Analytical Integration of a tunneling Wavefunction

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1 Setting up the problem

The problem is set up as follows, a guassian wavepacket intitially centered at xnot (in the first domain) travels in the postive x direction to interact with a barrier at x postion 0 to d (2nd section), some of the wavepacket is refelcted back into the first domain while some of the wavepacket tunnels through the barrier into section 3.

Section 1

$$\Psi(x,t) = \int\limits_{-\infty}^{\infty} A_I e^{ikx} + A_R e^{-ikx} \tag{1}$$

Section 2

$$\Psi(x,t) = \int\limits_{-\infty}^{\infty} B_I e^{\kappa x} + B_R e^{-\kappa x} \tag{2}$$

Section 3

$$\Psi(x,t) = \int\limits_{-\infty}^{\infty} A_T e^{ikx} \tag{3}$$

after apply continutiy at x=0 and x=d the following equations can be found.

2 First domain

Here basic integration techniques are used to yeild an exact result of the wavefunction in the first domain (before the barrier). seting up the equation

$$\Psi_1(x,t) = \int_{-\infty}^{\infty} (A_I e^{ik(x-x_0)} + w(k) A_I e^{-ik(x+x_0)}) \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \tag{4}$$

$$w(k) = \frac{k_0^2}{2k^2 - k_0^2 + 2ik\sqrt{k_0^2 - k^2} coth(\sqrt{k_0^2 - k^2}d)} \tag{5}$$

$$= \frac{A_I}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x_0)} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk + \frac{A_I}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} w(k) e^{-ik(x+x_0)} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \tag{6}$$

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$$k' = k - q \tag{7}$$

$$dk' = dk \tag{8}$$

$$\Psi_1 = \Psi_t + \Psi_r \tag{9}$$

$$\Psi_t = \frac{A_I e^{iq(x-x_0)} e^{\frac{-iq^2\hbar t}{2m}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik'(x-x_0)} e^{\frac{-(k')^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m} dk' \tag{10}$$

transmitted wavefunction

$$\Psi_t = \frac{A_I e^{iq(x-x_0)} e^{\frac{-iq^2\hbar t}{2m}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik'(x-x_0-\frac{q\hbar t}{m})} e^{k'^2(\frac{-1}{2\sigma^2}-\frac{i\hbar t}{2m})} dk' \tag{11}$$

$$= \frac{A_I e^{iq(x-x_0)} e^{\frac{-iq^2\hbar t}{2m}} e^{\frac{-(x-x_0-\frac{q\hbar t}{m})^2 m\sigma^2}{2(i\hbar\sigma^2 t+m)}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{k'\sqrt{i\hbar\sigma^2 t+m}}{\sqrt{2m\sigma^2}}+\frac{-i(x-x_0-\frac{q\hbar t}{m})\sqrt{m}\sigma}{\sqrt{2i\hbar\sigma^2 t+2m}})^2} dk' \tag{12}$$

$$= \frac{A_I e^{iq(x-x_0)} e^{\frac{-iq^2\hbar t}{2m}} e^{\frac{-(x-x_0-\frac{q\hbar t}{m})^2 m\sigma^2}{2(i\hbar\sigma^2 t+m)}}}{\sigma\sqrt{2\pi}\frac{\sqrt{i\hbar\sigma^2 t+m}}{\sqrt{2m\sigma^2}}} \int_{-\infty}^{\infty} e^{-k''^2} dk'' \tag{13}$$

$$\Psi_t = \frac{A_I\sqrt{m} e^{iq(x-x_0)} e^{\frac{-iq^2\hbar t}{2m}} e^{\frac{-(x-x_0-\frac{q\hbar t}{m})^2 m\sigma^2}{2(i\hbar\sigma^2 t+m)}}}{\sqrt{i\hbar\sigma^2 t+m}} \tag{14}$$

