Perceptron

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Contents

1 How works?:

1.1 A real neuron:

A real neuron, also known as a nerve cell, is a specialized cell that transmits electrical and chemical signals in the nervous system.

At the most basic level, a neuron consists of a cell body, dendrites, and an axon. The cell body contains the nucleus and other organelles, and serves as the metabolic center of the neuron. Dendrites are thin, branching extensions that receive signals from other neurons or sensory cells. The axon is a long, thin projection that carries signals away from the cell body to other neurons or target cells. Neurons communicate with each other through synapses, which are specialized
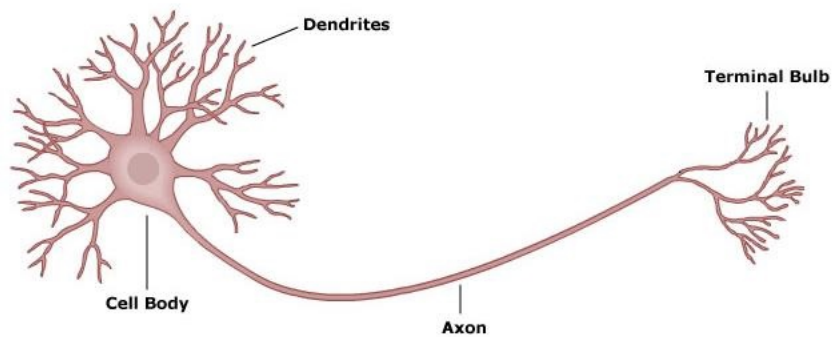
Figure 1: Cell Body with its axon

junctions between neurons. When an electrical signal, known as an action potential, reaches the end of an axon, it triggers the release of neurotransmitter molecules, which diffuse across the synapse and bind to receptors on the dendrites or cell body of the target neuron. This binding can cause the target neuron to generate its own action potential, which propagates down its axon to signal other neurons or target cells.

Figure 2: Synapse

Neurons talk to each other across synapses. When an action potential reaches the presynaptic terminal, it causes neurotransmitter to be released from the neuron into the synapti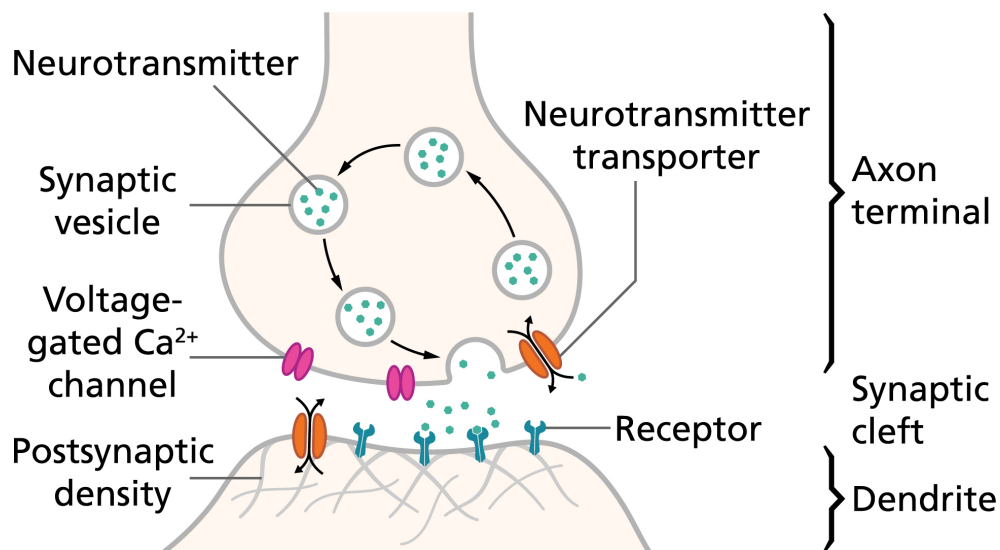c cleft, a 20–40nm gap between the presynaptic axon terminal and the postsynaptic dendrite (often a spine). After travelling across the synaptic cleft, the transmitter will attach to neurotransmitter receptors on the postsynaptic side, and depending on the neurotransmitter released (which is dependent on the type of neuron releasing it), particular positive (e.g. Na+, K+, Ca+) or negative ions (e.g. Cl-) will travel through channels that span the membrane.

