown below

Property	Density
Mass	ρ
Momentum	$\rho\mathbf{u}$
Energy	$\rho(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u})$
Passive scalar	ρY

When each of these densities is multiplied by $\mathbf{u}\cdot\mathbf{n}$, then the resulting terms are referred to as convective fluxes.

2.1 Mass

Lagrangian C.V. Approach: Amount of mass contained in a Lagrangian C.V attached to the fluid is conserved.

$$\frac{d}{dt}\int_{\Omega_u^+(t)} \rho\, dv = 0 \tag{2.1}$$

Using the identity in eq. (1.20), we obtain:

$$\left(\int_\Omega \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\, dv\right)_{\Omega=\Omega_u^+(t)} = 0, \tag{2.2}$$

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0} \tag{2.3}$$

Eulerian C.V. Approach: Change of mass in an Eulerian C.V (stationary) is equivalent to: mass flow rate in – mass flow rate out.

$$\frac{d}{dt}\int_\Omega \rho\, dv = -\int_{\partial\Omega} \rho\mathbf{u}\cdot\mathbf{n}\, ds \tag{2.4}$$

Since this volume is stationary, we move the derivative inside the integral for the term on the LHS and use Gauss's theorem for the term on the RHS,

$$\int_\Omega \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\, dv = 0. \tag{2.5}$$

Thus,

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.} \tag{2.6}$$

Einstein Notation:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0.} \tag{2.7}$$

Alternate Form: Expanding the second term above, the conservation of mass equation can be written as

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0. \tag{2.8}$$

2.2 Momentum

Lagrangian C.V. Approach: Change of momentum contained in a Lagrangian C.V. is equal to applied forces.

$$\frac{d}{dt} \int_{\Omega_u^+(t)} \rho \mathbf{u} \, dv = \int_{\partial \Omega_u^+(t)} \mathbf{T} \, ds + \int_{\Omega_u^+(t)} \rho \mathbf{f} \, dv \tag{2.9}$$

where $\mathbf{T}$ is the stress vector and $\mathbf{f}$ the body force. Since $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$, where $\boldsymbol{\sigma}$ is the stress tensor, the above is written as

$$\frac{d}{dt} \int_{\Omega_u^+(t)} \rho \mathbf{u} \, dv = \int_{\Omega_u^+(t)} \nabla \cdot \boldsymbol{\sigma} \, dv + \int_{\Omega_u^+(t)} \rho \mathbf{f} \, dv \tag{2.10}$$

We note that the ∇ operator is being applied to the second index of the stress tensor.

Using the identity in eq. (1.20), we obtain

$$\left(\int_{\Omega} \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) \, dv \right)_{\Omega = \Omega_u^+(t)} = \int_{\Omega_u^+(t)} \nabla \cdot \boldsymbol{\sigma} \, dv + \int_{\Omega_u^+(t)} \rho \mathbf{f} \, dv \tag{2.11}$$

which gives,

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}. \tag{2.12}$$

Eulerian C.V. Approach: Change of momentum over fixed volume is equal to: (1) convected amount of momentum through the borders of the C.V. *plus* (2) the forces acting on the volume.

$$\frac{d}{dt} \int_{\Omega} \rho \mathbf{u} \, dv = - \int_{\partial \Omega} \rho \mathbf{u}\mathbf{u} \cdot \mathbf{n} \, ds + \int_{\Omega} \nabla \cdot \boldsymbol{\sigma} \, dv + \int_{\Omega} \rho \mathbf{f} \, dv. \tag{2.13}$$

Since this volume is stationary, we move the derivative inside the integral for the first term on the LHS and use Gauss's theorem for the first term on the RHS,

$$\int_{\Omega} \frac{\partial \rho \mathbf{u}}{\partial t} \, dv + \int_{\Omega} \nabla \cdot (\rho \mathbf{u}\mathbf{u}) \, dv = \int_{\Omega} \nabla \cdot \boldsymbol{\sigma} \, dv + \int_{\Omega} \rho \mathbf{f} \, dv. \tag{2.14}$$

Thus,

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}. \tag{2.15}$$

Einstein Notation:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i \tag{2.16}$$

Stress tensor: The stress tensor is expressed as follows:

$$\begin{aligned}\boldsymbol{\sigma} &= -p\mathbf{I} + \mathbf{t} \\ \sigma_{ij} &= -p\delta_{ij} + t_{ij}\end{aligned}$$

where $\mathbf{t}$ is the deviatoric stress tensor.

With this expression the conservation of momentum becomes,

$$\boxed{\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \mathbf{t} + \rho \mathbf{f}} \tag{2.17}$$

$$\boxed{\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} + \rho f_i} \tag{2.18}$$

Alternate Form: Using the product rule for the left terms of the conservation of momentum equation, where $\rho u_i u_j$ is the product of ρu_j and u_i, we get:

$$\frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_j} u_i = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} + \rho f_i$$

The first and last terms on the left side vanish due to continuity, and the second and third terms amount to ρ times the material derivative. Thus, the alternate form of the conservation of momentum is

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} + \rho f_i \tag{2.19}$$

Viscous stress tensor: The Navier-Stokes equations are the conservation of momentum equations with the shear stress tensor given by:

$$t_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_\nu\right) \delta_{ij} S_{kk} \tag{2.20}$$

where μ is the coefficient of viscosity, μ_ν is the bulk coefficient of viscosity and S_{ij} is the rate of strain tensor.

If μ and μ_ν are constant, we can reformulate the derivative of the shear stress tensor as follows.

$$\begin{aligned}\frac{\partial}{\partial x_j}\left[2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_\nu\right)\delta_{ij} S_{kk}\right] &= \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \left(\frac{2}{3}\mu - \mu_\nu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] \\ &= \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_i \partial x_j} - \frac{2}{3}\mu\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu_\nu\frac{\partial^2 u_k}{\partial x_i \partial x_k} \\ &= \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\mu\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu_\nu\frac{\partial^2 u_k}{\partial x_i \partial x_k}\end{aligned}$$

The Navier-Stokes equations then become,

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\mu\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu_\nu\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \rho f_i$$

If the bulk viscosity is zero we obtain:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\mu\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \rho f_i$$

If the flow is incompressible, we obtain:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho f_i$$

2.3 Energy

Lagrangian C.V. Approach: The total energy per unit mass E is defined as $E = e + K$, where e and K are the internal energy per unit mass and the kinetic energy per unit mass, respectively. These are defined as $e = C_v T$ and $K = \frac{1}{2} U_i U_i$, where C_v is the specific heat at constant volume and T the temperature. The total change of E is equal to work done by forces on the C.V and heat transferred to the C.V.

$$\frac{d}{dt}\int_{\Omega_u^+(t)} \rho E \, dv = \int_{\partial\Omega_u^+(t)} \mathbf{T}\cdot\mathbf{u}\, ds + \int_{\Omega_u^+(t)} \rho\mathbf{f}\cdot\mathbf{u}\, dv - \int_{\partial\Omega_u^+(t)} \mathbf{q}\cdot\mathbf{n}\, ds. \tag{2.21}$$

The first and second terms on the RHS are work done by the surface and body forces, respectively, and the third term is the heat transfer away from the body. Since $\mathbf{T}\cdot\mathbf{u} = (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{u} = (\mathbf{u}\cdot\boldsymbol{\sigma})\cdot\mathbf{n}$, application of Gauss's law gives

$$\frac{d}{dt}\int_{\Omega_u^+(t)} \rho E \, dv = \int_{\Omega_u^+(t)} \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma})\, dv + \int_{\Omega_u^+(t)} \rho\mathbf{f}\cdot\mathbf{u}\, dv - \int_{\Omega_u^+(t)} \nabla\cdot\mathbf{q}\, dv. \tag{2.22}$$

Using the identity in eq. (1.20) we obtain,

$$\left(\int_\Omega \frac{\partial\rho E}{\partial t} + \nabla\cdot(\rho E\mathbf{u})\, dv\right)_{\Omega=\Omega_u^+(t)} = \int_{\Omega_u^+(t)} \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma})\, dv + \int_{\Omega_u^+(t)} \rho\mathbf{f}\cdot\mathbf{u}\, dv - \int_{\Omega_u^+(t)} \nabla\cdot\mathbf{q}\, dv, \tag{2.23}$$

which gives,

$$\frac{\partial\rho E}{\partial t} + \nabla\cdot(\rho E\mathbf{u}) = \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma}) + \rho\mathbf{f}\cdot\mathbf{u} - \nabla\cdot\mathbf{q}. \tag{2.24}$$

Eulerian C.V. Approach: Change of energy over fixed volume is equal to: (1) convected energy through the borders of the C.V., *plus* (2) energy transfer due to work done by forces acting on the volume *plus*, (3) heat transfer away from the C.V.

$$\frac{d}{dt}\int_\Omega \rho E\, dv = -\int_{\partial\Omega} \rho E\mathbf{u}\cdot\mathbf{n}\, ds + \int_\Omega \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma})\, dv + \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\, dv - \int_\Omega \nabla\cdot\mathbf{q}\, dv. \tag{2.25}$$

Since this volume is stationary, we move the derivative inside the integral for the first term on the LHS and use Gauss's theorem for the first term on the RHS,

$$\int_\Omega \frac{\partial\rho E}{\partial t}\, dv + \int_\Omega \nabla\cdot(\rho E\mathbf{u})\, dv = \int_\Omega \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma})\, dv + \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\, dv - \int_\Omega \nabla\cdot\mathbf{q}\, dv. \tag{2.26}$$

Thus

$$\frac{\partial\rho E}{\partial t} + \nabla\cdot(\rho E\mathbf{u}) = \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma}) + \rho\mathbf{f}\cdot\mathbf{u} - \nabla\cdot\mathbf{q}. \tag{2.27}$$

Einstein Notation:

$$\frac{\partial\rho E}{\partial t} + \frac{\partial\rho E u_j}{\partial x_j} = \frac{\partial u_i\sigma_{ij}}{\partial x_j} + \rho f_j u_j - \frac{\partial q_j}{\partial x_j} \tag{2.28}$$

Stress tensor: As for the momentum case, we separate the stress tensor into its isotropic and deviatoric components, and thus express the conservation of energy equation as

$$\boxed{\frac{\partial\rho E}{\partial t} + \nabla\cdot\left[\rho\left(E + \frac{p}{\rho}\right)\mathbf{u}\right] = \nabla\cdot(\mathbf{u}\cdot\mathbf{t}) + \rho\mathbf{f}\cdot\mathbf{u} - \nabla\cdot\mathbf{q}} \tag{2.29}$$

$$\boxed{\frac{\partial\rho E}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho\left(E + \frac{p}{\rho}\right)u_j\right] = \frac{\partial u_i t_{ij}}{\partial x_j} + \rho f_j u_j - \frac{\partial q_j}{\partial x_j}} \tag{2.30}$$

Alternate Forms: The definition of the enthalpy is

$$h = e + \frac{p}{\rho} \tag{2.31}$$

and the definition of the total enthalpy is

$$H = h + K. \tag{2.32}$$

Thus, since $E = e + K$, the energy equation can be re-expressed as

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}(\rho H u_j) = \frac{\partial u_i t_{ij}}{\partial x_j} + \rho f_j u_j - \frac{\partial q_j}{\partial x_j} \tag{2.33}$$

Moving the pressure back to the right hand side, and using the product rule on the terms on the left, where $\rho E u_j$ is the product of ρu_j and E, we get

$$\frac{\partial \rho}{\partial t}E + \rho\frac{\partial E}{\partial t} + \rho u_j \frac{\partial E}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_j}E = -\frac{\partial u_j p}{\partial x_j} + \frac{\partial u_i t_{ij}}{\partial x_j} + \rho f_j u_j - \frac{\partial q_j}{\partial x_j}. \tag{2.34}$$

The first and fourth terms on the left side vanish due to continuity, and the second and third terms amount to ρ times the material derivative of E. Thus, the alternate form of the conservation of energy is:

$$\rho\frac{DE}{Dt} = -\frac{\partial u_j p}{\partial x_j} + \frac{\partial u_i t_{ij}}{\partial x_j} + \rho f_j u_j - \frac{\partial q_j}{\partial x_j}. \tag{2.35}$$

Heat conduction: According to Fourier's law, the heat transfer vector is expressed as follows

$$q_i = -\kappa\frac{\partial T}{\partial x_i} \tag{2.36}$$

where κ is the thermal conductivity and T the temperature. The thermal conductivity is obtained using

$$\kappa = \frac{\mu C_p}{Pr} \tag{2.37}$$

where Pr is the Prandtl number. We note here that the thermal diffusivity d is expressed in terms of the thermal conductivity through the relation $d = \kappa/\rho C_p$.

