EngSci 263 Temperature ODE Derivation

We first start with the equation for the thermal energy in the reservoir,

$$E = McT$$

We then differentiate this to obtain

$$\frac{dE}{dt} = M_0 c \frac{dT}{dt} + \frac{dM}{dt} cT$$

where M_0 is the initial mass in the reservoir (which doesn't deviate much from its initial value). We now look to find equations for $\frac{dE}{dt}$ and $\frac{dM}{dt}$ in the reservoir and substitute these into this equation. We know that the rate of change of mass in the reservoir is equal to the sum of the rates at which water enters, q_{rchg}, and leaves, q (or q_{src}); that is,

$$\frac{dM}{dt} = -q + q_{rchg}$$

We also know (from the LPM notes) that

$$q_{rchg} = -\frac{b_p}{a_p}(P - P_0)$$

Substituting this into the previous equation we get that

$$\frac{dM}{dt} = -q - \frac{b_p}{a_p}(P - P_0)$$

We now look to find an equation for the rate of change of thermal energy in the reservoir; this is given by the sum of rate of energy leaving in the extracted water, the rate of energy entering in the cold water, and the rate of thermal energy being transferred to the reservoir water by the surrounding rock. The change in thermal energy in the reservoir when water is extracted is

$$E = -McT$$

where M is the mass of the extracted water and T is the reservoir temperature. The rate of change of energy of the reservoir due to water extraction is therefore

$$\frac{dE}{dt} = -qcT$$

The change in thermal energy in the reservoir when cold water enters is

$$E = M_{rchg} c T_{cold}$$

where M_{rchg} is the mass of the water that has entered and T_{cold} is the water temperature. The rate of change of energy of the reservoir due to cold water entering is therefore

$$\begin{aligned}\frac{dE}{dt} &= q_{rchg} c T_{cold} \\ &= -\frac{b_p}{a_p}(P - P_0) c T_{cold}\end{aligned}$$

Finally, the rate of change of thermal energy in the reservoir due heat exchange with the surrounding rock is proportional to $-(T - T_0)$, where T is the reservoir temperature and T_0 is the temperature of the rock. If we call this proportionality constant d, we can say that

$$\frac{dE}{dt} = -d(T - T_0)$$

Combining these together, the total rate of change of thermal energy in the reservoir is given by

$$\frac{dE}{dt} = -qcT - c\frac{b_p}{a_p}(P - P_0)T_{cold} - d(T - T_0)$$

We now substitute our equations for $\frac{dM}{dt}$ and $\frac{dE}{dt}$ into our initial equation for $\frac{dE}{dt}$, and simplify.

$$\begin{aligned}\frac{dE}{dt} &= M_0 c\frac{dT}{dt} + \frac{dM}{dt}cT \\ -qcT - c\frac{b_p}{a_p}(P - P_0)T_{cold} - d(T - T_0) &= M_0 c\frac{dT}{dt} + (-q - \frac{b_p}{a_p}(P - P_0))cT \\ -qcT - c\frac{b_p}{a_p}(P - P_0)T_{cold} - d(T - T_0) &= M_0 c\frac{dT}{dt} + -qcT - \frac{b_p}{a_p}(P - P_0)cT \\ -c\frac{b_p}{a_p}(P - P_0)(T_{cold} - T) - d(T - T_0) &= M_0 c\frac{dT}{dt} \\ -\frac{1}{M_0}\frac{b_p}{a_p}(P - P_0)(T_{cold} - T) - \frac{d}{M_0 c}(T - T_0) &= \frac{dT}{dt}\end{aligned}$$

Introducing lumped parameters

$$a_T = \frac{1}{M_0} \qquad b_T = \frac{d}{M_0 c}$$

gives us

$$\frac{dT}{dt} = -a_T \frac{b_p}{a_p}(P - P_0)(T_{cold} - T) - b_T(T - T_0)$$

as required.