Approximation of w

$$w(k) \approx w(q) + w'(q)(k-q) \tag{15}$$

$$w(q) = \frac{k_0^2}{2q^2 - k_0^2 + 2iq\sqrt{k_0^2-q^2}coth(\sqrt{k_0^2-q^2}d)} \tag{16}$$

$$w'(q) = \frac{2ik_0^2((k_0^2-2q^2)coth(\sqrt{k_0^2-q^2}d) + q\sqrt{k_0^2-q^2}(dqcsch^2(\sqrt{k_0^2-q^2}d)-2i))}{\sqrt{k_0^2-q^2}(2q\sqrt{k_0^2-q^2}coth(\sqrt{k_0^2-q^2}d)+i(k_0^2-2q^2))^2} \tag{17}$$

reflected wavefunction

$$\Psi_r = \int_{-\infty}^{\infty} w(k) A_I e^{-ik(x+x_0)} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \qquad (18)$$

$$\Psi_r \approx \frac{A_I e^{-iq(x+x_0)} e^{\frac{-iq^2\hbar t}{2m}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (w(q) + w'(q)k') e^{-ik'(x+x_0)} e^{\frac{-(k')^2}{2\sigma^2} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}} dk'$$
$$(19)$$

$$= \frac{A_I e^{-iq(x+x_0)} e^{\frac{-iq^2\hbar t}{2m}} e^{\frac{-(x+x_0+\frac{q\hbar t}{m})^2\sigma^2 m}{2i\hbar t\sigma^2+2m}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (w(q)+w'(q)k') e^{-(\sqrt{\frac{i\hbar t\sigma^2+m}{2\sigma^2 m}}k'+i(x+x_0+\frac{q\hbar t}{m})\sqrt{\frac{\sigma^2 m}{2i\hbar t\sigma^2+2m}})^2} dk'$$

$$= \frac{A_I e^{-iq(x+x_0)} e^{\frac{-iq^2\hbar t}{2m}} e^{\frac{-(x+x_0+\frac{q\hbar t}{m})^2\sigma^2 m}{2i\hbar t\sigma^2+2m}}}{\sigma\sqrt{2\pi}\sqrt{\frac{i\hbar t\sigma^2+m}{2\sigma^2 m}}} \int_{-\infty}^{\infty} (w(q)+w'(q)(\frac{k''-i(x+x_0+\frac{q\hbar t}{m})\sqrt{\frac{\sigma^2 m}{2i\hbar t\sigma^2+2m}}}{\sqrt{\frac{i\hbar t\sigma^2+m}{2\sigma^2 m}}}) e^{-(k'')^2} dk''$$

$$\Psi_r \approx \frac{A_I\sqrt{m} e^{-iq(x+x_0)} e^{\frac{-iq^2\hbar t}{2m}} e^{\frac{-(x+x_0+\frac{q\hbar t}{m})^2\sigma^2 m}{2i\hbar t\sigma^2+2m}} (w(q)-w'(q)(i(x+x_0+\frac{q\hbar t}{m})\frac{\sigma^2 m}{i\hbar t\sigma^2+m}))}{\sqrt{i\hbar\sigma^2 t+m}}$$
$$(20)$$

3 second domain

Here an approximative method is used to determine the wavefunction of the wavepacket in the second domain (within the barrier). x-x0

$$\Psi_2(x,t) = A_I \int_{-\infty}^{\infty} e^{-ikx_0}(((\kappa-ik)+w(k)(\kappa+ik))e^{-\kappa x}+((\kappa+ik)+w(k)(-ik+\kappa))e^{\kappa x})\frac{1}{2\kappa\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$
(21)

$$\Psi_2(x,t) = A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{1}{2})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{-ik}{2\kappa})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{w(k)}{2})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{iw(k)k}{2\kappa})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{1}{2})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{ik}{2\kappa})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{-iw(k)k}{2\kappa})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{w(k)}{2})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$
(22)

$$w(k) \approx w(q) + w'(q)(k-q) \tag{23}$$

$$\Psi_2(x,t) \approx A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{1}{2})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{-ik}{2\kappa})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{w(q)+w'(q)(k-q)}{2})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{i(w(q)+w'(q)(k-q))k}{2\kappa})e^{-\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{1}{2})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{ik}{2\kappa})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{-i(w(q)+w'(q)(k-q))k}{2\kappa})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{w(q)+w'(q)(k-q)}{2})e^{\kappa x}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk \tag{24}$$

$$\Psi_2(x,t) \approx \frac{A_I(1+w(q)-w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}e^{-\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$+\frac{A_I i(-1+w(q)-w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}\frac{k}{\kappa}e^{-\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}ke^{-\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$+\frac{A_I i w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}\frac{k^2}{\kappa}e^{-\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$+\frac{A_I(1+w(q)-w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}e^{\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$+\frac{A_I i(1-w(q)+w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}\frac{k}{\kappa}e^{\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$-\frac{A_I i w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}\frac{k^2}{\kappa}e^{\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}ke^{\kappa x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}\,dk \qquad (25)$$

