

Moment Integrations for 3D Printed Shapes



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Autodesk Research



[Prevost et al. 2013]

Overview

- Intro
- Integration over 3D domain
- Mass
- Center of Mass
- Moment of Inertia

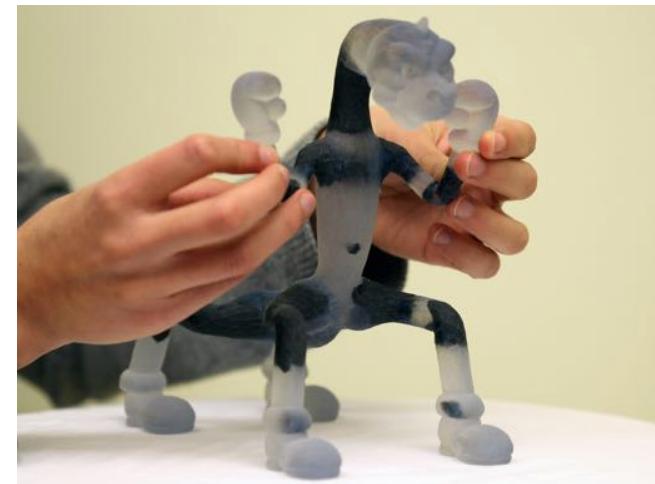
Inhomogeneous Material Distribution

color



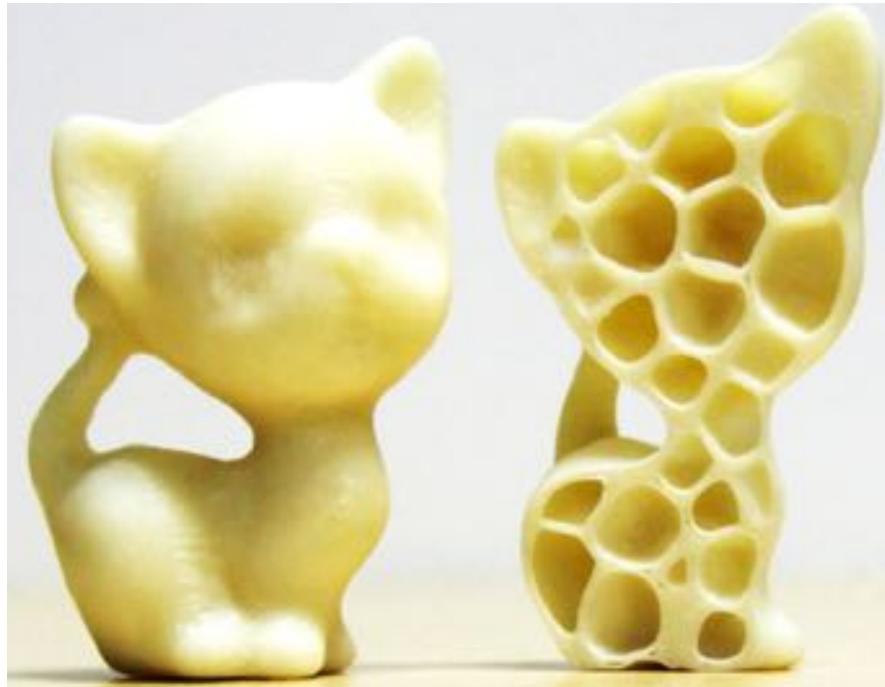
<http://www.stratasys.com/>

stiffness



[Skouras et al.2013]

Inhomogeneous Density Distribution



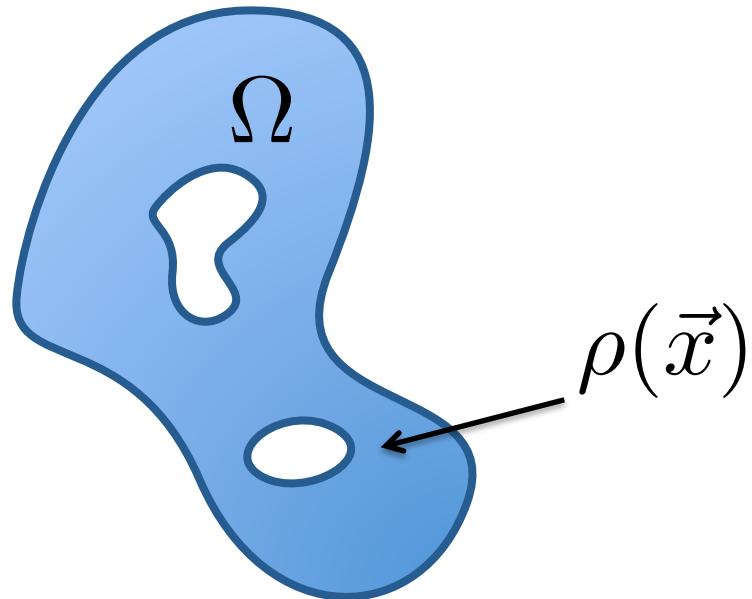
Built-to-last, [Lu et al. 2014]



Buoyancy Optimization, [Wang et al. 2016]

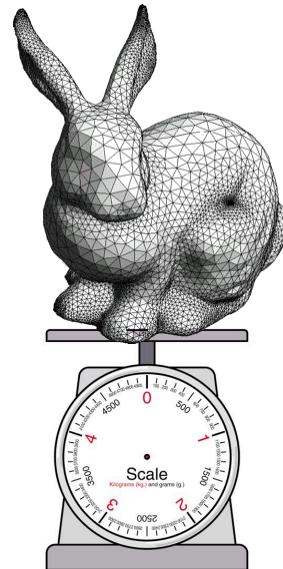
Measure of 3D Density Distribution

- How much density information can you still measure without seeing ‘inside’ of a 3D shape?

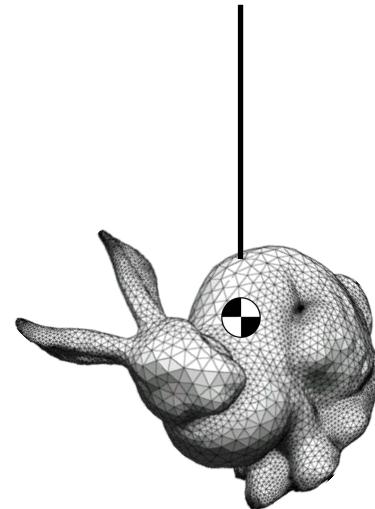


Invariants of 3D Density Distribution

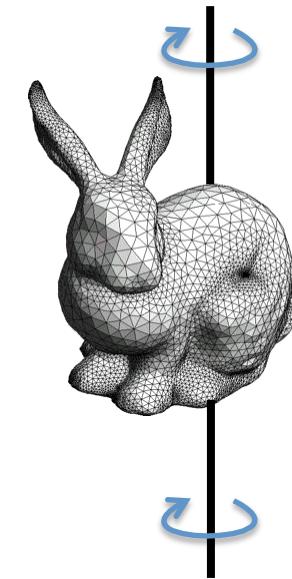
Put on the scale



Hang on a string

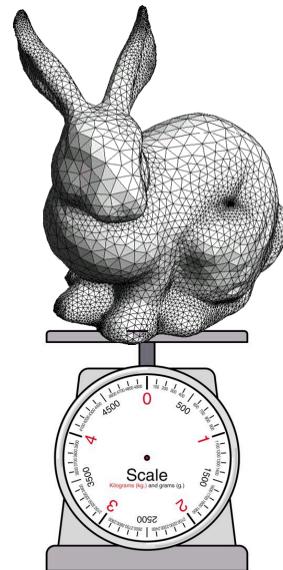


Rotate on an axis

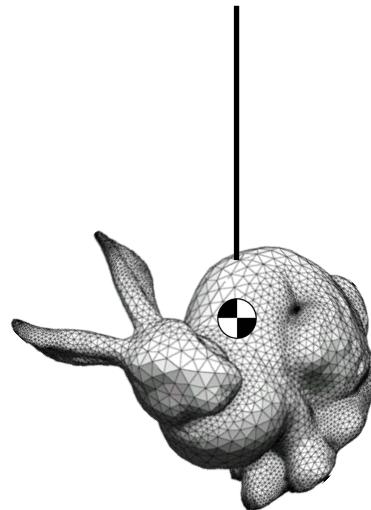


Invariants of 3D Density Distribution

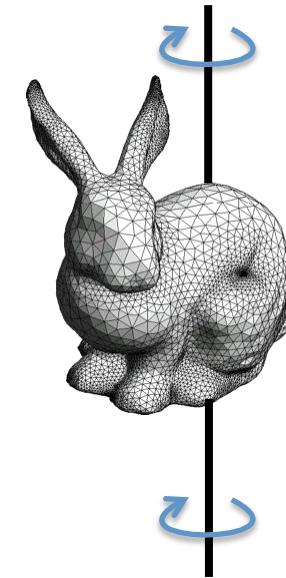
Mass



Center of Mass



Moment of Inertia



Invariants of 3D Density Distribution

Mass

$$\int_{\Omega} (\vec{x})^0 dv$$

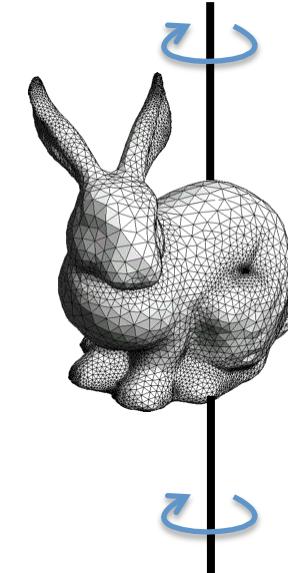
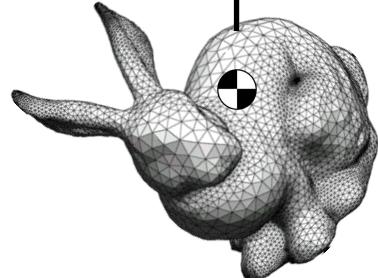
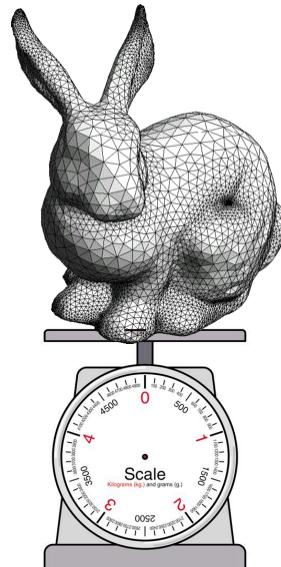
Center of Mass

$$\int_{\Omega} (\vec{x})^1 dv$$

Moment of Inertia

$$\int_{\Omega} (\vec{x})^2 dv$$

Integration characterize these properties



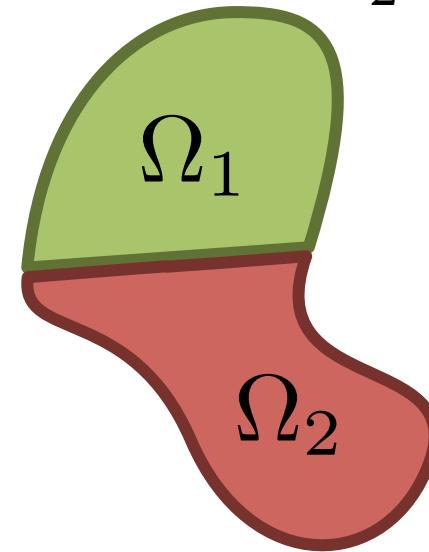
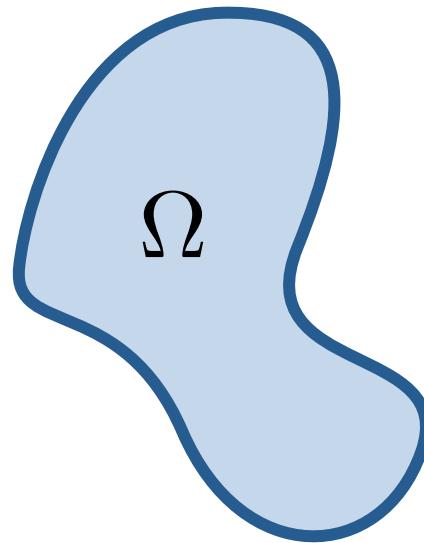
Overview

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Integration Formula is Convenient

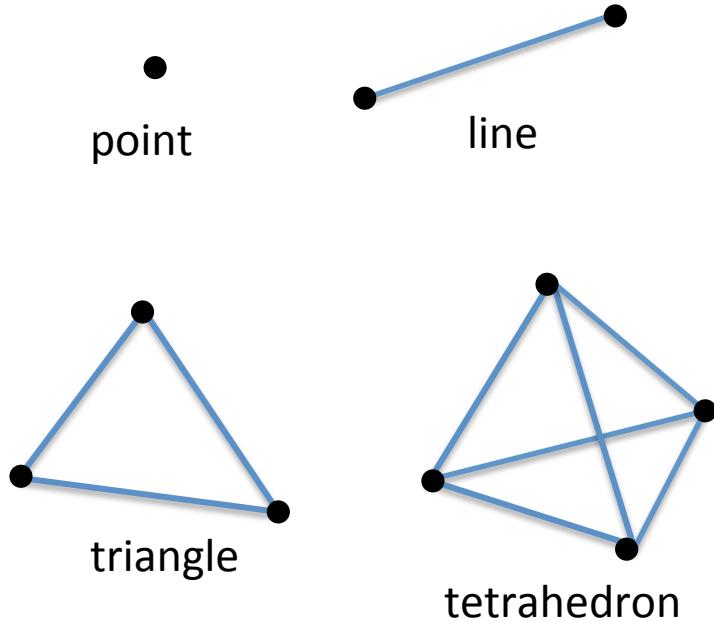
Decomposing the integration region into simpler shapes