1.2   The perceptron:

The perceptron is a type of artificial neural network that is loosely modeled after the structure and function of real neurons in the brain. The basic idea behind the perceptron is to use a mathematical algorithm to simulate the behavior of a simplified neuron, which can then be used to perform simple classification tasks.
The perceptron consists of an input layer, a set of weights, a summing or activation function, and an output. The input layer consists of one or more nodes, each of which represents a feature of the input data. The weights are values that are assigned to each input node, which control the strength of the connection between that node and the output.
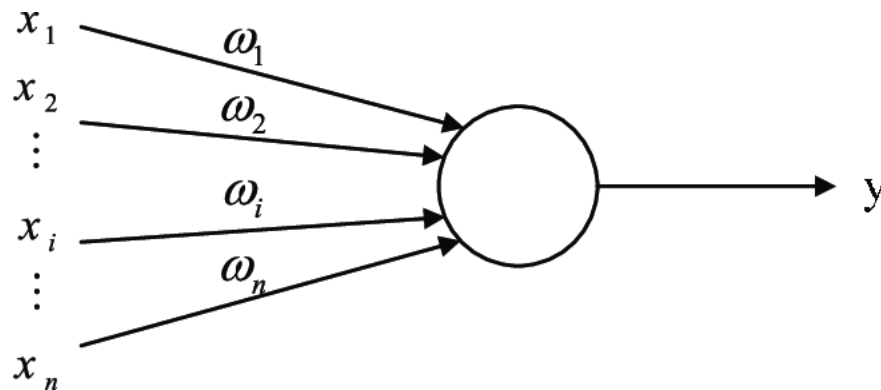

Figure 3: Perceptron

The summing or activation function takes the weighted inputs and adds them together to produce a single output value. This output value is then compared to a threshold value, which determines whether the perceptron will output a positive or negative value. If the output is positive, the perceptron classifies the input as belonging to one class; if it is negative, the input is classified as belonging to the other class.

1.3   Perceptron parts:

- **Input Nodes or Input Layer:**
  This is the primary component of Perceptron which accepts the initial data into the system for further processing. Each input node contains a real numerical value.

- **Wight and Bias:**
  Weight parameter represents the strength of the connection between units. This is another most important parameter of Perceptron components. Weight is directly proportional to

the strength of the associated input neuron in deciding the output. Further, Bias can be considered as the line of intercept in a linear equation.

- **Activation Function:**
  These are the final and important components that help to determine whether the neuron will fire or not. Activation Function can be considered primarily as a step function.
  Types of Activation functions:

  - Sign function.
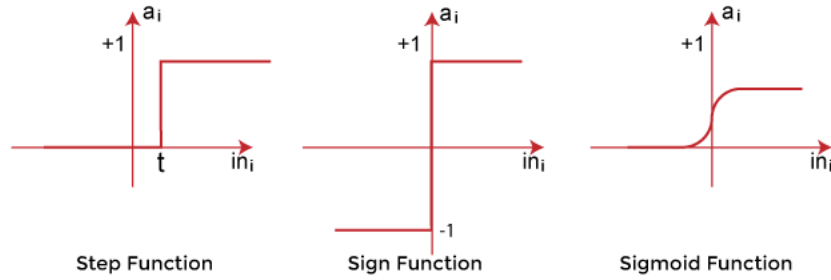 - Step function.
  - Sigmoid function.

Figure 4: Types of activation functions

2 Códing the perceptron:

2.1 The Perceptron Algorithm:

Perceptron model works in two important steps as follows.

1. In the first step first, multiply all input values with corresponding weight values and then add them to determine the weighted sum. Mathematically, we can calculate the weighted sum as follows.

$$\sum_{i=0}^{n}(weight_i \cdot input_i) = \sum_{i=0}^{n}(w_i \cdot x_i) = w_0 \cdot x_0 + w_1 \cdot x_1 + \cdots + w_n \cdot x_n$$

Add a special term called bias 'b' to this weighted sum to improve the model's performance.

$$\sum_{i=0}^{n}(w_i \cdot x_i) + b$$

2. In the second step, an activation function is applied with the above-mentioned weighted sum, which gives us output either in binary form or a continuous value as follows.

$$y = activation(\sum_{i=0}^{n}(w_i \cdot x_i) + b) = activation(\vec{w} \cdot \vec{x} + b)$$

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At the end what we are doing here is a simple dot product of two vectors.

$$\sum_{i=0}^{n}(w_i \cdot x_i) + b = \vec{w} \cdot \vec{x}$$

Where.

$$\vec{w} = (b, w_1, \cdots, w_n)$$

$$w_0 = b$$

$$\vec{x} = (1, x_1, \cdots, x_n)$$

$$x_0 = 1$$

At the end we can think that this is hiperplane.
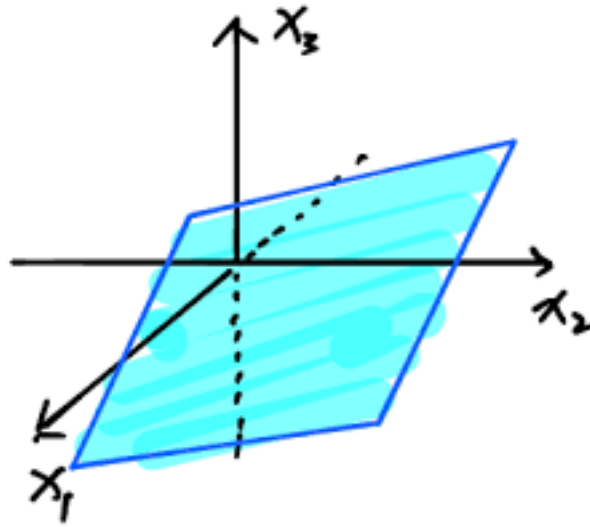
Figure 5: A hyperplane

