SGN – Assignment #1

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1 Periodic orbit

Exercise 1

Consider the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$.

1) Find the x-coordinate of the Lagrange point L_1 in the rotating, adimensional reference frame with at least 10-digit accuracy.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.08892819445324 \\ y_0 &= 0 \\ z_0 &= 0.0591799623455459 \\ v_{x0} &= 0 \\ v_{y0} &= 0.257888699435051 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit that passes through z_0; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM either approximated through finite differences or achieved by integrating the variational equation.

The periodic orbits in the CRTBP exist in families. These can be computed by continuing the orbits along one coordinate, e.g., z_0. This is an iterative process in which one component of the state is varied, while the other components are taken from the solution of the previous iteration.

3) By gradually decreasing z_0 and using numerical continuation, compute the families of halo orbits until $z_0 = 0.034$.

(8 points)

The collinear libration point L_1 is one of the Lagrange points lying on the x-axis of the rotating, adimensional reference frame of the system we are considering. Its coordinate can be found by analyzing the x partial derivative of the scalar potential function U and equating it to zero:

$$\frac{\partial U}{\partial x} = x - \frac{1-\mu}{r_1^3}(\mu + x) + \frac{\mu}{r_2^3}(1 - \mu - x) = 0 \tag{1}$$

The x coordinate is $x = 0.83691800731693$.

To obtain a periodic halo orbit, a set of initial conditions that once propagated will return to the same state are needed. For this initial condition, the orbit will be periodic if, after some integration time, another perpendicular x-z crossing is found. At that point the state will be:

$$\mathbf{x}(t) = [x, 0, z, 0, \dot{y}, 0]^T \tag{2}$$

However, unless the initial conditions were selected perfectly, when the trajectory is integrated until the point $y = 0$, there will be some velocity components in the x and z directions.

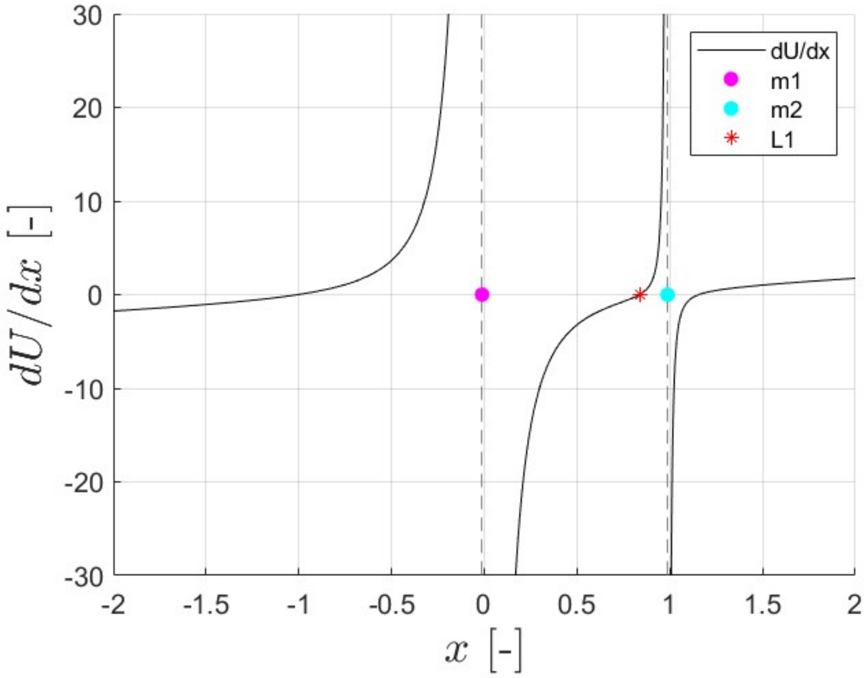
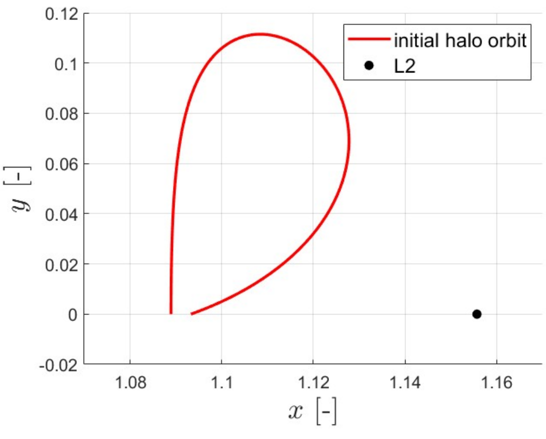