$$\int_{\Omega} f dv = \int_{\Omega_1} f dv + \int_{\Omega_2} f dv$$



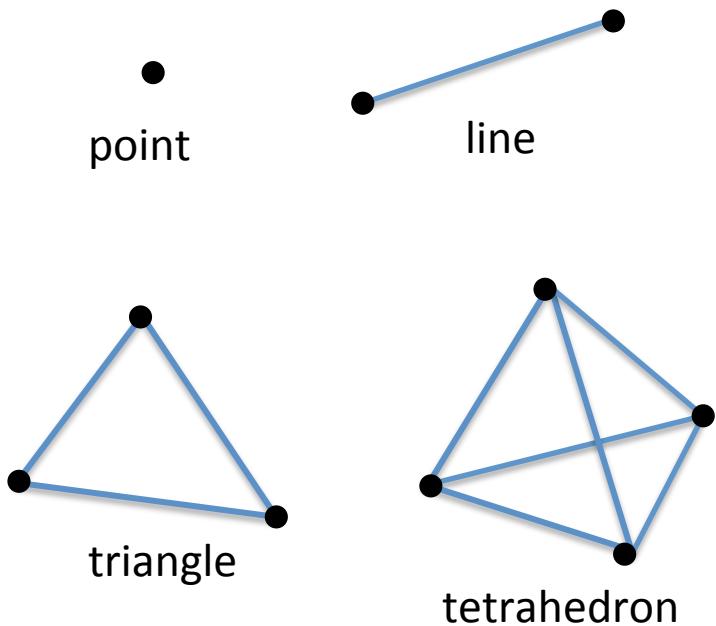
Integration Rules for the Simplex Shape

- Simplex shape



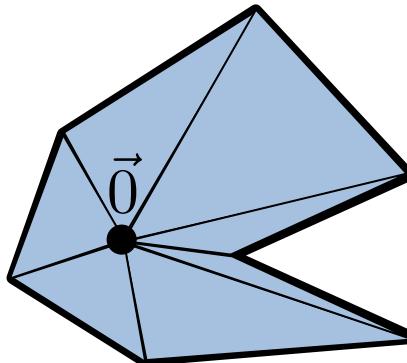
Integration Rules for the Simplex Shape

- Simplex shape

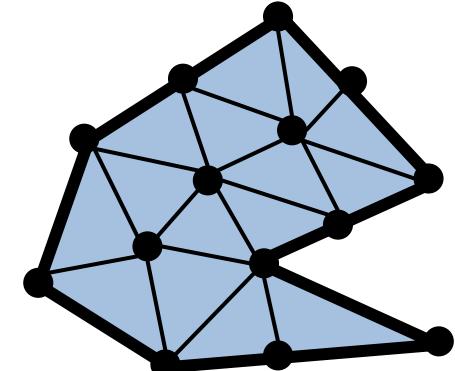


- Decomposition

simplest



FEM-style



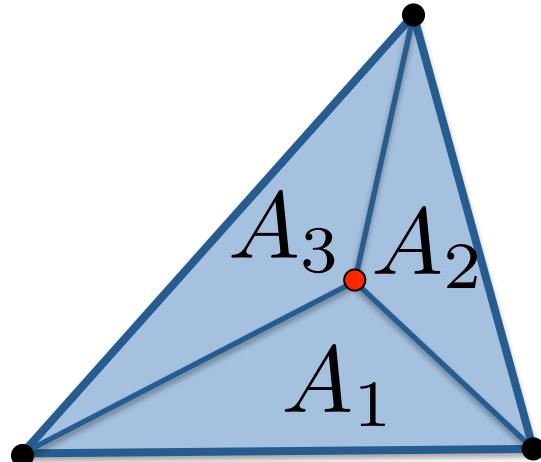
inverted or
skewed elements

Integration Rules: **Analytical**

Barycentric coordinate (L_1, L_2, L_3)

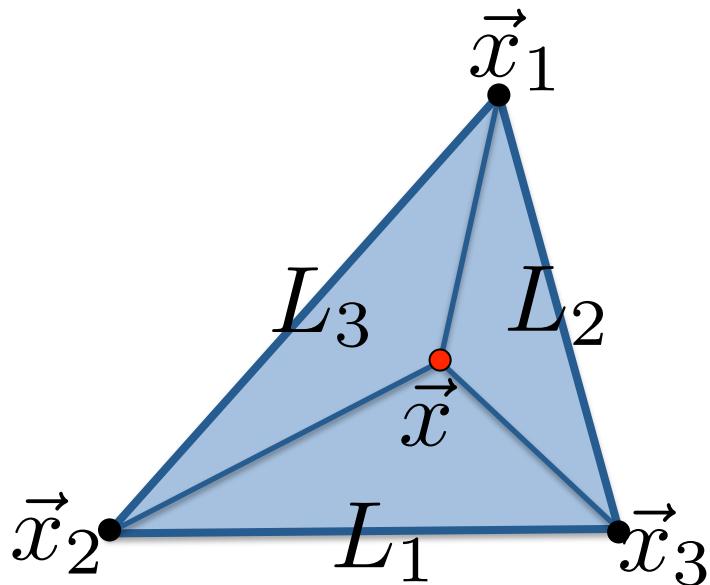
$$A = A_1 + A_2 + A_3$$

$$L_i = A_i/A$$



Integration Rules: Analytical

Position and integrand are both represented using the Barycentric coordinate



position

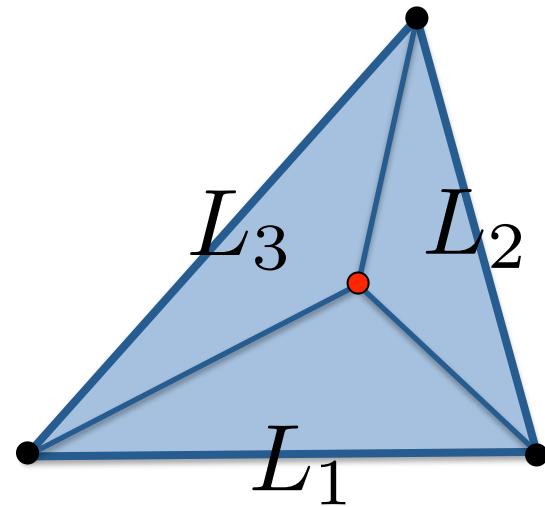
$$\vec{x} = \vec{x}_1 L_1 + \vec{x}_2 L_2 + \vec{x}_3 L_3$$

integrand

$$f(\vec{x}) = f(L_1, L_2, L_3)$$

Integration Rules: Analytical

For polynomial integrands, there is an analytic integration formula



$$f(L_1, L_2, L_3) = L_1^a L_2^b L_3^c$$



Integration over the triangle

$$\int_A L_1^a L_2^b L_3^c da = \frac{a!b!c!2!}{(a+b+c+2)!} A$$

Integration Rules: Analytical

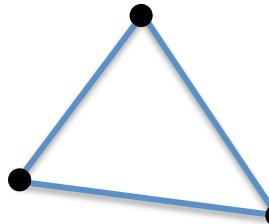
1D

$$\int_L L_1^a L_2^b dl = \frac{a!b!1!}{(a+b+1)!} L$$



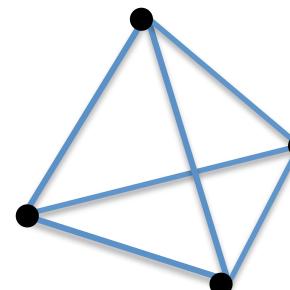
2D

$$\int_A L_1^a L_2^b L_3^c da = \frac{a!b!c!2!}{(a+b+c+2)!} A$$



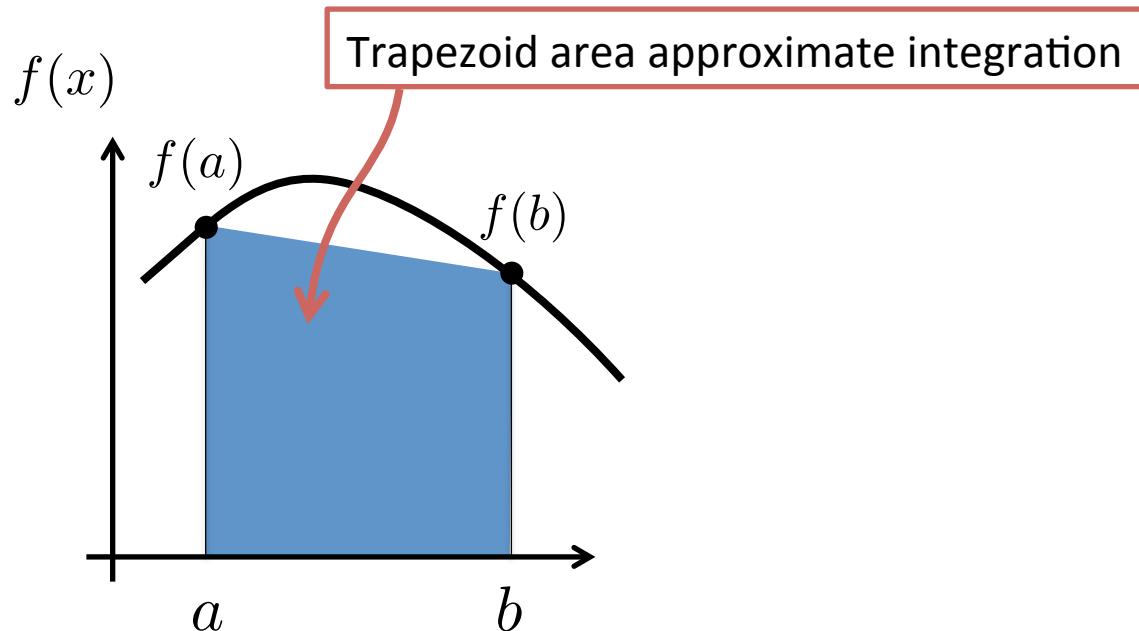
3D

$$\int_V L_1^a L_2^b L_3^c L_4^d dv = \frac{a!b!c!d!3!}{(a+b+c+d+3)!} V$$



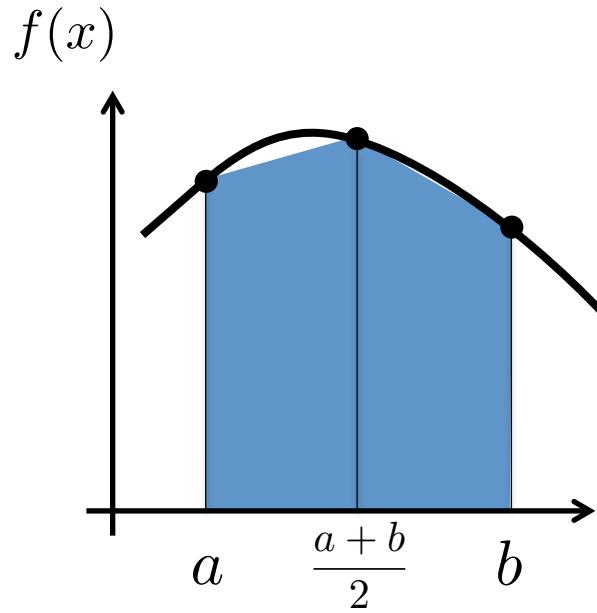
Integration Rules: Numerical

Trapezoid rule

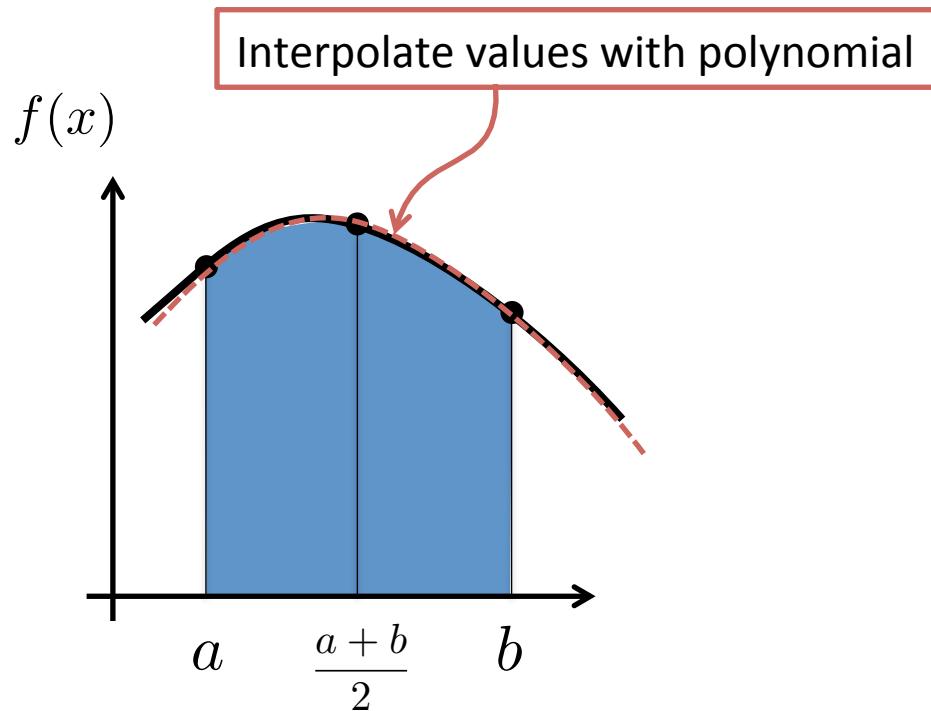


Integration Rules: Numerical

Trapezoid rule

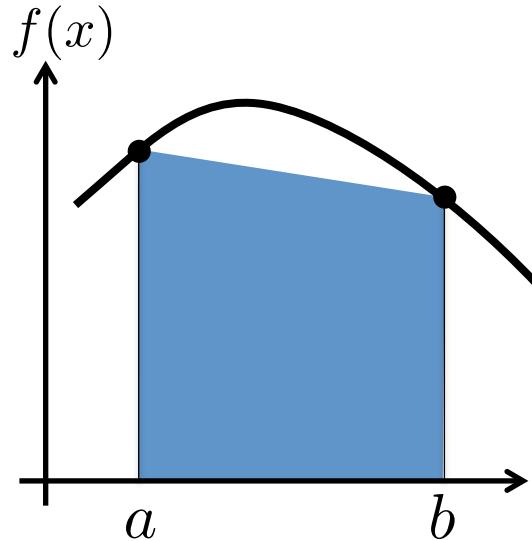


Newton-Cotes quadrature

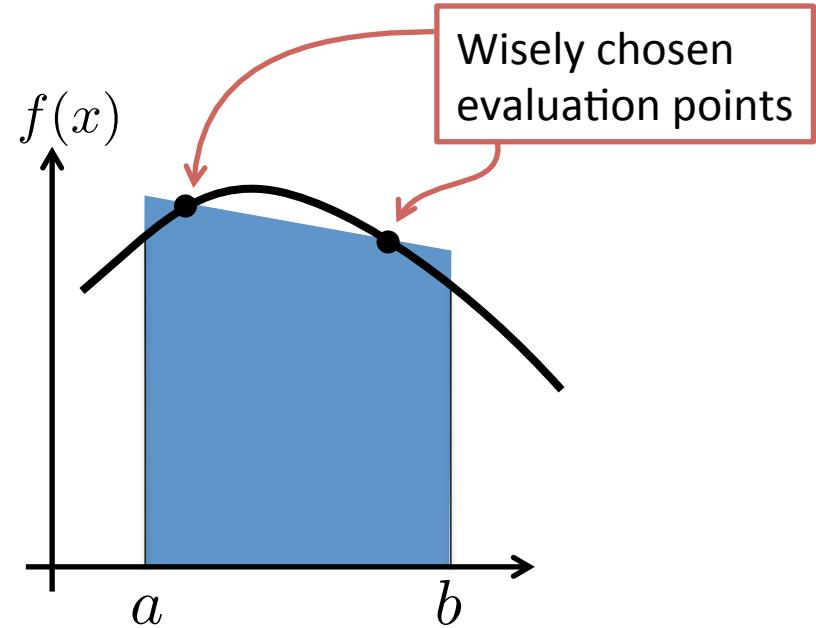


Integration Rules: Numerical

Trapezoid rule



Gauss quadrature

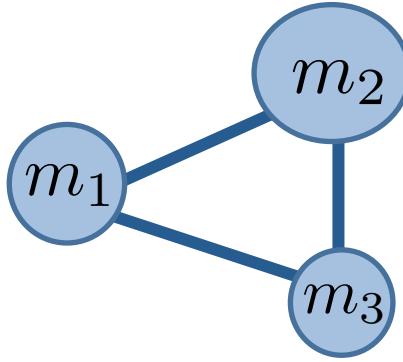


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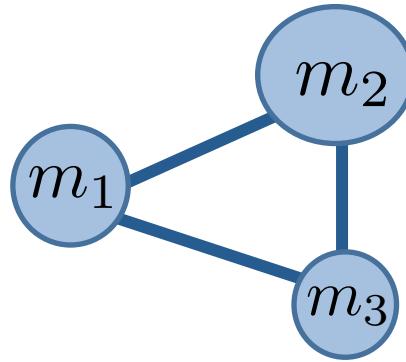
Mass

