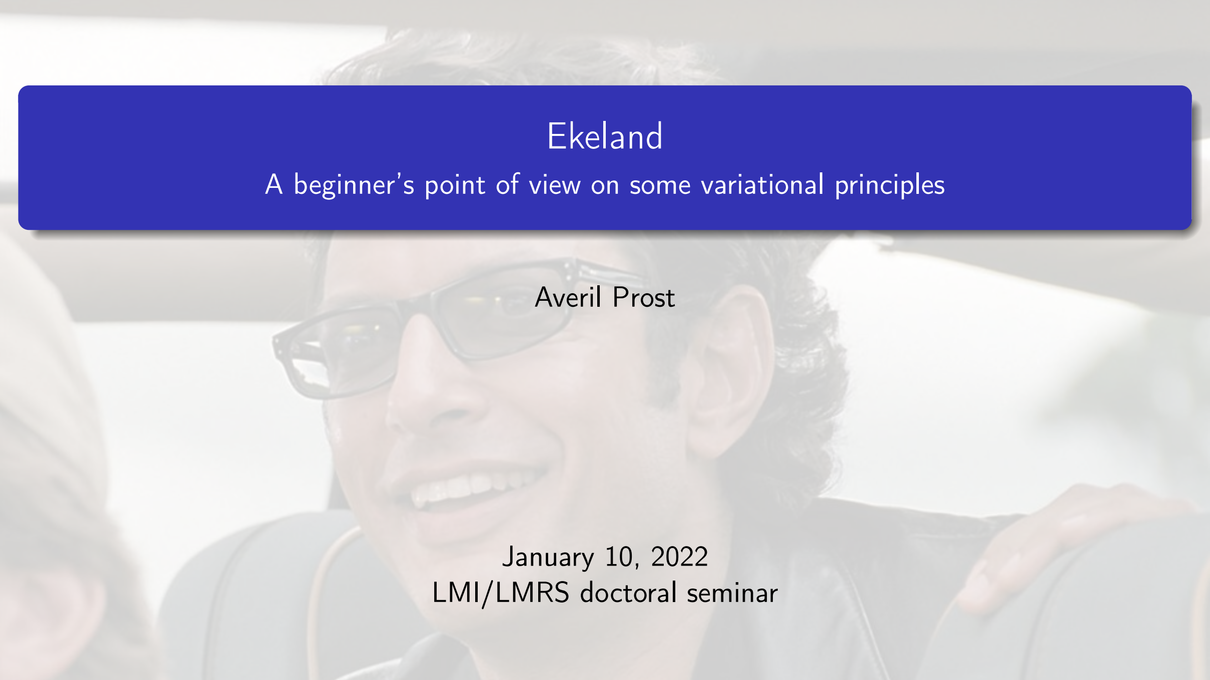

Ekeland

A beginner's point of view on some variational principles

Averil Prost

January 10, 2022
LMI/LMRS doctoral seminar

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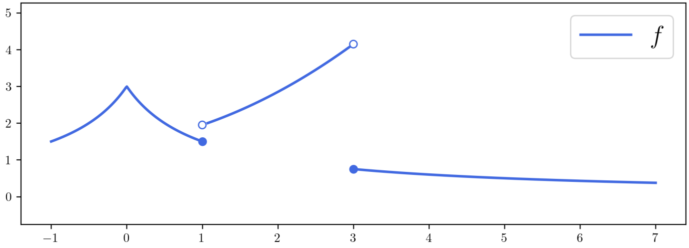
History (1/2)

$\boxed{1974}$ Ivar Ekeland's *On the variational principle* [Eke74], metric, distances.

Theorem – Ekeland [Eke74] Let (X, d) be a complete metric space. Let $f : X \mapsto \mathbb{R} \cup \{\infty\}$ be proper, lsc and lower bounded.