approxiamting kappa in the domain

$$\kappa \approx \sqrt{k_0^2 - q^2} - \frac{q(k-q)}{\sqrt{k_0^2 - q^2}} \tag{26}$$

$$\Psi_2(x,t) \approx \frac{A_I(1 + w(q) - w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{-(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i(-1 + w(q) - w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k}{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})} e^{-(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{-(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k^2}{\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}}} e^{-(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I(1 + w(q) - w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i(1 - w(q) + w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k}{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})} e^{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$-\frac{A_I i w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k^2}{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})} e^{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{(\sqrt{k_0^2-q^2} - \frac{q(k-q)}{\sqrt{k_0^2-q^2}})x} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \tag{27}$$

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$$\Psi_2(x,t) \approx \frac{A_I(1+w(q)-w'(q)q)e^{\frac{-q^2 x}{\sqrt{k_0^2-q^2}}}e^{-\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}e^{\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+\frac{A_I i(-1+w(q)-w'(q)q)e^{\frac{-q^2 x}{\sqrt{k_0^2-q^2}}}e^{-\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}\frac{k}{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})}e^{\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+\frac{A_I w'(q)e^{\frac{-q^2 x}{\sqrt{k_0^2-q^2}}}e^{-\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}ke^{\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+\frac{A_I iw'(q)e^{\frac{-q^2 x}{\sqrt{k_0^2-q^2}}}e^{-\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}\frac{k^2}{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})}e^{\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+\frac{A_I(1+w(q)-w'(q)q)e^{\frac{q^2 x}{\sqrt{k_0^2-q^2}}}e^{\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}e^{-\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+\frac{A_I i(1-w(q)+w'(q)q)e^{\frac{q^2 x}{\sqrt{k_0^2-q^2}}}e^{\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}\frac{k}{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})}e^{-\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$-\frac{A_I iw'(q)e^{\frac{q^2 x}{\sqrt{k_0^2-q^2}}}e^{\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}\frac{k^2}{(\sqrt{k_0^2-q^2}-\frac{qk}{\sqrt{k_0^2-q^2}})}e^{-\frac{qk}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk$$

$$+\frac{A_I w'(q)e^{\frac{q^2 x}{\sqrt{k_0^2-q^2}}}e^{\sqrt{k_0^2-q^2}x}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx_0}ke^{-\frac{q(k-q)}{\sqrt{k_0^2-q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}}dk \qquad (28)$$

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$$\frac{k}{\kappa} \approx \left(\frac{q}{\sqrt{k_0^2 - q^2}} - \frac{qk_0^2}{(k_0^2 - q^2)^{3/2}}\right) + k\left(\frac{k_0^2}{(k_0^2 - q^2)^{3/2}}\right) \tag{29}$$

$$\frac{k^2}{\kappa} \approx \left(\frac{q^2}{\sqrt{k_0^2 - q^2}} - \frac{q^4 + 2q^2(k_0^2 - q^2)}{(k_0^2 - q^2)^{3/2}}\right) + k\left(\frac{q^3 + 2q(k_0^2 - q^2)}{(k_0^2 - q^2)^{3/2}}\right) \tag{30}$$

$$A = (1 + w(q) - w'(q)q) \tag{31}$$

$$B = i(-1 + w(q) - w'(q)q) \tag{32}$$

$$C = \left(\frac{q}{\sqrt{k_0^2 - q^2}} - \frac{qk_0^2}{(k_0^2 - q^2)^{3/2}}\right) \tag{33}$$

$$D = \left(\frac{q^2}{\sqrt{k_0^2 - q^2}} - \frac{(q^4 + 2q^2(k_0^2 - q^2))}{(k_0^2 - q^2)^{3/2}}\right) \tag{34}$$

$$E = iw'(q)\left(\frac{q^3 + 2q(k_0^2 - q^2)}{(k_0^2 - q^2)^{3/2}}\right) \tag{35}$$

$$\Psi_2(x,t) \approx \frac{A_I(A + BC + w'(q)D)e^{\frac{-q^2 x}{\sqrt{k_0^2 - q^2}}}e^{-\sqrt{k_0^2 - q^2}x}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}e^{\frac{qk}{\sqrt{k_0^2 - q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}}\frac{-ik^2\hbar t}{2m}\,dk$$