2.2 From the perceptron algorithm to code:

Firstly declare the threshold variable or bias variable, this variable will be used as a threshold or bias for the activation function.

```
const threshold = 1.5
```

After that declare two arrays the weights and inputs, the weights array is always initialized with random values.

```
inputs = [1, 0, 1, 0, 1]
weights = [0.7, 0.6, 0.5, 0.3, 0.4]
```

The weighted sum of the input values and weights by initializing a variable called sum to zero, and then iterating over the input values and weights and adding the product of each input value and weight to sum.

```
sum = 0.0
for (i = 0, i < inputs.length, i++)
    sum += inputs[i] * weights[i]
```

And at the end just apply an activation function, a sign function to the weighted sum by checking if sum is greater than the threshold value of 1.5. If sum is greater than 1.5, the activate variable is set to 1, otherwise, it is set to -1.

```
activate = (sum > threshold) ? 1 : -1
```

Overall, this code represents a simple example of how a perceptron might work by calculating a weighted sum of inputs and weights and applying a threshold activation function to produce a binary output.

2.3 How to train a perceptron? and The perceptron learning algorithm:

To train a perceptron, we need to adjust the weights of the input connections so that the perceptron produces the desired output for a given set of inputs. We can do this by using an iterative algorithm called the perceptron learning algorithm.

The key idea behind the perceptron learning algorithm is that we can use the error between the predicted and desired outputs to adjust the weights so that the perceptron becomes better at producing the desired output. Specifically, we adjust the weights in proportion to the error, with larger errors resulting in larger weight updates.

Note that the perceptron learning algorithm assumes that the problem is linearly separable, which means that there is a hyperplane that can separate the positive and negative examples in the input space. If the problem is not linearly separable, the perceptron may not be able to converge to a solution.
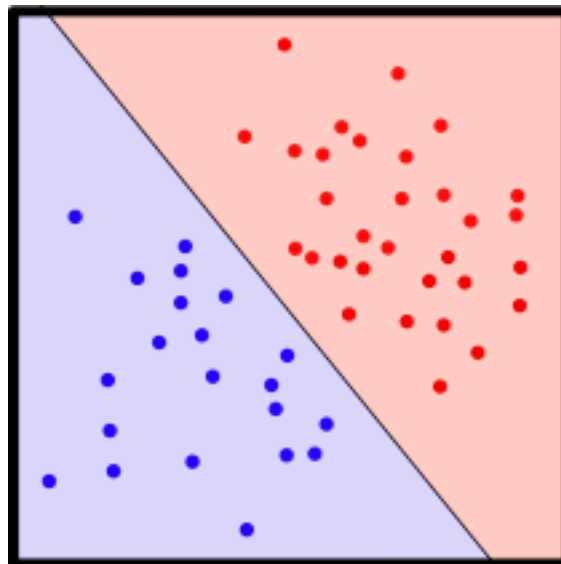

Figure 6: Linear separability

2.3.1 The Mathematical definition behind linear separability:

Let X_0 and X_1 be two sets of points in an n-dimensional Euclidean space. Then X_0 and X_1 are linearly separable, if there exist $n+1$ real numbers $w_1, w_2, \cdots, w_n, k$ such that every point $x \in X_0$ satisfies $\sum_{i=1}^{n}(w_i \cdot x_i) > k$ and every point $x \in X_1$ satisfies $\sum_{i=0}^{n}(w_i \cdot x_i) < k$ where x_i is the i-th component of x.

2.4 From the learning algorithm to code:

You must create a training function with these characteristics using the previous data.

1. The training function guesses the outcome based on the activate function.

2. Every time the guess is wrong, the perceptron should adjust the weights.

3. After many guesses and adjustments, the weights will be correct.

Define the activation function which at the end it is a simple binary function.

```
int activation(x)
    return (x > 0) ? 1 : -1
```

Define the learning rate and a number of epochs, which is the amount of times that we are going train the perceptron and the learning rate represents how fast our perceptron is going to learn.

```
learning_rate = 0.1
num_epochs = 100
```

Train the perceptron by iterating with all the training data which in this case it is an bidimensinal array and a simple threshold or bias.

```
for (i = 0, i < num_epochs, i++)
    for (j = 0, j < inputs.length, j++)
        sum = 0.0
        for (k = 0, k < weights.length, k++)
            sum += inputs[j][k] * weights[i]
        sum += b
        out = activation(sum)
```

And at the end we only need to adjust the weights and bias from the error, so the error is so simple it is a subtraction, then use the error and the learning rate to update the weights and bias.

```
error = label[j] - output
for (k = 0, k < weights.length, k++)
    weights[k] += learning_rate * error * inputs[j][k]
b += learning_rate * error
```