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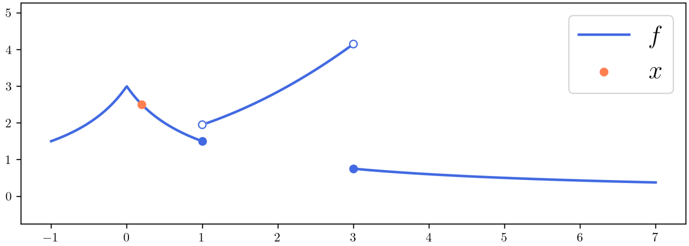

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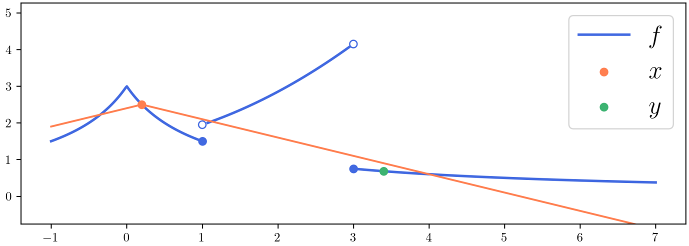
$$\begin{cases} f(y) \leqslant f(x) - \delta d(x, y), & \text{(1a)} \\ & \text{(1b)} \end{cases}$$



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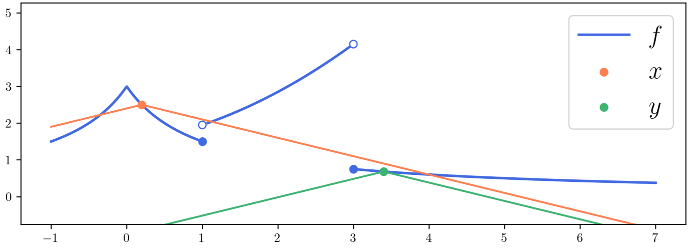

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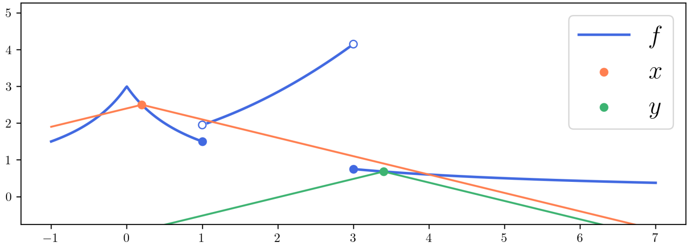

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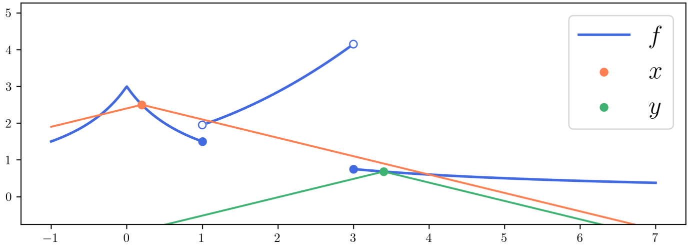

- No min when it exists
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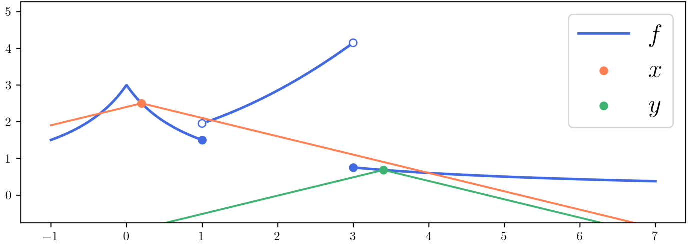

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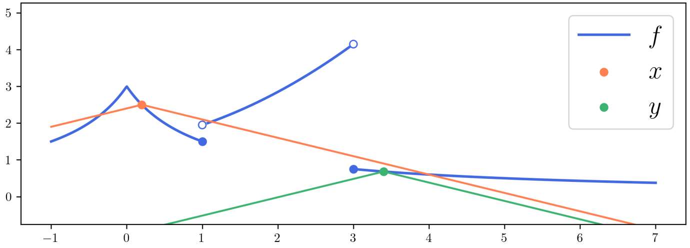

