1 Discrete Mathematics

2 Probability

Counting

- Experiment 1 has n outcomes, another has m. This gives $n \cdot m$ outcomes. N repitions of Experiment 1 gives n^N outcomes.
- Permutations: how many ways to order a set. $n!$, or if we count repeated items as indistinguishable, $\frac{n!}{r_1! \cdot \dots \cdot r_i!}$ where r_i is the number of times the number i was repeated.
- $P_{n,k} = \frac{n!}{(n-k)!}$ ($\binom{n}{k}$ but order matters), $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- The number of ways to choose something OR something else is addition. Choosing two things together (AND) is multiplication. If we are to find 'at least one (up to 3)', it would be the number of ways to choose 1 OR 2 OR 3.

Basic Probability

- $\bigcup_i E_i$ means at least one E_i and $\bigcap_i E_i$ means all of the E_i's.
- Mutex events (disjoint) can't happen at the same time. If $E \subseteq F$, E can't happen without F happening. $E^c = \Omega - E$.
- Axioms: for countably many *mutex* events, $P(\bigcup E_i) = \sum_i P(E_i)$. If not mutex, this equality is replaced with $\leq$.
- Elementary events are those such that all events have probability $\frac{1}{|\Omega|}$.
- $P(F - E) = P(F \cap E^c) = P(F) - P(F \cap E)$. This is the set difference law.
- De Morgan's: $(E \cap F)^c = E^c \cup F^c$ (flip intersection/union and take complement).
- $P(E|F) = \frac{P(E \cap F)}{P(F)}$. All other axioms work fine, just add the 'given' part to each.
- Multiplication rule: $P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 \cap E_2) \cdots P(E_n|E_1 \cap \dots \cap E_{n-1})$.
- Independent $\iff$ $P(F|E) = P(F)$ $\iff$ $P(E|F) = P(E)$ $\iff$ $P(E \cap F) = P(E) \cdot P(F)$.
- Three or more events are mutually independent if the above multiplication rule applies to all pairs, and all of them together.
- Murphy's Law: As $n \to \infty$ with fixed probability p, $P(\text{all } n \text{ experiments succeed}) = p^n \to 0$, $P(\text{at least one succeeds}) = 1 - P(\text{each one fails}) = 1 - (1-p)^n \to 1$, $P(\text{exactly } k \text{ succeed}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$.

Discrete Random Variables

- PMF: $p_X(x) = P(X = x)$, CDF: $F_X(x) = P(X \leq x)$. CDF is non-decreasing.
- $\mathbb{E}(X) = \sum_i x_i \cdot p(x_i)$. Needn't be a possible value.
- nth moment: $\mathbb{E}(X^n)$. Absolute moment: $\mathbb{E}(|X|^n)$.
- $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. $\text{SD}(X) = \sqrt{\text{Var}(X)}$.
- Binomial: there are n independent trials, each succeeding with probability p. X counts the number of successes: $X \sim \text{Bin}(n, p)$. If $n = 1$, this is Bernoulli where $P(X = 1) = p$ is the probability of the 'only' success. $p(x) = P(\text{number of successes is x}) = \binom{n}{x} p^x (1-p)^{n-x}$, $\mathbb{E}(X) = np$, $\text{Var}(X) = np(1-p)$.
- Poisson: $X \sim \text{Pois}(\lambda \in \mathbb{R}^+)$ if $p(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$, $\mathbb{E}(X) = \text{Var}(X) = \lambda$. Poisson measures the probability of a number of events happening in a space, based on an average (λ). For example, number of calls per hour, or number of typos on a page. $\text{Bin}(n, p) \approx \text{Pois}(np)$ for large n and small p.
- Geometric: $X \sim \text{Geom}(p)$ counts the number of independent trials repeated until we get a success (with probability p), with no memory. $p(x) = p(1-p)^{x-1}$, $\mathbb{E}(X) = \frac{1}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$.

Continuous Random Variables

- test