2.5   The math behind the learning algorithm:

Maybe now the main question is why we adjust the weights like that.

```
weights[k] += learning_rate * error * inputs[j][k]
```

And the bias or threshold.

```
b += learning_rate * error
```

To answer that we need to formulate a math function that represents our neuron which receives a vector as input.

$$f(\vec{x}) = \begin{cases} 1 & if \vec{w} \cdot \vec{x} > 0 \\ -1 & \text{otherwise} \end{cases}$$

Where $\vec{w} = (b, w_1, \cdots, w_n)$ is the coefficients or weights, $\vec{x} = (1, x_1, \cdots, x_n)$ the input vector, so from this idea we can come up with something like this.

$$y_t = 1 \implies \vec{w} \cdot \vec{x}_t > 0$$

$$y_t = -1 \implies \vec{w} \cdot \vec{x}_t \leq 0$$

Where $(\vec{x}_t, y_t) \in \mathbb{D}$ which is the whole dataset, then from these implications we can formulate two inequalities that represent when the perceptron is wrong and when it is correct.

$$\text{Correct} \implies y_t \cdot \vec{x}_t \cdot \vec{w} > 0$$

$$\text{Wrong} \implies y_t \cdot \vec{x}_t \cdot \vec{w} \leq 0$$

At the end what we want is desired vector $\vec{\theta^*}$ which fulfills a single characteristic.

$$\exists \vec{\theta^*} \in \mathbb{R}^d \quad | \quad y_t \vec{\theta^*} \vec{x}_t > 0, \quad \forall (\vec{x}_t, y_t) \in \mathbb{D}$$

Where $\vec{\theta^*}$ is the desired vector configuration which classifies the inputs perfectly, then when the perceptron gets an error this both inequalities are true.

$$y_t \vec{w}_k \vec{x} \leq 0 \Leftrightarrow y_t \neq y'$$

Where y' is the result obtained, then in the algorithm when that happens we update the $\vec{w}$ like this.

$$\vec{w}_{k+1} = \vec{w}_k + y_t \vec{x}_t$$

At the first time it doesn't make sense why it works like that, well think in this chart.

Again the $\vec{\theta}^*$ is the desired vector configuration which for any $(\vec{x}_t, y_t) \in \mathbb{D}$ returns an $y' = y_t$ and the $\vec{w}_k$ is the current k-th iteration of the weights, obviously we don't know where $\vec{\theta}$ is, but we can approximate $\vec{w}_k$ to $\vec{\theta}^*$ by a simple additions of vectors, then We can put another vector $y_t \cdot \vec{x}_t$ to this chart.
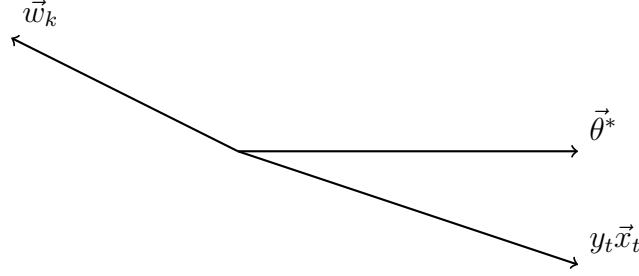

Then if we compute the dot product of these two vectors $y_t \vec{x}_t \cdot \vec{\theta}^*$ this inequalitie becomes true.
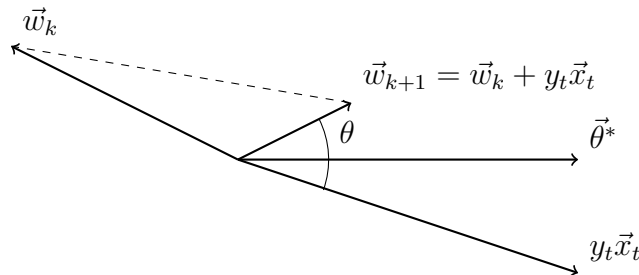
$$y_t \vec{\theta}^* \vec{x}_t > 0$$

Because again the $\vec{\theta}^*$ is the desired vector configuration, the meaning of the dot product of two vectors is how much they are pointing in the same direction, if the dot product of two vectors is positive, it means they are pointing in roughly the same direction, if it is negative, they are pointing in roughly opposite directions, and if it is zero, they are perpendicular (at a right angle) to each other, then the dot product of two Euclidean vectors $\vec{v}$ and $\vec{u}$ is defined by.

$$\vec{v} \cdot \vec{u} = ||\vec{v}|| \cdot ||\vec{u}|| \cdot \cos \theta$$

Where θ is the angle between $\vec{v}$ and $\vec{u}$, so from that idea by this chart we can perfectly infer that $y_t \vec{x}_t \cdot \vec{w}_k \leq 0$ by their angle.
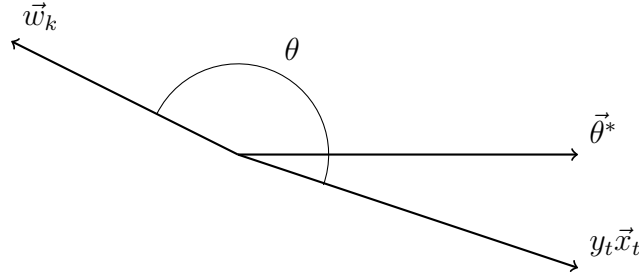
So think in the addition of vector as way to rotate a vector, look if I sum $\vec{w}_k + y_t \vec{x}_t$, and I get the next $k + 1$ vector $\vec{w}_{k+1}$.