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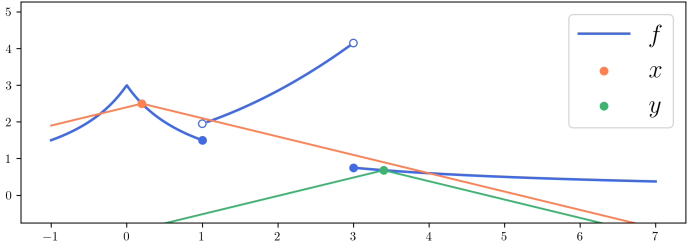
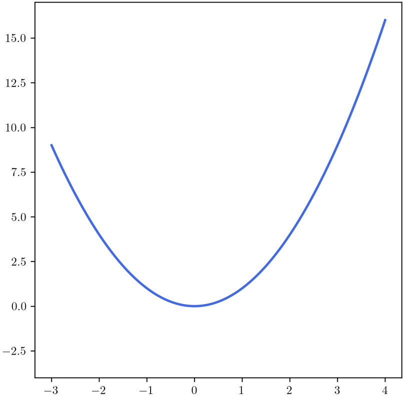


- No min when it exists
- δ is arbitrary
- y stays in X
- $\sim$ no $+\infty$ behavior
- No (local) compactness

The proof (1/3)

Let $S_0 :=$ dom f, and $\varepsilon > 0$.



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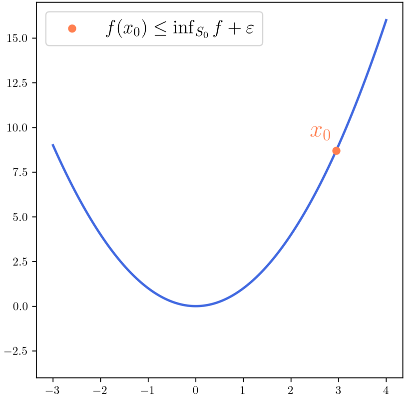
Let $S_0 := \operatorname{dom} f$, and $\varepsilon > 0$. Pick $x_0 \in S_0$ such that $f(x_0) \leqslant \inf_X f + \varepsilon$.



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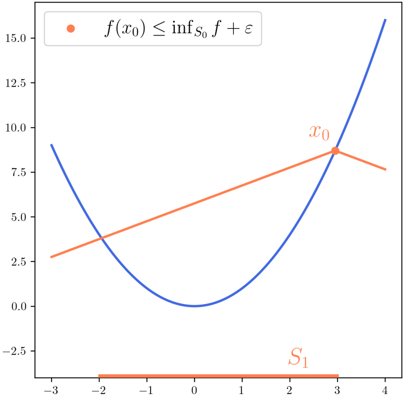

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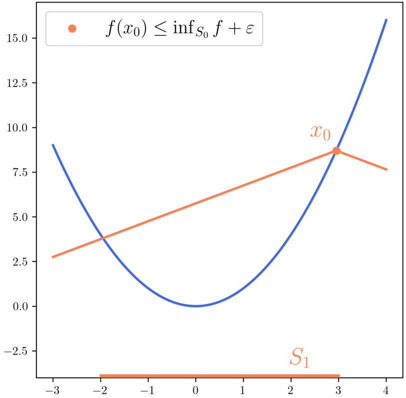

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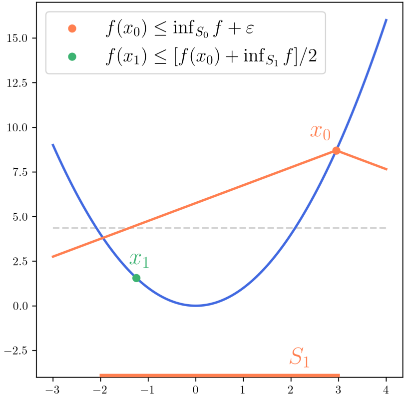