2.4 Passive Scalar

Lagrangian C.V. Approach: Define Y as some passive scalar per unit mass. The total amount of this passive scalar within a Lagrangian C.V. can change due to diffusion through the borders of the C.V., or due to sources and sinks within. Thus

$$\frac{d}{dt}\int_{\Omega_u^+(t)} \rho Y \, dv = -\int_{\partial\Omega_u^+(t)} \mathbf{J}\cdot\mathbf{n}\, ds + \int_{\Omega_u^+(t)} w \, dv. \tag{2.38}$$

$\mathbf{J}$ is the diffusive mass flux and w a volumetric source/sink. Using Gauss's theorem, the above is rewritten as

$$\frac{d}{dt}\int_{\Omega_u^+(t)} \rho Y \, dv = -\int_{\Omega_u^+(t)} \nabla\cdot\mathbf{J}\, dv + \int_{\Omega_u^+(t)} w \, dv. \tag{2.39}$$

Using the identity in eq. (1.20), we obtain

$$\left(\int_{\Omega} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\, dv\right)_{\Omega=\Omega_u^+(t)} = -\int_{\Omega_u^+(t)} \nabla\cdot\mathbf{J}\, dv + \int_{\Omega_u^+(t)} w \, dv \tag{2.40}$$

which gives

$$\boxed{\frac{\partial \rho Y}{\partial t} + \nabla\cdot(\rho Y\mathbf{u}) = -\nabla\cdot\mathbf{J} + w.} \tag{2.41}$$

Eulerian C.V. Approach: The change in the amount of a passive scalar in an Eulerian C.V. is equal to: (1) the convected amount of the passive scalar through the borders of the C.V., *plus* (2) the diffusion of the scalar through the borders of the C.V., *plus* (3) the creation or destruction of the scalar within the C.V.

$$\frac{d}{dt}\int_{\Omega} \rho Y \, dv = -\int_{\partial\Omega} \rho Y \mathbf{u}\cdot\mathbf{n}\, ds - \int_{\partial\Omega} \mathbf{J}\cdot\mathbf{n}\, ds + \int_{\Omega} w\, dv. \tag{2.42}$$

Since this volume is stationary, we move the derivative inside the integral for the first term on the LHS, and use Gauss's theorem for the second term in the LHS

$$\int_{v} \frac{\partial \rho Y}{\partial t}\, dv + \int_{s} \rho Y \mathbf{u}\cdot\mathbf{n}\, ds = -\int_{v} \nabla\cdot\mathbf{J}\, dv + \int_{v} w\, dv. \tag{2.43}$$

Thus,

$$\boxed{\frac{\partial \rho Y}{\partial t} + \nabla\cdot(\rho Y \mathbf{u}) = -\nabla\cdot\mathbf{J} + w.} \tag{2.44}$$

Einstein Notation:

$$\boxed{\frac{\partial \rho Y}{\partial t} + \frac{\partial \rho Y u_i}{\partial x_i} = -\frac{\partial J_i}{\partial x_i} + w.} \tag{2.45}$$

Alternate Form: Expanding the second term above, the conservation of scalar equation can be written as

$$\rho\frac{DY}{Dt} = -\frac{\partial J_i}{\partial x_i} + W. \tag{2.46}$$

Diffusive mass flux: According to Fick's law of diffusion, the diffusive mass flux is given by

$$J_i = -\rho D\frac{\partial Y}{\partial x_i}, \tag{2.47}$$

where D is the diffusivity. It is obtained from

$$D = \frac{\mu}{\rho Sc}, \tag{2.48}$$

where Sc is the Schmidt number.

2.5 Additional relations

So far we have five conservation equations, one for ρ, three for the three components of $\mathbf{u}$ and one for E. However, we have additional unknowns, such as p, T, μ and κ. Thus, additional equations are required to close the system. Some of these are the thermal equation of state

$$p = \rho R T, \tag{2.49}$$

and the caloric equation of state

$$e = C_v T. \tag{2.50}$$

In the above R is the gas constant, which is defined as $R = R_u/M$, where R_u is the universal gas constant and M the molar mass. C_v is the specific heat at constant volume. If we introduce the specific heat at constant pressure C_p, we have

$$R = C_p - C_v. \tag{2.51}$$

Table 2.1: List of variables used in this section.

Symbol	Description	Units	Value
p	pressure	$Pa = N/m^2 = J/m^3$	n/a
v	volume	m^3	n/a
T	temperature	K	n/a
e	specific internal energy	J/kg	n/a
U	energy density	J/m^3	n/a
ρ	mass density	kg/m^3	n/a
n	particle density	$1/m^3$	n/a
m	mass	kg	n/a
M	molar mass	kg/mol	n/a
o	# of moles	mol	n/a
N	# of particles	n/a	n/a
k_B	Boltzman constant	J/K	$1.38064852 \times 10^{-23}$
N_a	Avogadro constant	mol^{-1}	$6.022140857 \times 10^{23}$
R_u	universal gas constant	$J/(mol \cdot K)$	8.3144598
R	specific gas constant	$J/(kg \cdot K)$	n/a
C_p	specific heat at const p	$J/(kg \cdot K)$	n/a
C_v	specific heat at const V	$J/(kg \cdot K)$	n/a

We also introduce γ, the ratio of specific heats

$$\gamma = \frac{C_p}{C_v}. \tag{2.52}$$

It's value is equal to 5/3 for monoatomic gases and 7/5 for diatomic gasses. Combining eqs. (2.51) and (2.52) leads to the following expressions for the specific heats

$$C_v = \frac{R}{\gamma - 1}, \tag{2.53}$$

$$C_p = \frac{\gamma R}{\gamma - 1}. \tag{2.54}$$

Using eqs. (2.50) and (2.53), we can express the caloric equation of state in terms of the energy density $U = \rho e$ as follows

$$U = \frac{1}{\gamma - 1}\rho RT. \tag{2.55}$$

We now list a few useful relationships: $\rho = m/v$, $m = oM$, $R_u = k_B N_a$, $N = N_a o$, and $n = N/v$, where m is the mass of the system, v the volume of the system, o the number of moles, k_B the Boltzman constant, N_a Avogadro's number, n the particle density, and N the number of particles in the system. Using this, one shows that

$$\rho R = \frac{m}{v}\frac{R_u}{M} = \frac{o}{v}N_a k_B = \frac{N}{v}k_B = nk_B. \tag{2.56}$$

Thus, we can make the following equivalences

$$p = \rho RT \quad \longleftrightarrow \quad p = nk_B T, \tag{2.57}$$

$$U = \frac{1}{\gamma - 1}\rho RT \quad \longleftrightarrow \quad U = \frac{1}{\gamma - 1}nk_B T. \tag{2.58}$$

Combining the relationships above we get $U = \frac{1}{\gamma-1}P$, which is the same as $P = \rho(\gamma - 1)e$.

2.6 Summary of Navier-Stokes equations for ideal gas

We summarize the governing equations below.

2.6.1 Conservation equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \tag{2.59}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} + \rho f_i \tag{2.60}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_i}\left[\rho\left(E + \frac{p}{\rho}\right)u_i\right] = \frac{\partial u_i t_{ij}}{\partial x_j} + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} \tag{2.61}$$

$$\frac{\partial \rho Y}{\partial t} + \frac{\partial \rho Y u_i}{\partial x_i} = -\frac{\partial J_i}{\partial x_i} + w \tag{2.62}$$

2.6.2 Transport models

Shear stress, heat flux, diffusive scalar flux:

$$t_{ij} = 2\mu S^*_{ij} \tag{2.63}$$

$$q_i = -\kappa \frac{\partial T}{\partial x_i} \tag{2.64}$$

$$J_i = -\rho D \frac{\partial Y}{\partial x_i} \tag{2.65}$$

Transport coefficients:

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^n \tag{2.66}$$

$$\kappa = \frac{\mu C_p}{Pr} \tag{2.67}$$

$$D = \frac{\mu}{\rho Sc} \tag{2.68}$$

2.6.3 Equation of state

$$p = \rho R T \tag{2.69}$$

$$R = \frac{R_u}{M} \tag{2.70}$$

$$e = C_v T \tag{2.71}$$

2.6.4 Additional relations

$$E = e + K \tag{2.72}$$

$$K = \frac{1}{2} u_i u_i \tag{2.73}$$

$$S^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{2.74}$$

Chapter 3

Alternate forms of the fluid equations

3.1 Vorticity Equation

The vorticity vector $\mathbf{w}$ is defined as

$$\mathbf{w} = \nabla \times \mathbf{u}. \tag{3.1}$$

Using tensor notation, the cross product above can be reformulated as follows

$$\begin{aligned} w_i &= \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \\ &= \epsilon_{ijk}(S_{kj} + W_{kj}) \\ &= \frac{1}{2}(\epsilon_{ijk} S_{kj} - \epsilon_{ikj} S_{jk}) + \epsilon_{ijk} W_{kj} \\ &= \epsilon_{ijk} W_{kj}. \end{aligned} \tag{3.2}$$

A further identity is derived as follows

$$\begin{aligned} \frac{1}{2}\epsilon_{jit} w_t &= \frac{1}{2}\epsilon_{jit}\epsilon_{tpq}\frac{\partial u_q}{\partial x_p} \\ &= \frac{1}{2}(\delta_{jp}\delta_{iq} - \delta_{jq}\delta_{ip})\frac{\partial u_q}{\partial x_p} \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \\ &= W_{ij}, \end{aligned} \tag{3.3}$$

where standard relations for the levi-cevita tensors were used to expand their product.

The transport equation for vorticity is obtained by taking the curl of the momentum equation (2.19). This leads to

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = -\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \mathbf{t}\right). \tag{3.4}$$

For the second term on the left hand side, we use a vector identity to write

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{3.5}$$

Thus,

$$\nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = -\nabla \times (\mathbf{u} \times \mathbf{w}). \tag{3.6}$$

Using a second vector identity, we have

$$\begin{aligned}\nabla \times (\mathbf{u} \times \mathbf{w}) &= \mathbf{u}(\nabla \cdot \mathbf{w}) - \mathbf{w}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u} \\ &= -\mathbf{w}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u}. \end{aligned} \tag{3.7}$$

Thus, the vorticity equation can be expressed as

$$\frac{D\mathbf{w}}{Dt} + \mathbf{w}(\nabla \cdot \mathbf{u}) - (\mathbf{w} \cdot \nabla)\mathbf{u} = -\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \mathbf{t}\right). \tag{3.8}$$

For incompressible flows, the vorticity equation becomes

$$\frac{D\mathbf{w}}{Dt} - (\mathbf{w} \cdot \nabla)\mathbf{u} = \nu\Delta\mathbf{w}. \tag{3.9}$$

For inviscid flows, the vorticity equation becomes

$$\frac{D\mathbf{w}}{Dt} + \mathbf{w}(\nabla \cdot \mathbf{u}) - (\mathbf{w} \cdot \nabla)\mathbf{u} = -\nabla \times \left(\frac{1}{\rho}\nabla p\right). \tag{3.10}$$

Explain stream tubes, vortex tubes, vortex lines and line vortex (filament). NOTE: A vortex line is not a line vortex!

