Question 27: Conical Paper Cup Problem

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1 Introduction

This document contains the mathematical derivation and calculations for solving the Conical Paper Cup Problem. The goal is to maximize the volume of the conical cup formed by removing a sector of length x from a circular waxed paper of radius R.

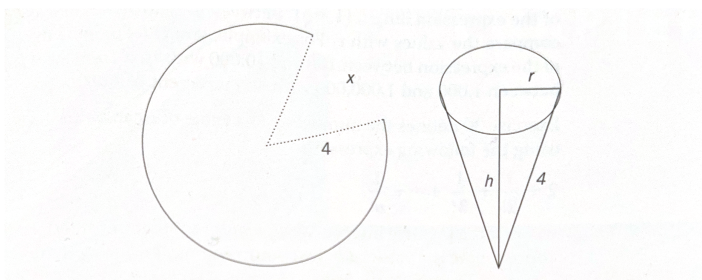

Figure 1: Diagram of the conical paper cup problem showing a circle (radius 4), cut arc (x), and inscribed cone (height h, radius r).

2 Variables

- R : Radius of the circular waxed paper (user inputted as `circleRadius`).
- r : Base radius of the resulting cone.
- h : Height of the resulting cone.
- s : Slant height of the cone, which is equal to R (since the remaining sector's arc forms the cone's circumference).
- x : Length of the removed sector.

3 Formulation and Derivation

When a sector of length is removed, the remaining arc length becomes:

$$2\pi R - x \tag{1}$$

This arc forms the circumference of the base of the resulting cone, so:

$$2\pi r = 2\pi R - x \tag{2}$$

Solving for r:

$$r = \frac{2\pi R - x}{2\pi} \tag{3}$$

Using the Pythagorean theorem, since the slant height is R:

$$h = \sqrt{R^2 - r^2} \tag{4}$$

The volume of the cone is given by:

$$V = \frac{1}{3}\pi r^2 h \tag{5}$$

Substituting $r = \frac{2\pi R - x}{2\pi}$ and $h = \sqrt{R^2 - r^2}$:

$$V = \frac{1}{3}\pi \left(\frac{2\pi R - x}{2\pi}\right)^2 \sqrt{R^2 - \left(\frac{2\pi R - x}{2\pi}\right)^2} \tag{6}$$

4 Maximization using Calculus

To maximize V, differentiate V with respect to x and set $\frac{dV}{dX} = 0$.

$$\frac{dV}{dx} = \frac{d}{dx}\left[\frac{1}{3}\pi \left(\frac{2\pi R - x}{2\pi}\right)^2 \sqrt{R^2 - \left(\frac{2\pi R - x}{2\pi}\right)^2}\right] = 0 \tag{7}$$

4.1 Define u and v

Let $u = \left(\frac{2\pi R - x}{2\pi}\right)^2$ and $v = \sqrt{R^2 - \left(\frac{2\pi R - x}{2\pi}\right)^2}$.

Then:

$$V(x) = \frac{\pi}{3} * u * v \tag{8}$$

Using the product rule:

$$V'(x) = \frac{\pi}{3}\left(u'v + uv'\right) \tag{9}$$

where u' and v' are derivatives of u and v.

4.2 Differentiate u'

$$u = \left(\frac{2\pi R - x}{2\pi}\right)^2 = (r)^2$$

Using the chain rule:

$$\begin{aligned}\frac{du}{dx}\left[\left(\frac{2\pi R - x}{2\pi}\right)^2\right] &= 2\left(\frac{2\pi R - x}{2\pi}\right) * \frac{d}{dx}\left(\frac{2\pi R - x}{2\pi}\right) \\ &= 2\left(\frac{2\pi R - x}{2\pi}\right) * \left(-\frac{1}{2\pi}\right) \\ &= -\frac{2\pi R - x}{2\pi^2}\end{aligned}$$

So we have:

$$u' = -\frac{2\pi R - x}{2\pi^2}$$

4.3 Differentiate v'

$$v = \sqrt{R^2 - \left(\frac{2\pi R - x}{2\pi}\right)^2} = \sqrt{R^2 - r^2}$$

Using the chain rule:

$$
\begin{aligned}
\frac{dv}{dx}\left[\sqrt{R^2-r^2}\right] &= \frac{1}{2\sqrt{R^2-r^2}} * \frac{d}{dx}\left(R^2-r^2\right)\\
&= \frac{1}{2\sqrt{R^2-r^2}} * \frac{d}{dx}\left(R^2-u\right)\\
&= \frac{1}{2\sqrt{R^2-r^2}} * (-u')\\
&= \frac{1}{2\sqrt{R^2-r^2}} * \left(\frac{2\pi R - x}{2\pi^2}\right)\\
&= \frac{2\pi R - x}{4\pi^2\sqrt{R^2-r^2}}
\end{aligned}
$$