$$M = \sum m_i$$



Mass

$$M = \sum m_i$$



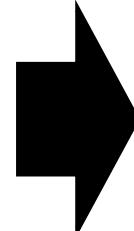
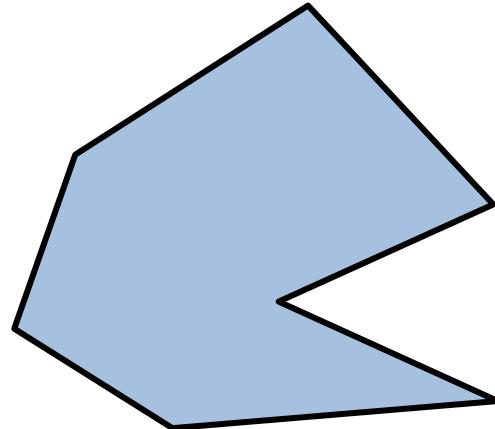
$$M = \int_V \rho(\vec{x}) dV \xrightarrow{\text{Homogeneous density}} M = \rho \int_V dV$$

Volume of a Triangle Mesh I

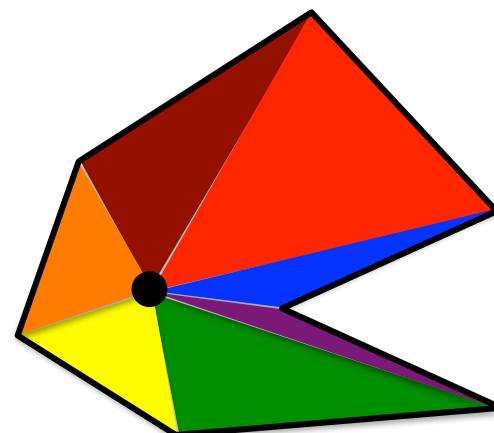
Computing volume integral using volume discretization

$$\text{Area}(\text{Region}) = \text{area}(\text{orange triangle}) + \text{area}(\text{red triangle}) + \text{area}(\text{dark red triangle}) + \dots$$

Volume integral



Divide the region into simplexes



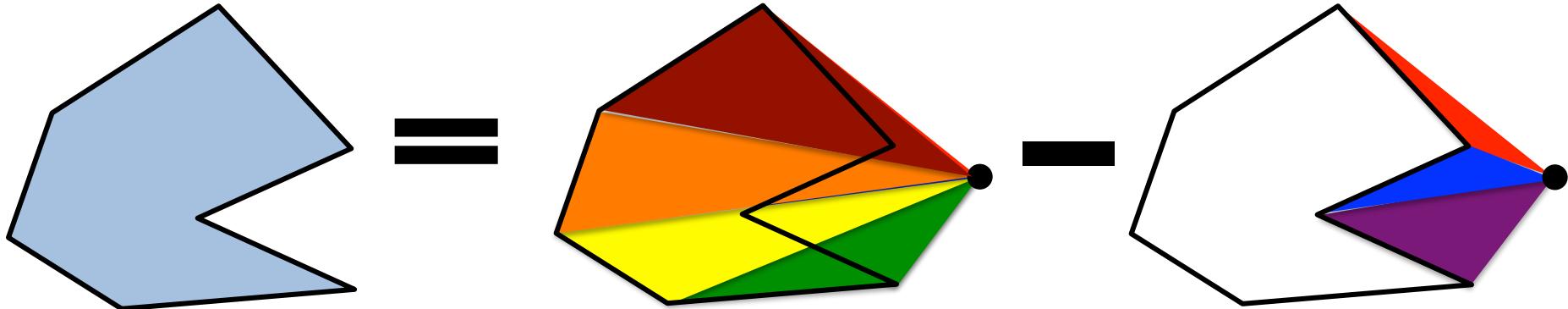
Volume of a Triangle Mesh I

Computing volume integral using volume discretization

$$\text{Area}(\text{ } \text{ } \text{ } \text{ } \text{ } \text{ }) = \text{area}(\textcolor{orange}{\triangle}) + \text{area}(\textcolor{red}{\triangle}) + \text{area}(\textcolor{darkred}{\triangle}) + \dots$$

Volume integral

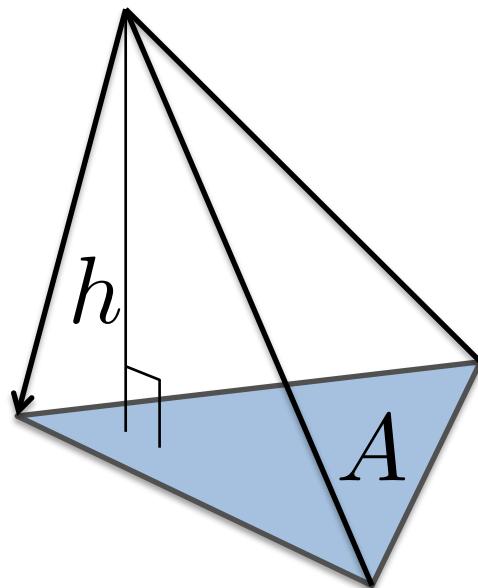
OK to be inverted. The area can be minus



Volume of a Triangle Mesh I

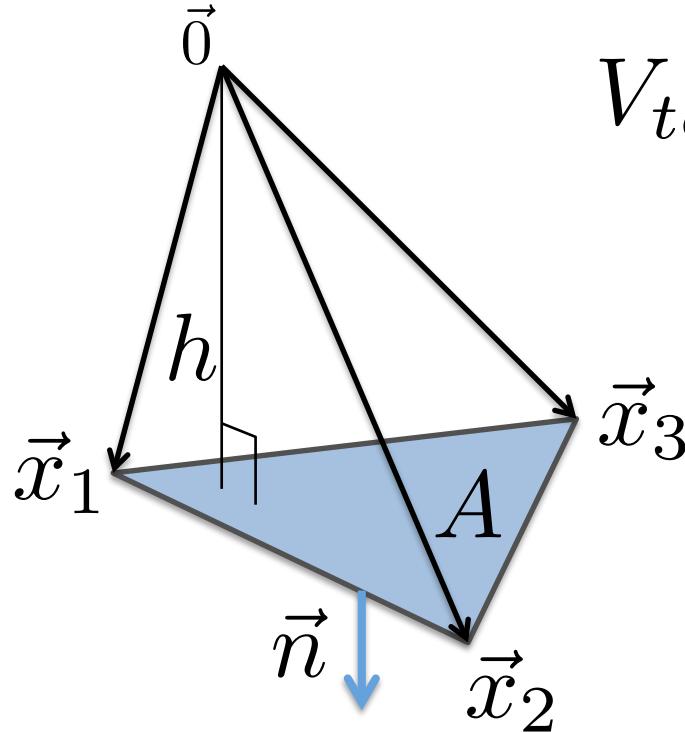
Volume of a tetrahedron

$$V_{tet} = \frac{1}{3}Ah$$



Volume of a Triangle Mesh I

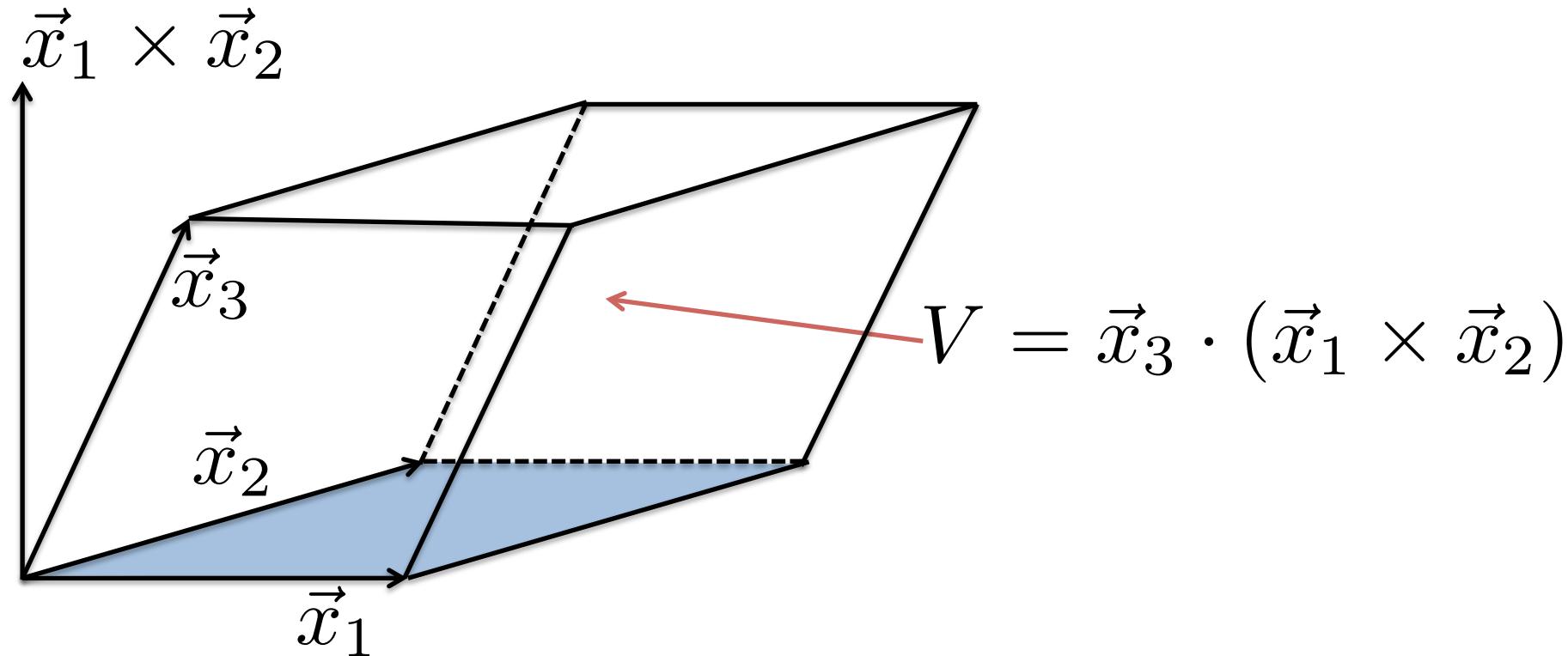
Volume of a tetrahedron



$$\begin{aligned} V_{tet} &= \frac{1}{3} Ah \\ &= \frac{1}{3} A(\vec{x}_1 \cdot \vec{n}) \\ &= \frac{1}{3} A \frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{3} \cdot \vec{n} \end{aligned}$$