3.2 Kinetic energy equation

The kinetic energy is defined as $K = \frac{1}{2}u_i u_i$. To derive its transport equation, multiply the conservation of momentum eq. (2.18) by u_i. For example, for the left-hand side, one would proceed as follows

$$\begin{aligned} u_i\frac{\partial \rho u_i}{\partial t} + u_i\frac{\partial \rho u_i u_j}{\partial x_j} =& \frac{1}{2}\left(u_i\frac{\partial \rho u_i}{\partial t} + u_i\frac{\partial \rho u_i u_j}{\partial x_j}\right) + \frac{1}{2}\left(u_i\frac{\partial \rho u_i}{\partial t} + u_i\frac{\partial \rho u_i u_j}{\partial x_j}\right) \\ =& \frac{1}{2}\left(u_i\frac{\partial \rho u_i}{\partial t} + u_i\frac{\partial \rho u_i u_j}{\partial x_j}\right) \\ &+ \frac{1}{2}\left(\rho u_i\frac{\partial u_i}{\partial t} + \rho u_i u_j\frac{\partial u_i}{\partial x_j} + u_i u_i\frac{\partial \rho}{\partial t} + u_i u_i\frac{\partial \rho u_j}{\partial x_j}\right) \\ =& \frac{\partial \rho K}{\partial t} + \frac{\partial \rho K u_j}{\partial x_j} + K\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right). \end{aligned} \tag{3.11}$$

Thus the transport equation for the TKE is as follows

$$\frac{\partial \rho K}{\partial t} + \frac{\partial \rho K u_j}{\partial x_j} = -\frac{\partial u_j p}{\partial x_j} + \frac{\partial u_i t_{ij}}{\partial x_j} + p\frac{\partial u_i}{\partial x_i} - t_{ij}\frac{\partial u_i}{\partial x_j} + \rho f_j u_j. \tag{3.12}$$

3.3 Internal energy equation

The internal enegy is defined as $e = C_v T$. To derive its transport equation, we subtract the kinetic energy eq. (3.12) from the total energy eq. (2.30) to obtain

$$\frac{\partial \rho e}{\partial t} + \frac{\partial \rho e u_j}{\partial x_j} = -\frac{\partial q_j}{\partial x_j} - p\frac{\partial u_i}{\partial x_i} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.13}$$

The last two terms are referred to as the dissipative terms, since they represent a transfer of energy away from K towards e.

Alternate forms of the internal energy equation can be derived, these are shown in the subsections below.

3.3.1 Pressure equation

We begin with the equation for internal energy, eq. (3.13), but express the internal energy in terms of the pressure using

$$\rho e = \rho C_v T = \frac{C_v}{R} p = \frac{1}{\gamma - 1} p. \tag{3.14}$$

Thus, the internal energy eq. (3.13) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{\gamma-1}p\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{\gamma-1}pu_j\right) = -\frac{\partial q_j}{\partial x_j} - p\frac{\partial u_i}{\partial x_i} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.15}$$

which we re-write as

$$\frac{1}{\gamma-1}\left(\frac{\partial p}{\partial t} + u_j\frac{\partial p}{\partial x_j} + \gamma p\frac{\partial u_i}{\partial x_i}\right) = -\frac{\partial q_j}{\partial x_j} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.16}$$

We will now re-write the left-hand side of the pressure equation above into a more convenient form. Using eq. (2.8), we first show that

$$\frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t} - \frac{1}{\rho}\frac{\partial \rho}{\partial x_i}u_i = -\frac{\partial \ln \rho}{\partial t} - \frac{\partial \ln \rho}{\partial x_i}u_i, \tag{3.17}$$

and thus

$$\gamma\frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial t} - \frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial x_i}u_i. \tag{3.18}$$

Using the above, one can then write

$$\begin{aligned}\frac{\partial p}{\partial t} + u_j\frac{\partial p}{\partial x_j} + \gamma p\frac{\partial u_i}{\partial x_i} &= \frac{\partial p}{\partial t} - p\frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial t} + u_j\frac{\partial p}{\partial x_j} - p\frac{1}{\rho^\gamma}\frac{\partial \rho^\gamma}{\partial x_j}u_j \\ &= \rho^\gamma\left[\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u_j\frac{\partial}{\partial x_j}\left(\frac{p}{\rho^\gamma}\right)\right].\end{aligned} \tag{3.19}$$

Thus, the equation for pressure is re-written as

$$\frac{1}{\gamma-1}\rho^\gamma\left[\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u_j\frac{\partial}{\partial x_j}\left(\frac{p}{\rho^\gamma}\right)\right] = -\frac{\partial q_j}{\partial x_j} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.20}$$

3.3.2 Enthalpy equation

Add to both sides of the internal energy eq. (3.13) the following terms

$$\frac{\partial p}{\partial t} + \frac{\partial p u_j}{\partial x_j}. \tag{3.21}$$

This allows us to write the internal energy equation as

$$\frac{\partial \rho h}{\partial t} + \frac{\partial \rho h u_j}{\partial x_j} = \frac{\partial p}{\partial t} + u_i\frac{\partial p}{\partial x_i} - \frac{\partial q_j}{\partial x_j} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.22}$$

3.3.3 Entropy equation

We first write the transport equation for internal energy eq. (3.13) in non-conservative form

$$\rho \frac{De}{Dt} = -\frac{\partial q_j}{\partial x_j} - p\frac{\partial u_i}{\partial x_i} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.23}$$

To proceed we first note that

$$\frac{D\rho\vartheta}{Dt} = 0 \tag{3.24}$$

where $\vartheta = 1/\rho$, and thus

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\vartheta}\frac{D\vartheta}{Dt}. \tag{3.25}$$

Using this in the non-conservative form of the continuity eq. (2.8) gives

$$\frac{\partial u_i}{\partial x_i} = \frac{1}{\vartheta}\frac{D\vartheta}{Dt}. \tag{3.26}$$

Thus, eq. (3.23) becomes

$$\rho\left(\frac{De}{Dt} + p\frac{D\vartheta}{Dt}\right) = -\frac{\partial q_j}{\partial x_j} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.27}$$

We now make use of the Gibbs equation

$$Tds = de + pd\vartheta \tag{3.28}$$

to obtain

$$\rho T\frac{Ds}{Dt} = -\frac{\partial q_j}{\partial x_j} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.29}$$

3.4 Kelvin's Theorem

3.5 Bernoulli's equation

We'll assume the flow is steady, inviscid, and the volume force is conservative. Thus, eq. (2.19) in vector notation becomes

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla G. \tag{3.30}$$

Using the vector identity $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u})$ we have

$$\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nabla G. \tag{3.31}$$

It can be shown that

$$\frac{1}{\rho}\nabla p = \nabla\int\frac{1}{\rho}dp. \tag{3.32}$$

See, for example, Fundamental Mechanics of Fluids by I.G. Currie. Using this and dotting by $\mathbf{u}$, one obtains

$$\mathbf{u}\cdot\nabla\left(\int\frac{1}{\rho}dp + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - G\right) = 0. \tag{3.33}$$

The above is equivalent to

$$\int\frac{1}{\rho}dp + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - G = \text{constant along a streamline.} \tag{3.34}$$

This equation is referred to as the Bernoulli equation.

A similar result can be obtain if one focuses on the energy equation instead of the momentum equation. If the flow is steady, inviscid, and the volume force is conservative, the energy eq. (2.35) in vector notation becomes

$$\mathbf{u} \cdot \nabla E = -\frac{1}{\rho} \nabla \cdot (\mathbf{u} p) + \nabla G \cdot \mathbf{u}. \tag{3.35}$$

For the first term on the right-hand side, we can write

$$\frac{1}{\rho} \nabla \cdot (\mathbf{u} p) = \frac{1}{\rho} \nabla \cdot \left(\rho \mathbf{u} \frac{p}{\rho} \right) = \mathbf{u} \cdot \nabla \left(\frac{p}{\rho} \right), \tag{3.36}$$

where we have used conservation of mass for the last equality. Thus, we now have

$$\mathbf{u} \cdot \nabla \left(E + \frac{p}{\rho} - G \right) = 0, \tag{3.37}$$

or in terms of enthalpy

$$\mathbf{u} \cdot \nabla \left(h + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - G \right) = 0, \tag{3.38}$$

The above is equivalent to

$$h + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - G = \text{constant along a streamline.} \tag{3.39}$$

3.6 Arbitrary Lagrangian Eulerian equations

3.6.1 Differential form

Consider an arbitrary particle—not a fluid particle—with position $\mathbf{x}^+$ and velocity field $\mathbf{v}$. Using eq. (1.4) for the density function $\rho^+ = \rho(t, \mathbf{x}^+)$ we have

$$\frac{\partial \rho^+}{\partial t} = \left(\frac{\partial \rho}{\partial t} \right)_{\mathbf{x}=\mathbf{x}^+} + (\mathbf{v} \cdot \nabla \rho)_{\mathbf{x}=\mathbf{x}^+}. \tag{3.40}$$

We evaluate eq. (2.6) at the particle's location $\mathbf{x}^+$, which we express as

$$\left(\frac{\partial \rho}{\partial t} \right)_{\mathbf{x}=\mathbf{x}^+} + (\mathbf{u} \cdot \nabla \rho)_{\mathbf{x}=\mathbf{x}^+} + (\rho \nabla \cdot \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} = 0.$$

Using eq. (3.40), this can be written as

$$\frac{\partial \rho^+}{\partial t} + [(\mathbf{u} - \mathbf{v}) \cdot \nabla \rho]_{\mathbf{x}=\mathbf{x}^+} + (\rho \nabla \cdot \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} = 0.$$

Manipulating, we have

$$\frac{\partial \rho^+}{\partial t} + \{\nabla \cdot [\rho(\mathbf{u} - \mathbf{v})]\}_{\mathbf{x}=\mathbf{x}^+} + (\rho \nabla \cdot \mathbf{v})_{\mathbf{x}=\mathbf{x}^+} = 0.$$

Using eq. (1.28), the above becomes

$$\frac{\partial \rho^+}{\partial t} + \{\nabla \cdot [\rho(\mathbf{u} - \mathbf{v})]\}_{\mathbf{x}=\mathbf{x}^+} + \rho^+ \frac{1}{J^+} \frac{\partial J^+}{\partial t} = 0.$$

Multiplying by J^+ we finally obtain

$$\frac{\partial J^+\rho^+}{\partial t} + J^+\left\{\nabla\cdot[\rho(\mathbf{u}-\mathbf{v})]\right\}_{\mathbf{x}=\mathbf{x}^+} = 0. \tag{3.41}$$

We follow the same procedure detailed above but for the momentum equation. Using eq. (1.4), we have

$$\frac{\partial \rho^+\mathbf{u}^+}{\partial t} = \left(\frac{\partial \rho\mathbf{u}}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+} + (\mathbf{v}\cdot\nabla\rho\mathbf{u})_{\mathbf{x}=\mathbf{x}+}. \tag{3.42}$$

