ALE finite-element hydrodynamics

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1 Lagrangian governing equations

We consider Lagrangian fluid particles, for which we define the position $\mathbf{x}^+ = \mathbf{x}^+(t, \mathbf{y})$, the density $\rho^+ = \rho^+(t, \mathbf{y})$, the velocity $\mathbf{u}^+ = \mathbf{u}^+(t, \mathbf{y})$, and the internal energy $e^+ = e^+(t, \mathbf{y})$, where $\mathbf{y}$ is the location of each fluid particle at time zero. The Eulerian counterparts for the density, velocity, and internal energy are, respectively, $\rho = \rho(t, \mathbf{x})$, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, and $e = e(t, \mathbf{x})$. Also consider the volume Ω_0 as the set of all $\mathbf{y}$ vectors that make up the initial domain. The control volume $\Omega^+ = \Omega^+(t, \Omega_0)$ is then defined by

$$\Omega^+ = \{\mathbf{x}^+ : \mathbf{y} \in \Omega_0\}. \tag{1}$$

Note that $\Omega^+(0, \Omega_0) = \Omega_0$.

The governing equations for the Lagrangian fluid particles are derived in my fluid-mechanics notes (see section on kinematics, Lagrangian governing equations, etc.). These are shown below

$$\frac{\partial \mathbf{x}^+}{\partial t} = \mathbf{u}^+, \tag{2}$$

$$\frac{\partial J^+ \rho^+}{\partial t} = 0, \tag{3}$$

$$\rho^+ \frac{\partial \mathbf{u}^+}{\partial t} = \left(\nabla \cdot \boldsymbol{\sigma}\right)_{\mathbf{x}=\mathbf{x}^+}, \tag{4}$$

$$\rho^+ \frac{\partial e^+}{\partial t} = \left(\boldsymbol{\sigma} : \nabla \mathbf{u}\right)_{\mathbf{x}=\mathbf{x}^+}. \tag{5}$$

In the above, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t, \mathbf{x})$ is the stress tensor, and $J^+ = J^+(t, \mathbf{y})$ is the determinant of the Jacobian matrix $\mathbf{J}^+ = \mathbf{J}^+(t, \mathbf{y})$, which itself is defined as $\mathbf{J}^+ = \partial \mathbf{x}^+ / \partial \mathbf{y}$.

A note on notation. The products that involve a tensor $\boldsymbol{\tau}$ can be expressed in Einstein notation as

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}, \tag{6}$$

$$\boldsymbol{\tau} \cdot \nabla f = \tau_{ij} \frac{\partial f}{\partial x_j}, \tag{7}$$

$$\mathbf{g} \cdot \boldsymbol{\tau} \cdot \nabla f = g_i \tau_{ij} \frac{\partial f}{\partial x_j}, \tag{8}$$

$$\boldsymbol{\tau} : \nabla \mathbf{g} = \tau_{ij} \frac{\partial g_i}{\partial x_j}. \tag{9}$$

where f is a scalar and $\mathbf{g}$ a vector. In these notes we'll mostly be using indices i and j for FE expansions, rather than for Einstein notation.

2 Lagrangian finite elements

We introduce a Lagrangian basis function $\Phi_i^+ = \Phi_i^+(t, \mathbf{y})$ and an Eulerian basis function $\Phi_i = \Phi_i(t, \mathbf{x})$. These are related to each other as any other Lagrangian-Eulerian pair, namely

$$\Phi_i^+(t, \mathbf{y}) = \Phi_i(t, \mathbf{x}^+(t, \mathbf{y})). \tag{10}$$

We now introduce the Lagrangian variable $f^+ = f^+(t, \mathbf{y})$ and the Eulerian counterpart $f = f(t, \mathbf{x})$, and they also satisfy

$$f^+(t, \mathbf{y}) = f(t, \mathbf{x}^+(t, \mathbf{y})). \tag{11}$$

The expansion of an Eulerian variable in terms of basis functions is as follows

$$f = \sum_i^n F_i \Phi_i, \tag{12}$$

where $F_i = F_i(t)$. Plugging in $\mathbf{x}^+$ for $\mathbf{x}$ in the above, and using eqs. (10) and (11) gives

$$f^+ = \sum_i^n F_i \Phi_i^+. \tag{13}$$

Thus, both the Lagrangian and Eulerian variables share the same finite-element coefficients F_i.