$$+ \frac{A_I(B\frac{k_0^2}{(k_0^2 - q^2)^{3/2}} + w'(q) + E)e^{\frac{-q^2 x}{\sqrt{k_0^2 - q^2}}}e^{-\sqrt{k_0^2 - q^2}x}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}ke^{\frac{qk}{\sqrt{k_0^2 - q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}}\frac{-ik^2\hbar t}{2m}\,dk$$

$$+ \frac{A_I(A + BC - iw'(q)D)e^{\frac{q^2 x}{\sqrt{k_0^2 - q^2}}}e^{\sqrt{k_0^2 - q^2}x}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}e^{-\frac{qk}{\sqrt{k_0^2 - q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}}\frac{-ik^2\hbar t}{2m}\,dk$$

$$+ \frac{A_I(-B\frac{k_0^2}{(k_0^2 - q^2)^{3/2}} + w'(q) - E)e^{\frac{q^2 x}{\sqrt{k_0^2 - q^2}}}e^{\sqrt{k_0^2 - q^2}x}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0}ke^{-\frac{qk}{\sqrt{k_0^2 - q^2}}x}e^{\frac{-(k-q)^2}{2\sigma^2}}\frac{-ik^2\hbar t}{2m}\,dk \tag{36}$$

$$\Psi_2(x,t) \approx \frac{A_I(A + BC + w'(q)D)e^{\frac{-iq^2\hbar t}{2m} - \sqrt{k_0^2-q^2}\,x}e^{-iqx_0}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik'x_0} e^{\frac{qk'x}{\sqrt{k_0^2-q^2}}} e^{\frac{-k'^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}\,dk$$

$$+\frac{A_I(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}} + w'(q) + E)e^{\frac{-iq^2\hbar t}{2m} - \sqrt{k_0^2-q^2}\,x}e^{-iqx_0}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} k' e^{-ik'x_0} e^{\frac{qk'x}{\sqrt{k_0^2-q^2}}} e^{\frac{-k'^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}\,dk$$

$$+\frac{A_I q(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}} + w'(q) + E)e^{\frac{-iq^2\hbar t}{2m} - \sqrt{k_0^2-q^2}\,x}e^{-iqx_0}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik'x_0} e^{\frac{qk'x}{\sqrt{k_0^2-q^2}}} e^{\frac{-k'^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}\,dk$$

$$+\frac{A_I(A + BC - iw'(q)D)e^{\sqrt{k_0^2-q^2}\,x \frac{-iq^2\hbar t}{2m}}e^{-iqx_0}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik'x_0} e^{-\frac{qk'x}{\sqrt{k_0^2-q^2}}} e^{\frac{-k'^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}\,dk$$

$$+\frac{A_I(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}} + w'(q) - E)e^{\sqrt{k_0^2-q^2}\,x \frac{-iq^2\hbar t}{2m}}e^{-iqx_0}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} k' e^{-ik'x_0} e^{-\frac{qk'x}{\sqrt{k_0^2-q^2}}} e^{\frac{-k'^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}\,dk$$

$$(37)$$

$$+\frac{A_I q(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}} + w'(q) - E)e^{\sqrt{k_0^2-q^2}\,x \frac{-ik^2\hbar t}{2m}}e^{-iqx_0}}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik'x_0} e^{-\frac{qk'x}{\sqrt{k_0^2-q^2}}} e^{\frac{-k'^2}{2\sigma^2}} \frac{-ik'^2\hbar t}{2m} \frac{-ik'q\hbar t}{m}\,dk$$

$$(38)$$

$$\Psi_2(x,t) \approx \frac{A_I(A+BC+w'(q)D)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x}e^{-iqx_0}e^{\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-((\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k'+\cdots}$$

$$+\frac{A_I(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)+E)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x}e^{-iqx_0}e^{\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}k'e^{-((\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k'+\frac{(i\cdots}{}}$$

$$+\frac{A_Iq(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)+E)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x}e^{-iqx_0}e^{\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-((\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k'+\frac{(ix\cdots}{}}$$