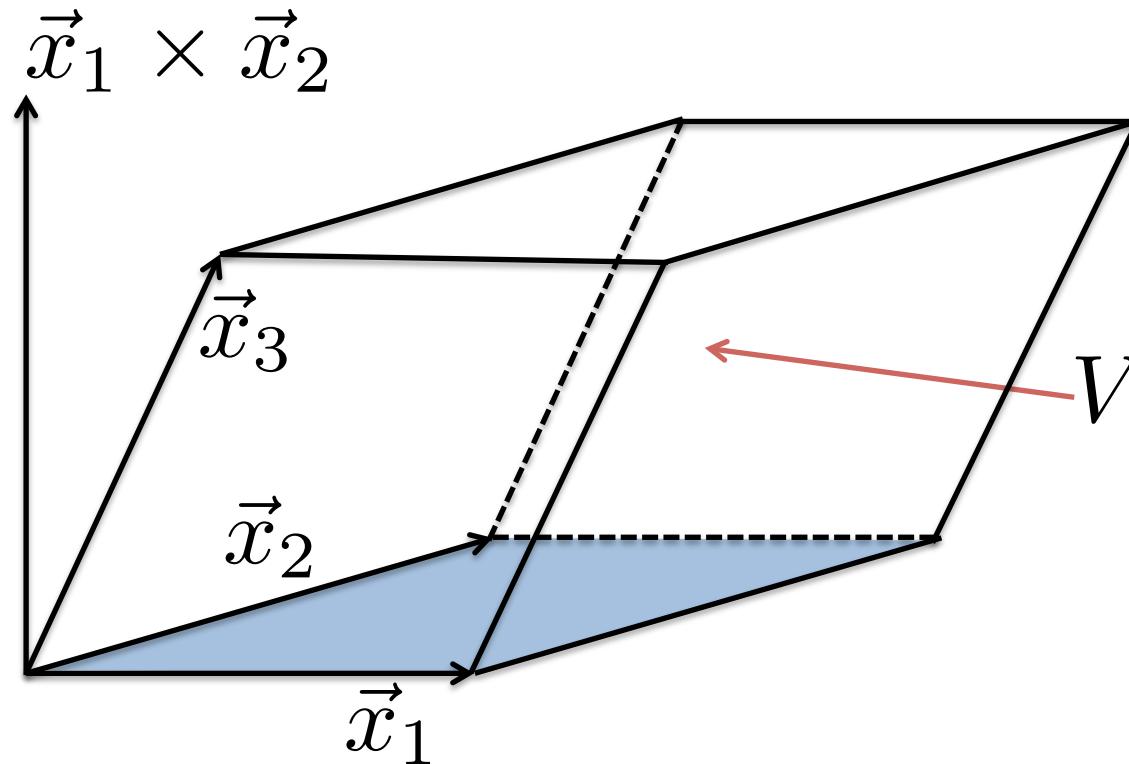
Volume of a Triangle Mesh I

Volume of a tetrahedron can be computed from a parallelepiped



Volume of a Triangle Mesh I

Volume of a tetrahedron can be computed from a parallelepiped

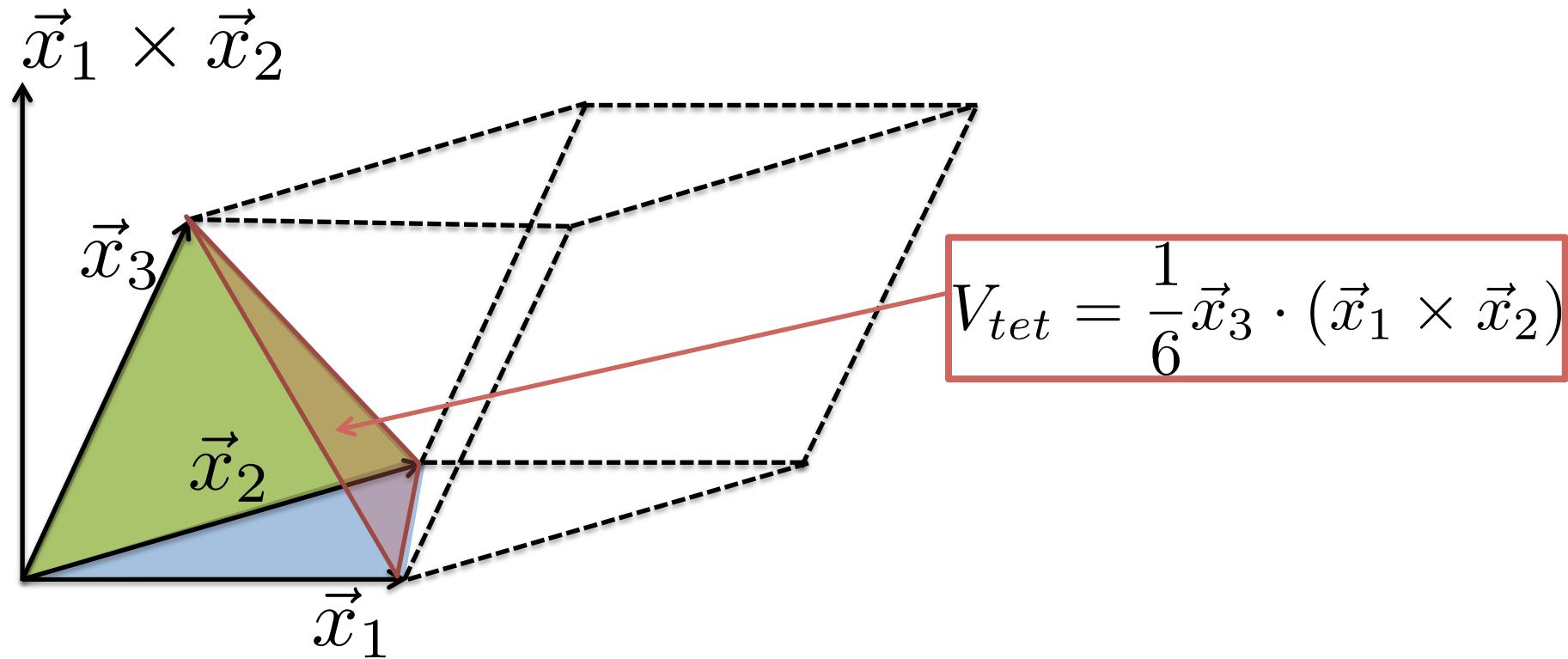


vector triple product

$$\begin{aligned} V &= \vec{x}_3 \cdot (\vec{x}_1 \times \vec{x}_2) \\ &= \vec{x}_1 \cdot (\vec{x}_2 \times \vec{x}_3) \\ &= \vec{x}_2 \cdot (\vec{x}_3 \times \vec{x}_1) \end{aligned}$$

Volume of a Triangle Mesh I

Volume of a tetrahedron can be computed from a parallelepiped



Volume of a Triangle Mesh II

Computing volume integration using **surface integration**

The Gauss's divergence theorem

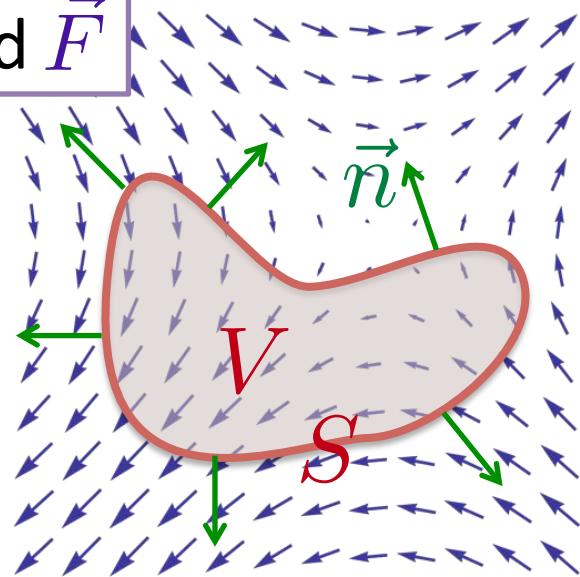
$$\int_{\Omega} (\nabla \cdot \vec{F}) dv = \int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds$$

volume integration surface integration

Volume of a Triangle Mesh II

$$\int_{\Omega} (\nabla \cdot \vec{F}) dv = \int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds$$

vector field \vec{F}



measures total outward flow through V 's boundary

Volume of a Triangle Mesh II

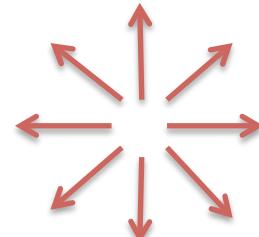
add up little bits of outward flow in V

$$\boxed{\int_{\Omega} (\nabla \cdot \vec{F}) dv = \int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds}$$

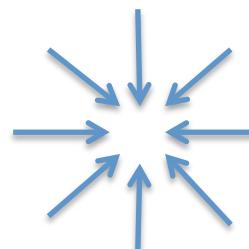
divergence

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot \vec{F} > 0$$



$$\nabla \cdot \vec{F} < 0$$



$$\nabla \cdot \vec{F} = 0$$



Volume of a Triangle Mesh II

$$\int_{\Omega} dv = \int_{\Omega} (\nabla \cdot \vec{F}) dv = \int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds$$

$\nabla \cdot \vec{F} = 1 \rightarrow$ What's the \vec{F} looks like?

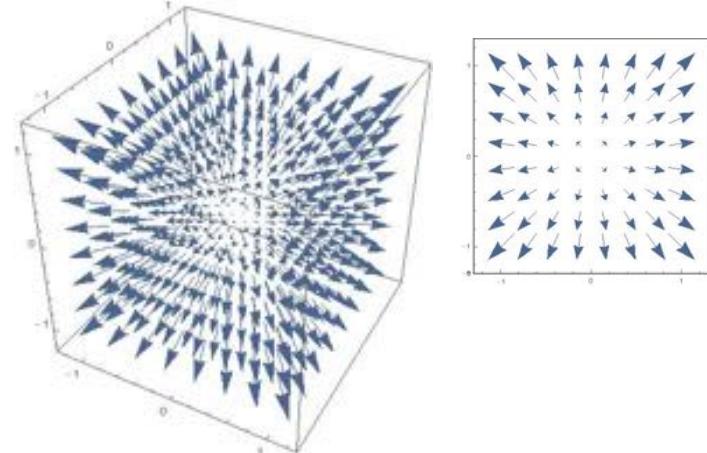
Volume of a Triangle Mesh II

One example for \vec{F}

$$\vec{F} = \frac{1}{3}\vec{x} = \frac{1}{3}(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)$$



$$\int_{\Omega} dv = \int_{\partial\Omega} \left(\frac{1}{3} \vec{x} \cdot \vec{n} \right) ds = \sum_{t \in T_{ri}} \int_{\partial\Omega_t} \left(\frac{1}{3} \vec{x} \cdot \vec{n} \right) ds$$



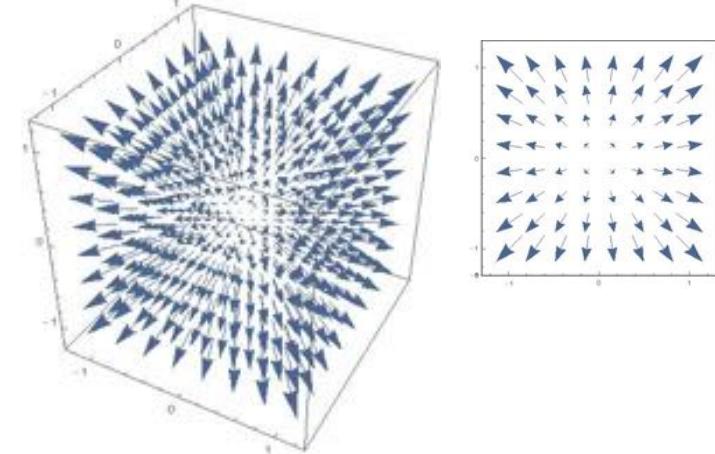
Volume of a Triangle Mesh II

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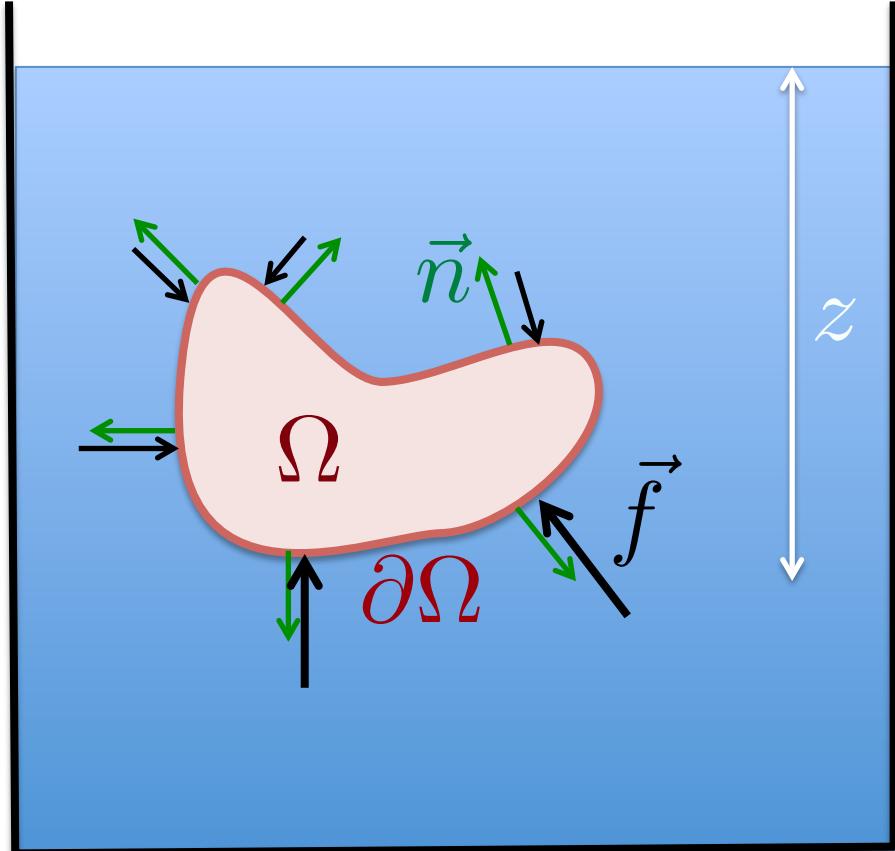


$$\int_{\Omega} dv = \int_{\partial\Omega} \left(\frac{1}{3}\vec{x} \cdot \vec{n} \right) ds = \sum_{t \in T_{ri}} \int_{\partial\Omega_t} \left(\frac{1}{3}\vec{x} \cdot \vec{n} \right) ds$$



$$\int_{\partial\Omega_t} \frac{1}{3}(L_1\vec{x}_1 + L_2\vec{x}_2 + L_3\vec{x}_3) \cdot \vec{n} ds = \frac{1}{3}A \frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{3} \cdot \vec{n}$$

Application of Mass: Buoyancy



Force on the surface is always in
normal direction

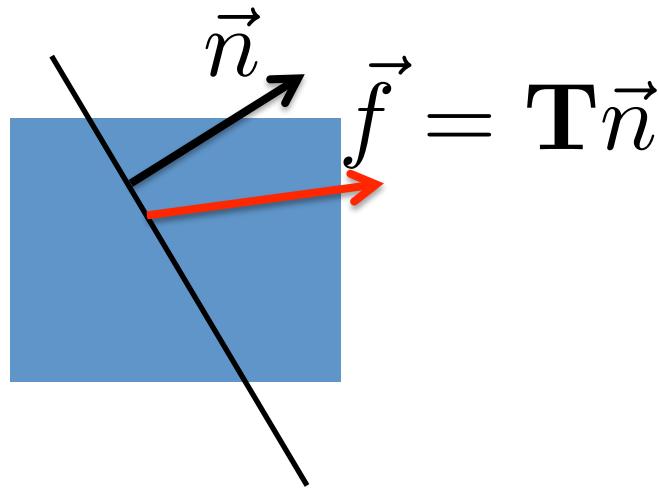
$$\begin{aligned}\vec{f} &= -\rho g z \vec{n} \\ &= T \vec{n}\end{aligned}$$

Cauchy stress

$$T = -\rho g \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix}$$

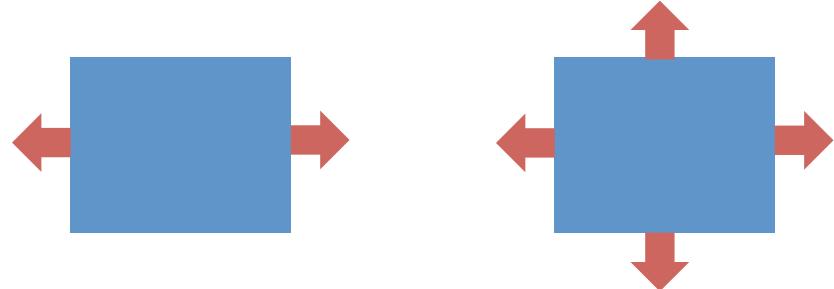
Cauchy Stress Tensor

Symmetric matrix that relate cutting plane normal \vec{n} and force on the cutting plane \vec{f}



$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

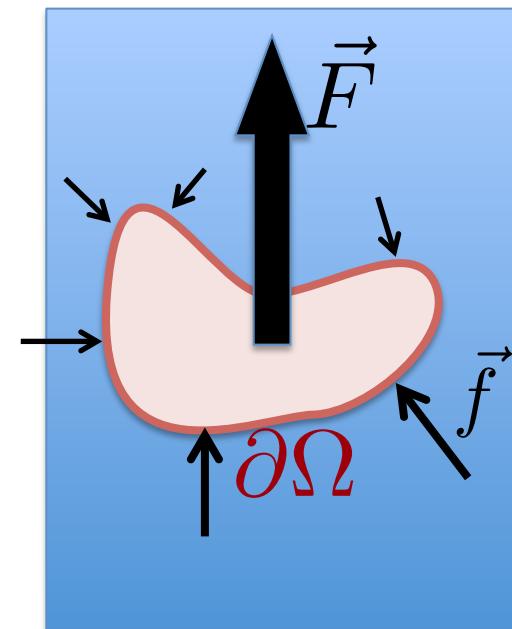
$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Application of Mass: Buoyancy