We evaluate eq. (2.17) at the particle's location, which we express as

$$\left(\frac{\partial \rho\mathbf{u}}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+} + [\mathbf{u}\cdot\nabla(\rho\mathbf{u})]_{\mathbf{x}=\mathbf{x}^+} + (\rho\mathbf{u}\nabla\cdot\mathbf{u})_{\mathbf{x}=\mathbf{x}^+} = (-\nabla p + \nabla\cdot\mathbf{t} + \rho\mathbf{f})_{\mathbf{x}=\mathbf{x}^+}.$$

Using eq. (3.42), this can be written as

$$\frac{\partial \rho^+\mathbf{u}^+}{\partial t} + [(\mathbf{u}-\mathbf{v})\cdot\nabla(\rho\mathbf{u})]_{\mathbf{x}=\mathbf{x}^+} + (\rho\mathbf{u}\nabla\cdot\mathbf{u})_{\mathbf{x}=\mathbf{x}^+} = (-\nabla p + \nabla\cdot\mathbf{t} + \rho\mathbf{f})_{\mathbf{x}=\mathbf{x}^+}.$$

Manipulating, we have

$$\frac{\partial \rho^+\mathbf{u}^+}{\partial t} + \left\{\nabla\cdot[\rho\mathbf{u}(\mathbf{u}-\mathbf{v})]\right\}_{\mathbf{x}=\mathbf{x}^+} + (\rho\mathbf{u}\nabla\cdot\mathbf{v})_{\mathbf{x}=\mathbf{x}^+} = (-\nabla p + \nabla\cdot\mathbf{t} + \rho\mathbf{f})_{\mathbf{x}=\mathbf{x}^+}.$$

Using eq. (1.28), the above becomes

$$\frac{\partial \rho^+\mathbf{u}^+}{\partial t} + \left\{\nabla\cdot[\rho\mathbf{u}(\mathbf{u}-\mathbf{v})]\right\}_{\mathbf{x}=\mathbf{x}^+} + \rho^+\mathbf{u}^+\frac{1}{J^+}\frac{\partial J^+}{\partial t} = (-\nabla p + \nabla\cdot\mathbf{t} + \rho\mathbf{f})_{\mathbf{x}=\mathbf{x}^+}.$$

Multiplying by J^+ we finally obtain

$$\frac{\partial J^+\rho^+\mathbf{u}^+}{\partial t} + J^+\left\{\nabla\cdot[\rho\mathbf{u}(\mathbf{u}-\mathbf{v})]\right\}_{\mathbf{x}=\mathbf{x}^+} = J^+(-\nabla p + \nabla\cdot\mathbf{t} + \rho\mathbf{f})_{\mathbf{x}=\mathbf{x}^+}. \tag{3.43}$$

We now follow the previous procedure but for the energy equation. Using eq. (1.4), we have

$$\frac{\partial \rho^+E^+}{\partial t} = \left(\frac{\partial \rho E}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+} + (\mathbf{v}\cdot\nabla\rho E)_{\mathbf{x}=\mathbf{x}+}. \tag{3.44}$$

We evaluate eq. (2.29) at the particle's location, which we express as

$$\left(\frac{\partial \rho E}{\partial t}\right)_{\mathbf{x}=\mathbf{x}^+} + [\mathbf{u}\cdot\nabla(\rho E)]_{\mathbf{x}=\mathbf{x}^+} + (\rho E\nabla\cdot\mathbf{u})_{\mathbf{x}=\mathbf{x}^+} = \\ [-\nabla\cdot(p\mathbf{u}) + \nabla\cdot(\mathbf{u}\cdot\mathbf{t}) + \rho\mathbf{f}\cdot\mathbf{u} - \nabla\cdot\mathbf{q}]_{\mathbf{x}=\mathbf{x}^+}.$$

Using eq. (3.44), this can be written as

$$\frac{\partial \rho^+E^+}{\partial t} + [(\mathbf{u}-\mathbf{v})\cdot\nabla(\rho E)]_{\mathbf{x}=\mathbf{x}^+} + (\rho E\nabla\cdot\mathbf{u})_{\mathbf{x}=\mathbf{x}^+} = \\ [-\nabla\cdot(p\mathbf{u}) + \nabla\cdot(\mathbf{u}\cdot\mathbf{t}) + \rho\mathbf{f}\cdot\mathbf{u} - \nabla\cdot\mathbf{q}]_{\mathbf{x}=\mathbf{x}^+}.$$

Manipulating, we have

$$\frac{\partial \rho^+E^+}{\partial t} + \left\{\nabla\cdot[(\rho E)(\mathbf{u}-\mathbf{v})]\right\}_{\mathbf{x}=\mathbf{x}^+} + (\rho E\nabla\cdot\mathbf{v})_{\mathbf{x}=\mathbf{x}^+} = \\ [-\nabla\cdot(p\mathbf{u}) + \nabla\cdot(\mathbf{u}\cdot\mathbf{t}) + \rho\mathbf{f}\cdot\mathbf{u} - \nabla\cdot\mathbf{q}]_{\mathbf{x}=\mathbf{x}^+}.$$

Using eq. (1.28), the above becomes

$$\frac{\partial \rho^+ E^+}{\partial t} + \{\nabla \cdot [(\rho E)(\mathbf{u} - \mathbf{v})]\}_{\mathbf{x}=\mathbf{x}^+} + \rho^+ E^+ \frac{1}{J^+} \frac{\partial J^+}{\partial t} = \\ [-\nabla \cdot (p\mathbf{u}) + \nabla \cdot (\mathbf{u} \cdot \mathbf{t}) + \rho \mathbf{f} \cdot \mathbf{u} - \nabla \cdot \mathbf{q}]_{\mathbf{x}=\mathbf{x}^+}.$$

Multiplying by J^+ we finally obtain

$$\frac{\partial J^+ \rho^+ E^+}{\partial t} + J^+ \{\nabla \cdot [(\rho E)(\mathbf{u} - \mathbf{v})]\}_{\mathbf{x}=\mathbf{x}^+} = \\ J^+ [-\nabla \cdot (p\mathbf{u}) + \nabla \cdot (\mathbf{u} \cdot \mathbf{t}) + \rho \mathbf{f} \cdot \mathbf{u} - \nabla \cdot \mathbf{q}]_{\mathbf{x}=\mathbf{x}^+}. \quad (3.45)$$

3.6.2 Integral form

Consider an arbitrary volume—not a material volume—whose volume is given by $\Omega^+(t)$, and its boundary motion is given by $\mathbf{v}$. Using eq. (1.19) for the density we have

$$\frac{d}{dt} \int_{\Omega^+(t)} \rho \, dv = \int_{\Omega^+(t)} \frac{\partial \rho}{\partial t} \, dv + \int_{\Omega^+(t)} \nabla \cdot (\rho \mathbf{v}) \, dv. \quad (3.46)$$

We integrate eq. (2.6) over the volume $\Omega^+(t)$, which we express as

$$\int_{\Omega^+(t)} \frac{\partial \rho}{\partial t} \, dv + \int_{\Omega^+(t)} \nabla \cdot (\rho \mathbf{u}) \, dv = 0.$$

Using eq. (3.46) and the divergence theorem, this can be written as

$$\frac{d}{dt} \int_{\Omega^+(t)} \rho \, dv + \int_{\partial\Omega^+(t)} \rho(\mathbf{u} - \mathbf{v}) \cdot \mathbf{n} \, ds. \quad (3.47)$$

We follow the same procedure as detailed above but for the momentum equations. Using eq. (1.19) for momentum we have

$$\frac{d}{dt} \int_{\Omega^+(t)} \rho \mathbf{u} \, dv = \int_{\Omega^+(t)} \frac{\partial \rho \mathbf{u}}{\partial t} \, dv + \int_{\Omega^+(t)} \nabla \cdot (\rho \mathbf{u}\mathbf{v}) \, dv. \quad (3.48)$$

We integrate eq. (2.17) over the volume $\Omega^+(t)$, which we express as

$$\int_{\Omega^+(t)} \frac{\partial \rho \mathbf{u}}{\partial t} \, dv + \int_{\Omega^+(t)} \nabla \cdot (\rho \mathbf{u}\mathbf{u}) \, dv = \int_{\Omega^+(t)} -\nabla p + \nabla \cdot \mathbf{t} + \rho \mathbf{f} \, dv.$$

Using eq. (3.48) and the divergence theorem, this can be written as

$$\frac{d}{dt} \int_{\Omega^+(t)} \rho \mathbf{u} \, dv + \int_{\partial\Omega^+(t)} \rho \mathbf{u}(\mathbf{u} - \mathbf{v}) \cdot \mathbf{n} \, ds = \int_{\partial\Omega^+(t)} (-p\mathbf{I} + \mathbf{t}) \cdot \mathbf{n} \, ds + \int_{\Omega^+(t)} \rho \mathbf{f} \, dv. \quad (3.49)$$

We repeat the same procedure as above but for the energy equation. Using eq. (1.19) for energy we have

$$\frac{d}{dt} \int_{\Omega^+(t)} \rho E \, dv = \int_{\Omega^+(t)} \frac{\partial \rho E}{\partial t} \, dv + \int_{\Omega^+(t)} \nabla \cdot (\rho E \mathbf{v}) \, dv. \quad (3.50)$$

We integrate eq. (2.29) over the volume $\Omega^+(t)$, which we express as

$$\int_{\Omega^+(t)} \frac{\partial \rho E}{\partial t} \, dv + \int_{\Omega^+(t)} \nabla \cdot (\rho \mathbf{u} E) \, dv = \int_{\Omega^+(t)} -\nabla \cdot (p\mathbf{u}) + \nabla \cdot (\mathbf{u} \cdot \mathbf{t}) + \rho \mathbf{f} \cdot \mathbf{u} - \nabla \cdot \mathbf{q} \, dv.$$

Using eq. (3.50) and the divergence theorem, this can be written as

$$\frac{d}{dt}\int_{\Omega^+(t)} \rho E \, dv + \int_{\partial\Omega^+(t)} \rho E(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}\, ds = \int_{\partial\Omega^+(t)} -(p\mathbf{u}+\mathbf{u}\cdot\mathbf{t})\cdot\mathbf{n}\, ds + \int_{\Omega^+(t)} \rho\mathbf{f}\cdot\mathbf{u}\, ds - \int_{\partial\Omega^+(t)} \mathbf{q}\cdot\mathbf{n}\, ds. \quad (3.51)$$

Chapter 4

Thermodynamics of fluid flows

4.1 Thermodynamic variables

We will focus on a basic set of thermodynamic variables, namely

ρ	density
p	pressure
T	temperature
e	internal energy
s	entropy.

Additional thermodynamic variables that are defined in terms of those above will also be used, namely the enthalphy $h = e + p/\rho$ and the specific volume $\vartheta = 1/\rho$.

There are two more relevant thermodynamic variables that need to be introduced, which are referred to as heat capacities or specific heats. The specific heat at constant volume is

$$C_v = \left(\frac{\partial e}{\partial T}\right)_\vartheta \tag{4.1}$$

and the specific heat at constant pressure is

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p . \tag{4.2}$$

Their ratio is labelled as γ.

4.2 Equation of state

According to the state principle only two thermodynamic variables are required to know any of the thermodynamic variables (as long as the chemical composition of the fluid is not changed by mixing or diffusion). Thus, for example, if P and T are the independent variables, then $e = \xi(P, T)$, or if ρ and s are the independent variables, then $e = \chi(\rho, s)$. These two expressions can be equated in the following manner

$$e = \xi(P, T) = \chi(\rho(P, T), s(P, T)). \tag{4.3}$$

Expression in which a thermodynamic variable is written in terms of two others are known as equations of state. We will focus on two particular equations of state, namely a thermal equation of state

$$p = \phi(\rho, T), \tag{4.4}$$

and a caloric equations of state

$$e = \psi(\rho, T). \tag{4.5}$$

Given that ρ and e are provided by the conservation of mass and conservation of energy equations, we now have four equations for the four unknowns ρ, p, T, and e.