$$+\frac{A_I(A+BC-iw'(q)D)e^{\sqrt{k_0^2-q^2}x\frac{-iq^2\hbar t}{2m}}e^{-iqx_0}e^{\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-((\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k'+\frac{(ix_0m\sqrt{k_0^2-q^2}}{\sqrt{2m(\cdots}}}$$

$$+\frac{A_I(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)-E)e^{\sqrt{k_0^2-q^2}x\frac{-iq^2\hbar t}{2m}}e^{-iqx_0}e^{\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}k'e^{-((\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k'+\frac{(\cdots}{}}$$

$$(39)$$

$$+\frac{A_Iq(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)-E)e^{\sqrt{k_0^2-q^2}x\frac{-ik^2\hbar t}{2m}}e^{-iqx_0}e^{\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-((\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k'+\frac{(ix\cdots}{}}$$

$$(40)$$

$$k'' = (\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})k' + \frac{(ix_0m\sqrt{k_0^2-q^2}\pm qxm+iq\hbar t\sqrt{k_0^2-q^2})\sigma}{\sqrt{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}) \qquad (41)$$

$$dk'' = (\sqrt{\frac{m+i\hbar t\sigma^2}{2\sigma^2m}})dk' \qquad (42)$$

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$$\Psi_2(x,t) \approx \frac{A_I(A+BC+w'(q)D)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}(\sqrt{\frac{2\sigma^2 m}{m+i\hbar t\sigma^2}})}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$+\frac{A_I(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)+E)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}(\sqrt{\frac{2\sigma^2 m}{m+i\hbar t\sigma^2}})}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(k''-\frac{(ix_0m}{}$$

$$+\frac{A_Iq(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)+E)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}(\sqrt{\frac{2\sigma^2 m}{m+i\hbar t\sigma^2}})}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$+\frac{A_I(A+BC-iw'(q)D)e^{\sqrt{k_0^2-q^2}x\frac{-iq^2\hbar t}{2m}-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}(\sqrt{\frac{2\sigma^2 m}{m+i\hbar t\sigma^2}})}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$+\frac{A_I(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)-E)e^{\sqrt{k_0^2-q^2}x\frac{-iq^2\hbar t}{2m}-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}(\sqrt{\frac{2\sigma^2 m}{m+i\hbar t\sigma^2}})}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(k''-\frac{(ix_0m}{}$$

$$(43)$$

$$+\frac{A_Iq(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)-E)e^{\sqrt{k_0^2-q^2}x\frac{-ik^2\hbar t}{2m}-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}(\sqrt{\frac{2\sigma^2 m}{m+i\hbar t\sigma^2}})}{2\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$(44)$$

$$\Psi_2(x,t) \approx \frac{A_I\sqrt{m}(A+BC+w'(q)D)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sqrt{m+i\hbar t\sigma^2}}$$

$$-\frac{A_I\sqrt{m}(ix_0m\sqrt{k_0^2-q^2}+qxm+iq\hbar t\sqrt{k_0^2-q^2})\sigma^2(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)+E)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}+iq}{2m(k_0^2-q^2}}}{2\sqrt{(k_0^2-q^2)(m+i\hbar t\sigma^2)}}$$

$$+\frac{A_Iq\sqrt{m}(B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)+E)e^{\frac{-iq^2\hbar t}{2m}-\sqrt{k_0^2-q^2}x-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}+iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sqrt{m+i\hbar t\sigma^2}}$$

$$+\frac{A_I\sqrt{m}(A+BC-iw'(q)D)e^{\sqrt{k_0^2-q^2}x\frac{-iq^2\hbar t}{2m}-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sqrt{m+i\hbar t\sigma^2}}$$

$$-\frac{A_I\sqrt{m}(ix_0m\sqrt{k_0^2-q^2}+qxm+iq\hbar t\sqrt{k_0^2-q^2})\sigma^2(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)-E)e^{\sqrt{k_0^2-q^2}x\frac{-iq^2\hbar t}{2m}-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}-i}{2m(k_0^2-q}}}{2\sqrt{(k_0^2-q^2)(m+i\hbar t\sigma^2)}}$$

$$\tag{45}$$

$$+\frac{A_Iq\sqrt{m}(-B\frac{k_0^2}{(k_0^2-q^2)^{3/2}}+w'(q)-E)e^{\sqrt{k_0^2-q^2}x\frac{-ik^2\hbar t}{2m}-iqx_0\frac{-(x_0m\sqrt{k_0^2-q^2}-iqxm+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(m+i\hbar t\sigma^2)}}}{2\sqrt{m+i\hbar t\sigma^2}}$$