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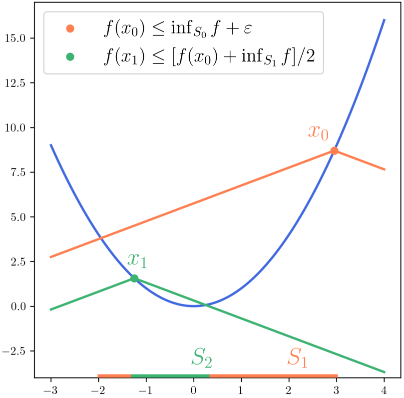

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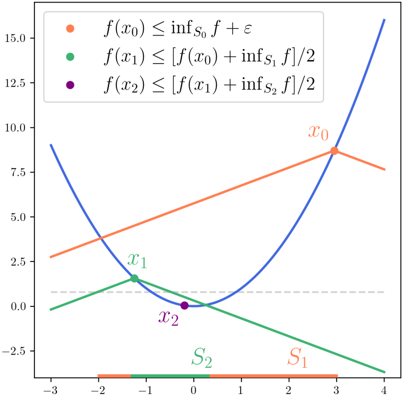

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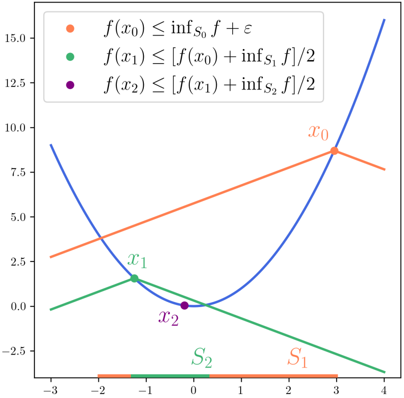

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and pick $x_i \in S_i$ such that

$$f(x_i) \leqslant \frac{f(x_{i-1}) + \inf_{y \in S_i} f(y)}{2}.$$

S_i nonempty and closed.
Let us show that $S_{i+1} \subset S_i$, and diam $S_i \xrightarrow[i \to \infty]{} 0$.

The proof (2/3)

Let $x \in S_{i+1}$:

$$f(x) \underset{x \in S_{i+1}}{\leqslant} f(x_i) - \delta d(x, x_i)$$

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\begin{aligned}
f(x) &\underset{x \in S_{i+1}}{\leqslant} f(x_i) - \delta d(x, x_i) \\
&\underset{x_i \in S_i}{\leqslant} f(x_{i-1}) - \delta \left[d(x_i, x_{i-1}) + d(x, x_i) \right]
\end{aligned}
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and $x \in S_i$, so that $S_{i+1} \subset S_i$.

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Let $x \in S_{i+1}$:

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$$\underset{\triangle\text{ ineq}}{\leqslant} f(x_{i-1}) - \delta d(x, x_{i-1}),$$

and $x \in S_i$, so that $S_{i+1} \subset S_i$.

On the other hand, since $\inf_{S_{i+1}} f \geqslant \inf_{S_i} f$,

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$$\delta d(x, x_i) \leqslant f(x_i) - f(x) \leqslant f(x_i) - \inf_{S_{i+1}} f \leqslant \frac{\varepsilon}{2^i}.$$

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Let $x \in S_{i+1}$:

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so that $d(x, x_{i-1}) \leqslant \frac{\varepsilon}{\delta 2^i}$, and $\operatorname{diam} S_i \underset{i \to \infty}{\longrightarrow} 0$.