4.3 The Gibb's equations

4.3.1 Definitions and additional relationships

An axiom of thermodynamics is the Gibbs equation, which is

$$Tds = de + pd\vartheta. \tag{4.6}$$

Using the definition of enthalpy, the above can also be expressed as

$$Tds = dh - \vartheta dp. \tag{4.7}$$

From the Gibb's equation, a wide variety of relationships between the thermodynamics variables can be derived. One of these is

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T. \tag{4.8}$$

To derive the above we start with

$$\gamma = \frac{C_p}{C_v} = \frac{\left(\frac{\partial h}{\partial T}\right)_p}{\left(\frac{\partial e}{\partial T}\right)_\vartheta} = \frac{\left(\frac{\partial h}{\partial s}\right)_p \left(\frac{\partial s}{\partial T}\right)_p}{\left(\frac{\partial e}{\partial s}\right)_\vartheta \left(\frac{\partial s}{\partial T}\right)_\vartheta} \tag{4.9}$$

Equations (4.6) and (4.7) can be used to show that

$$\left(\frac{\partial h}{\partial s}\right)_p = \left(\frac{\partial e}{\partial s}\right)_\vartheta. \tag{4.10}$$

We then write

$$\gamma = \frac{\left(\frac{\partial s}{\partial T}\right)_p}{\left(\frac{\partial s}{\partial T}\right)_\rho} = \frac{\left(\frac{\partial s}{\partial \rho}\right)_p \left(\frac{\partial \rho}{\partial T}\right)_p}{\left(\frac{\partial s}{\partial p}\right)_\rho \left(\frac{\partial p}{\partial T}\right)_\rho} = \frac{\left(\frac{\partial s}{\partial \rho}\right)_p \left(\frac{\partial p}{\partial s}\right)_\rho}{\left(\frac{\partial T}{\partial \rho}\right)_p \left(\frac{\partial p}{\partial T}\right)_\rho}. \tag{4.11}$$

By calculus

$$\left(\frac{\partial s}{\partial \rho}\right)_p \left(\frac{\partial \rho}{\partial p}\right)_s \left(\frac{\partial p}{\partial s}\right)_\rho = -1, \tag{4.12}$$

$$\left(\frac{\partial T}{\partial \rho}\right)_p \left(\frac{\partial \rho}{\partial p}\right)_T \left(\frac{\partial p}{\partial T}\right)_\rho = -1. \tag{4.13}$$

Thus,

$$\gamma = \frac{\left(\frac{\partial p}{\partial \rho}\right)_s}{\left(\frac{\partial p}{\partial \rho}\right)_T}, \tag{4.14}$$

as required.

4.3.2 Applicaitons of the Gibb's equation

Crocco's equation

Consider the momentum eq. (2.19), which for inviscid flows with no body forces becomes

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p. \tag{4.15}$$

Using the Gibb's equation, the above becomes

$$\frac{D\mathbf{u}}{Dt} = T\nabla s - \nabla h. \tag{4.16}$$

The above is referred to as Crocco's equation. Using the vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times(\nabla\times\mathbf{u}), \tag{4.17}$$

Crocco's equation is typically written as

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + h\right) = \mathbf{u}\times\mathbf{w} + T\nabla s. \tag{4.18}$$

Vorticity equation

Consider the vorticity eq. (3.10) for inviscid flows, which we re-write below

$$\frac{D\mathbf{w}}{Dt} + \mathbf{w}(\nabla\cdot\mathbf{u}) - (\mathbf{w}\cdot\nabla)\mathbf{u} = -\nabla\times\left(\frac{1}{\rho}\nabla p\right). \tag{4.19}$$

The right-hand side can be re-written using the alternate form of the Gibbs equation, namely eq. (4.7). Thus,

$$\nabla\times\left(\frac{1}{\rho}\nabla p\right) = -\nabla\times(T\nabla s) = -\nabla T\times\nabla s. \tag{4.20}$$

Additionally, we note that

$$\frac{1}{\rho}(\nabla\cdot\mathbf{u}) = \frac{1}{\rho}\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) = \frac{D}{Dt}\left(\frac{1}{\rho}\right). \tag{4.21}$$

and thus

$$\frac{D\mathbf{w}}{Dt}\frac{1}{\rho} + \frac{\mathbf{w}}{\rho}(\nabla\cdot\mathbf{u}) = \frac{D\mathbf{w}}{Dt}\frac{1}{\rho} + \mathbf{w}\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{D}{Dt}\left(\frac{\mathbf{w}}{\rho}\right). \tag{4.22}$$

Using eqs. (4.20) and (4.22) the vorticity eq. (4.19) for inviscid flows becomes

$$\frac{D}{Dt}\left(\frac{\mathbf{w}}{\rho}\right) = \left(\frac{\mathbf{w}}{\rho}\cdot\nabla\right)\mathbf{u} + \frac{1}{\rho}\nabla T\times\nabla s. \tag{4.23}$$

For flows where $\nabla s = 0$ everywhere, we obtain

$$\frac{D}{Dt}\left(\frac{\mathbf{w}}{\rho}\right) = \left(\frac{\mathbf{w}}{\rho}\cdot\nabla\right)\mathbf{u} \tag{4.24}$$

If we evaluate the above at the position $\mathbf{x}_u^+$ of a fluid particle, we can re-write it as follows

$$\frac{\partial}{\partial t}\left(\frac{w_{u,i}^+}{\rho_u^+}\right) = \frac{w_{j,u}^+}{\rho_u^+}\left(\frac{\partial u_i}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}_u^+}. \tag{4.25}$$

This equation is identical to that shown on the left of (1.42), and thus its solution is

$$\frac{w_{u,i}^+(t,\mathbf{y})}{\rho_u^+(t,\mathbf{y})} = \frac{w_{u,j}^+(0,\mathbf{y})}{\rho_u^+(0,\mathbf{y})}J_{ij}^+. \tag{4.26}$$

4.4 Ideal gasses

An ideal gas is defined as one whose thermal equation of state, i.e. eq. (4.4), is

$$p = \rho RT. \tag{4.27}$$

For the caloric equation of state, one can use the same reasoning used for eq. (4.3) to write

$$e = \psi(\rho, T) = \theta(P(\rho, T), T). \tag{4.28}$$

Thus,

$$\frac{\partial \psi}{\partial \rho} = \frac{\partial \theta(P,T)}{\partial P} \frac{\partial P(\rho, T)}{\partial \rho}. \tag{4.29}$$

As shown in Thompson pg. 70, $\partial\theta(P,T)/\partial P = 0$. Thus,

$$e = e(T). \tag{4.30}$$

Since the enthalphy for an ideal gas can be written as $h = e + RT$,

$$h = h(T). \tag{4.31}$$

The specific heats can now be expressed as

$$C_v = \frac{de}{dT}, \tag{4.32}$$

$$C_p = \frac{dh}{dT}. \tag{4.33}$$

The internal energy and enthalpy can be computed from

$$e(T) - e(T_0) = \int_{T_0}^{T} C_v(\alpha) d\alpha, \tag{4.34}$$

$$h(T) - h(T_0) = \int_{T_0}^{T} C_p(\alpha) d\alpha \tag{4.35}$$

We can also now show that

$$C_p - C_v = \frac{dh}{dT} - \frac{de}{dT} = \frac{d}{dT}(h - e) = R \tag{4.36}$$

For perfect gasses, C_v and C_p are constant. Thus, $e(T) - e(T_0) = C_v(T - T_0)$ and $h(T) - h(T_0) = C_p(T - T_0)$. If one assumes that at $T = 0$, $e = 0$, we obtain the familiar expressions

$$e = C_v T, \tag{4.37}$$

$$h = C_p T. \tag{4.38}$$

4.5 Isentropic flow

4.5.1 Definitions and governing equations

We begin with the equation for entropy derived in section 3.3, namely eq. (3.29), which is repeated below

$$\rho T \frac{Ds}{Dt} = -\frac{\partial q_j}{\partial x_j} + t_{ij}\frac{\partial u_i}{\partial x_j}. \tag{3.29}$$

We re-write the following term as follows

$$\frac{1}{T}\frac{\partial q_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) + \frac{q_j}{T^2}\frac{\partial T}{\partial x_j},$$

to thus obtain

$$\rho\frac{Ds}{Dt} - \frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) = \frac{q_j}{T^2}\frac{\partial T}{\partial x_j} + \frac{t_{ij}}{T}\frac{\partial u_i}{\partial x_j}.$$

Using the continuity equation this is equivalent to

$$\frac{\partial \rho s}{\partial t} + \frac{\partial \rho s u_j}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) = \frac{q_j}{T^2}\frac{\partial T}{\partial x_j} + \frac{t_{ij}}{T}\frac{\partial u_i}{\partial x_j}.$$

Integrating this equation over a control volume $\Omega^+ = \Omega^+(t, \Omega_0)$ and using the Reynolds transport theorem (eq. (1.19)) allows us to write

$$\frac{d}{dt}\int_{\Omega^+} \rho s \, dv - \int_{\Omega^+} \frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) dv = \int_{\Omega^+} \frac{\partial}{\partial x_j}\rho s \left(v_j - u_j\right) dv + \int_{\Omega^+} \frac{q_j}{T^2}\frac{\partial T}{\partial x_j} + \frac{t_{ij}}{T}\frac{\partial u_i}{\partial x_j}\, dv.$$

Using Gauss's theorem this is equivalent to

$$\frac{d}{dt}\int_{\Omega^+(t)} \rho s \, dv - \int_{\partial\Omega^+(t)} \frac{q_j}{T} n_j \, ds = \int_{\partial\Omega^+(t)} \rho s \left(v_j - u_j\right) n_j \, dv + \int_{\Omega^+(t)} \frac{q_j}{T^2}\frac{\partial T}{\partial x_j} + \frac{t_{ij}}{T}\frac{\partial u_i}{\partial x_j}\, dv.$$

Consider now a control volume $\Omega_u^+ = \Omega_u^+(t, \Omega_0)$ moving with the flow. Then

$$\frac{d}{dt}\int_{\Omega_u^+} \rho s \, dv - \int_{\partial\Omega_u^+} \frac{q_j}{T} n_j \, ds = \int_{\Omega_u^+} \frac{q_j}{T^2}\frac{\partial T}{\partial x_j} + \frac{t_{ij}}{T}\frac{\partial u_i}{\partial x_j}\, dv. \tag{4.39}$$

We define a reversible flow as one for which the right-hand side above is zero. We define an adiabatic flow as one for which q_j is zero at the boundary, and thus the second term on the left-hand side above is zero. An isentropic flow is a reversible adiabatic flow, and thus satisfies

$$\frac{d}{dt}\int_{\Omega_u^+} \rho s \, dv = 0. \tag{4.40}$$

An alternate definition of an isentropic flow can be given by focusing on a fluid particle, rather than a material volume. If q_j and $t_{ij}\partial u_i/\partial x_j$ can be neglected, then eq. (3.29) gives $Ds/Dt = 0$. That is, the entropy is constant along a streamline. This is an isentropic flow. If *in addition* the entropy is the same in all directions (i.e. $\nabla s = 0$), then the flow is homentropic.

$$\text{isentropic flow: } \frac{Ds}{Dt} = 0, \tag{4.41}$$

$$\text{homentropic flow: } \frac{\partial s}{\partial t} = \nabla s = 0. \tag{4.42}$$

We also note that for an inviscid fluid $\mu = \mu_\nu = \kappa = 0$. Thus, if the stress tensor is given by

$$t_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_\nu\right)\delta_{ij}S_{kk} \tag{4.43}$$

and the heat conduction by

$$q_i = -\kappa\frac{\partial T}{\partial x_i} \tag{4.44}$$

then both of the above are zero for an inviscid fluid, and thus the flow is isentropic.