As shown in my fluid mechanics notes, we also have

$$\frac{\partial \Phi_i^+}{\partial t} = \left(\frac{\partial \Phi_i}{\partial t} + \mathbf{u} \cdot \nabla \Phi_i \right)_{\mathbf{x}=\mathbf{x}^+}, \tag{14}$$

where $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the Eulerian counterpart to $\mathbf{u}^+$. We'll introduce the restriction that Φ_i^+ is constant in time, that is $\partial \Phi_i^+ / \partial t = 0$, which gives

$$\frac{\partial \Phi_i}{\partial t} + \mathbf{u} \cdot \nabla \Phi_i = 0. \tag{15}$$

Thus, F_i in eq. (13) accounts for the time dependence of F^+, whereas Φ_i^+ accounts for the dependence on $\mathbf{y}$.

3 Finite element expansion

We introduce the coefficients $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_i(t)$, $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(t)$ and $\hat{e}_i = \hat{e}_i(t)$, as well as the Lagrangian basis functions $\phi_i^+ = \phi_i^+(\mathbf{y}) \in L^2$, and $w_i^+ = w_i^+(\mathbf{y}) \in H^1$. We note that $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{u}}_i$ are each vectors, e.g., the components of $\hat{\mathbf{u}}_i$ are $\hat{u}_{i,\alpha} = \hat{u}_{i,\alpha}(t)$ for $\alpha = x, y, z$. We also note that ϕ_i^+ and w_i^+ have Eulerian counterparts $\phi_i = \phi_i(t, \mathbf{x})$ and $w_i = w_i(t, \mathbf{x})$, respectively. The coefficients are used in the following expansions

$$\mathbf{x}^+ = \sum_j^{N_w} \hat{\mathbf{x}}_j w_j^+, \tag{16}$$

$$\mathbf{u}^+ = \sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+, \tag{17}$$

$$e^{+} = \sum_{j}^{N_\phi} \hat{e}_j \phi_j^{+}. \tag{18}$$

We note that the expansion coefficients are the same for the Lagrangian and Eulerian variables, as shown in section 2. For example, for the Eulerian velocity, we have

$$\mathbf{u} = \sum_{j}^{N_w} \hat{\mathbf{u}}_j w_j. \tag{19}$$

4 Semi-discrete Lagrangian governing equations

4.1 Position and Jacobian

Plugging in eqs. (16) and (17) in eq. (2) gives

$$\sum_{j}^{N_w} \frac{d\hat{\mathbf{x}}_j}{dt} w_j^{+} = \sum_{j}^{N_w} \hat{\mathbf{u}}_j w_j^{+}. \tag{20}$$

To satisfy the equation above, we'll require

$$\frac{d\hat{\mathbf{x}}_j^{+}}{dt} = \hat{\mathbf{u}}_j. \tag{21}$$

We now introduce the vectors $\mathbf{X}$ and $\mathbf{U}$, whose components are $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{u}}_i$, respectively. Thus, the above is written as

$$\frac{d\mathbf{X}}{dt} = \mathbf{U}. \tag{22}$$

To obtain $\mathbf{J}^{+}$ we plug in eq. (16) into its definition, that is

$$\mathbf{J}^{+} = \frac{\partial}{\partial \mathbf{y}} \sum_{j}^{N_w} \hat{\mathbf{x}}_j w_j^{+} = \sum_{j}^{N_w} \hat{\mathbf{x}}_j \nabla_{\mathbf{y}} w_j^{+}. \tag{23}$$

Note that for any function $\mathbf{x}^{+}$, whether it be an exact analytical expression or a finite-element expansion as given by eq. (16), one can derive the following equation for the determinant of the Jacobian

$$\frac{\partial J^{+}}{\partial t} = J^{+} \left(\frac{\partial u_k}{\partial x_k} \right)_{\mathbf{x}=\mathbf{x}^{+}}, \tag{24}$$

In the above $\mathbf{u}$ is the Eulerian counterpart to $\mathbf{u}^{+}$, which is given by eq. (2).