$$\tag{46}$$

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4 third domain

$$\Psi_3(x,t) = A_I \int_{-\infty}^{\infty} (e^{-ikx_0}((\kappa-ik)+w(k)(\kappa+ik))e^{-\kappa d-ikd+ikx}+((\kappa+ik)+w(k)(-ik+\kappa))e^{\kappa d-ikd+ikx}) \frac{1}{2\kappa\sigma\sqrt{2\pi}} e^{\frac{-(k-q)}{2\sigma^2}}$$

$$(47)$$

$$\Psi_3(x,t) = A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{1}{2})e^{-\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{-ik}{2\kappa})e^{-\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{w(k)}{2})e^{-\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{iw(k)k}{2\kappa})e^{-\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{1}{2})e^{\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{ik}{2\kappa})e^{\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{-iw(k)k}{2\kappa})e^{\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+A_I \int_{-\infty}^{\infty} e^{-ikx_0}(\frac{w(k)}{2})e^{\kappa d-ikd+ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \qquad (48)$$

$$w(k) \approx w(q) + w'(q)(k - q) \tag{49}$$

$$\Psi_3(x,t) \approx \frac{A_I(1 + w(q) - w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{-\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i(-1 + w(q) - w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k}{\kappa} e^{-\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{-\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k^2}{\kappa} e^{-\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I(1 + w(q) - w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i(1 - w(q) + w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k}{\kappa} e^{\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$-\frac{A_I i w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} \frac{k^2}{\kappa} e^{\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{\kappa d - ikd + ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \tag{50}$$

$$\kappa \approx \sqrt{k_0^2 - q^2} - \frac{q(k-q)}{\sqrt{k_0^2 - q^2}} \tag{51}$$

$$\frac{k}{\kappa} \approx \left(\frac{q}{\sqrt{k_0^2 - q^2}} - \frac{qk_0^2}{(k_0^2 - q^2)^{3/2}} \right) + k\left(\frac{k_0^2}{(k_0^2 - q^2)^{3/2}} \right) \tag{52}$$

$$\frac{k^2}{\kappa} \approx \left(\frac{q^2}{\sqrt{k_0^2 - q^2}} - \frac{q^4 + 2q^2(k_0^2 - q^2)}{(k_0^2 - q^2)^{3/2}} \right) + k\left(\frac{q^3 + 2q(k_0^2 - q^2)}{(k_0^2 - q^2)^{3/2}} \right) \tag{53}$$

$$\Psi_3(x,t) \approx \frac{A_I(1+w(q)-w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{-(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i(-1+w(q)-w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} ((\frac{q}{\sqrt{k_0^2-q^2}} - \frac{qk_0^2}{(k_0^2-q^2)^{3/2}}) + k(\frac{k_0^2}{(k_0^2-q^2)^{3/2}})) e^{-(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{-(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I iw'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} ((\frac{q^2}{\sqrt{k_0^2-q^2}} - \frac{q^4+2q^2(k_0^2-q^2)}{(k_0^2-q^2)^{3/2}}) + k(\frac{q^3+2q(k_0^2-q^2)}{(k_0^2-q^2)^{3/2}})) e^{-(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2}}$$

$$+\frac{A_I(1+w(q)-w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I i(1-w(q)+w'(q)q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} ((\frac{q}{\sqrt{k_0^2-q^2}} - \frac{qk_0^2}{(k_0^2-q^2)^{3/2}}) + k(\frac{k_0^2}{(k_0^2-q^2)^{3/2}})) e^{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k}{2}}$$

$$-\frac{A_I iw'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} ((\frac{q^2}{\sqrt{k_0^2-q^2}} - \frac{q^4+2q^2(k_0^2-q^2)}{(k_0^2-q^2)^{3/2}}) + k(\frac{q^3+2q(k_0^2-q^2)}{(k_0^2-q^2)^{3/2}})) e^{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2}}$$

$$+\frac{A_I w'(q)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2} \frac{-ik^2\hbar t}{2m}} dk \tag{54}$$