We can now proof Archimedes' principle.

$$\vec{F} = \int_{\partial\Omega} \vec{f} ds = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds = ?$$



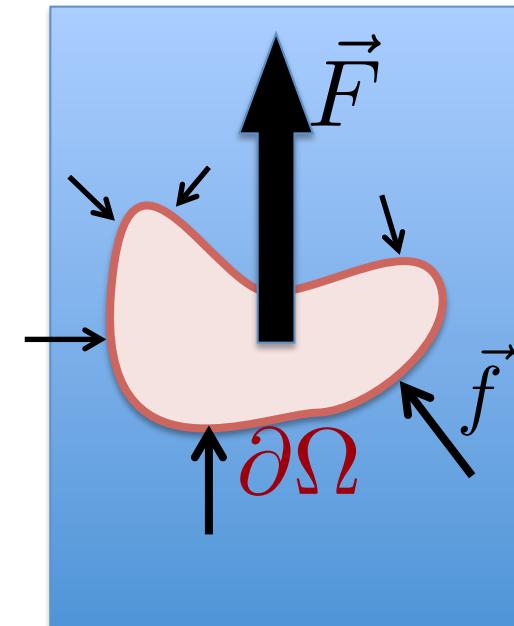
Application of Mass: Buoyancy

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The divergence theorem for tensor

$$\int_{\Omega} (\nabla \cdot \mathbf{T}) dv = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds$$



Interesting Object: Cartesian Diver



<https://www.youtube.com/watch?v=sNOXFjJ4IDU>

Research in CG: Buoyancy Optimization

Buoyancy Optimization for Computational Fabrication

Lingfeng Wang Emily Whiting

Dartmouth College, USA

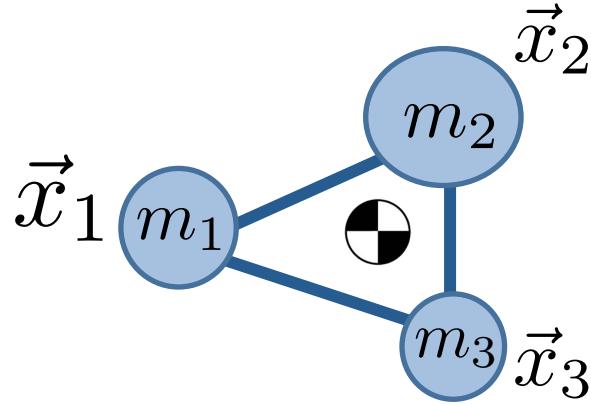
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The Center of Mass

weighted average of the position

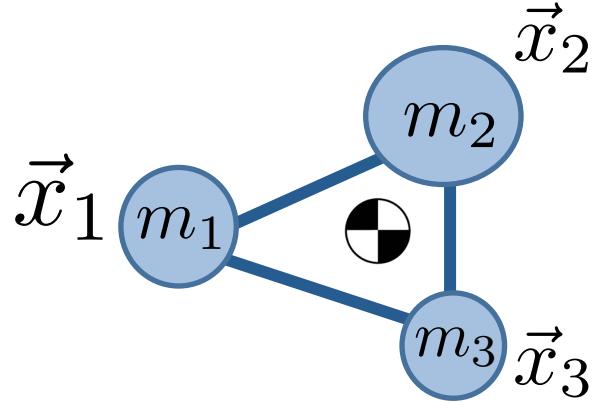
$$\vec{C} = \frac{1}{M} \sum m_i \vec{x}_i$$



The Center of Mass

weighted average of the position

$$\vec{C} = \frac{1}{M} \sum m_i \vec{x}_i$$

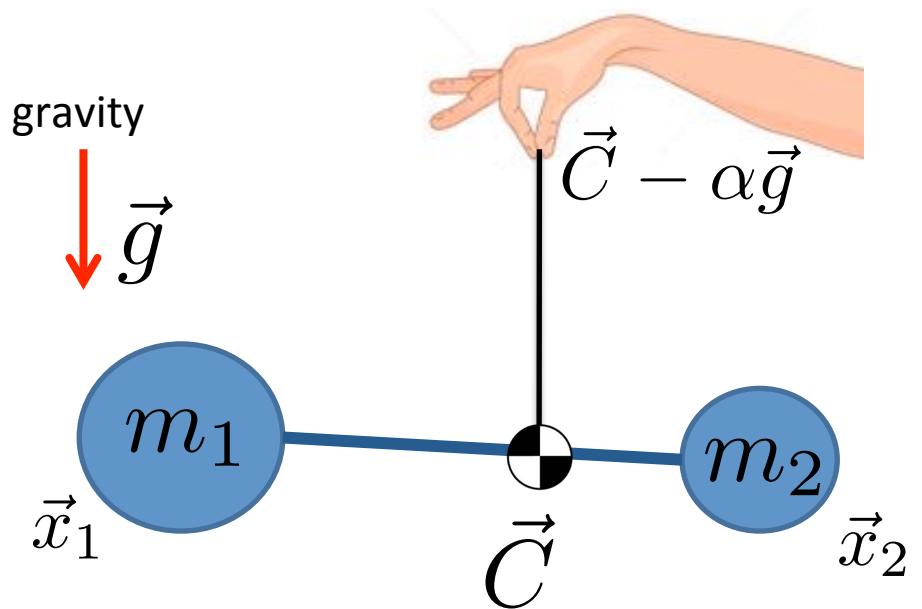


$$\vec{C} = \frac{1}{M} \int_{\Omega} \rho(\vec{x}) \vec{x} dv$$

Homogeneous density \longrightarrow $\vec{C} = \frac{\rho}{M} \int_{\Omega} \vec{x} dv$

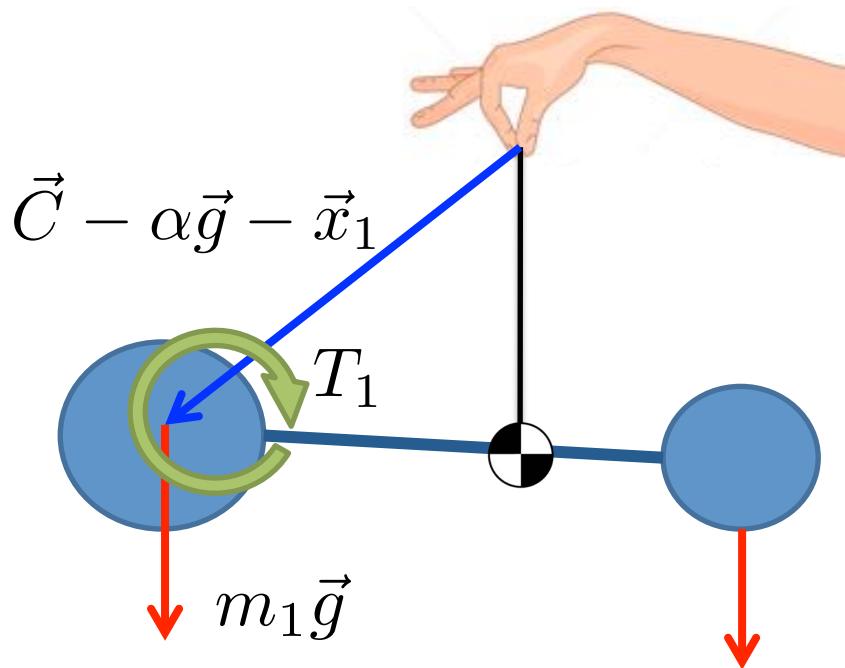
The Center of Mass and Balance

We can hang an object static by putting string above CM



The Center of Mass and Balance

We can hang an object static by putting string above CM

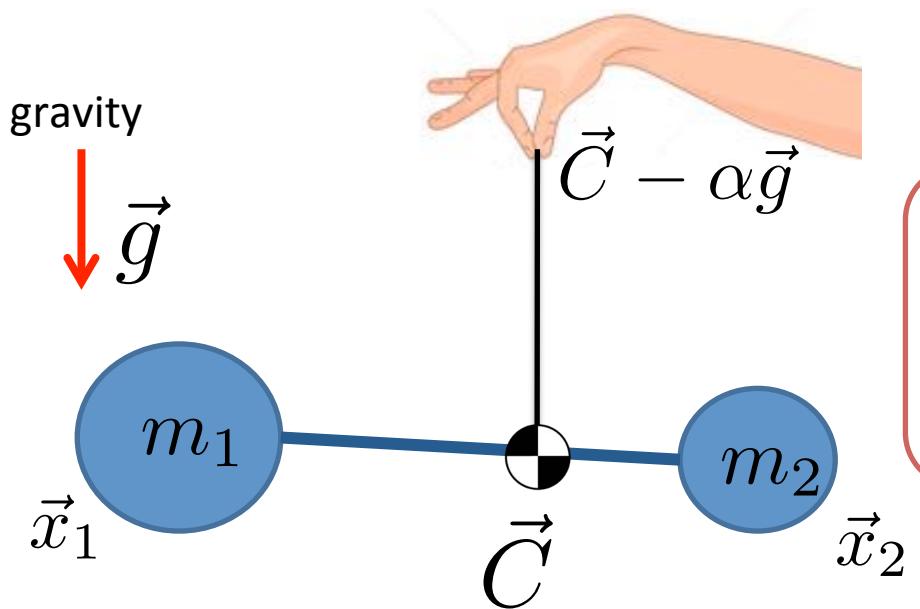


Torque from the right side of the object

$$T_1 = (m_1 \vec{g}) \times (\vec{C} - \alpha \vec{g} - \vec{x}_1)$$

The Center of Mass and Balance

We can hang an object static by putting string above CM



Sum of the torque is always zero

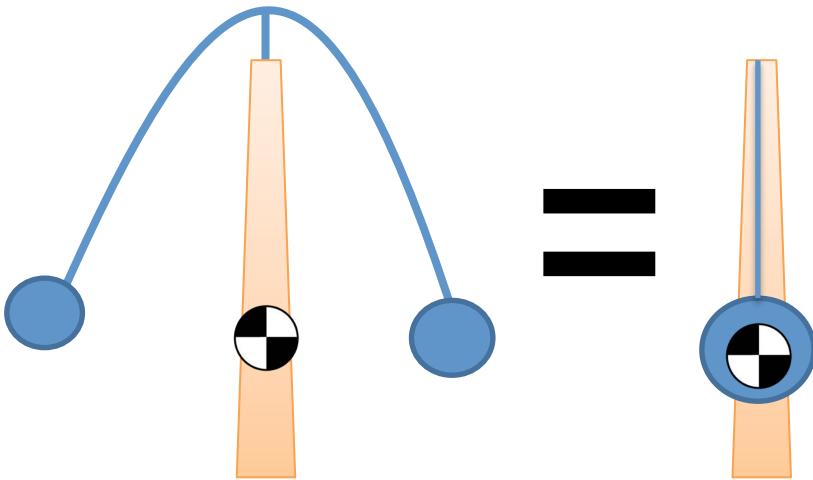
$$\sum (m_i \vec{g}) \times (\vec{C} - \alpha \vec{g} - \vec{x}_i) = 0$$