4.2 Density

Equation (3) allows us to write

$$\rho^{+} = \frac{\rho_0^{+}}{J^{+}}, \tag{25}$$

where $\rho_0^{+} = \rho^{+}(0, \mathbf{y})$.

4.3 Velocity

Plugging in eq. (25) in eq. (4) we get

$$\rho_0^+ \frac{\partial \mathbf{u}^+}{\partial t} = (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} J^+. \tag{26}$$

We then multiply both sides of the above by the basis functions for velocity and integrate over all space to obtain

$$\int_{\Omega_0} \rho_0^+ \frac{\partial \mathbf{u}^+}{\partial t} w_i^+ \, dV_y = \int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} w_i^+ J^+ \, dV_y. \tag{27}$$

For the left-hand side we have

$$\begin{aligned}
\int_{\Omega_0} \rho_0^+ \frac{\partial \mathbf{u}^+}{\partial t} w_i^+ \, dV_y &= \int_{\Omega_0} \rho_0^+ \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} w_j^+ w_i^+ \, dV_y, \\
&= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} \int_{\Omega_0} \rho_0^+ w_i^+ w_j^+ \, dV_y, \\
&= \sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)},
\end{aligned} \tag{28}$$

where

$$m_{ij}^{(w)} = \int_{\Omega_0} \rho_0^+ w_i^+ w_j^+ \, dV_y \tag{29}$$

is a mass bilinear form (which is independent of time). For the right-hand side we have

$$\begin{aligned}
\int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma})_{\mathbf{x}=\mathbf{x}^+} w_i^+ J^+ \, dV_y &= \int_{\Omega_0} (\nabla \cdot \boldsymbol{\sigma} w_i)_{\mathbf{x}=\mathbf{x}^+} J^+ \, dV_y \\
&= \int_{\Omega^+} \nabla \cdot \boldsymbol{\sigma} w_i \, dV_x \\
&= -\int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \, dV_x.
\end{aligned} \tag{30}$$

The second equality above follows from integration by substitution. Combining results we have

$$\sum_j^{N_w} \frac{d\hat{\mathbf{u}}_j}{dt} m_{ij}^{(w)} = -\int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \, dV_x. \tag{31}$$

We introduce the matrix $\mathbf{M}^{(w)}$ whose components are $m_{ij}^{(w)}$. Thus, the left-hand side of eq. (31) can be written as $\mathbf{M}^{(w)} \, d\mathbf{U}/dt$. We also introduce the vector bilinear form

$$\mathbf{f}_{ij} = \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV_x. \tag{32}$$

This is a *vector* bilinear form since $\mathbf{f}_{ij}$ has components $f_{ij,\alpha} = f_{ij,\alpha}(t)$, for $\alpha = x, y, z$, where α denotes the first index of $\boldsymbol{\sigma}$. We introduce the force matrix $\mathbf{F}$, whose components are $\mathbf{f}_{ij}$. We also expand the field with constant value of one as follows

$$1 = \sum_{i}^{N_\phi} \hat{c}_i \phi_i. \tag{33}$$

If we define the vector $\mathbf{C}$ as that with components $\hat{c}_i$, we can show that

$$\begin{aligned}
\mathbf{FC} &= \sum_{j}^{N_\phi} \mathbf{f}_{ij} \hat{c}_j \\
&= \sum_{j}^{N_\phi} \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV_x \hat{c}_j \\
&= \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \left(\sum_{j}^{N_\phi} \hat{c}_j \phi_j \right) dV_x \\
&= \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \, dV_x.
\end{aligned} \tag{34}$$

The above is the negative of the right-hand side of eq. (31). Thus, combining all together we get

$$\mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} = -\mathbf{FC}. \tag{35}$$

We note that since both the Lagrangian and Eulerian velocities share the same coefficients $\mathbf{U}$, we now have a solution for both.