4.5.2 Isentropic stagnation variables

An isentropic stagnation variable is used to describe the value a flow variable would reach as the flow decelerates to a stagnation point in an isentropic fashion.

We re-write eq. (3.39), without a body force, below

$$h + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = \text{constant along a streamline.} \tag{4.45}$$

This equation was obtained by assuming steady inviscid flow along a streamline, i.e. isentropic flow. If a fluid particle decelerates to stagnation isentropically, we have

$$h_t = h + \frac{1}{2}u_i u_i, \tag{4.46}$$

where h_t is the stagnation enthalpy. This definition of stagnation enthalpy holds true even if we relax the assumptions of eq. (3.39), namely inviscid steady-state flow. This is because at any point in the domain, one can always imagine a hypothetical path that a particle follows as it decelerates to stagnation isentropically. At the end of this hypothetical path, the stagnation enthalpy would be computed as in eq. (4.46). It is also interesting to note that the isentropic enthalpy reached as the particle isentropically decelerates to stagnation is independent of the path taken by the particle.

4.5.3 Isentropic flow for a perfect gas

If we compare eq. (3.20) (the pressure evolution equation for a perfect gas) with eq. (3.29) (the entropy evolution equation) we can write

$$\frac{1}{\gamma-1}\rho^\gamma\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \rho T\frac{Ds}{Dt}.$$

We now substitute $p = \rho RT$ on the right-hand side to obtain,

$$\frac{1}{\gamma-1}\rho^\gamma\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{p}{R}\frac{Ds}{Dt}.$$

Re-arranging, we have

$$\frac{\rho^\gamma}{p}\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{\gamma-1}{R}\frac{Ds}{Dt},$$

or

$$\frac{D}{Dt}\ln\left(\frac{p}{\rho^\gamma}\right) = \frac{\gamma-1}{R}\frac{Ds}{Dt}.$$

The solution to the PDE above is

$$\ln\left(\frac{p}{\rho^\gamma}\right) = c + \frac{\gamma-1}{R}s, \tag{4.47}$$

where $c = c(\mathbf{x}, t)$ satisfies $Dc/Dt = 0$. One can confirm this is the solution by simply taking D/Dt of both sides.

Let's assume isentropic flow, that is

$$\frac{Ds}{Dt} = 0.$$

Taking the D/Dt derivative of eq. (4.47) gives

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0. \tag{4.48}$$

The solution to the above is $p = C\rho^\gamma$, where $C = C(\mathbf{x}, t)$ satisfies $DC/Dt = 0$. One can confirm this is the solution by simply substituting it in the equation above.

We now assume homentropic flow, that is

$$\frac{\partial s}{\partial t} = \nabla s = 0. \tag{4.49}$$

Additionally assume that flow enters through the domain boundary, and that at each point of the boundary we have

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right)\bigg|_{\partial\Omega} = \nabla\left(\frac{p}{\rho^\gamma}\right)\bigg|_{\partial\Omega} = 0.$$

Using each of the two equalities above in eq. (4.47) we get

$$\frac{\partial c}{\partial t}\bigg|_{\partial\Omega} = \nabla c|_{\partial\Omega} = 0.$$

Since $Dc/Dt = 0$, we can make the stronger statement that

$$\frac{\partial c}{\partial t} = \nabla c = 0. \tag{4.50}$$

Using eqs. (4.49) and (4.50) in eq. (4.47) we finally obtain

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) = \nabla\left(\frac{p}{\rho^\gamma}\right) = 0. \tag{4.51}$$

The solution to the above is $p = C\rho^\gamma$, where C is a constant across space and time.

If we have isentropic flow, then the flow variables at two different states can be linked as follows

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^\gamma. \tag{4.52}$$

Using the equation of state, this can be rewritten as

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}, \tag{4.53}$$

and

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{1/(\gamma-1)}. \tag{4.54}$$

Equation (4.46) for a perfect gas can be written as

$$C_p T_t = C_p T + \frac{1}{2}u_i u_i, \tag{4.55}$$

which we further re-write as

$$C_pT_t = C_pT + \frac{1}{2}\gamma RTM^2. \tag{4.56}$$

Dividing both sides by C_pT gives

$$\frac{T_t}{T} = 1 + \frac{1}{2}\frac{\gamma R}{C_p}M^2, \tag{4.57}$$

which, given $R = C_p(\gamma - 1)/\gamma$, finally becomes

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2}M^2. \tag{4.58}$$

Using eqs. (4.53) and (4.54) we also obtain

$$\frac{P_t}{P} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}, \tag{4.59}$$

$$\frac{\rho_t}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{1/(\gamma-1)}. \tag{4.60}$$

4.6 Acousitc waves

Express the density, pressure, and velocity as

$$\rho = \rho_0 + \hat{\rho}, \tag{4.61}$$

$$p = p_0 + \hat{p}, \tag{4.62}$$

$$\mathbf{u} = \mathbf{u}_0 + \hat{\mathbf{u}}. \tag{4.63}$$

For the above, ρ_0 and p are constant, and $\mathbf{u}_0 = 0$. Additionally, $\hat{\rho}$, $\hat{p}$, and $\hat{\mathbf{u}}$ are small.

We assume the flow to be inviscid and homentropic. Since the entropy is constant, the state principle (section 4.2) states that the thermodynamic variables will depend on one quantity only. A Taylor expansion for the pressure then gives

$$p = p_0 + \left[\left(\frac{\partial p}{\partial \rho}\right)_s\right]_{\rho=\rho_0}(\rho - \rho_0) + h.o.t.. \tag{4.64}$$

We rewrite the above as

$$\hat{p} = c_0^2\hat{\rho}, \tag{4.65}$$

where

$$c_0^2 = \left[\left(\frac{\partial p}{\partial \rho}\right)_s\right]_{\rho=\rho_0}. \tag{4.66}$$

The assumption that $\hat{\rho}$ be small can be formally stated as $\hat{\rho} \ll \rho_0$. Using eq. (4.65), the assumption that $\hat{p}$ is small is formally stated as $\hat{p} \ll \rho_0 c_0^2$.

Assuming products of two small quantities can be neglected, the density equation gives

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= \frac{\partial(\rho_0 + \hat{\rho})}{\partial t} + \nabla\cdot[(\rho_0 + \hat{\rho})(\mathbf{u}_0 + \hat{\mathbf{u}})] \\ &= \frac{\partial \hat{\rho}}{\partial t} + \nabla\cdot(\rho_0\hat{\mathbf{u}}) = 0.\end{aligned} \tag{4.67}$$

The momentum equation gives

$$\begin{aligned}
\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= (\rho_0+\hat{\rho})\frac{\partial(\mathbf{u}_0+\hat{\mathbf{u}})}{\partial t} + (\rho_0+\hat{\rho})[(\mathbf{u}_0+\hat{\mathbf{u}})\cdot\nabla](\mathbf{u}_0+\hat{\mathbf{u}}) + \nabla(p_0+\hat{p}) \\
&= (\rho_0+\hat{\rho})\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\rho_0+\hat{\rho})(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}} + \nabla(p_0+\hat{p}) \\
&= \rho_0\frac{\partial \hat{\mathbf{u}}}{\partial t} + \nabla\hat{p} \\
&= \rho_0\frac{\partial \hat{\mathbf{u}}}{\partial t} + c_0^2\nabla\hat{\rho} = 0. \qquad (4.68)
\end{aligned}$$

Taking the time derivative of eq. (4.67) and using eq. (4.68) leads to the wave equation for density

$$\frac{\partial^2\hat{\rho}}{\partial t^2} - c_0^2\nabla^2\hat{\rho} = 0. \qquad (4.69)$$

Similarly, taking the time derivative of eq. (4.68) and using eq. (4.67) leads to

$$\frac{\partial^2\hat{\mathbf{u}}}{\partial t^2} - c_0^2\nabla(\nabla\cdot\hat{\mathbf{u}}) = 0. \qquad (4.70)$$

Using a vector identity, we have

$$\nabla(\nabla\cdot\hat{\mathbf{u}}) = \nabla\times(\nabla\times\hat{\mathbf{u}}) + \nabla^2\hat{\mathbf{u}}. \qquad (4.71)$$

Taking the curl of eq. (4.68) we see that if the flow is initially irrotational it remains so for all time. Thus, the wave equation for velocity takes the form

$$\frac{\partial^2\hat{\mathbf{u}}}{\partial t^2} - c_0^2\nabla^2\hat{\mathbf{u}} = 0. \qquad (4.72)$$

In fact, one can show that all flow variables satisfy the wave equation given the conditions specified above: inviscid, homentropic, irrotational, with $\hat{\rho}$, $\hat{p}$, and $\hat{\mathbf{u}}$ small.

A more general expression for eq. (4.66) is

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s, \qquad (4.73)$$

which represents the speed of sound. Using eq. (4.8), and assuming an ideal gas, we have

$$c^2 = \gamma RT. \qquad (4.74)$$

Chapter 5

Shock waves and related discontinuities

Shock jump relations; Prandtl relation and Shock equations; Oblique shocks, theta-beta curve; Shock wave boundary layer interaction.

5.1 Shock waves

Shock waves are thin regions through which flow variables change drastically. For inviscid flows, these thin regions become actual discontinuities. We will label flow variables before the shock with the underscript 1, and variables after the shock with the underscript 2. We also define the unit vector normal to the shock and pointing towards the outgoing flow as $\mathbf{n}$, and the shock velocity as $\mathbf{b}$. Then, we can define the relative velocities normal to the shock as

$$\begin{aligned} w_1 &= (\mathbf{u}_1 - \mathbf{b}) \cdot \mathbf{n}, \\ w_2 &= (\mathbf{u}_2 - \mathbf{b}) \cdot \mathbf{n}. \end{aligned} \tag{5.1}$$

The relative velocities tangent to the shock are given by

$$\begin{aligned} v_1 &= (\mathbf{u}_1 - \mathbf{b}) \cdot (1 - \mathbf{nn}), \\ v_2 &= (\mathbf{u}_2 - \mathbf{b}) \cdot (1 - \mathbf{nn}). \end{aligned} \tag{5.2}$$

The shock jump conditions, assuming negligible viscous stresses and heat flux on either side of the shock, are then given by

$$\rho_1 w_1 = \rho_2 w_2, \tag{5.3}$$

$$\rho_1 w_1^2 + P_1 = \rho_2 w_2^2 + P_2, \tag{5.4}$$

$$v_1 = v_2, \tag{5.5}$$

$$h_1 + \frac{1}{2} w_1^2 = h_2 + \frac{1}{2} w_2^2. \tag{5.6}$$

Combining eqs. (5.3), (5.4) and (5.6) (see Thompson [1988]), the Rankine-Hugoniot equation is obtained

$$h_2 - h_1 = \frac{1}{2} (P_2 - P_1) (\vartheta_2 + \vartheta_1). \tag{5.7}$$

Given an equation of state, we can write $h_2 = \xi(P_2, \vartheta_2)$ and $h_1 = \xi(P_1, \vartheta_1)$. Thus, the Rankine-Hugoniot equation is an expression that relates ϑ_1, P_1, ϑ_2, and P_2. If we assume that ϑ_1 and

P_1 are known, the Rankinge-Hugoniot equation is simply an expression of the form $P_2 = f(\vartheta_2)$. The function f is known as a shock adiabat, and is different for each equation of state used. Lets consider for example a perfect gas. Then, eq. (5.7) becomes

$$\frac{P_2}{P_1} = \frac{\dfrac{\gamma+1}{\gamma-1} - \dfrac{\vartheta_2}{\vartheta_1}}{\dfrac{\gamma+1}{\gamma-1}\dfrac{\vartheta_2}{\vartheta_1} - 1}. \tag{5.8}$$

In fig. 5.1 an arbitrary shock adiabat $P_2 = f(\vartheta_2)$ is shown. One thing to note is that for $\vartheta_2 = \vartheta_1$ (no real shock), the shock adiabat naturally gives $P_2 = P_1$. Thus, the state corresponding to the flow upstream of the shock, i.e. (P_1, ϑ_1), is a point in the shock adiabat line. One can then draw a straight line to various possible states after the shock, as shown in fig. 5.1. The slope of any of these lines is $(P_2 - P_1)/(\vartheta_2 - \vartheta_1)$, which as shown in Thompson [1988] is equal to $-J^2$, where $J = \rho_1 w_1 = \rho_2 w_2$.

