

# Blast

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## 1 Governing equations

We consider Lagrangian fluid particles, for which we define the position  $\mathbf{x}^+ = \mathbf{x}^+(t, \mathbf{y})$ , the determinant of the Jacobian  $J^+ = J^+(t, \mathbf{y})$ , density  $\rho^+ = \rho^+(t, \mathbf{y})$ , velocity  $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$ , and internal energy  $e^+ = e^+(t, \mathbf{y})$ . The Eulerian counterparts for the density, velocity, and internal energy are, respectively,  $\rho = \rho(t, \mathbf{x})$ ,  $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ , and  $e = e(t, \mathbf{x})$ . The governing equations for the Lagrangian fluid particles are derived in my hydrodynamics notes (see section on kinematics, Lagrangian governing equations, etc.). These are shown below

$$\frac{\partial \mathbf{x}^+}{\partial t} = \mathbf{u}^+, \quad (1)$$

$$\frac{\partial J^+}{\partial t} = J^+ (\nabla \cdot \mathbf{u})_{\mathbf{x}=\mathbf{x}^+}, \quad (2)$$

$$\frac{\partial J^+ \rho^+}{\partial t} = 0, \quad (3)$$

$$\rho^+ \frac{\partial \mathbf{u}^+}{\partial t} = (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+}, \quad (4)$$

$$\rho^+ \frac{\partial e^+}{\partial t} = (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+}. \quad (5)$$

A note on notation. The products that involve a tensor  $\boldsymbol{\tau}$  can be expressed in Einstein notation as

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}, \quad (6)$$

$$\boldsymbol{\tau} \cdot \nabla \alpha = \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \quad (7)$$

$$\mathbf{f} \cdot \boldsymbol{\tau} \cdot \nabla \alpha = f_i \tau_{ij} \frac{\partial \alpha}{\partial x_j}, \quad (8)$$

$$\boldsymbol{\tau} : \nabla \mathbf{f} = \tau_{ij} \frac{\partial f_i}{\partial x_j}. \quad (9)$$

where  $\alpha$  is a scalar and  $\mathbf{f}$  a vector. In these notes we'll mostly be using indices  $i$  and  $j$  for FE expansions, rather than for Einstein notation.

## 2 Finite element expansion

We introduce the coefficients  $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_i(t)$ ,  $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(t)$  and  $\hat{e}_i = \hat{e}_i(t)$ , as well as the Lagrangian basis functions  $\phi_i^+ = \phi_i^+(\mathbf{y}) \in L^2$ , and  $w_i^+ = w_i^+(\mathbf{y}) \in H^1$ . We note that  $\hat{\mathbf{x}}_i$  and  $\hat{\mathbf{u}}_i$  are each vectors, e.g., the components of  $\hat{\mathbf{u}}_i$  are  $\hat{u}_{i,\alpha} = \hat{u}_{i,\alpha}(t)$  for  $\alpha = x, y, z$ . We also note that  $\phi_i^+$  and  $w_i^+$  have Eulerian counterparts  $\phi_i = \phi_i(t, \mathbf{x})$  and  $w_i = w_i(t, \mathbf{x})$ , respectively (see more details in section on finite elements in my notes for numerical methods). The coefficients are used in the following expansions

$$\mathbf{x}^+ = \sum_j^{N_w} \hat{\mathbf{x}}_j w_j^+, \quad (10)$$

$$\mathbf{u}^+ = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+, \quad (11)$$

$$e^+ = \sum_j^{N_\phi} \hat{e}_j \phi_j^+. \quad (12)$$

We note that the expansion coefficients are the same for the Lagrangian and Eulerian variables. For example, for the Eulerian velocity, we have

$$\mathbf{u} = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j. \quad (13)$$

## 3 Semi-discrete equations for $\mathbf{x}^+$ and $\mathbf{J}^+$

## 4 Semi-discrete equation for $\rho^+$

## 5 Semi-discrete equation for $\mathbf{u}^+$

We begin by showing that

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \sum_j^{N_w} \left( \frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \frac{\partial w_j}{\partial t} \right) + \mathbf{u} \cdot \left( \sum_j^{N_w} \hat{\mathbf{u}}_j \nabla w_j \right) \\ &= \sum_j^{N_w} \left[ \frac{d\hat{\mathbf{u}}_j}{dt} w_j + \hat{\mathbf{u}}_j \left( \frac{\partial w_j}{\partial t} + \mathbf{u} \cdot \nabla w_j \right) \right] \\ &= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j. \end{aligned} \quad (14)$$

The finite element formulation of the momentum equation is thus

$$\int_{\Omega} \rho \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j w_i dV = - \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i dV \quad \text{for } i = 1, \dots, N_w. \quad (15)$$