Balancing Toy

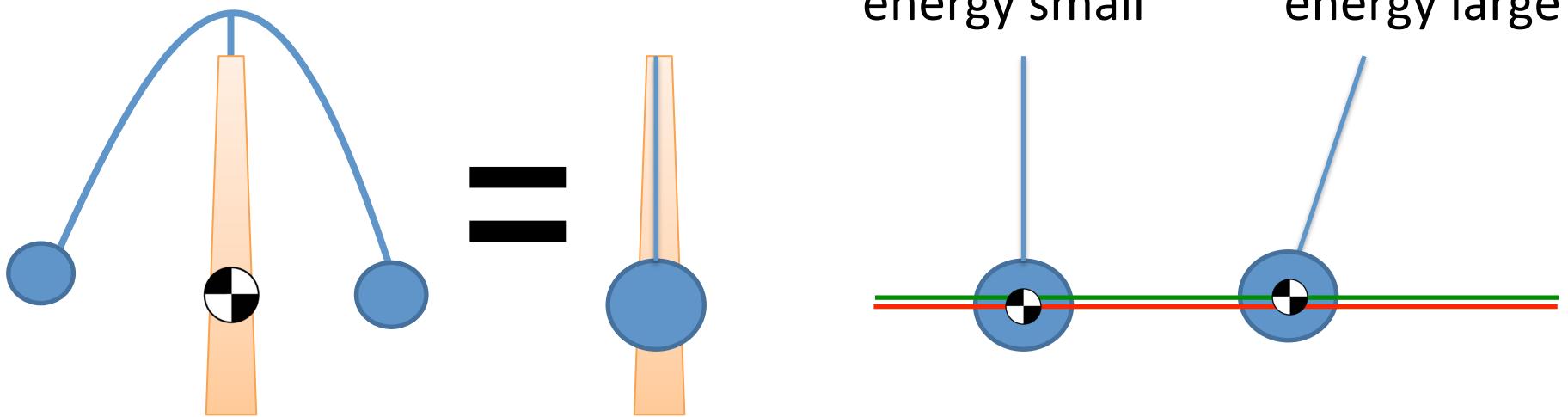


<https://www.youtube.com/watch?v=wKvk-Q7U3nM>

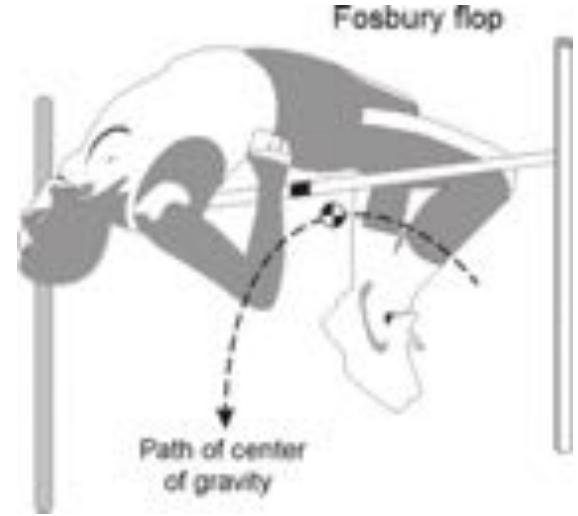
Criteria for Stability: Point Support



Criteria for Stability: Point Support



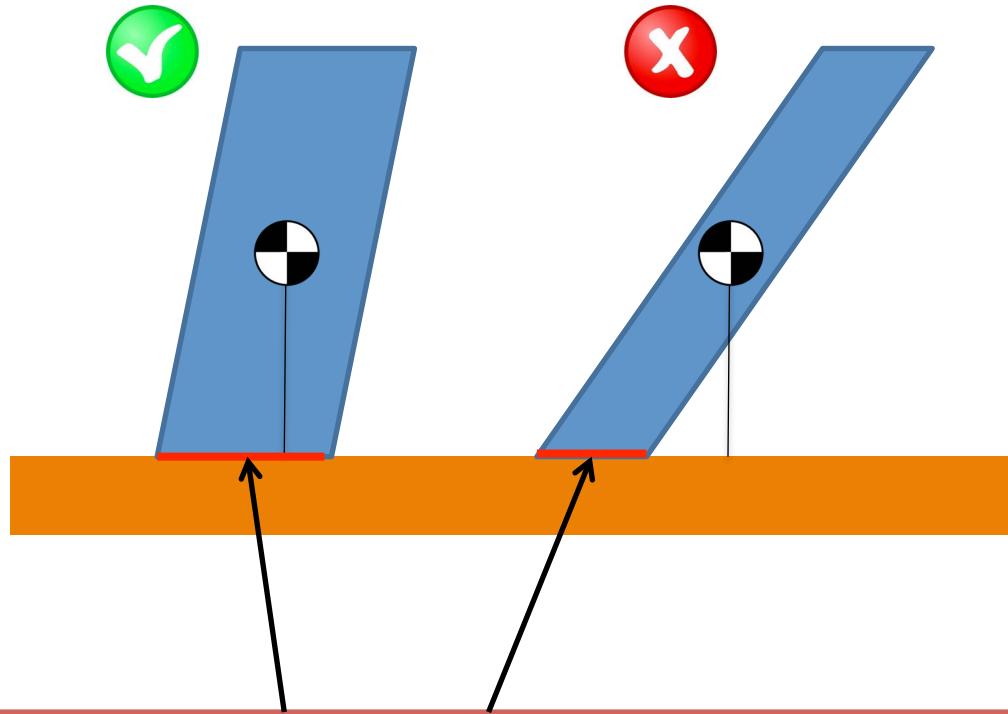
CM Can be Outside: The Fosbury Flop



1968 summer Olympic in Mexico City

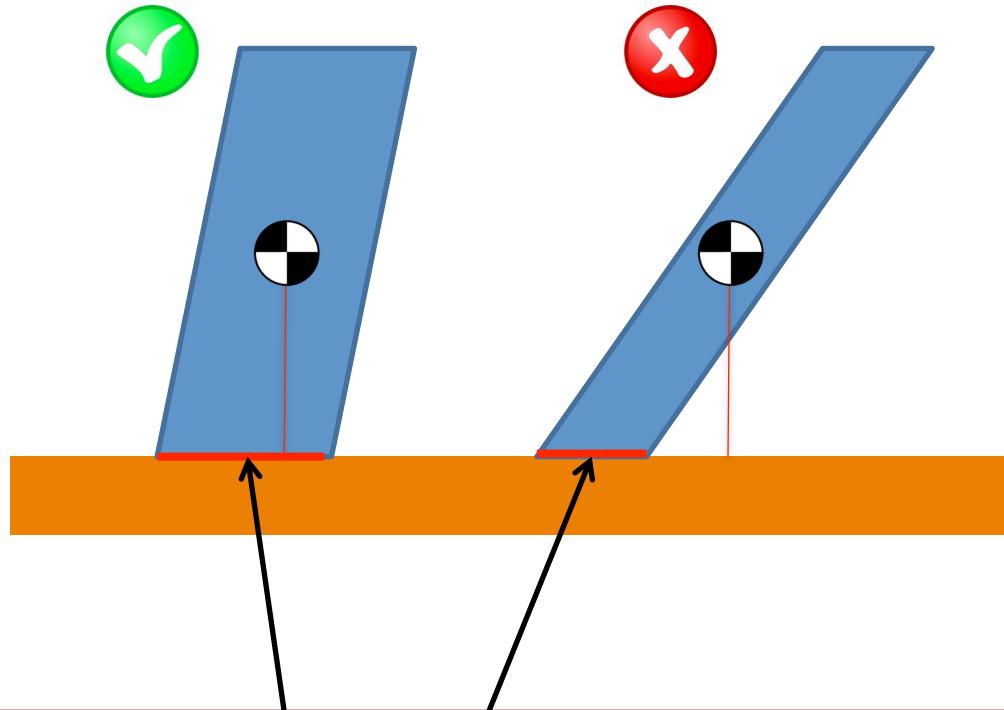


Criteria for Stability: Standing

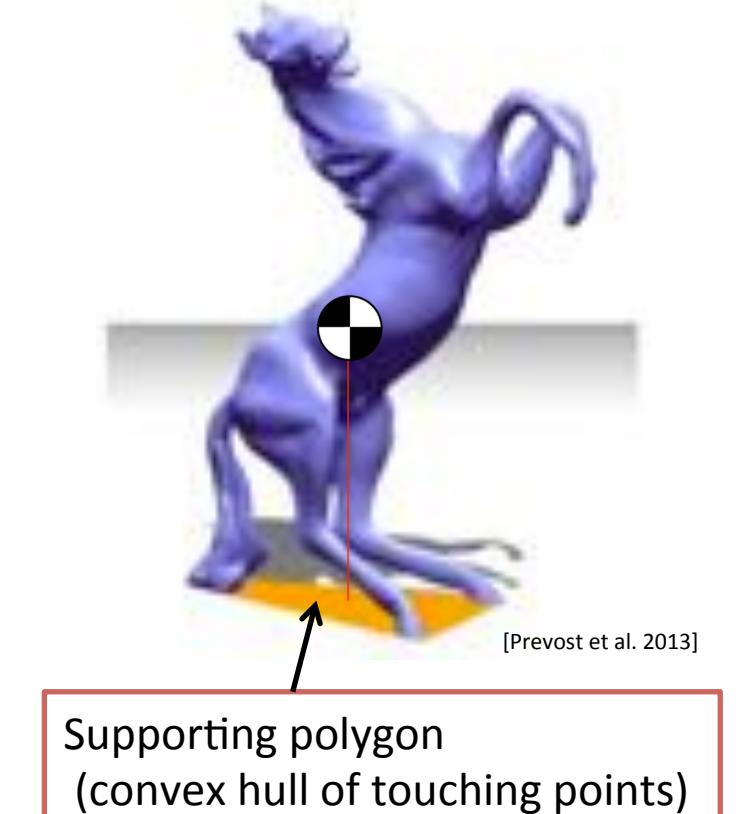


Center of mass should be above the support region

Criteria for Stability: Standing



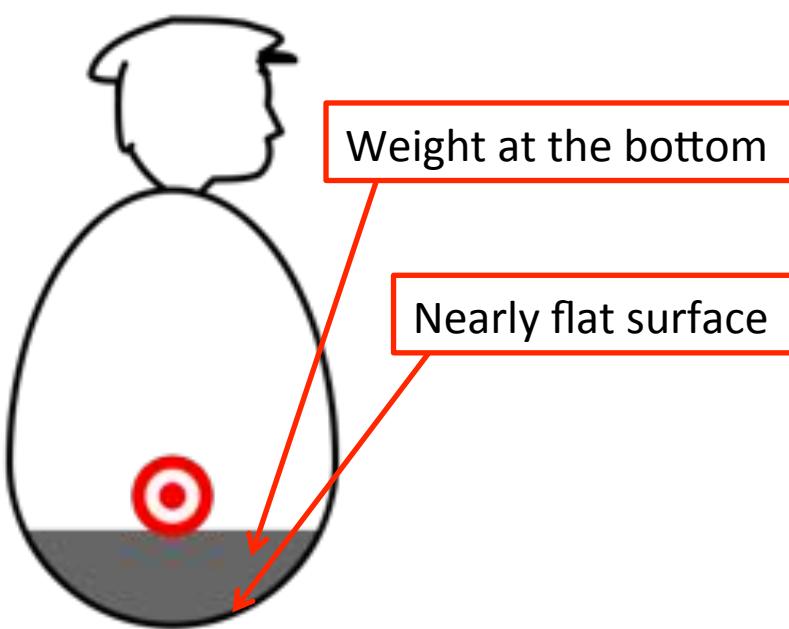
Center of mass should be above the support region



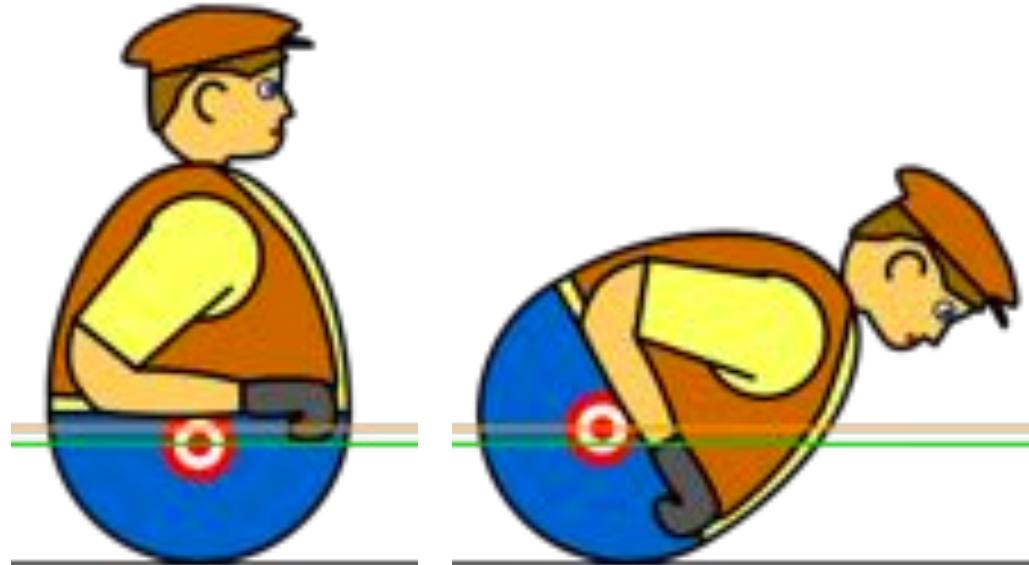
Supporting polygon
(convex hull of touching points)

Criteria for Stability: Smooth Surface

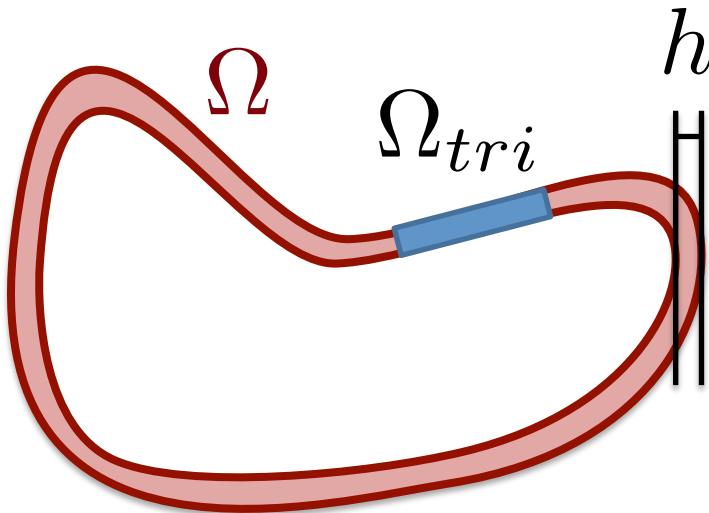
Roly-poly toy



energy small

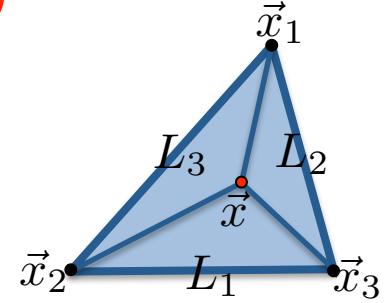


Center of Mass for **Shell** Triangle Mesh

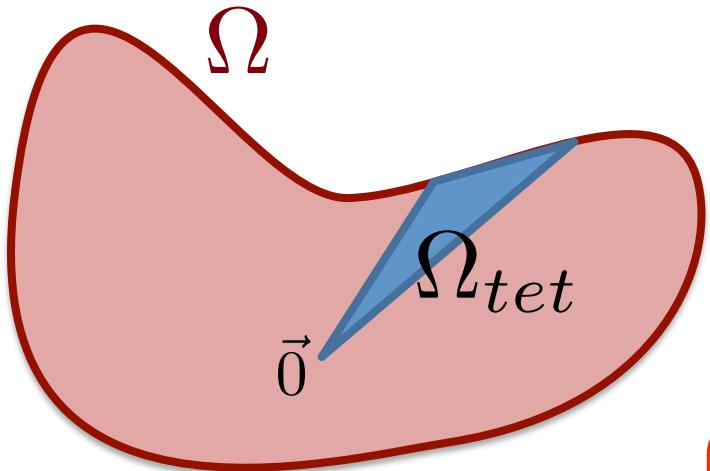


$$\int_{\Omega} \vec{x} dv = \sum_{t \in T} h \int_{\Omega_{tri}} \vec{x} ds$$

$$\begin{aligned}\int_{\Omega_{tri}} \vec{x} ds &= \int_{\Omega_{tri}} (L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3) ds \\ &= A \frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{3}\end{aligned}$$

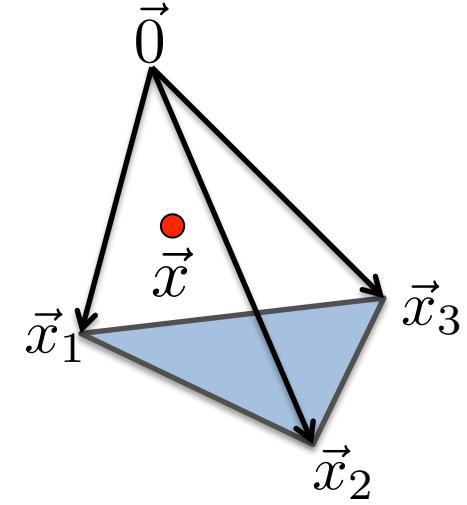


Center of Mass for Solid Triangle Mesh



$$\int_{\Omega} \vec{x} dv = \sum_{t \in T} \int_{\Omega_{tet}} \vec{x} dv$$

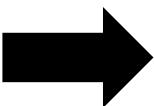
$$\begin{aligned}\int_{\Omega_{tet}} \vec{x} dv &= \int_{\Omega_{tet}} (L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0}) dv \\ &= V \frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{4}\end{aligned}$$



Center of Mass for **Solid** Triangle Mesh II

The divergence theorem for tensor

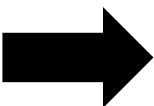
$$\int_{\Omega} \vec{x} dv = \int_{\Omega} (\nabla \cdot \mathbf{T}) dv = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds$$

$\nabla \cdot \mathbf{T} = \vec{x}$  What's the \mathbf{T} looks like?

