Tropical heat

The eikonal equation as a (max,+) version of the Poisson equation

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The (max,+) idempotent calculus

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Then $(\Omega, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is a semiring (ring without additive inverse).

Integrals

Endow Ω with the application $\mathbb{d}(a,b) := \left|e^a - e^b\right|$, and let $f : \Omega \to \Omega$ be continuous. Then

$$\sum_{i \in \mathbb{Z} \cup \{-\infty\}}^{\oplus} f(hi) = \max_{i \in \mathbb{Z} \cup \{-\infty\}} f(hi) \xrightarrow[h \searrow 0]{} \sup_{x \in \Omega = \mathbb{R} \cup \{-\infty\}} f(x).$$

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In particular, the scalar product becomes

$$\langle f, g\rangle_{\oplus} = \int_{x\in\Omega}^{\oplus} f(x)\otimes g(x) = \sup_{x\in\Omega} f(x) + g(x).$$

People and motivation

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Motivation Ease the study of **optimization problems** by directly working with the "natural" operations of (here) **maximization and sum** of gains.

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An elementary result

Proposition – Let $f > g$. Then

$$\lim_{h \searrow 0} h \log\left(e^{f/h} + e^{g/h}\right) = \lim_{h \searrow 0} h \log\left(e^{f/h} - e^{g/h}\right) = f.$$

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Going further, for any upper bounded and continuous f, there holds

$$\lim_{h \searrow 0} h \log\left(\int_{x \in \mathbb{R}^d} \exp\left(\frac{f(x)}{h}\right) dx\right) = \sup_{x \in \mathbb{R}^d} f(x).$$

Logarithm transform

Def 3 – Logarithm trick For $h > 0$, consider the operations

$$a \otimes_h b := h \log\left(e^{a/h} \cdot e^{b/h}\right), \qquad a \oplus_h b := h \log(e^{a/h} + e^{b/h}).$$

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Then

$$a \otimes_h b = a + b, \qquad a \oplus_h b \xrightarrow[h \searrow 0]{} \max(a, b).$$

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We could go further with this game, for instance with

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$$a \otimes_h b = a + b, \qquad a \oplus_h b \underset{h \searrow 0}{\longrightarrow} \max(a, b).$$

We could go further with this game, for instance with

$$a \hat{}^{\oplus} b := \lim_{h \searrow 0} h \log\left(\exp(a/h) \hat{}\ \exp(b/h)\right) = \begin{cases} a & b = 0 \\ \mathbb{1} = 0 & b < 0 \\ +\infty & b > 0. \end{cases}$$

Dialog between $(+,\cdot)$ and $(\max,+)$

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Linearity

Let $\mathrm{I} : \mathbb{R}^d \to \mathbb{R}^d$ be linear, and denote

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The limit operator $\Phi := \lim_{h \searrow 0} \Phi_h$ is $(\max, +)-$linear.

Example

Let $d = 2$ and $I(x) = Ax$, where $A := \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. Then

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Setting

Consider the $(\max, +)$ system

$$\Pi \otimes \xi = \beta, \tag{1}$$

where $\Pi \in \mathbb{M}_{2,2}$ is a matrix, $\xi, \beta \in \mathbb{R}^2$ are vectors with β given, and for each $i \in [\![1, 2]\!]$,

$$(\Pi \otimes \xi)_i = \sum_{j \in [\![1,2]\!]}^{\oplus} \Pi_{ij} \otimes \xi_j = \max_{j \in [\![1,2]\!]} \Pi_{ij} + \xi_j.$$

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In which cases can we get a solution to (1) by using the link with classical algebra?

Transformation into linear variables

For each $h > 0$, define $P \in \mathbb{M}_{2,2}$ and $x, b \in \mathbb{R}^2$ by

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The linear system $P^h x^h = b^h$ has a solution if $\det(P^h) \neq 0$, i.e.

$$P_{11}^h P_{22}^h - P_{12}^h P_{21}^h \neq 0 \qquad \Longleftrightarrow \qquad \exp\left(\frac{\Pi_{11} + \Pi_{22}}{h}\right) \neq \exp\left(\frac{\Pi_{12} + \Pi_{21}}{h}\right).$$

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Assume that $\Pi_{11}+\Pi_{22} \neq \Pi_{12}+\Pi_{21}$, and consider $x^h = (P^h)^{-1}b^h$ solving $P^h x^h = b^h$.