With this new vector $\vec{w}_{k+1}$ we generate a new angle θ that makes this inequalitie $\vec{w}_{k+1} \cdot y_t \vec{x}_t \leq 0$ false which implies that the perceptron have learned a new pattern, basically we adjust the weights to new direction.

$$\neg(\vec{w}_{k+1} \cdot y_t \vec{x}_t \leq 0) \implies \vec{w}_{k+1} \cdot y_t \vec{x}_t > 0 \quad | \quad y' = y_t, \quad \vec{w}_k \to \vec{\theta^*}$$

We are arriving to a conjecture which is that it looks like we need a finite amount of k iterations to perfectly separate the points or vectors $\vec{x}_t$ in $\mathbb{D}$ making $\vec{w}_k \to \vec{\theta^*}$, creating that perfectly hyperplane that separates the points, then this is important because if that amount of iterations is infinite then the algorithm is useless, but if it is not then becomes useful, spoiler, it is finite amounts of iterations for any finite set of tuples $(\vec{x}_t, y_t) \in \mathbb{D}$, lets proof that.

2.6 Perceptron Convergence Proof:

Lets think in that variable k which represents the amount of iterations, It should be a finite natural number.

$$k \in \mathbb{N}$$

Then before to continue with the proof we need to do an assumption, we are going to scale our $\vec{\theta^*}$ in another words we are going to change it for an easy manipulation, so for this section we are going to work with a new vector.

$$\vec{\theta^*} = \alpha \cdot \vec{\theta} \quad | \quad ||\vec{\theta}|| = 1$$

So we scale the desired vector $\vec{\theta^*}$ for getting its unitary vector $\vec{\theta}$, At the end the inequality is still being true.

$$y_t \vec{\theta^*} \vec{x}_t > 0$$
$$y_t \alpha \cdot \vec{\theta} \vec{x}_t > 0$$
$$\frac{1}{\alpha} \cdot y_t \alpha \vec{\theta} \vec{x}_t > 0 \cdot \frac{1}{\alpha}$$
$$y_t \vec{\theta} \vec{x}_t > 0$$

Obviously all the computed values with this new vector $\vec{\theta}$ are going to be lesser than the comptued values with $\vec{\theta^*}$, to be exact is going to be α times greater, and actually if you think we don't care how big these outputs are, we only care that they are big enoguh to make that statement true the whole time.

$$y_t \vec{\theta^*} \vec{x}_t > y_t \vec{\theta} \vec{x}_t > 0$$

To just finished with this idea think on this statement and lets continue with the proof of the convergence.

$$\exists \vec{\theta^*} = \alpha \cdot \vec{\theta} \quad | \quad y_t \vec{\theta^*} \vec{x}_t > 0 \wedge y_t \vec{\theta} \vec{x}_t > 0, \quad \forall (\vec{x}_t, y_t) \in \mathbb{D}$$