4.4 Energy

Plugging in eq. (25) in eq. (5) we get

$$\rho_0^+ \frac{\partial e^+}{\partial t} = (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} J^+. \tag{36}$$

We then multiply both sides of the above by the basis functions for energy and integrate over all space to obtain

$$\int_{\Omega_0} \rho_0^+ \frac{\partial e^+}{\partial t} \phi_i^+ \, dV_y = \int_{\Omega_0} (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} \phi_i^+ J^+ \, dV_y. \tag{37}$$

For the left-hand side we have

$$\begin{aligned}
\int_{\Omega_0} \rho_0^+ \frac{\partial e^+}{\partial t} \phi_i^+ \, dV_y &= \int_{\Omega_0} \rho_0^+ \sum_{j}^{N_\phi} \frac{d\hat{e}_j}{dt} \phi_j^+ \phi_i^+ \, dV_y, \\
&= \sum_{j}^{N_\phi} \frac{d\hat{e}_j}{dt} \int_{\Omega_0} \rho_0^+ \phi_j^+ \phi_i^+ \, dV_y, \\
&= \sum_{j}^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)}
\end{aligned} \tag{38}$$

where

$$m_{ij}^{(\phi)} = \int_{\Omega_0} \rho_0^+ \phi_j^+ \phi_i^+ \, dV_y \tag{39}$$

is a mass bilinear form (which is independent of time). For the right-hand side we have

$$\begin{aligned}\int_{\Omega_0} (\boldsymbol{\sigma} : \nabla \mathbf{u})_{\mathbf{x}=\mathbf{x}^+} \phi_i^+ J^+ \, dV_y &= \int_{\Omega_0} (\boldsymbol{\sigma} : \nabla \mathbf{u} \phi_i)_{\mathbf{x}=\mathbf{x}^+} J^+ \, dV_y \\ &= \int_{\Omega^+} \boldsymbol{\sigma} : \nabla \mathbf{u} \phi_i \, dV_x. \end{aligned} \tag{40}$$

Combining results we have

$$\sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)} = \int_{\Omega^+} \boldsymbol{\sigma} : \nabla \mathbf{u} \phi_i \, dV_x. \tag{41}$$

We no show that

$$\boldsymbol{\sigma} : \nabla \mathbf{u} = \boldsymbol{\sigma} : \nabla \left(\sum_k^{N_w} \hat{\mathbf{u}}_k w_k \right) = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \boldsymbol{\sigma} \cdot \nabla w_k, \tag{42}$$

and hence the previous result is written as

$$\sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_k \phi_i \, dV_x. \tag{43}$$

The above is finally re-written as

$$\sum_j^{N_\phi} \frac{d\hat{e}_j}{dt} m_{ij}^{(\phi)} = \sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki}. \tag{44}$$

Note that in the above there is a dot product in the right-hand side, that is, the right-hand side expanded out is

$$\sum_k^{N_w} \hat{\mathbf{u}}_k \cdot \mathbf{f}_{ki} = \sum_k^{N_w} \sum_{\alpha=x,y,z} \hat{u}_{k,\alpha} f_{ki,\alpha}. \tag{45}$$

We now introduce the vector $\mathbf{E}$ whose components are $\hat{e}_i$. We also introduce the matrix $\mathbf{M}^{(\phi)}$ whose components are $m_{ij}^{(\phi)}$. Thus, eq. (44) can be succinctly written as

$$\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{U}. \tag{46}$$

Note again that on the right-hand side above there is a matrix-vector product *and* a dot product. We also note that since both the Lagrangian and Eulerian internal energies share the same coefficients $\mathbf{E}$, we now have a solution for both.

5 Momentum and energy conservation

We'll now define the internal energy $IE = IE(t)$, the kinetic energy $KE = KE(t)$, and the momentum $P_{\mathbf{n}} = P_{\mathbf{n}}(t)$ along a constant $\mathbf{n}$ direction.