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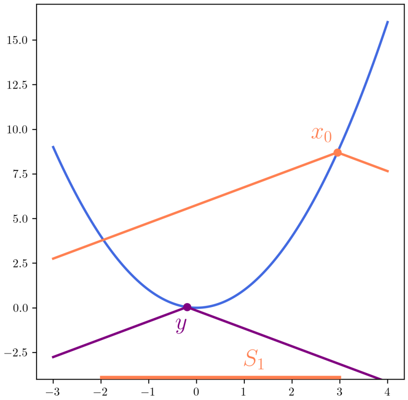
so that $d(x, x_{i-1}) \leqslant \frac{\varepsilon}{\delta 2^i}$, and $\operatorname{diam} S_i \xrightarrow[i\to\infty]{} 0$.

Since X is closed, by Cantor's intersection theorem, there exists an unique $y \in \bigcap_{i=0}^{\infty} S_i$.

The proof (3/3)



- Since $y \in S_1$, $f(y) \leqslant f(x_0) - \delta d(y, x_0)$, hence (1a).

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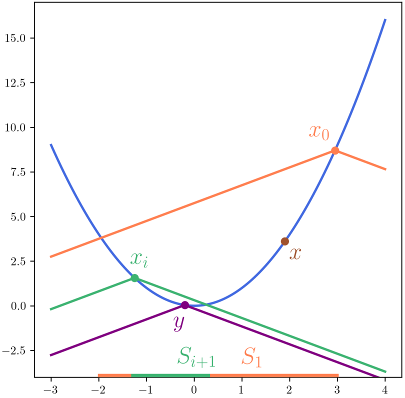

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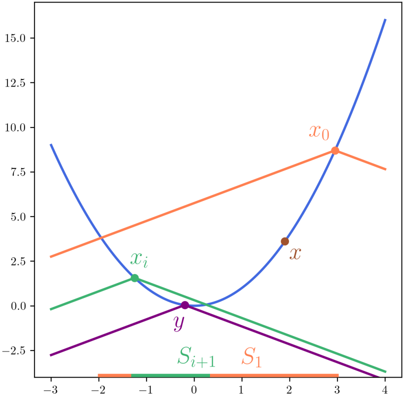

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$$f(x) \underset{x \notin S_{i+1}}{>} f(x_i) - \delta d(x, x_i)$$

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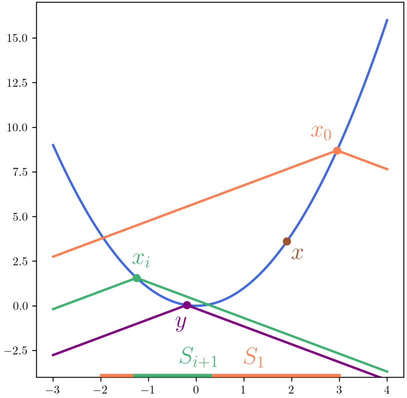

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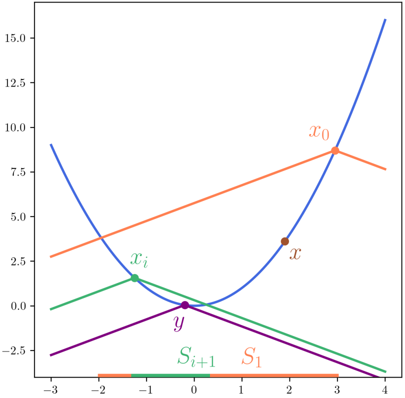

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$$\begin{aligned} f(x) &\underset{x \notin S_{i+1}}{>} f(x_i) - \delta d(x, x_i) \\ &\underset{y \in S_{i+1}}{\geqslant} f(y) + \delta d(y, x_i) - \delta d(x, x_i), \\ &\underset{\triangle \text{ ineq.}}{\geqslant} f(y) - \delta d(y, x), \end{aligned}$$

hence (1b). □

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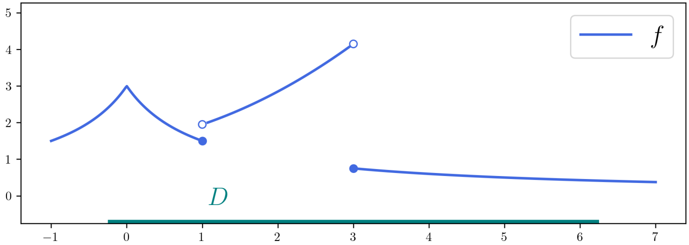