So we have:

$$
v' = \frac{2\pi R - x}{4\pi^2\sqrt{R^2-r^2}}
$$

4.4 Applying the Product Rule for $V(x)$

Using the product rule:

$$
V'(x) = \frac{\pi}{3}\left(u'v + uv'\right)
$$

Substituting u' and v':

$$
V'(x) = \frac{\pi}{3}\left[\left(-\frac{2\pi R - x}{2\pi^2}\right) * \sqrt{R^2-r^2} + \left(\frac{2\pi R - x}{2\pi}\right)^2 * \frac{2\pi R - x}{4\pi^2\sqrt{R^2-r^2}}\right]
$$

5 Maximum Volume

Set V'(x) = 0:

$$
V'(x) = \frac{\pi}{3}\left[\left(-\frac{2\pi R - x}{2\pi^2}\right) * \sqrt{R^2-r^2} + \left(\frac{2\pi R - x}{2\pi}\right)^2 * \frac{2\pi R - x}{4\pi^2\sqrt{R^2-r^2}}\right]
$$

Expand Equation:

$$
\begin{aligned}
\frac{\pi}{3} * (2\pi R - x)\left[\left(\frac{\sqrt{R^2-r^2}}{2\pi^2}\right) + \left(\frac{4\pi^2R^2 - x^2}{16\pi^4\sqrt{R^2-r^2}}\right)\right] &= 0\\
\frac{\pi}{3} * (2\pi R - x)\left(\frac{\left(8\pi^2 * \left(R^2-r^2\right)\right) + 4\pi^2R^2 - x^2}{16\pi^4\sqrt{R^2-r^2}}\right) &= 0
\end{aligned}
$$

We can remove $\frac{\pi}{3} * (2\pi R - x)$ and solve within the brackets:

$$
\begin{aligned}
\frac{8\pi^2 * \left(R^2 - \left(\frac{2\pi R - x}{2\pi}\right)^2\right) + 4\pi^2R^2 - x^2}{16\pi^4\sqrt{R^2 - \left(\frac{2\pi R - x}{2\pi}\right)^2}} &= 0\\
\frac{8\pi^2 * \left(R^2 - \left(R^2 - \frac{x^2}{4\pi^2}\right)\right) + 4\pi^2R^2 - x^2}{16\pi^4\sqrt{R^2 - \left(R^2 - \frac{x^2}{4\pi^2}\right)}} &= 0\\
\frac{8\pi^2 * \left(-\frac{x^2}{4\pi^2}\right) + 4\pi^2R^2 - x^2}{16\pi^4\sqrt{-\frac{x^2}{4\pi^2}}} &= 0\\
\frac{8\pi^2 * \left(-\frac{x^2}{4\pi^2}\right) + 4\pi^2R^2 - x^2}{16\pi^4 - \frac{x}{2\pi}} &= 0
\end{aligned}
$$

$$\frac{-2x^2 + 4\pi^2 R^2 - x^2}{\frac{32\pi^5 - x}{2\pi}} = 0$$
$$\frac{-3x^2 + 4\pi^2 R^2}{\frac{32\pi^5 - x}{2\pi}} = 0$$
$$\frac{-6\pi x^2 + 8\pi^3 R^2}{32\pi^5 - x} = 0$$

Since this equals zero and assuming the denominator is not zero, we can focus on the numerator:

$$-6\pi x^2 + 8\pi^3 R^2 = 0$$
$$-6\pi x^2 = -8\pi^3 R^2$$
$$x^2 = \frac{8\pi^3 R^2}{6\pi}$$
$$x^2 = \frac{4\pi^2 R^2}{3}$$
$$x = \sqrt{\frac{4\pi^2}{3}} * R$$

6 Conclusion

The solution $x = R\sqrt{\frac{4\pi^2}{3}}$ represents the optimal cut length that maximizes the cone's volume. We can verify this solution by examining boundary conditions:

- When $x = 0$, no sector is removed, resulting in a flat disc with zero volume.
- When $x = 2\pi R$, we remove the entire circumference, again resulting in zero volume.

Our solution falls between these two extremes.