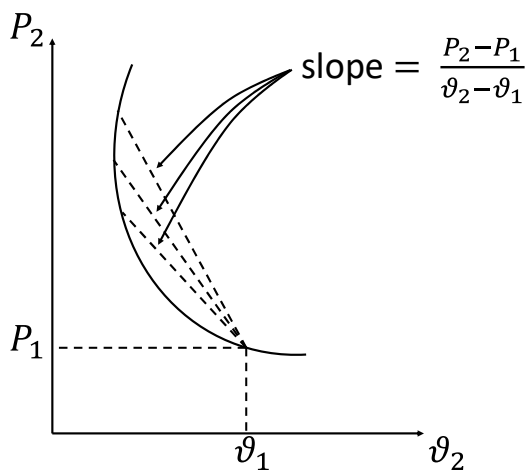

Figure 5.1: Shock adiabat for an arbitrary equation of state.

A few extra notes on shock waves:

- The shock Mach number M_{1n} is defined as

$$M_{1n} = \frac{w_1}{c_1}. \tag{5.9}$$

- The speed of sound is independent of the reference frame. Thus, whether you are in the shock reference frame, or in a stationary reference frame, the speeds of sound before and after the shock are invariant.

- The relative velocities satisfy

$$\begin{aligned} w_1 &\geq c_1 \\ w_2 &\leq c_2. \end{aligned} \tag{5.10}$$

- The shock strength Π is defined as (see Thompson [1988] for derivation):

$$\Pi = \frac{P_2 - P_1}{\rho_1 c_1^2}. \tag{5.11}$$

- Strong and weak shocks are defined according to

$$\begin{aligned} \Pi &\ll 1 \quad \text{weak shock} \\ \Pi &\gg 1 \quad \text{strong shock.} \end{aligned} \tag{5.12}$$

- For weak shocks the entropy increase across a shock is so weak, they might as well be considered isentropic.
- In the limit of vanishing strength, shock waves become acoustic discontinuities, propagating with speed c relative to the fluid.

5.1.1 Normal shocks

Following the derivations in Thompson [1988] for a perfect gas, one can express shock jump conditions as a function of γ and M_{1n} only. These are

$$\frac{P_2}{P_1} = \frac{\frac{2\gamma}{\gamma-1}M_{1n}^2 - 1}{\frac{\gamma+1}{\gamma-1}}, \tag{5.13}$$

$$\frac{w_2}{w_1} = \frac{1 + \frac{\gamma-1}{2}M_{1n}^2}{\frac{\gamma+1}{2}M_{1n}^2}, \tag{5.14}$$

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{2}M_{1n}^2}{1 + \frac{\gamma-1}{2}M_{1n}^2}. \tag{5.15}$$

Additionally, combining the three above one obtains

$$M_{2n}^2 = \frac{M_{1n}^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_{1n}^2 - 1}. \tag{5.16}$$

According to eq. (5.6) $h_{t2} = h_{t1}$ (this holds not only for normal shocks but all shocks). Thus, $T_{t2} = T_{t1}$. For the stagnation pressure, we have

$$\frac{P_{t2}}{P_{t1}} = \frac{P_{t2}}{P_2}\frac{P_1}{P_{t1}}\frac{P_2}{P_1}. \tag{5.17}$$

Using eqs. (4.59), (5.13) and (5.16), and rearranging, one obtains

$$\frac{P_{t2}}{P_{t1}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_{1n}^2 - 1}\right)^{1/(\gamma-1)} \left(\frac{\frac{\gamma+1}{2}M_{1n}^2}{1 + \frac{\gamma-1}{2}M_{1n}^2}\right)^{\gamma/(\gamma-1)}. \tag{5.18}$$

Finally, entropy change across a normal shock is

$$s_2 - s_1 = -R \ln \frac{P_{t2}}{P_{t1}}, \tag{5.19}$$

as shown in Thompson [1988].

5.1.2 Oblique shocks

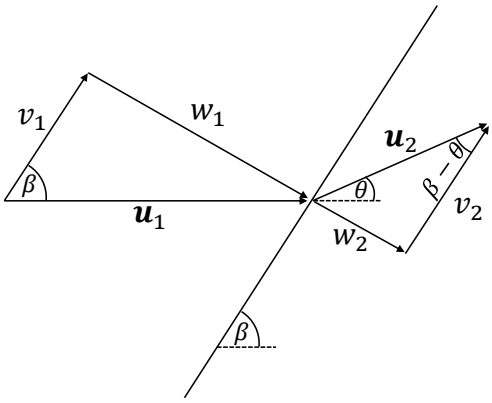

Figure 5.2: Oblique-shock geometry, β is the shock angle and θ the turning angle.

The normal shock jump conditions given by eqs. (5.13) to (5.16) and (5.18) still apply, but it is again emphasized that w_1 and w_2 are the components normal to the shock, as defined in eq. (5.1). For the tangential components, defined by eq. (5.2), we have $v_1 = v_2$ as mentioned in section 5.1. As shown in fig. 5.2, the relationship between w_1, w_2 and $\mathbf{u}_1$, $\mathbf{u}_2$ is

$$\begin{aligned} w_1 &= |\mathbf{u}_1| \sin\beta, \\ w_2 &= |\mathbf{u}_2| \sin(\beta - \theta). \end{aligned} \tag{5.20}$$

Similarly, the relationship between v_1, v_2 and $\mathbf{u}_1$, $\mathbf{u}_2$ is

$$\begin{aligned} v_1 &= |\mathbf{u}_1| \cos\beta, \\ v_2 &= |\mathbf{u}_2| \cos(\beta - \theta). \end{aligned} \tag{5.21}$$

If the upstream state is known (ρ_1, $\mathbf{u}_1$, P_1, P_{t1}), along with the shock angle β, then the downstream state can be determined using the above relationships and the normal shock jump conditions.

Using the shock jump condition for velocity (eq. (5.14)), and some trigonometric identities, one can derive an equation for θ in terms of β, for a given inflow Mach number $M_1 = |\mathbf{u}_1|/c_1$, (see Thompson [1988]). This relationship is

$$\tan\theta = \frac{\cot\beta \left(M_1^2 \sin^2\beta - 1\right)}{1 + \left(\frac{\gamma+1}{2}\right) M_1^2 - M_1^2 \sin^2\beta}. \tag{5.22}$$

The information contained in the above relationship is quite vast, and can best be understood by looking at θ profiles as a function of β—for different Mach numbers—obtained from the equation above. A plot of these profiles is given in fig. 5.3. Starting from the right-most point on the x-axis, labelled as "a", is a normal shock with a shock angle of 90°. Moving along the blue line as the shock angle decreases, we see that the turning angle increases until a maximum, labeled as "b", is reached. A further decrease in shock angle leads to point "c", which corresponds to turning angles for which the flow behind the shock becomes subsonic. Smaller shock angles lead to even smaller turning angles, until point "d" is reached, which corresponds to a Mach wave, to be described in a subsequent section. It is important to note that there are two shock angles that can give the same turning angle. The turning angle corresponding to the smaller shock

angle is referred to as the weak solution, whereas the turning angle corresponding to the larger shock angle is the strong solution. This nomenclature has no direct connection to that defined in eq. (5.12). The black dashed line in fig. 5.3 corresponds to the peak value for each Mach number, and thus demarcates the weak and strong solutions. A more pictorial representation of this behavior, for the specific case of a shock in front of a cylinder, is given in fig. 5.4.

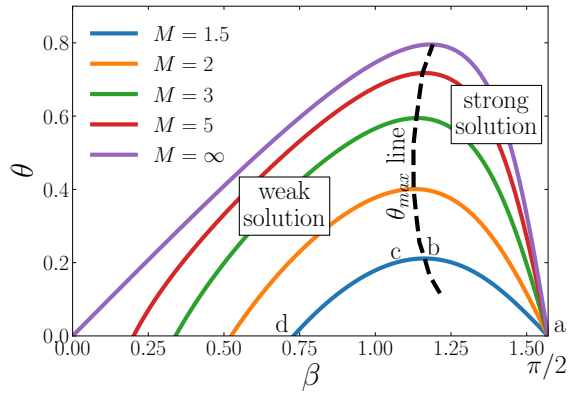

Figure 5.3: θ–β curve for a perfect gas with $\gamma = 1.4$.

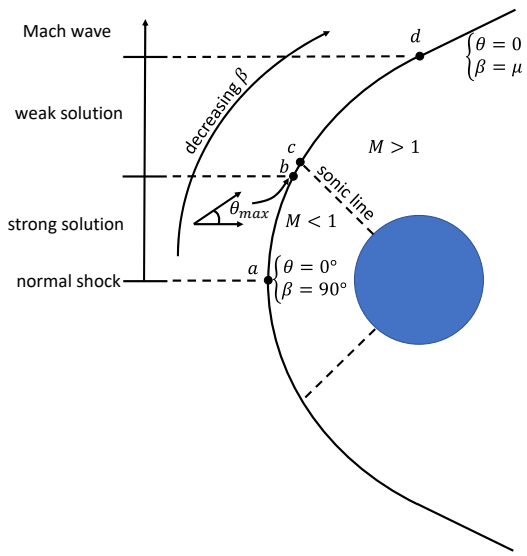

Figure 5.4: Representation of different shock waves for different shock angles β.

Given the information above, it is relevant to ask what happens when a compressible fluid flows over a wedge or a some similar object, whose wedge angle is so large that the turning angle needs to be larger than θ_{max}. Given that, as fig. 5.3 shows, there is no turning angle greater than θ_{max}, it seems like an inconsistency has been found. In reality, when the flow encounters a wedge whose angle is not that large and the turning angle can be lower than θ_{max}, then the weak solution is the one that occurs and the shock is attached to the leading edge of the wedge. As the wedge angle increases and the flow needs to be deflected by an angle greater than θ_{max},

the shock detaches from the wedge leading edge, as shown in fig. 5.4, and both strong and weak solutions occur along the shock. For this case, there will be a section behind the shock that will be subsonic, and thus the flow there can be turned by any angle greater than θ_{max}.

5.1.3 Weak shocks

By definition, weak shocks are those for which Π, defined in eq. (5.11), is very small. Assuming that $\rho_1 c_1^2$ is not excessively large, the definition of Π indicates that $P_2 - P_1$ is small for weak shocks. We can use the fact that $P_2 - P_1$ is small for weak shocks to show that $(s_2 - s_1) \propto (P_2 - P_1)^3$, which then indicates weak shocks have negligible entropy changes.