Then we need to think in this dot product $\vec{w}_{k+1} \cdot \vec{\theta}$ because this dot product will grow but it will never decrease and why is that?, well because of the algorithm we only rotate or adjust $\vec{w}_k$ when $\vec{w}_k \cdot \vec{x}_t y_t \leq 0$, look this chart.
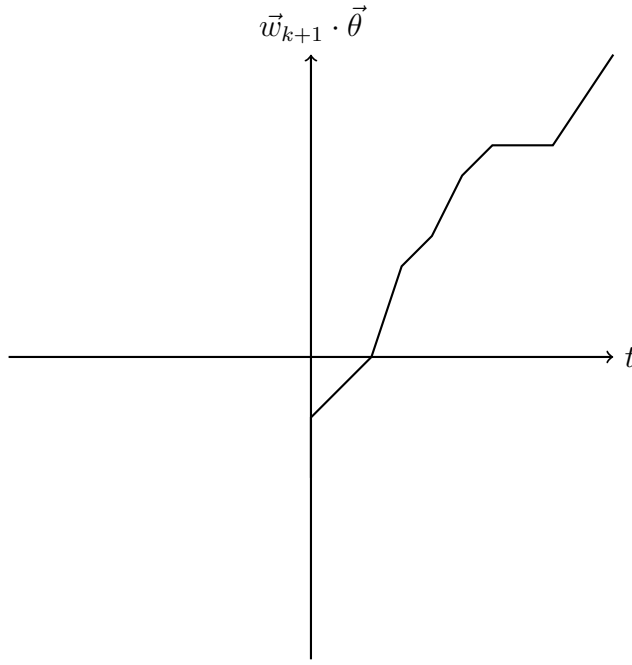
You can imagine something like this, it never decreases, so why this dot product is useful?, well because this dot product represents how the weights are being adjust or how the perceptron learns, it would start in some negative real number but it will finish in some positive value, and as I said before there is going to be a moment where $\vec{w}_{k+1} \cdot \vec{\theta}$ will never increase anymore, at the end we would find inserting how this dot product grows through the instances of $(\vec{x}_t, y_t)$.
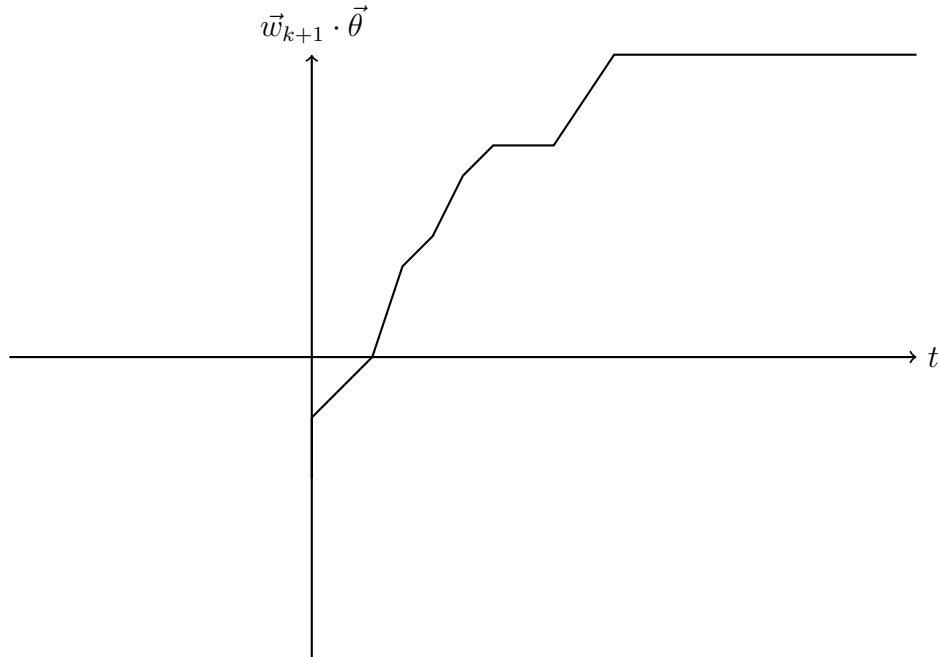
Let's imagine that we already have made a some k iteration, then we want to know what would be the minimum difference between this k iterations $\vec{w}_{k+1} \cdot \vec{\theta}$ with It's previous iteration $\vec{w}_k \cdot \vec{\theta}$, well.

$$\vec{w}_{k+1} \cdot \vec{\theta} = (\vec{w}_k + y_t \vec{x}_t)\vec{\theta} = \vec{w}_k \vec{\theta} + y_t \vec{x}_t \vec{\theta}$$

$$\vec{w}_{k+1} \cdot \vec{\theta} - \vec{w}_k \vec{\theta} = y_t \vec{x}_t \vec{\theta}$$

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So basically this is the difference $y_t\vec{x}_t\vec{\theta}$ which is $y_t\vec{x}_t\vec{\theta} > 0$ and so from this expression we can create constant gamma which represents the minimum difference of some pair of values $\vec{w}_{k+t} \cdot \vec{\theta} - \vec{w}_k \cdot \vec{\theta}$ by any k iteration.

$$\gamma = minimum(y_t\vec{x}_t\vec{\theta}) \quad | \quad \forall(\vec{x}_t, y_t) \in \mathbb{D}$$

We know that $y_t \in \{1, -1\}$, so we can forget it and use a simple absolute value.

$$\gamma = minimum(|\vec{x}_t\vec{\theta}|) \quad | \quad \forall(\vec{x}_t, y_t) \in \mathbb{D}$$

$$|\vec{w}_{k+1} \cdot \vec{\theta} - \vec{w}_k\vec{\theta}| \geq \gamma \quad | \quad \forall\vec{w}_{k+1}$$