The above is re-written as

$$\sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)} = - \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i dV \quad \text{for } i = 1, \dots, N_w. \quad (16)$$

where the mass bilinear form  $m_{ij}^{(w)}$  is given by

$$m_{ij}^{(w)} = \int_{\Omega} \rho w_i w_j dV. \quad (17)$$

We now introduce the vector  $\mathbf{U}$  whose components are  $\hat{\mathbf{u}}_i$ . We also introduce the matrix  $\mathbf{M}^{(w)}$  whose components are  $m_{ij}^{(w)}$ . Thus, the left-hand side of eq. (16) can be written as  $\mathbf{M}^{(w)} d\mathbf{U}/dt$ . We also introduce the vector bilinear form

$$\mathbf{f}_{ij} = \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j dV. \quad (18)$$

This is a *vector* bilinear form since  $\mathbf{f}_{ij}$  has components  $f_{ij,\alpha} = f_{ij,\alpha}(t)$ , for  $\alpha = x, y, z$ , where  $\alpha$  denotes the first index of  $\boldsymbol{\sigma}$ . We introduce the matrix  $\mathbf{F}$ , whose components are  $\mathbf{f}_{ij}$ . We also expand the field with constant value of one as follows

$$1 = \sum_i^{N_{\phi}} \hat{c}_i \phi_i. \quad (19)$$

If we define the vector  $\mathbf{C}$  as that with components  $\hat{c}_i$ , we can show that

$$\begin{aligned} \mathbf{FC} &= \sum_j^{N_{\phi}} \mathbf{f}_{ij} \hat{c}_j && \text{for } i = 1, \dots, N_w \\ &= \sum_j^{N_{\phi}} \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j dV \hat{c}_j && \text{for } i = 1, \dots, N_w \\ &= \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i \left( \sum_j^{N_{\phi}} \hat{c}_j \phi_j \right) dV && \text{for } i = 1, \dots, N_w \\ &= \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_i dV && \text{for } i = 1, \dots, N_w \end{aligned} \quad (20)$$

The above is the negative of the right-hand side of eq. (16). Thus, combining all together we get

$$\mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} = -\mathbf{FC}. \quad (21)$$

We note that since both the Lagrangian and Eulerian velocities share the same coefficients  $\mathbf{U}$ , we now have a solution for both.

## 6 Semi-discrete equation for $e^+$

As with momentum conservation, we have

$$\begin{aligned}
\frac{\partial \mathbf{e}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{e} &= \sum_j^{N_\phi} \left( \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \frac{\partial \phi_j}{\partial t} \right) + \mathbf{u} \cdot \left( \sum_j^{N_\phi} \hat{\mathbf{e}}_j \nabla \phi_j \right) \\
&= \sum_j^{N_\phi} \left[ \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j + \hat{\mathbf{e}}_j \left( \frac{\partial \phi_j}{\partial t} + \mathbf{u} \cdot \nabla \phi_j \right) \right] \\
&= \sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j.
\end{aligned} \tag{22}$$

For the right-hand side of the energy conservation equation, we have

$$\boldsymbol{\sigma} : \nabla \mathbf{u} = \boldsymbol{\sigma} : \nabla \left( \sum_k^{N_w} \hat{\mathbf{u}}_k w_k \right) = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k. \tag{23}$$

The finite element formulation of the energy equation is thus

$$\begin{aligned}
\int_{\Omega} \rho \sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} \phi_j \phi_i dV &= \int_{\Omega} \left( \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k \right) \phi_i dV && \text{for } i = 1, \dots, N_w \\
&= \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla w_k \phi_i dV && \text{for } i = 1, \dots, N_w
\end{aligned} \tag{24}$$

The above is re-written as

$$\sum_j^{N_\phi} \frac{d\hat{\mathbf{e}}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} \quad \text{for } i = 1, \dots, N_w. \tag{25}$$

where the mass bilinear form  $m_{ij}^{(\phi)}$  is given by

$$m_{ij}^{(\phi)} = \int_{\Omega} \rho \phi_i \phi_j dV. \tag{26}$$

Note that in eq. (25) there is a dot product in the right-hand side, that is, the right-hand side expanded out is

$$\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} = \sum_k^{N_w} \sum_{\alpha=x,y,z} \hat{u}_{k,\alpha} f_{ki,\alpha}. \tag{27}$$

We now introduce the vector  $\mathbf{E}$  whose components are  $\hat{e}_i$ . We also introduce the matrix  $\mathbf{M}^{(\phi)}$  whose components are  $m_{ij}^{(\phi)}$ . Thus, eq. (25) can be succinctly written as

$$\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{U}. \tag{28}$$

Note again that on the right-hand side above there is a matrix-vector product *and* a dot product. We also note that since both the Lagrangian and Eulerian internal energies share the same coefficients  $\mathbf{E}$ , we now have a solution for both.