$$
\begin{aligned}
IE &= \int_{\Omega^+} \rho e \, dV_x \\
&= \int_{\Omega_0} \rho^+ e^+ J^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ e^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ \sum_j^{N_\phi} \hat{e}_j \phi_j^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ \sum_j^{N_\phi} \hat{e}_j \phi_j^+ \left(\sum_i^{N_\phi} \hat{c}_i \phi_i^+ \right) dV_y \\
&= \sum_i^{N_\phi} \sum_j^{N_\phi} \hat{c}_i \int_{\Omega_0} \rho_0^+ \phi_i^+ \phi_j^+ \, dV_y \hat{e}_j \\
&= \sum_i^{N_\phi} \sum_j^{N_\phi} \hat{c}_i m_{ij}^{(\phi)} \hat{e}_j \\
&= \mathbf{C} \mathbf{M}^{(\phi)} \mathbf{E}
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
KE &= \int_{\Omega^+} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \, dV_x \\
&= \int_{\Omega_0} \frac{1}{2} \rho^+ \mathbf{u}^+ \cdot \mathbf{u}^+ J^+ \, dV_y \\
&= \int_{\Omega_0} \frac{1}{2} \rho_0^+ \mathbf{u}^+ \cdot \mathbf{u}^+ \, dV_y \\
&= \int_{\Omega_0} \frac{1}{2} \rho_0^+ \left(\sum_i^{N_w} \hat{\mathbf{u}}_i w_i^+ \right) \cdot \left(\sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+ \right) dV_y \\
&= \sum_i^{N_w} \sum_j^{N_w} \frac{1}{2} \hat{\mathbf{u}}_i \cdot \int_{\Omega_0} \rho_0^+ w_i^+ w_j^+ \, dV_y \hat{\mathbf{u}}_j \\
&= \sum_i^{N_w} \sum_j^{N_w} \frac{1}{2} \hat{\mathbf{u}}_i \cdot m_{ij}^{(w)} \hat{\mathbf{u}}_j \\
&= \frac{1}{2} \mathbf{U} \cdot \mathbf{M}^{(w)} \mathbf{U}.
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
P_{\mathbf{n}} &= \int_{\Omega^+} \rho \mathbf{u} \cdot \mathbf{n} \, dV_x \\
&= \int_{\Omega_0} \rho^+ \mathbf{u}^+ \cdot \mathbf{n} J^+ \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ \mathbf{u}^+ \cdot \mathbf{n} \, dV_y \\
&= \int_{\Omega_0} \rho_0^+ \left(\sum_j^{N_w} \hat{\mathbf{u}}_j w_j^+ \right) \cdot \left(\sum_i^{N_w} \hat{\mathbf{n}}_i w_i^+ \right) \, dV_y \\
&= \sum_i^{N_w} \sum_j^{N_w} \hat{\mathbf{n}}_i \cdot \int_{\Omega_0} \rho_0^+ w_i^+ w_j^+ \, dV_y \hat{\mathbf{u}}_j \\
&= \sum_i^{N_w} \sum_j^{N_w} \hat{\mathbf{n}}_i \cdot m_{ij}^{(w)} \hat{\mathbf{u}}_j \\
&= \mathbf{N} \cdot \mathbf{M}^{(w)} \mathbf{U}.
\end{aligned} \tag{49}
$$

The total energy is conserved, as shown below

$$
\begin{aligned}
\frac{d}{dt}(IE + KE) &= \mathbf{C}\mathbf{M}^{(\phi)} \frac{d\mathbf{E}}{dt} + \mathbf{U} \cdot \mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} \\
&= \mathbf{C}\mathbf{F}^T \cdot \mathbf{U} - \mathbf{U} \cdot \mathbf{F}\mathbf{C} \\
&= 0.
\end{aligned} \tag{50}
$$

The momentum along a constant direction is conserved, as shown below

$$
\begin{aligned}
\frac{dP_{\mathbf{n}}}{dt} &= \mathbf{N} \cdot \mathbf{M}^{(w)} \frac{d\mathbf{U}}{dt} \\
&= -\mathbf{N} \cdot \mathbf{F}\mathbf{C} \\
&= -\sum_i^{N_w} \sum_j^{N_\phi} \hat{\mathbf{n}}_i \cdot \mathbf{f}_{ij} \hat{c}_j \\
&= -\sum_i^{N_w} \sum_j^{N_\phi} \hat{\mathbf{n}}_i \cdot \int_{\Omega^+} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV_x \hat{c}_j \\
&= -\int_{\Omega^+} \boldsymbol{\sigma} : \nabla \mathbf{n} \, dV_x \\
&= 0.
\end{aligned} \tag{51}
$$