$$A = (1+w(q)-w'(q)q) \tag{55}$$

$$B = i(-1+w(q)-w'(q)q) \tag{56}$$

$$C = (\frac{q}{\sqrt{k_0^2-q^2}} - \frac{qk_0^2}{(k_0^2-q^2)^{3/2}}) \tag{57}$$

$$D = (\frac{q^2}{\sqrt{k_0^2-q^2}} - \frac{(q^4+2q^2(k_0^2-q^2))}{(k_0^2-q^2)^{3/2}}) \tag{58}$$

$$E = iw'(q)(\frac{q^3+2q(k_0^2-q^2)}{(k_0^2-q^2)^{3/2}}) \tag{59}$$

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$$\Psi_3(x,t) \approx \frac{A_I(A+BC+iw'(q)D)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{-(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B+w'(q)+E)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{-(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I(A+BC-iw(q)D)}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} e^{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}} dk$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B-E+w'(q))}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx_0} k e^{(\sqrt{k_0^2-q^2}-\frac{q(k-q)}{\sqrt{k_0^2-q^2}})d-ikd+ikx} e^{\frac{-(k-q)^2}{2\sigma^2}\frac{-ik^2\hbar t}{2m}} dk$$

$$\tag{60}$$

$$k' = k - q \tag{61}$$

$$\Psi_3(x,t) \approx \frac{A_I(A+BC+iw'(q)D)}{2\sigma\sqrt{2\pi}} e^{-iqx_0-\sqrt{k_0^2-q^2}d-iqd+iqx-\frac{iq^2\hbar t}{2m}} \int_{-\infty}^{\infty} e^{-ik'x_0+\frac{qk'd}{\sqrt{k_0^2-q^2}}-ik'd+ik'x\frac{-k'^2}{2\sigma^2}\frac{-ik'q\hbar t}{m}\frac{-ik'^2\hbar t}{2m}} d$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B+w'(q)+E)}{2\sigma\sqrt{2\pi}} e^{-iqx_0-\sqrt{k_0^2-q^2}d-iqd+iqx-\frac{iq^2\hbar t}{2m}} \int_{-\infty}^{\infty} (k'+q) e^{-ik'x_0+\frac{qk'd}{\sqrt{k_0^2-q^2}}-ik'd+ik'x\frac{-k'^2}{2\sigma^2}\frac{-ik'q\hbar t}{m}\frac{-ik'^2\hbar t}{2m}}$$

$$+\frac{A_I(A+BC-iw(q)D)}{2\sigma\sqrt{2\pi}} e^{-iqx_0+\sqrt{k_0^2-q^2}d-iqd+iqx-\frac{iq^2\hbar t}{2m}} \int_{-\infty}^{\infty} e^{-ik'x_0-\frac{qk'd}{\sqrt{k_0^2-q^2}}-ik'd+ik'x\frac{-k'^2}{2\sigma^2}\frac{-ik'q\hbar t}{m}\frac{-ik'^2\hbar t}{2m}} dk'$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B-E+w'(q))}{2\sigma\sqrt{2\pi}} e^{-iqx_0+\sqrt{k_0^2-q^2}d-iqd+iqx-\frac{iq^2\hbar t}{2m}} \int_{-\infty}^{\infty} (k'+q) e^{-ik'x_0-\frac{qk'd}{\sqrt{k_0^2-q^2}}-ik'd+ik'x\frac{-k'^2}{2\sigma^2}\frac{-ik'q\hbar t}{m}\frac{-ik'^2\hbar t}{2m}}$$

$$\tag{62}$$

$$e^{g1} = e^{\frac{-(x_0 m\sqrt{k_0^2-q^2}+iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}} \tag{63}$$

$$e^{g2} = e^{\frac{-(x_0 m\sqrt{k_0^2-q^2}-iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})^2\sigma^2}{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}} \tag{64}$$

$$e^{\phi_1} = e^{-iqx_0-\sqrt{k_0^2-q^2}d-iqd+iqx-\frac{iq^2\hbar t}{2m}} \tag{65}$$

$$e^{\phi_2} = e^{-iqx_0+\sqrt{k_0^2-q^2}d-iqd+iqx-\frac{iq^2\hbar t}{2m}} \tag{66}$$