We begin by adding and subtracting $(P_2 - P_1)\vartheta_1$ on the right-hand side of the Rankine-Hugoniot eq. (5.7) to obtain

$$h_2 - h_1 = (P_2 - P_1)\vartheta_1 + \frac{1}{2}(P_2 - P_1)(\vartheta_2 - \vartheta_1). \tag{5.23}$$

We now use Taylor-series expansions for h and ϑ, each as a function of s and P, and plug them into the equation above to obtain the scaling of s as a function of P. For an arbitrary function $f(s, P)$ the Taylor-series expansion can symbolically be expressed as

$$\begin{aligned} f_2 - f_1 = &\left[(s_2 - s_1)\left(\frac{\partial}{\partial s}\right)_1 + (P_1 - P_1)\left(\frac{\partial}{\partial P}\right)_1\right] f \\ &+\frac{1}{2!}\left[(s_2 - s_1)\left(\frac{\partial}{\partial s}\right)_1 + (P_1 - P_1)\left(\frac{\partial}{\partial P}\right)_1\right]^2 f \\ &+\frac{1}{3!}\left[(s_2 - s_1)\left(\frac{\partial}{\partial s}\right)_1 + (P_1 - P_1)\left(\frac{\partial}{\partial P}\right)_1\right]^3 f + ... \end{aligned}$$

Since we are interested in the leading-order expression for $(s_2 - s_1)$, terms of higher-order than $(s_2 - s_1)$ will be neglected, i.e. $(s_2 - s_1)^2$, $(s_2 - s_1)^3$, $(s_2 - s_1)(P_2 - P_1)$, etc.

The Taylor-series expansion for enthalpy is

$$h_2 - h_1 = (s_2 - s_1)\left(\frac{\partial h}{\partial s}\right)_1 + (P_2 - P_1)\left(\frac{\partial h}{\partial P}\right)_1 + \frac{1}{2}(P_2 - P_1)^2\left(\frac{\partial^2 h}{\partial P^2}\right)_1 + \frac{1}{6}(P_2 - P_1)^3\left(\frac{\partial^3 h}{\partial P^3}\right)_1 + ... \tag{5.24}$$

Using the Gibbs equation shown in eq. (4.7), we have $(\partial h/\partial s)_p = T$ and $(\partial h/\partial P)_s = \vartheta$. Using this in the above gives

$$h_2 - h_1 = (s_2 - s_1)T_1 + (P_2 - P_1)\vartheta_1 + \frac{1}{2}(P_2 - P_1)^2\left(\frac{\partial \vartheta}{\partial P}\right)_1 + \frac{1}{6}(P_2 - P_1)^3\left(\frac{\partial^2 \vartheta}{\partial P^2}\right)_1 + ... \tag{5.25}$$

The Taylor-series expansion for the specific volume is

$$\vartheta_2 - \vartheta_1 = (s_2 - s_1)\left(\frac{\partial \vartheta}{\partial s}\right)_1 + (P_2 - P_1)\left(\frac{\partial \vartheta}{\partial P}\right)_1 + \frac{1}{2}(P_2 - P_1)^2\left(\frac{\partial^2 \vartheta}{\partial P^2}\right)_1 + \frac{1}{6}(P_2 - P_1)^3\left(\frac{\partial^3 \vartheta}{\partial P^3}\right)_1 + ... \tag{5.26}$$

Plugging in the two Taylor-series expansions above in eq. (5.23), gives

$$\begin{aligned} (s_2 - s_1)T_1 + (P_2 - P_1)\vartheta_1 + \frac{1}{2}(P_2 - P_1)^2\left(\frac{\partial \vartheta}{\partial P}\right)_1 + \frac{1}{6}(P_2 - P_1)^3\left(\frac{\partial^2 \vartheta}{\partial P^2}\right)_1 = \\ (P_2 - P_1)\vartheta_1 + \frac{1}{2}(P_2 - P_1)^2\left(\frac{\partial \vartheta}{\partial P}\right)_1 + \frac{1}{4}(P_2 - P_1)^3\left(\frac{\partial^2 \vartheta}{\partial P^2}\right)_1 + ... \end{aligned} \tag{5.27}$$

Simplifying the above finally gives

$$s_2 - s_1 = \frac{1}{12T_1}\left(\frac{\partial^2 \vartheta}{\partial P^2}\right)_1 (P_2 - P_1)^3 + ... \tag{5.28}$$

that is, $(s_2 - s_1) \propto (P_2 - P_1)^3$.

5.1.4 Strong shocks

Strong shocks are defined as those for which $\Pi \gg 1$. In general, it seems that $P < \rho c^2$, and thus

$$\Pi = \frac{P_2}{\rho_1 c_1^2} - \frac{P_1}{\rho_1 c_1^2} \gg 1 \tag{5.29}$$

becomes

$$\frac{P_2}{\rho_1 c_1^2} \gg 1, \tag{5.30}$$

or

$$P_2 \gg \rho_1 c_1^2. \tag{5.31}$$

The above in turn implies $P_2 \gg P_1$, that is, P_1 is negligible.

Consider the normal shock jump condition for pressure given by eq. (5.13). It can be rearranged to give

$$P_2 = \frac{\frac{2\gamma}{\gamma-1}\frac{\rho_1 w_1^2}{\gamma} - P_1}{\frac{\gamma+1}{\gamma-1}}. \tag{5.32}$$

Since P_1 can be neglected, this gives

$$P_2 = \frac{2}{\gamma+1}\rho_1 w_1^2. \tag{5.33}$$

Additionally, since $M_{1n}^2 = \rho_1 w_1^2/\gamma P_1$, a negligible P_1 leads to $M_{1n}^2 \gg 1$. Thus, eqs. (5.14) to (5.16) give

$$\frac{w_2}{w_1} = \frac{\gamma-1}{\gamma+1}, \tag{5.34}$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}, \tag{5.35}$$

and

$$M_{2n}^2 = \frac{\gamma-1}{2\gamma}. \tag{5.36}$$

5.2 Mach Waves

Assume a vehicle is moving at a Mach number greater than one. In fig. 5.5, the black line represents a section of the surface of the vehicle moving at supersonic speeds. At each instance in time, each infinitesimal point on this surface slightly distorts the stationary fluid and thus produces acoustic waves (blue line). These waves, by definition, expand at the speed of sound. The green line, which is aligned with the front of the acoustic waves, is referred to as a Mach wave or Mach line. The angle of inclination of this Mach wave, labeled μ, is called the Mach angle and satisfies

$$\sin(\mu) = \frac{1}{M}. \tag{5.37}$$

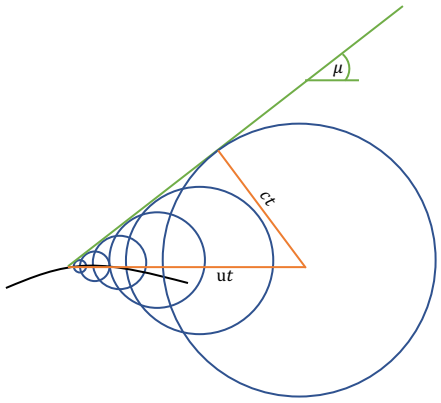

Figure 5.5: Mach wave

5.3 Contact discontinuities

A contact discontinuity is defined as a surface that separates two fluids of different properties or at different states. A defining condition is that there is no flow of matter across a contact discontinuity, that is, $w_1 = w_2 = 0$. Using this in the integral momentum conservation equation for a thin box bounding the discontinuity, and assuming negligible viscous stresses and heat fluxes, leads to $P_1 = P_2$. However, other flow properties, such as ρ, T, and v can change across the discontinuity.

Chapter 6

Quasi 1-D steady and unsteady flow

Quasi 1-D eqs; mass balance, f(M), and PAFMT; Fanno flow, Rayleigh flow.
Local acoustic speed, Expansion and Compression wave, Shock tube.

Chapter 7

2-D Compressible flow

Prandtl-Meyer (Compressive, Expansive); Shock-Expansion Theory: 2D Airfoil Calculations, Shock Reflection, Isentropic flow in curved channel, Nozzle exit flows.

Chapter 8

Hydrodynamic Instabilities

8.1 Linear Stability

Consider the following system of PDEs for a two-dimensional problem

$$\frac{\partial w}{\partial t} = \frac{\partial \phi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial w}{\partial y}, \tag{8.1}$$

$$w = \nabla^2 \phi. \tag{8.2}$$

In the above, $\phi = \phi(x, y, t)$ is the electrostatic potential and $w = w(x, y, t)$ the vorticity.

The analysis begins by splitting the variables into equilibrium and fluctuating components, namely

$$\phi = \phi_0 + \phi_1, \tag{8.3}$$

$$w = w_0 + w_1. \tag{8.4}$$

For the above, $\phi_0 = \phi_0(x)$, $w_0 = w_0(x)$, and $\phi_1 = \phi_1(x, y, t)$, $w_1 = w_1(x, y, t)$. We now introduce a Fourier series decomposition for the fluctuating variables, and focus on a single Fourier mode as follows

$$\phi_1(x, y, t) = F(x, k_y, t)e^{ik_y y} = \tilde{\phi}(x, k_y)e^{\gamma t + ik_y y}, \tag{8.5}$$

$$w_1(x, y, t) = G(x, k_y, t)e^{ik_y y} = \tilde{w}(x, k_y)e^{\gamma t + ik_y y}. \tag{8.6}$$

In the above, γ, which can be complex, is the growth rate factor, and $\tilde{\phi} = \tilde{\phi}(x, k_y)$, $\tilde{w} = \tilde{w}(x, k_y)$ are the remaining part of the Fourier coefficient.

We then plug in the decompositions for ϕ and w in the governing PDEs. Collecting the lowest order terms leads to equations for the equilibrium solution. For example, the Poisson equation to lowest order is

$$w_0 = \nabla^2 \phi_0 = \frac{\partial^2 \phi_0}{\partial x^2}. \tag{8.7}$$

Combining terms up to next order gives

$$\frac{\partial w_1}{\partial t} = \frac{\partial \phi_0}{\partial y}\frac{\partial w_1}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial w_0}{\partial x} - \frac{\partial \phi_0}{\partial x}\frac{\partial w_1}{\partial y} - \frac{\partial \phi_1}{\partial x}\frac{\partial w_0}{\partial y}, \tag{8.8}$$

$$w_1 = \nabla^2 \phi_1. \tag{8.9}$$

Using the expression for ϕ_1 in the Poisson equation above leads to

$$w_1 = \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi}\right) e^{\gamma t + ik_y y}, \tag{8.10}$$

or

$$\tilde{w} = \frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi}. \tag{8.11}$$

We'll now evaluate each of the terms in eq. (8.8).

$$\begin{aligned}
\frac{\partial w_1}{\partial t} &= \gamma \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) e^{\gamma t + i k_y y}, \\
\frac{\partial \phi_0}{\partial y} \frac{\partial w_1}{\partial x} &= 0, \\
\frac{\partial \phi_1}{\partial y} \frac{\partial w_0}{\partial x} &= i k_y \tilde{\phi} e^{\gamma t + i k_y y} \frac{\partial^3 \phi_0}{\partial x^3}, \\
\frac{\partial \phi_0}{\partial x} \frac{\partial w_1}{\partial y} &= \frac{\partial \phi_0}{\partial x} i k_y \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) e^{\gamma t + i k_y y}, \\
\frac{\partial \phi_1}{\partial x} \frac{\partial w_0}{\partial y} &= 0.
\end{aligned} \tag{8.12}$$

Combining all of the above, we obtain

$$\gamma \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) = i k_y \tilde{\phi} \frac{\partial^3 \phi_0}{\partial x^3} - \frac{\partial \phi_0}{\partial x} i k_y \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right). \tag{8.13}$$

This is re-written as

$$\gamma \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) = -i k_y \left[\frac{\partial \phi_0}{\partial x} \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\phi} \right) - \frac{\partial^3 \phi_0}{\partial x^3} \tilde{\phi} \right]. \tag{8.14}$$

Bibliography

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