Center of Mass for **Solid** Triangle Mesh II

The divergence theorem for tensor

$$\int_{\Omega} \vec{x} dv = \int_{\Omega} (\nabla \cdot \mathbf{T}) dv = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds$$

$\nabla \cdot \mathbf{T} = \vec{x}$  What's the \mathbf{T} looks like?

One example:

$$\mathbf{T} = \frac{1}{2} \begin{pmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix}$$

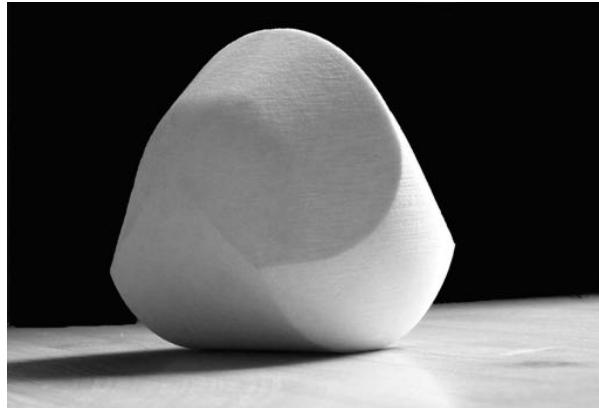
Interesting Object: Sphericon



<https://www.youtube.com/watch?v=L4KQmk7QEVA>

Interesting Object: Gomboc

- Convex and homogeneous object that has a single stable configuration



<https://www.youtube.com/watch?v=p96YazGhIY4>

Research in CG: Make it Stand

Make It Stand: Balancing Shapes for 3D Fabrication

Romain Prévost¹ Emily Whiting¹ Sylvain Lefebvre² Olga Sorkine-Hornung¹

¹ETH Zurich ²INRIA

(contains audio)

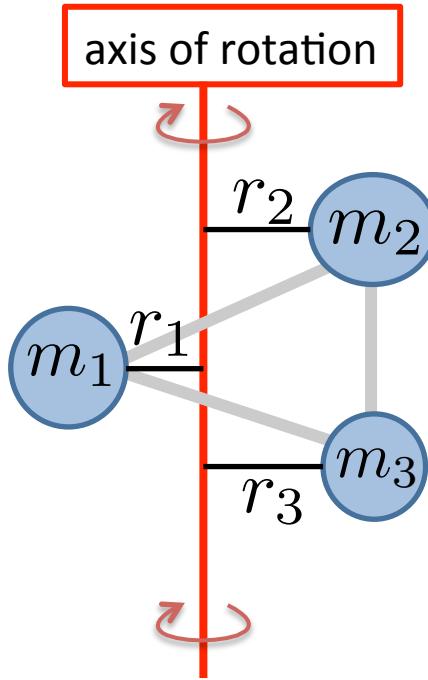
Overview

- Intro
- Integration over 3D domain
- Mass
- Center of Mass
- Moment of Inertia

The Moment of Inertia

We can define moment & energy of rotating object using MI

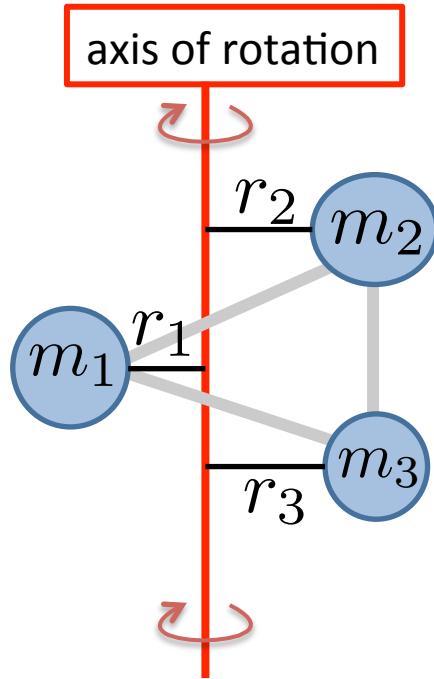
$$I = \sum m_i r_i^2$$



The Moment of Inertia

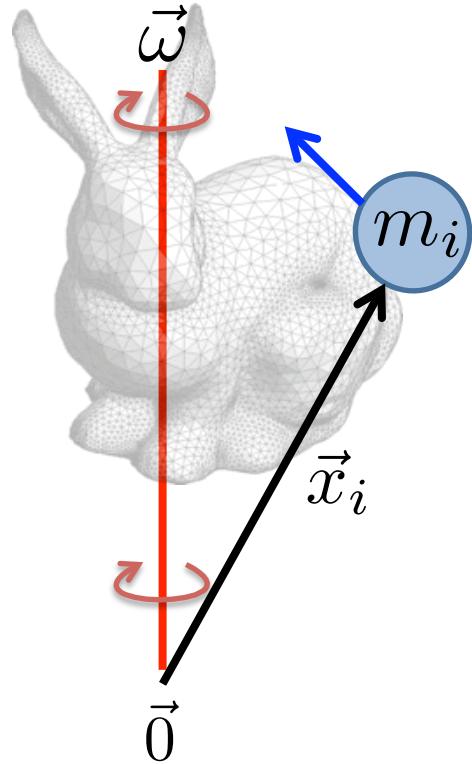
We can define moment & energy of rotating object using MI

$$I = \sum m_i r_i^2$$



$$I = \int_{\Omega} \rho(\vec{x}) r(\vec{x})^2 dv \xrightarrow{\text{Homogeneous density}} I = \rho \int_{\Omega} r(\vec{x})^2 dv$$

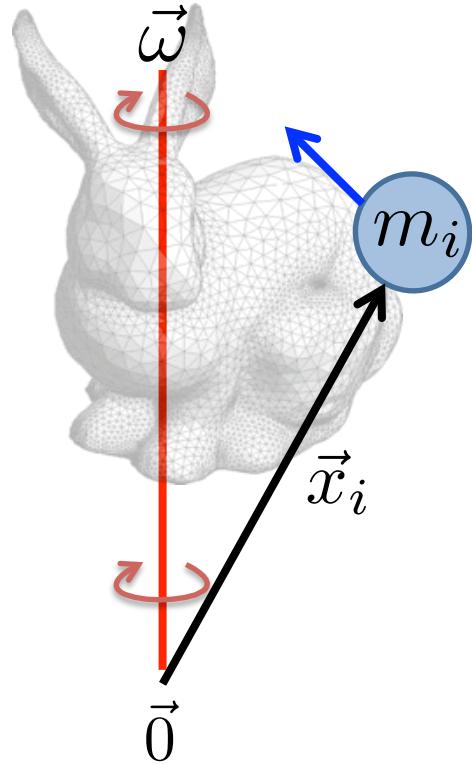
Angular Momentum and MI



angular momentum

$$\vec{L} = \sum \vec{x}_i \times (m_i \vec{v}_i)$$

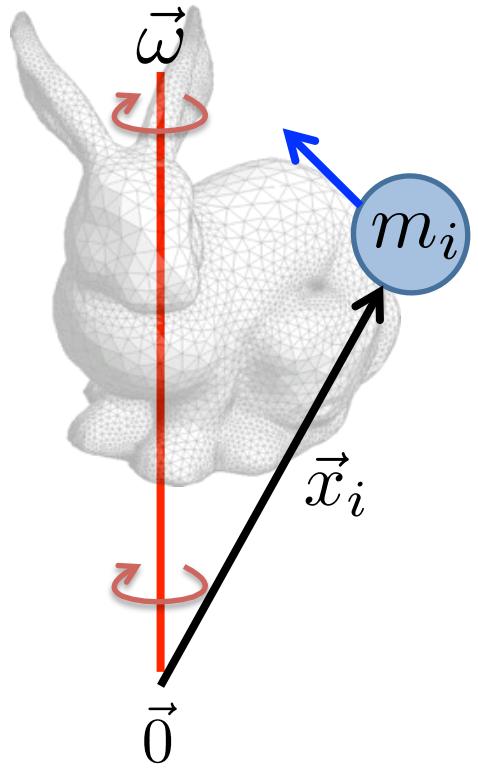
Angular Momentum and MI



angular momentum

$$\begin{aligned}\vec{L} &= \sum \vec{x}_i \times (m_i \vec{v}_i) \\ &= \sum m_i \vec{x}_i \times (\vec{\omega} \times \vec{x}_i)\end{aligned}$$

Angular Momentum and MI



angular momentum

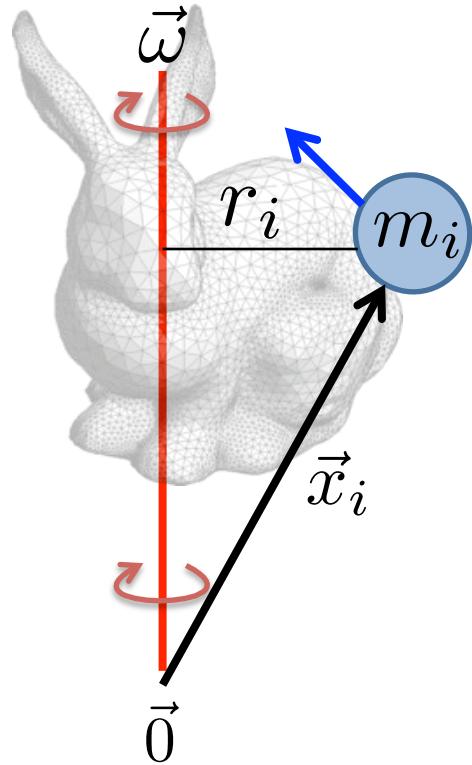
$$\begin{aligned}\vec{L} &= \sum \vec{x}_i \times (m_i \vec{v}_i) \\ &= \sum m_i \vec{x}_i \times (\vec{\omega} \times \vec{x}_i) \\ &\quad \downarrow \\ &\quad \boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}} \\ &= I \vec{\omega}\end{aligned}$$

Conservation of Angular Momentum



<https://www.youtube.com/watch?v=UZIW1a63Kzs>

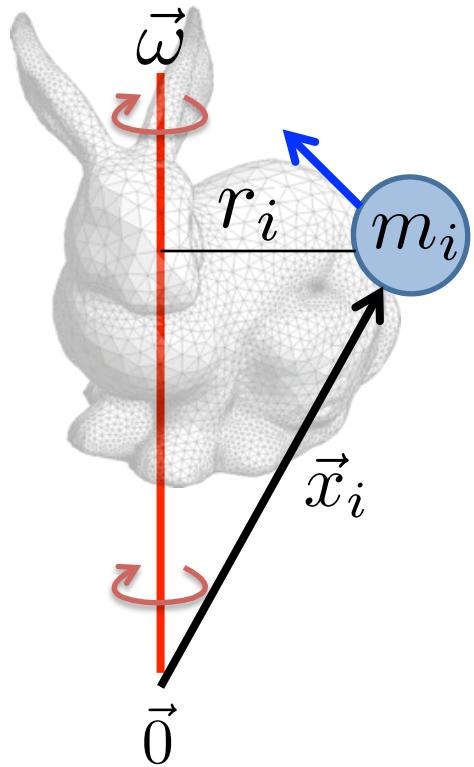
Kinetic Energy and MI



Kinetic Energy

$$E = \sum \frac{1}{2} m_i |\vec{v}_i|^2$$

Kinetic Energy and MI



Kinetic Energy

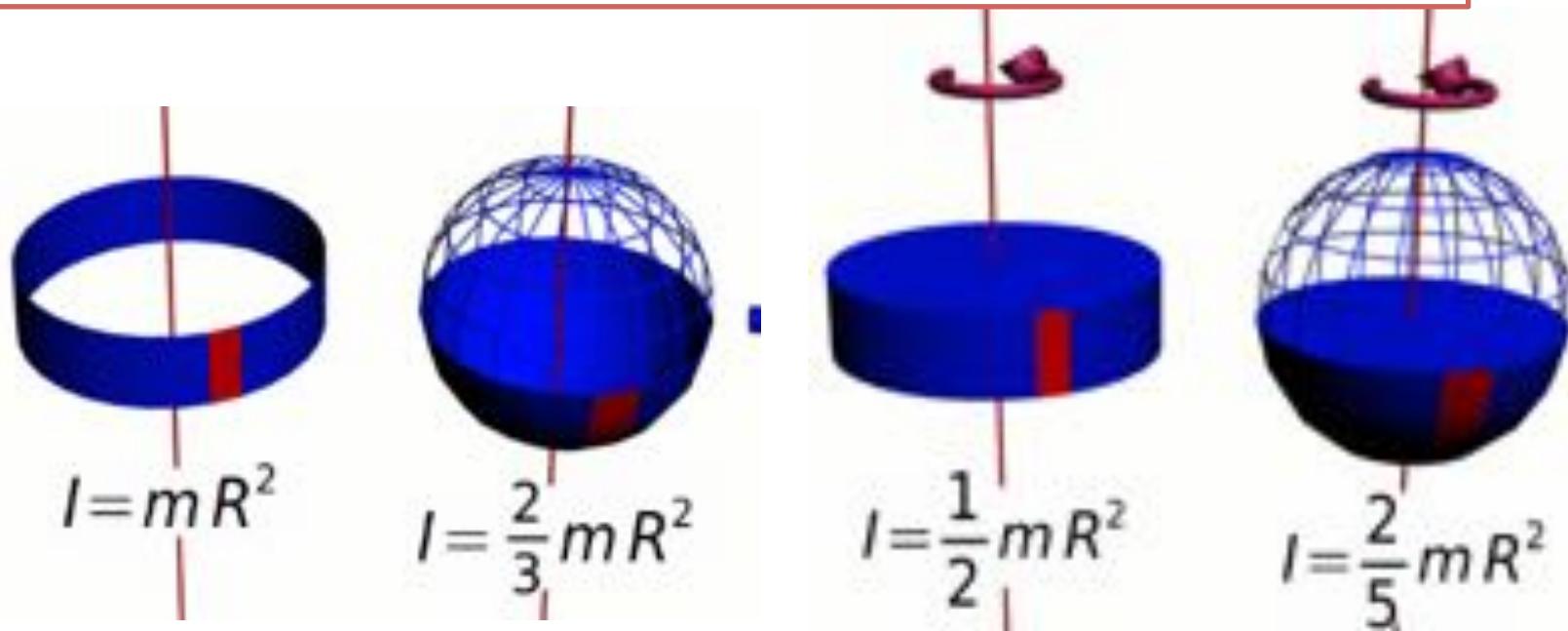
$$E = \sum \frac{1}{2} m_i |\vec{v}_i|^2$$