6 The reference element

We introduce the reference element as the unit square in 2D or the unit cube in 3D. The domain of this reference element is labelled as Ω_z and it doesn't change with time. We introduce the function

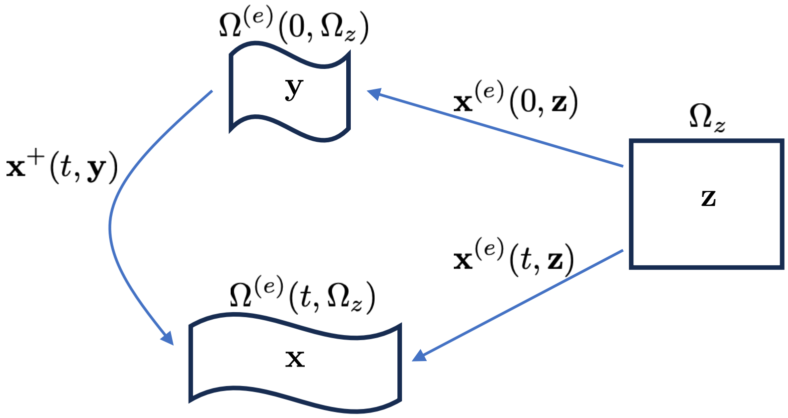

Figure 1: Schematic of the three domains Ω_z, $\Omega^{(e)}(t, \Omega_z)$, $\Omega^{(e)}(0, \Omega_z)$.

$\mathbf{x}^{(e)} = \mathbf{x}^{(e)}(t, \mathbf{z})$, which maps from points $\mathbf{z}$ in Ω_z to points in the finite element e of the mesh. The evolving domain of the finite element e is giving by the function $\Omega^{(e)} = \Omega^{(e)}(t, \Omega_z)$. A depiction of these domains and their mappings is shown in fig. 1. Whereas for Ω^+ we had $\Omega^+(0, \Omega_0) = \Omega_0$, for $\Omega^{(e)}$ the analogue does not hold, that is, $\Omega^{(e)}(0, \Omega_z) \neq \Omega_z$.

The mapping functions $\mathbf{x}^{(e)}$ and $\mathbf{x}^+$ are related to each other as follows

$$\mathbf{x}^{(e)}(t, \mathbf{z}) = \mathbf{x}^+(t, \mathbf{x}^{(e)}(0, \mathbf{z})). \tag{52}$$

The Jacobian $\mathbf{J}^{(e)} = \mathbf{J}^{(e)}(t, \mathbf{z})$ is defined as $\mathbf{J}^{(e)} = \partial \mathbf{x}^{(e)} / \partial \mathbf{z}$, and label its determinant as $J^{(e)} = J^{(e)}(t, \mathbf{z})$. Using eq. (52) in the definition of $\mathbf{J}^{(e)}$ we get

$$\begin{aligned} \mathbf{J}^{(e)} &= \left(\frac{\partial \mathbf{x}^+}{\partial \mathbf{y}} \right)_{\mathbf{y}=\mathbf{x}^{(e)}(0,\mathbf{z})} \frac{\partial \mathbf{x}^{(e)}(0, \mathbf{z})}{\partial \mathbf{z}} \\ &= \left(\mathbf{J}^+ \right)_{\mathbf{y}=\mathbf{x}^{(e)}(0,\mathbf{z})} \mathbf{J}_0^{(e)}, \end{aligned} \tag{53}$$

where $\mathbf{J}_0^{(e)} = \mathbf{J}^{(e)}(0, \mathbf{z})$. Taking the determinant of the above gives

$$J^{(e)} = \left(J^+ \right)_{\mathbf{y}=\mathbf{x}^{(e)}(0,\mathbf{z})} J_0^{(e)}, \tag{54}$$

where $J_0^{(e)} = J^{(e)}(0, \mathbf{z})$.

