Question 29: Cardboard Box Problem

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1 Introduction

This document contains the mathematical derivation and calculations for solving the cardboard box problem. The goal is to maximize the volume of the cardboard box formed by cutting 4 squares of x length on each corner and given area A.

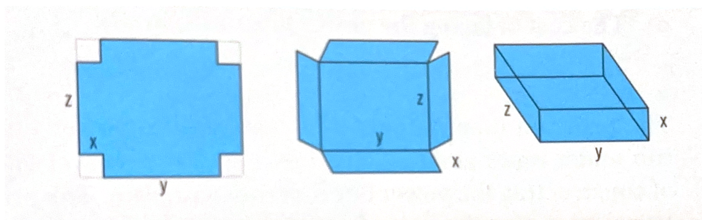

Figure 1: Diagram of the cardboard box problem showing 3 figures where we cut a square of x size of a cardboard box with a width z and length y.

2 Variables

- A : Area of the flat cardboard (user inputted as `area`).
- l : Length of the flat cardboard.
- w : Width of the flat cardboard.
- x : Side of the square being cut from the cardboard.
- y : Length of the cardboard box.
- z : Width of the cardboard box.
- h : Height of the cardboard box, which is equal to x (since the size of the square determines the height of the box).

3 Formulation and Derivation

The area of the flat cardboard is given by:

$$A = l \cdot w \tag{1}$$

To find l and w:

$$l = \frac{A}{w}, \quad w = \frac{A}{l} \tag{2}$$

The new dimensions after cutting squares of side x are:

$$y = l - 2x, \quad z = w - 2x \tag{3}$$

The volume of the resulting box is:

$$V = y \cdot z \cdot x \tag{4}$$

Expanding:

$$\begin{aligned}V &= (l-2x)(w-2x)(x)\\ &= lwx - 2lx^2 - 2wx^2 + 4x^3\\ &= Ax - 2lx^2 - 2wx^2 + 4x^3\end{aligned}$$

4 Maximization using Calculus

To maximize V, differentiate V with respect to x and set $\frac{dV}{dX} = 0$.

$$\frac{dV}{dx} = V' = \frac{d}{dx}\left[4x^3 - 2lx^2 - 2wx^2 + Ax\right] = 0 \tag{5}$$

$$V' = 12x^2 - 4lx - 4wx + A = 0 \tag{6}$$

Solving for x using the quadratic formula:

$$x = \frac{-(-4l-4w) \pm \sqrt{(-4l-4w)^2 - 4(12)(A)}}{2(12)} \tag{7}$$

Simplifying:

$$x = \frac{4l + 4w \pm \sqrt{16l^2 + 16w^2 - 48A}}{24} \tag{8}$$

However, this approach requires knowing l and w in advance, which is not possible without additional constraints (e.g., a given aspect ratio). Testing in code revealed that in many cases, the maximum volume could not be determined correctly.

5 Revised Iterative Approach

Since l and w must satisfy $A = l \cdot w$, we iterate over factored pairs of A, checking each valid configuration. For every divisor i of A, there exists a corresponding divisor $\frac{A}{i}$. We only check up to $\sqrt{A}$, reducing complexity to $O(\sqrt{A})$.

For each factor pair, we iterate over possible x values to maximize V. The additional iteration over x increases complexity beyond $O(\sqrt{A})$, making it dependent on the step size and range of x. If iterating over a range R with step size s, the additional cost is roughly $O(R/s)$, leading to an overall complexity of approximately:

$$O(\sqrt{A} \cdot R/s) \tag{9}$$

This adds overhead but ensures a more precise maximum volume calculation.

6 Conclusion

The calculus approach failed due to the unknown ratio of l and w. The revised iterative method, though slightly more complex, guarantees an accurate solution by considering all factorable pairs of A and iterating over possible values of x. While the additional loop increases runtime, the tradeoff ensures correctness.