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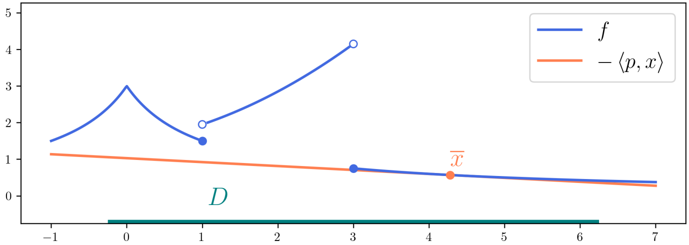

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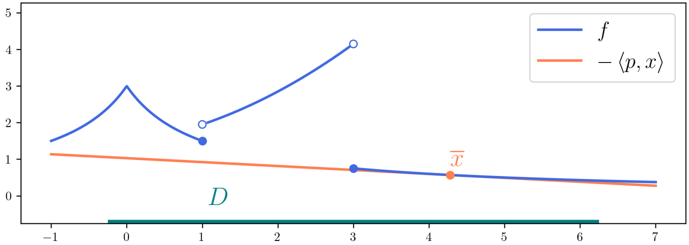

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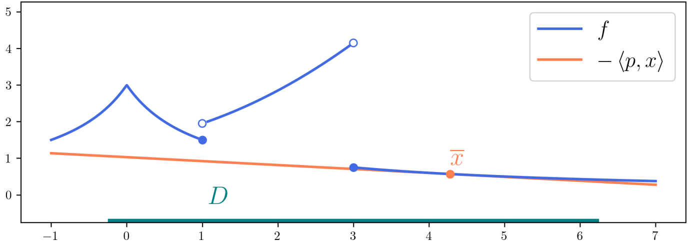

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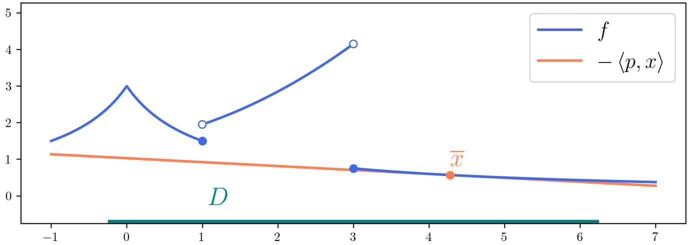

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The proof is quite different.

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Let (X, d) be a complete metric space.

gauge-type functions Any lower semicontinuous $\rho : X \times X \mapsto [0, \infty]$ satisfying $\rho(x, x) = 0$ for all $x \in X$, and $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\rho(x, y) \leqslant \eta$ implies $d(x, y) \leqslant \varepsilon$.

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Illustration of Borwein-Preiss