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$$\Pi = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}$$

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Then $P^{-1} = \mathbb{I}_d$. As (2) is satisfied for all $b = \exp(\beta/h)$, we obtain that $\xi = \beta$ solves (1).

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Since $\det P^h = e^{4/h} - e^{6/h} < 0$, the condition (2) over β becomes

$$e^{\frac{3+\beta_1}{h}} - e^{\frac{4+\beta_2}{h}} \leqslant 0, \quad \text{and} \quad - e^{\frac{2+\beta_1}{h}} + e^{\frac{1+\beta_2}{h}} \leqslant 0.$$

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so that after dividing by $\frac{\exp(v/h)}{h} > 0$, we get again $\partial_t v(t,x) - \langle \nabla v(t,x), b(t,x) \rangle = 0$.

More transformations

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or after simplification,

$$\boxed{\partial_t v(t,x) - h\Delta v(t,x) - \left|\nabla v(t,x)\right|^2 = 0.}$$

A link with viscosity solutions

Def 4 – (Historical intuition, see [CL83]) The *vanishing viscosity* solution of

$$\partial_t v - |\nabla v|^2 = 0$$

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It may be characterized by two sign inequalities that maintain the validity of the comparison principle coming from the elliptic perturbation.

The heat kernel

Def 5 – Heat kernel Let C_d be a normalizing constant, and for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, define

$$\mu^h_{t,x} := \frac{1}{\sqrt{S_d t h}} \exp\left(-\frac{|\cdot - x|^2}{2th}\right) \mathcal{L}_{\mathbb{R}^d} \in \mathscr{P}_2(\mathbb{R}^d).$$

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In particular, $\mu^h_{t,x} \xrightarrow[t \searrow 0]{} \delta_x$ narrowly and in the Wasserstein topology.

Proposition – Kernel representation The (weak) solution of the heat equation $\partial_t u(t,x) - h\Delta u(t,x) = 0$ with initial value u_0 is given by

$$u(t,x) = \int_{y \in \mathbb{R}^d} u_0(y) d\mu^h_{t,x}(y).$$

Going through the transformation

Let $u_0 > 0$ upper bounded and continuous.

Going through the transformation

Let $u_0 > 0$ upper bounded and continuous. Denoting again $v = h \log(u)$, and $v_0 = h \log(u_0)$,

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Control interpretation

Let

$$\hat{m}_{t,x} := -\frac{|\cdot - x|^2}{2t}, \qquad \hat{\mu}_{t,x}(B) := \sup_{y \in B} \hat{m}_{t,x}(y) \quad \forall B \subset \mathbb{R}^d.$$

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Proposition – Value function [Lio82] The function

$$V(t,x) := \int_{y \in \mathbb{R}^d}^{\oplus} v_0 \otimes \hat{\mu}_{t,x} = \sup_{y \in \mathbb{R}^d} \left[v_0(y) - \frac{|y-x|^2}{2t} \right] \tag{4}$$

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is the unique viscosity solution of the Hamilton-Jacobi equation $\partial_t V - |\nabla V|^2 = 0$ such that $V(0,\cdot) = v_0$. The formula (4) is known as the Hopf-Lax formula.

Going further

- In the case of the heat equation, **Hopf-Cole** transform. But Lax-Hopf formula valid for a larger class of HJ equations of the type

$$\partial_t V(t,x) + H\left(x, \nabla V(t,x)\right) = 0,$$

provided the Hamiltonian H is *concave* in its second variable (+ regularity conditions).

Going further

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- Using the Hopf-Lax semigroup, Maslov defined weak solution by "duality", in the spirit of

$$\langle u, \varphi \rangle_{\oplus} = \langle u_0, S_t^* \varphi \rangle_{\oplus},$$

where S_t^* is a "dual" semigroup acting on test functions φ [KM97, Definition 3.1].

Conclusion

$(+,\cdot)$		$(\oplus_h, \otimes_h)$		$(\max,+)$
Classical operations	$\underset{\exp(\cdot/h)}{\overset{h\log(\cdot)}{\rightleftharpoons}}$	Intermediate operations	$\xrightarrow{h \to 0}$	Idempotent operations
Linear systems		Intermediate values		$(\max,+)$ systems
Heat equation	$\xrightarrow{\text{Hopf-Cole}}$	Viscous Eikonal equation		Eikonal equation

Thank you!

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