$$= \sum \frac{1}{2} m_i (|\vec{w}| r_i)^2$$

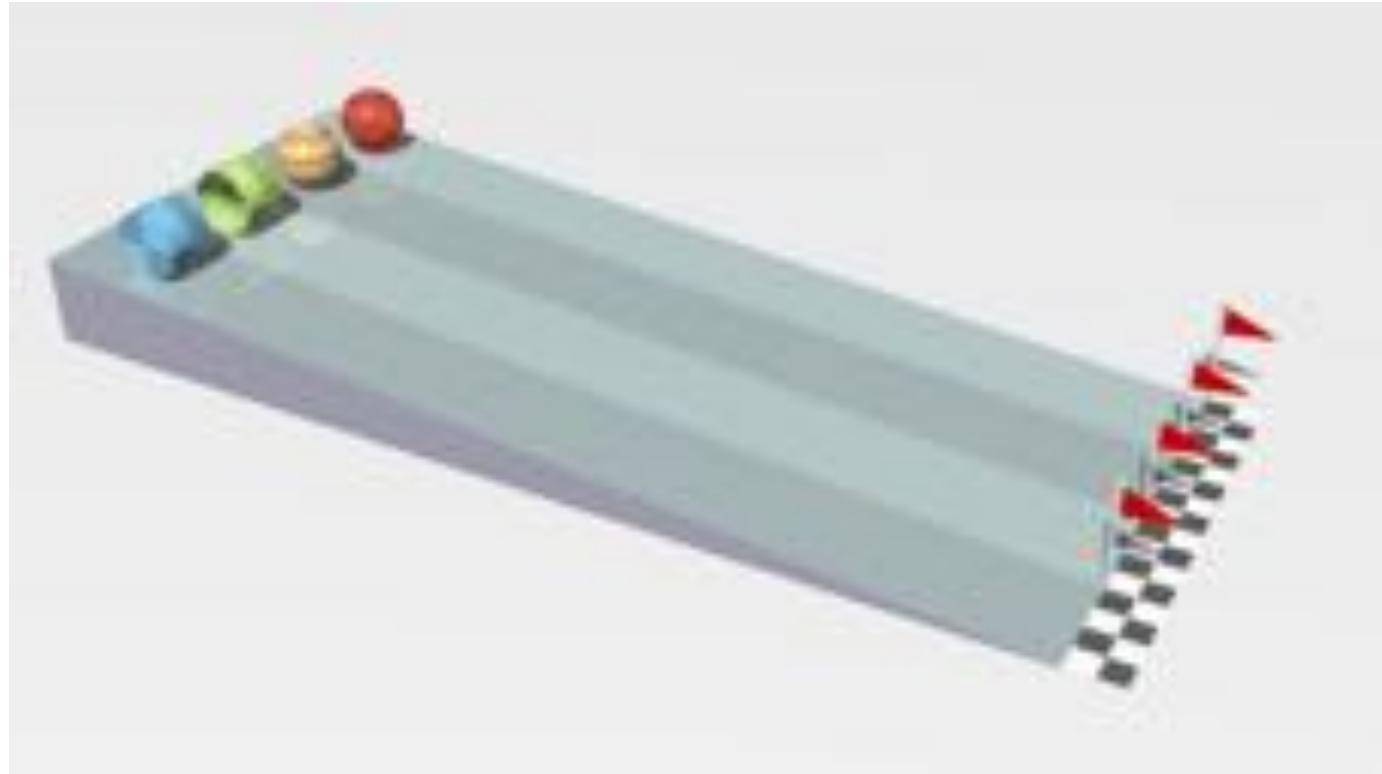
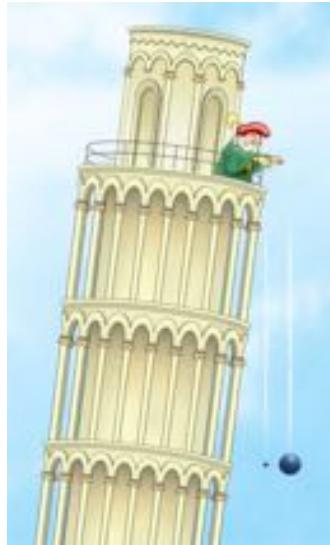
$$= \frac{1}{2} I |\vec{\omega}|^2$$

Example of Moment of Inertia

- Same weight give a different Moment of Inertia
- Putting weight outer ward give large Moment of Inertia



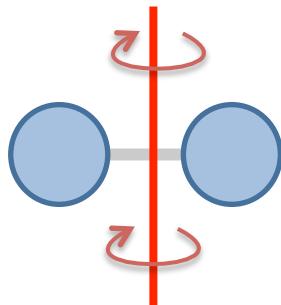
Slope Version of Galileo's Pisa Experiment



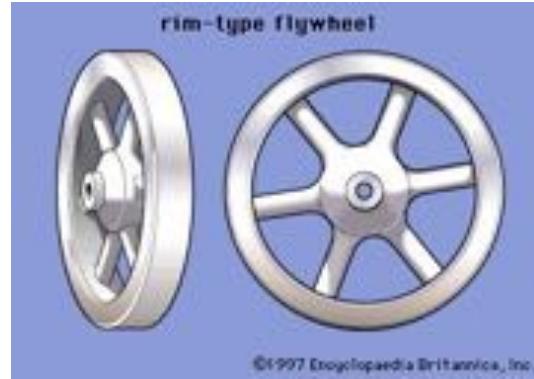
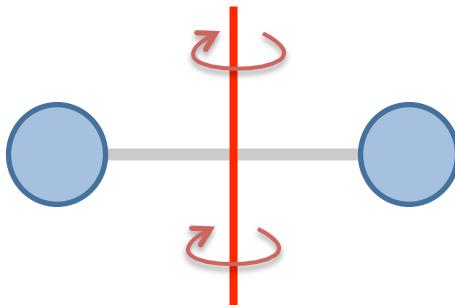
https://en.wikipedia.org/wiki/Moment_of_inertia

Application: Flywheel

small I



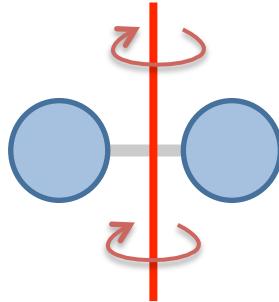
large I



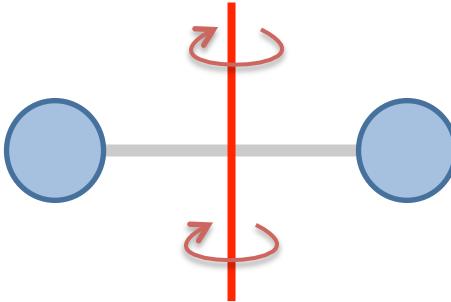
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Application: Flywheel

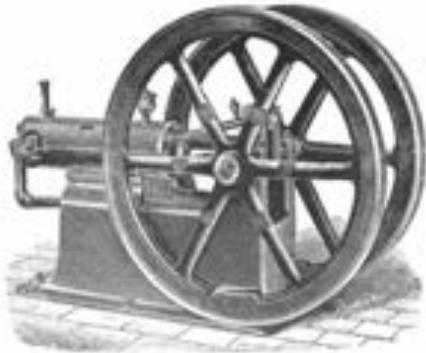
small I



large I



Smoothing output of engine



Energy storage

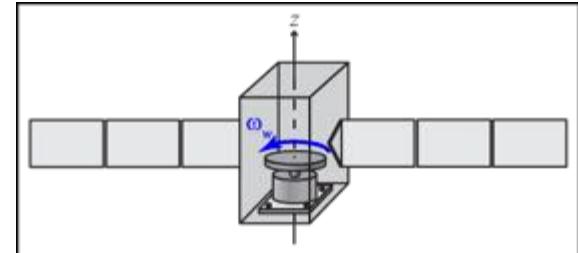


rim-type flywheel

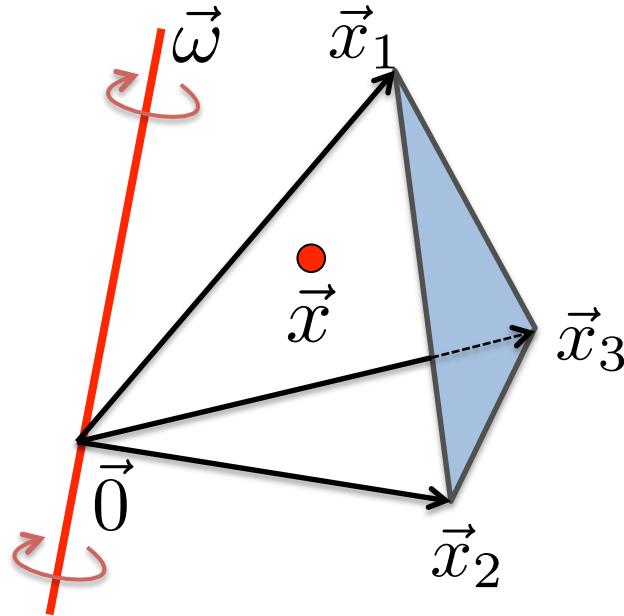


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Reaction wheel

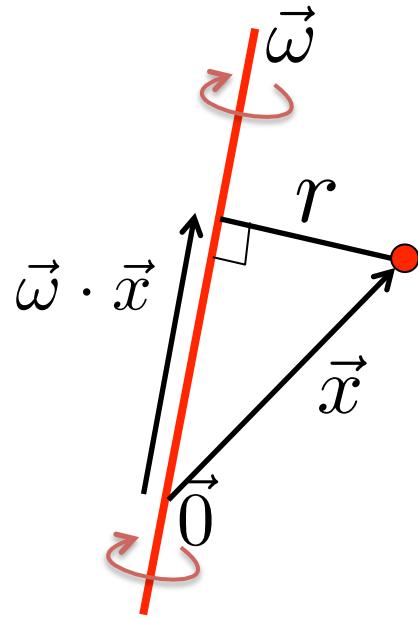


Moment of Inertia for a Tetrahedron



$$\begin{aligned}\vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\ &= L_i x_i\end{aligned}$$

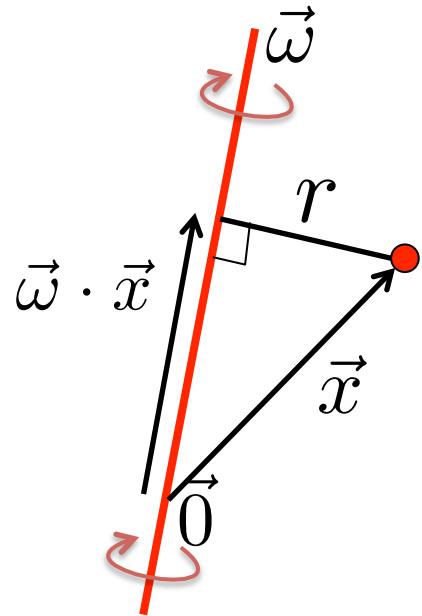
Moment of Inertia for a Tetrahedron



$$r^2 = (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2$$

$$\begin{aligned}\vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\ &= L_i \vec{x}_i\end{aligned}$$

Moment of Inertia for a Tetrahedron

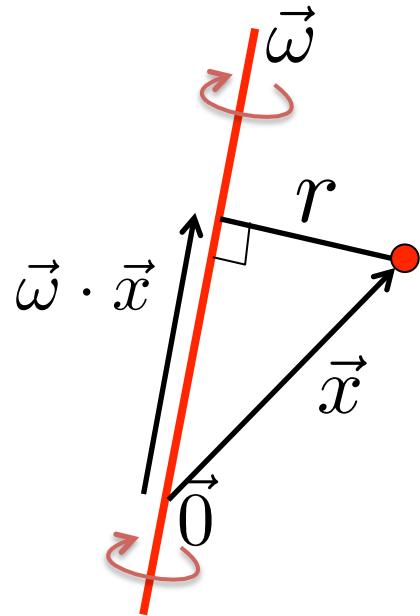


$$r^2 = (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2$$

$$= (L_i \vec{x}_i) \cdot (L_j \vec{x}_j) - (L_i \vec{x}_i \cdot \vec{\omega}) \cdot (L_j \vec{x}_j \cdot \vec{\omega})$$

$$\begin{aligned}\vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\ &= L_i \vec{x}_i\end{aligned}$$

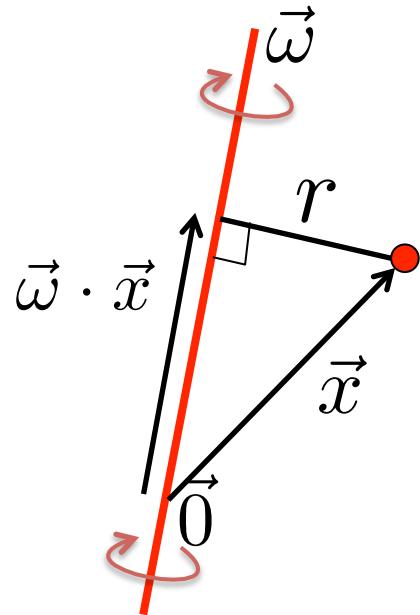
Moment of Inertia for a Tetrahedron



$$\begin{aligned} r^2 &= (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2 \\ &= (L_i \vec{x}_i) \cdot (L_j \vec{x}_j) - (L_i \vec{x}_i \cdot \vec{\omega}) \cdot (L_j \vec{x}_j \cdot \vec{\omega}) \\ &= L_i L_j (\vec{x}_i \cdot \vec{x}_j) - L_i L_j (\vec{x}_i \cdot \vec{\omega})(\vec{x}_j \cdot \vec{\omega}) \end{aligned}$$

$$\begin{aligned} \vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\ &= L_i \vec{x}_i \end{aligned}$$

Moment of Inertia for a Tetrahedron



$$\begin{aligned}\vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\ &= L_i \vec{x}_i\end{aligned}$$

$$r^2 = (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2$$

$$= (L_i \vec{x}_i) \cdot (L_j \vec{x}_j) - (L_i \vec{x}_i \cdot \vec{\omega}) \cdot (L_j \vec{x}_j \cdot \vec{\omega})$$

$$= \boxed{L_i L_j} (\vec{x}_i \cdot \vec{x}_j) - L_i L_j (\vec{x}_i \cdot \vec{\omega})(\vec{x}_j \cdot \vec{\omega})$$

integration

$$\int_V L_1^a L_2^b L_3^c L_4^d dv = \frac{a! b! c! d! 3!}{(a+b+c+d+3)!} V$$

$$\frac{V}{20} (i \neq j), \quad \frac{V}{10} (i = j)$$

Research in CG: Spin-It

Spin-It: Optimizing Moment of Inertia for Spinnable Objects

Moritz Bächer

Disney Research Zurich

Emily Whiting

ETH Zurich

Bernd Bickel

Disney Research Zurich

Olga Sorkine-Hornung

ETH Zurich



Disney Research

ETH zürich



Take aways

- Integration formula: divide region into **simplexes**
- Switch a volume/surface integration using the **divergence theorem**
- Three moments of shapes
 - mass, center of mass, moment of inertia
- Many physics are given by the moments
 - floating, standing, rotation