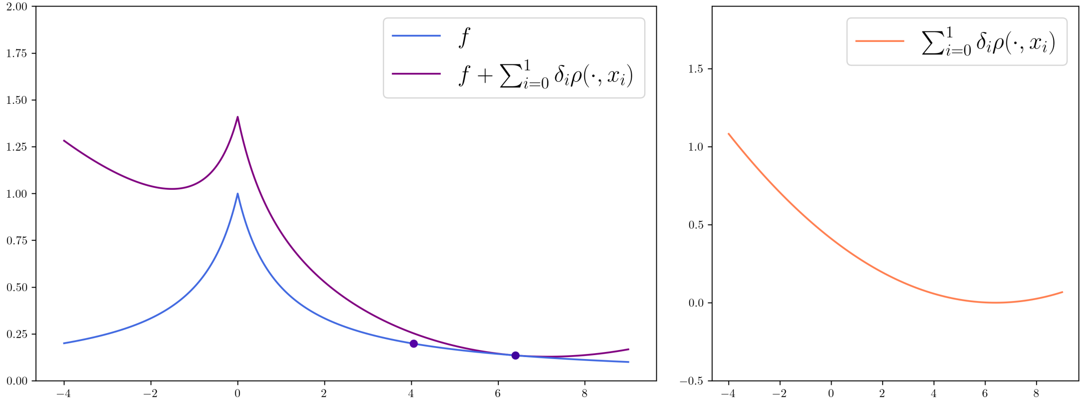

Figure: Iterative construction with $f(x) = (1 + |x|)^{-1}$, $\delta_i = 0.01/(1+i)^2$, $\rho(x, y) = |x - y|^2$.

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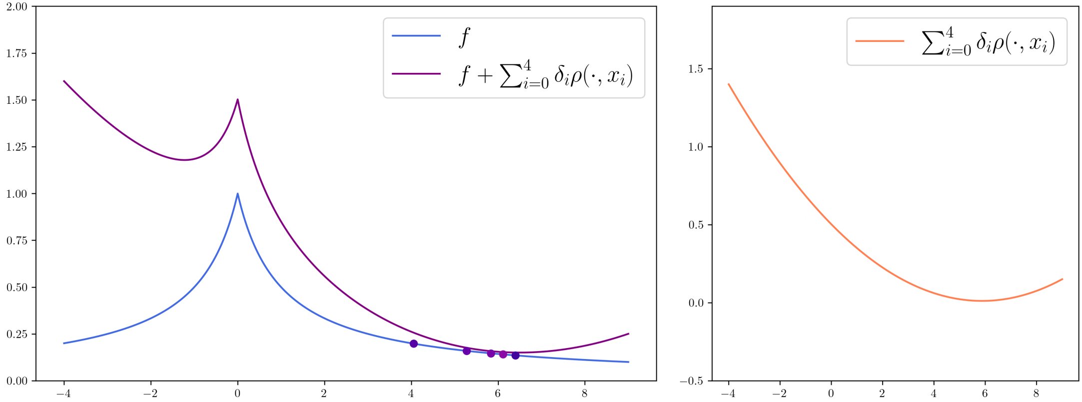

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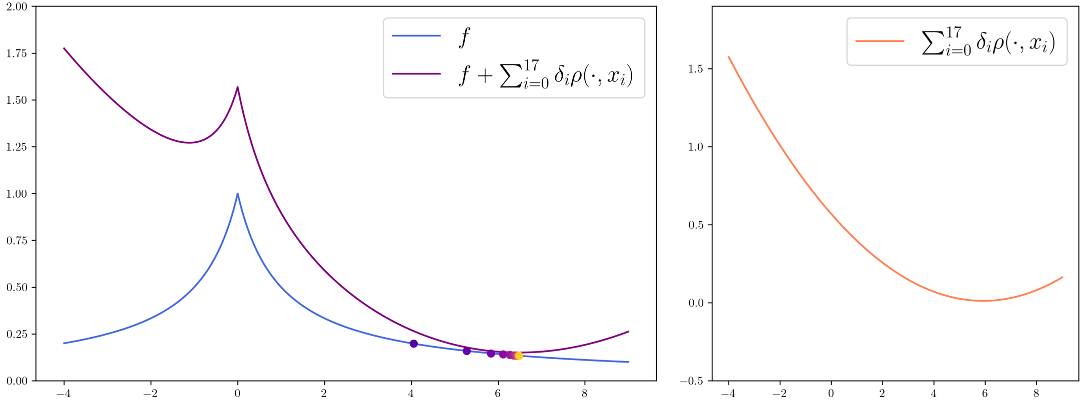

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Thank you!

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