Finally we can abstract this idea of γ to an inequalitie for all the k iterations.
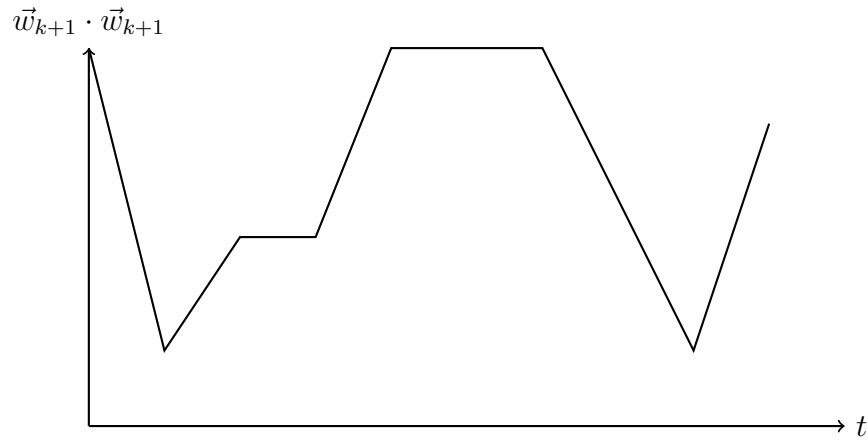
$$\gamma \cdot k \leq |\vec{w}_{k+1} \cdot \vec{\theta}| \leq ||\vec{w}_{k+1}|| \cdot ||\vec{\theta}|| = ||\vec{w}_{k+1}|| \quad | \quad \forall\vec{w}_{k+1}$$

$$\gamma \cdot k \leq ||\vec{w}_{k+1}|| \quad | \quad \forall\vec{w}_{k+1}$$

As you can see we replace $||\vec{\theta}|| = 1$, because if unitary vector and $\gamma \cdot k$ works as lower bound for any $|\vec{w}_{k+1} \cdot \vec{\theta}|$, this lead us to think if there exist a higher bound, spoiler, It exists, to find it now we need to think in another dot product which is this.

$$\vec{w}_{k+1} \cdot \vec{w}_{k+1}$$

In some way this dot product is an scalar way to represent each state of our weights, It is chaotic because this dot product would decrease and increase and something to take not It is that always this dot product will be $\vec{w}_{k+1} \cdot \vec{w}_{k+1} > 0$, this is because is the same vector, so you can think in a chart like this.

Now we are thinking what is the larger difference between any k iteration $\vec{w}_{k+1} \cdot \vec{w}_{k+1}$ with It's previous iteration $\vec{w}_k \cdot \vec{w}_k$, well.

$$\vec{w}_{k+1} \cdot \vec{w}_{k+1} = (\vec{w}_k + y_t\vec{x}_t) \cdot (\vec{w}_k + y_t\vec{x}_t) = \vec{w}_k\vec{w}_k + 2y_t\vec{w}_k\vec{x}_t + y_t^2\vec{x}_t\vec{x}_t$$

$$\vec{w}_{k+1} \cdot \vec{w}_{k+1} - \vec{w}_k \cdot \vec{w}_k = 2y_t\vec{w}_k\vec{x}_t + y_t^2\vec{x}_t\vec{x}_t$$

Then this is the difference $2y_t\vec{w}_k\vec{x}_t + y_t^2\vec{x}_t\vec{x}_t$, so we can rewrite this expression like this.

$$2y_t\vec{w}_k\vec{x}_t + \vec{x}_t\vec{x}_t$$

Because $y_t \in \{1, -1\}$ so $y_t^2 = 1$ for all $(\vec{x}_t, y_t) \in \mathbb{D}$, so what is going to be the maximum difference between any $\vec{w}_{k+1} \cdot \vec{w}_{k+1}$, so thinking in this expression $2y_t\vec{w}_k\vec{x}_t + \vec{x}_t\vec{x}_t$ to be specific thinking on the first quotient, this quotient always is going to be $2y_t\vec{w}_k\vec{x}_t \le 0$, so we eliminate this expression and create an inequalitie, because we want the largest posible value.

$$\vec{w}_{k+1} \cdot \vec{w}_{k+1} - \vec{w}_k \cdot \vec{w}_k \le \vec{x}_t\vec{x}_t$$

Then we can create our constant R^2 which represents the largest difference of some pair of values $\vec{w}_{k+t} \cdot \vec{w}_{k+1} - \vec{w}_k \cdot \vec{w}_k$ by any k iteration and why R^2 ?, well.

$$R = maximum(||\vec{x}_t||) \quad | \quad \forall(\vec{x}_t, y_t) \in \mathbb{D}$$

$$R = ||\vec{x}_t|| = \sqrt{\sum_{i=0}^{n} x_i^2}$$

$$R^2 = ||\vec{x}_t||^2 = \sqrt{\sum_{i=0}^{n} x_i^2}^2 = \sum_{i=0}^{n} x_i^2 = \vec{x}_t \cdot \vec{x}_t$$

Finally, we chose this constant R^2, and we don't choose $maximum(|\vec{x}_t\vec{\theta}|)$, because we are searching to a higher boundary, and as $||\vec{\theta}|| \le ||\vec{x}_t||$ always is going to be lesser to R^2, so finally we can mix everything together, so we can assume this.

$$\gamma^2 k^2 \le ||\vec{w}_{k+1}||^2 = \vec{w}_{k+1} \cdot \vec{w}_{k+1}$$

So we can take that and create our finall inequalitie, which at the end is this.