A Lagrangian variable $f^+ = f^+(t, \mathbf{y})$ has a corresponding reference-element function $f^{(e)} = f^{(e)}(t, \mathbf{z})$, which satisfies

$$f^{(e)}(t, \mathbf{z}) = f^+(t, \mathbf{x}^{(e)}(0, \mathbf{z})). \tag{55}$$

Now, $f^+(t, \mathbf{x}^{(e)}(0, \mathbf{z})) = f(t, \mathbf{x}^+(t, \mathbf{x}^{(e)}(0, \mathbf{z})))$. Using eq. (52), we also get

$$f^{(e)}(t, \mathbf{z}) = f(t, \mathbf{x}^{(e)}(t, \mathbf{z})) \tag{56}$$

Examples of these reference-element functions include those for density $\rho^{(e)} = \rho^{(e)}(t, \mathbf{z})$, velocity $\mathbf{u}^{(e)} = \mathbf{u}^{(e)}(t, \mathbf{z})$, and internal energy $e^{(e)} = e^{(e)}(t, \mathbf{z})$. Using integration by substitution and then eq. (56) we show

$$\int_{\Omega^{(e)}} f dV_x = \int_{\Omega_z} f(t, \mathbf{x}^{(e)}(t, \mathbf{z}))J^{(e)}\, dV_z$$
$$= \int_{\Omega_z} f^{(e)} J^{(e)}\, dV_z. \tag{57}$$

In other words, integrals over elements at any time can be computed as integrals over the reference space.

If the integrand contains a derivative, a bit of extra care is required. To show this, we'll use index notation for the sake of clarity. Consider as an example a term of the form

$$(\boldsymbol{\sigma} \cdot \nabla f)_{\mathbf{x}=\mathbf{x}^{(e)}} = \left(\sigma_{ij} \frac{\partial f}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}^{(e)}} = \sigma_{ij}^{(e)} \left(\frac{\partial f}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}^{(e)}}. \tag{58}$$

We first note that

$$\frac{\partial f^{(e)}}{\partial z_k} = \left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}^{(e)}} \frac{\partial x_i^{(e)}}{\partial z_k} = \left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}^{(e)}} J_{ik}^{(e)}. \tag{59}$$

Upon multiplying both sides by the inverse of $\mathbf{J}^{(e)}$, we get

$$\left(\frac{\partial f}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}^{(e)}} = \frac{\partial f^{(e)}}{\partial z_k} \left(J^{(e)}\right)_{kj}^{-1}. \tag{60}$$

Thus, we now have

$$(\boldsymbol{\sigma} \cdot \nabla f)_{\mathbf{x}=\mathbf{x}^{(e)}} = \sigma_{ij}^{(e)} \frac{\partial f^{(e)}}{\partial z_k} \left(J^{(e)}\right)_{kj}^{-1} = \sigma_{ij}^{(e)} \left[\left(J^{(e)}\right)^{-1}\right]_{jk}^{T} \frac{\partial f^{(e)}}{\partial z_k}. \tag{61}$$

In tensor notation, the above is written as

$$(\boldsymbol{\sigma} \cdot \nabla f)_{\mathbf{x}=\mathbf{x}^{(e)}} = \boldsymbol{\sigma}^{(e)} \cdot \left[\left(\mathbf{J}^{(e)}\right)^{-1}\right]^{T} \cdot \nabla_{\mathbf{z}} f^{(e)}. \tag{62}$$

Thus, for the force matrix $\mathbf{f}_{ij}$ we can now write

$$\int_{\Omega^{(e)}} \boldsymbol{\sigma} \cdot \nabla w_i \phi_j \, dV_x = \int_{\Omega_{\mathbf{z}}} (\boldsymbol{\sigma} \cdot \nabla w_i \phi_j)_{\mathbf{x}=\mathbf{x}^{(e)}} J^{(e)}\, dV_z$$
$$= \int_{\Omega_{\mathbf{z}}} \boldsymbol{\sigma}^{(e)} \cdot \left[\left(\mathbf{J}^{(e)}\right)^{-1}\right]^{T} \cdot \nabla_{\mathbf{z}} w_i^{(e)} \phi_j^{(e)} J^{(e)}\, dV_z. \tag{63}$$

Finally, we note that we can evaluate eq. (25) at $\mathbf{y} = \mathbf{x}^{(e)}(0, \mathbf{z})$ to obtain

$$\rho^{(e)} = \frac{\rho_0^{(e)} J_0^{(e)}}{J^{(e)}}. \tag{64}$$