$$\Psi_3(x,t) \approx \frac{A_I(A+BC+iw'(q)D)}{2\sigma\sqrt{2\pi}}e^{\phi_1}e^{g1}\int_{-\infty}^{\infty}e^{-((\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})k'+\frac{i(x_0 m\sqrt{k_0^2-q^2}+iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})\sigma}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}})}$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B+w'(q)+E)}{2\sigma\sqrt{2\pi}}e^{\phi_1}e^{g1}\int_{-\infty}^{\infty}(k'+q)e^{-((\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})k'+\frac{i(x_0 m\sqrt{k_0^2-q^2}+iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-}}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}}}$$

$$+\frac{A_I(A+BC-iw(q)D)}{2\sigma\sqrt{2\pi}}e^{\phi_2}e^{g2}\int_{-\infty}^{\infty}e^{-((\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})k'+\frac{i(x_0 m\sqrt{k_0^2-q^2}-iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})\sigma}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}})^2}dk'$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B-E+w'(q))}{2\sigma\sqrt{2\pi}}e^{\phi_2}e^{g2}\int_{-\infty}^{\infty}(k'+q)e^{-((\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})k'+\frac{i(x_0 m\sqrt{k_0^2-q^2}-iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-}}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}}} \tag{67}$$

$$\Psi_3(x,t) \approx \frac{A_I(A + BC + iw'(q)D)}{2\sigma\sqrt{2\pi}(\sqrt{\frac{i\hbar t\sigma^2 + m}{2m\sigma^2}})}e^{\phi_1}e^{g1}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B + w'(q) + E)}{2\sigma\sqrt{2\pi}(\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})^2}e^{\phi_1}e^{g1}\int_{-\infty}^{\infty}(k'' - \frac{i(x_0m\sqrt{k_0^2-q^2}+iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}}$$

$$+\frac{A_Iq((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B + w'(q) + E)}{2\sigma\sqrt{2\pi}(\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})}e^{\phi_1}e^{g1}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$+\frac{A_I(A + BC - iw(q)D)}{2\sigma\sqrt{2\pi}(\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})}e^{\phi_2}e^{g2}\int_{-\infty}^{\infty}e^{-k''^2}dk''$$

$$+\frac{A_I((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B - E + w'(q))}{2\sigma\sqrt{2\pi}(\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})^2}e^{\phi_2}e^{g2}\int_{-\infty}^{\infty}(k'' - \frac{i(x_0m\sqrt{k_0^2-q^2}-iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}}$$

(68)

$$+\frac{A_Iq((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B - E + w'(q))}{2\sigma\sqrt{2\pi}(\sqrt{\frac{i\hbar t\sigma^2+m}{2m\sigma^2}})}e^{\phi_2}e^{g2}\int_{-\infty}^{\infty}e^{-k''^2}dk'' \qquad (69)$$

$$F1 = \frac{i(x_0m\sqrt{k_0^2-q^2}+iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})\sigma}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}} \qquad (70)$$

$$F2 = \frac{i(x_0m\sqrt{k_0^2-q^2}-iqdm+d\sqrt{k_0^2-q^2}m-m\sqrt{k_0^2-q^2}+q\hbar t\sqrt{k_0^2-q^2})\sigma}{\sqrt{2m(k_0^2-q^2)(i\hbar t\sigma^2+m)}} \qquad (71)$$

$$\Psi_3(x,t) \approx \frac{A_I\sqrt{m}(A+BC+iw'(q)D)}{2\sqrt{i\hbar t\sigma^2 + m}}e^{\phi_1}e^{g1}$$

$$-\frac{A_I m\sigma((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B+w'(q)+E)}{\sqrt{2}(i\hbar t\sigma^2 + m)}e^{\phi_1}e^{g1}F1$$

$$+\frac{A_I\sqrt{m}q((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B+w'(q)+E)}{2\sqrt{i\hbar t\sigma^2 + m}}e^{\phi_1}e^{g1}$$

$$+\frac{A_I\sqrt{m}(A+BC-iw(q)D)}{2\sqrt{i\hbar t\sigma^2 + m}}e^{\phi_2}e^{g2}$$

$$-\frac{A_I m\sigma((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B-E+w'(q))}{\sqrt{2}(i\hbar t\sigma^2 + m)}e^{\phi_2}e^{g2}F2 \tag{72}$$

$$+\frac{A_I q\sqrt{m}((\frac{k_0^2}{(k_0^2-q^2)^{3/2}})B-E+w'(q))}{2\sqrt{i\hbar t\sigma^2 + m}}e^{\phi_2}e^{g2} \tag{73}$$