$$\gamma^2 k^2 \le \vec{w}_{k+1} \cdot \vec{w}_{k+1} \le k \cdot R^2 \quad | \quad \forall\vec{w}_{k+1}$$

Here we are boundary the amount of change, if you want to see this like that, of the weights of the perceptron for all the instances of $\vec{w}_{k+1}$, from this two bounds we can get a finite number of steps or times that the perceptron needs for update its weights to converge, just like this.

$$\gamma^2 k^2 \le k \cdot R^2$$

$$\frac{1}{\gamma^2 \cdot k} \cdot \gamma^2 k^2 \le k \cdot R^2 \cdot \frac{1}{\gamma^2 \cdot k}$$

$$k \le \frac{R^2}{\gamma^2}$$

That's it, k is a real number which means that takes at least $\frac{R^2}{\gamma^2}$ updates or steps to converge.

3 Example:

This is an example of how we can use the perceptron as a binary classifier, what I'm going to do is to create a program which identifies when is a circle and when is a rectangle.

3.1 Perceptron module in fortran:

Firstly I made a simple perceptron module in fortran which contains the weights, a bias and with a simple step activation function.

```fortran
#include <assertf.h>

module mod_perceptron
    use iso_fortran_env, only : real32, int32

    use assertf
    implicit none

    private
    public p_init, p_free, p_train, p_test, perceptron

    ! The type where we are going to contain the weights and bieas
    type perceptron
        real(real32) ::  b
        integer(int32) :: n      ! The amount of weights
        real(real32), pointer :: w(:) ! The weights
    end type perceptron
contains

    ! step: A simple activation function
    pure function step(output)
        real(real32), intent(in) :: output
        real(real32) :: step
        if (output > 0.0) then
            step = 1.0
        else
            step = 0.0
        end if
    end function step


    ! p_init: Receives a perceptron type and adds to it the weights
    ! and bias
    subroutine p_init(per, w, b)
        real(real32), intent(in) :: w(:), b
        type(perceptron), intent(out) :: per

        per%n = size(w)            ! Fetch the size of the array of weights

        ! Allocate the array of weights
        allocate(per%w(per%n))
        per%w(:) = w(:)
        per%b = b
```

```fortran
    end subroutine p_init

    ! p_free: To free the perceptrons weights and set the size and bias
    ! to zero
    subroutine p_free(per)
        type(perceptron), intent(in out) :: per

        deallocate(per%w)
        per%n = 0
        per%b = 0
    end subroutine p_free
end module mod_perceptron
```

3.2   Test and trainning function:

Then the **test** and **training** functions.

```fortran
    ! p_train: To train the perceptron with some data
    subroutine p_train(per, inputs, outputs, lrate, nepochs)
        ! inout Because we are going to modify
        type(perceptron), intent(in out) :: per
        ! The inputs and outputs to be trained
        real(real32), intent(in) :: inputs(:,:) ! Bindimensional array
        real(real32), intent(in) :: outputs(:), lrate ! Learning rate
        integer(int32), intent(in) :: nepochs        ! Num of epochs

        ! The size and the iterators
        integer(int32) :: n, i, j, k
        real(real32) :: output, error, delta
        ! Catch the amount of inputs, sizeof(inputs[1])
        n = size(inputs, 1)

        do i = 1, nepochs              ! Start iterating
            do j = 1, n ! Concurrent iteration
                ! Computing the perceptron calculation
                output = per%b
                do concurrent(k = 1 : per%n)
                    output = output + per%w(k) * inputs(j, k)
                end do

                ! The step function
                output = step(output)

                ! Compute the error
                error = outputs(j) - output
                delta = lrate * error
```

```fortran
                ! Update the perceptron
                per%b = per%b + delta
                do concurrent(k = 1 : per%n)
                    per%w(k) = per%w(k) + delta * inputs(j, k)
                end do
            end do
        end do
    end subroutine p_train

    ! p_test: To test the perceptron with some inputs
    subroutine p_test(per, inputs, results)
        type(perceptron), intent(in) :: per
        real(real32), intent(in) :: inputs(:, :)
        real(real32), intent(out) :: results(:)

        ! The size and outputs
        integer(int32) :: n, i, j

        ! Test the perceptron
        n = size(inputs, 1)
        do concurrent(i = 1 : n)
            results(i) = per%b
            do concurrent(j = 1 : per%n)
                results(i) = results(i) + per%w(j) * inputs(i, j)
            end do
            results(i) = step(results(i))
        end do
    end subroutine p_test
```

3.3   Main program:

4   Conclusion:

Perceptrons are limited to linearly separable problems, which means that they can only learn to classify data that can be separated by a hyperplane. However, they have been used as building blocks for more complex neural network architectures that can handle more complex problems. Overall, perceptrons are a simple and powerful tool for building basic binary classifiers, and they have contributed significantly to the development of artificial intelligence and machine learning.

5   References:

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