Figure 1: Position of the L1 point in the Earth-Moon system

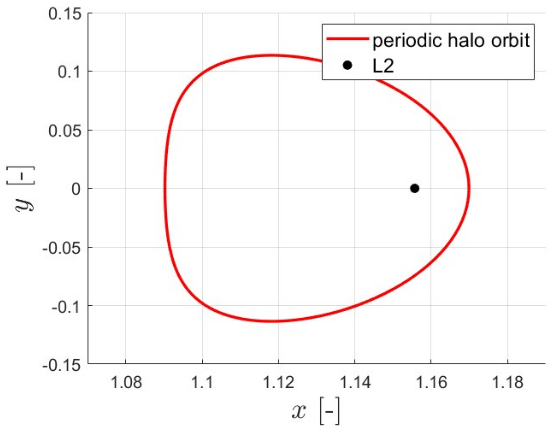
These values can be decreased by updating the magnitude of two of the three initial conditions coordinates and integrating again. The corrections can be achieved using the State Transition Matrix $\mathbf{\Phi}$, which is used to relate the state at time t (y crossing) to the state at time t_0. Propagation of the STM is performed using an analytical approach. To change x_0 and $\dot{y_0}$ and keep z_0 fixed, the following correction on the initial conditions is implemented:

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{z} \end{bmatrix} = \begin{pmatrix} \Phi_{43}\Phi_{45} \\ \Phi_{61}\Phi_{65} \end{pmatrix} \begin{bmatrix} \delta x_0 \\ \delta\dot{y_0} \end{bmatrix} \tag{3}$$

The process needs to be iterated until it converges on a periodic orbit.



(a) Initial halo orbit



(b) periodic halo orbit

Figure 2: Correction of the initial conditions to obtain a periodic orbit

The new initial conditions found are here presented:

$$
\begin{aligned}
x_0 &= 1.09027817963494 \\
y_0 &= 0 \\
z_0 &= 0.0591799623455459 \\
v_{x0} &= 0 \\
v_{y0} &= 0.260348873992119 \\
v_{z0} &= 0
\end{aligned}
$$

Starting from the initial conditions found previously and exploiting numerical continuation in the same correction scheme, a family of periodic halo orbits can be constructed by gradually decreasing z_0:

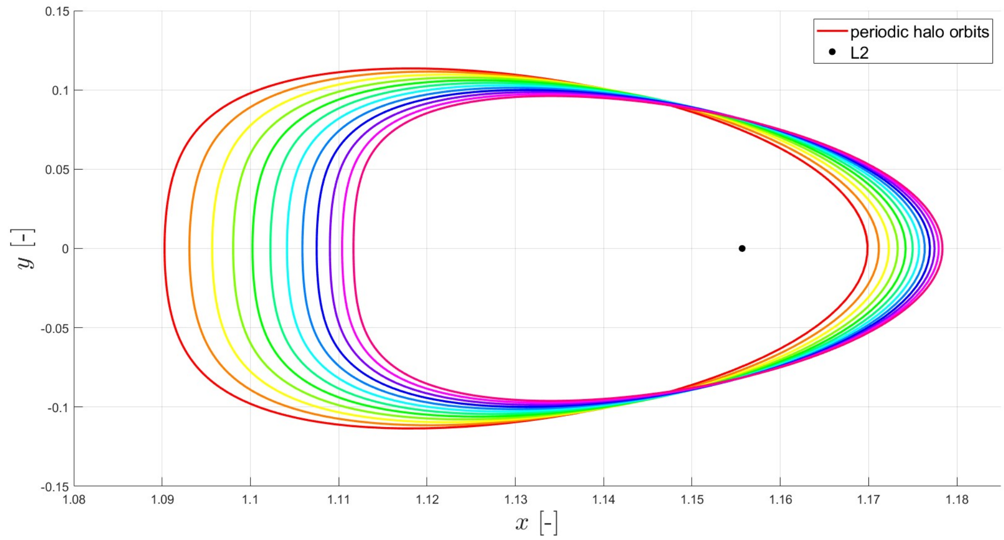


Figure 3: Family of L2 periodic halo orbits in the Earth-Moon system (x-y view)

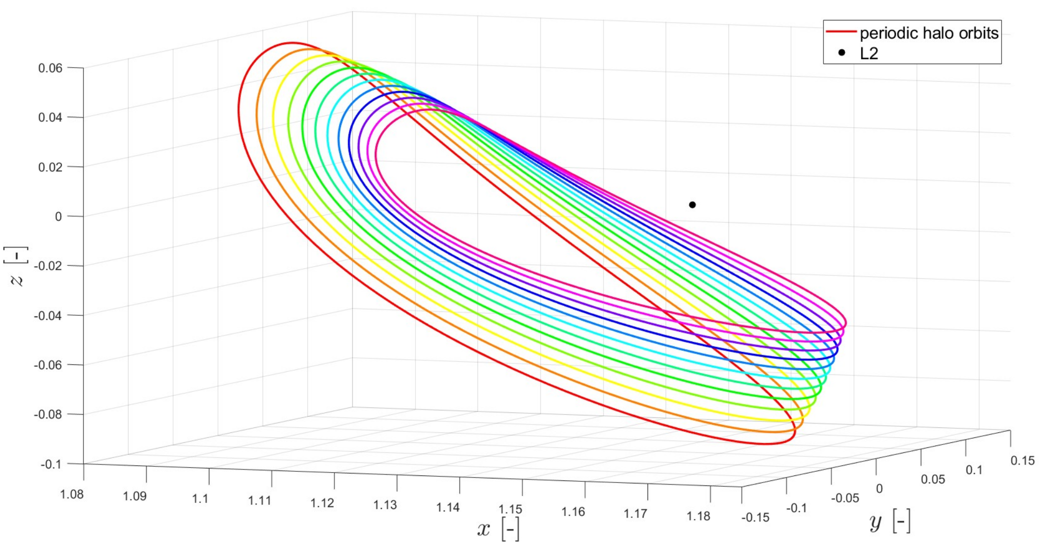


Figure 4: Family of L2 periodic halo orbits in the Earth-Moon system (x-y-z view)

2 Impulsive guidance

Exercise 2

The Aphophis close encounter with Earth will occur on April 2029. You shall design a planetary protection guidance solution aimed at reducing the risk of impact with the Earth. The mission shall be performed with an impactor spacecraft, capable of imparting a $\Delta\mathbf{v} = 0.00005\,\mathbf{v}(t_{\text{imp}})$, where $\mathbf{v}$ is the spacecraft velocity and t_{imp} is the impact time. The spacecraft is equipped with a chemical propulsion system that can perform impulsive manoeuvres up to a total Δv of 5 km/s.
The objective of the mission is to maximize the distance from the Earth at the time of the closest approach. The launch shall be performed between 2024-10-01 (LWO, Launch Window Open) and 2025-02-01 (LWC, Launch Window Close), while the impact with Apophis shall occur between 2028-08-01 and 2029-02-28. An additional Deep-Space Maneuver (DSM) can be performed between LWO+6 and LWC+18 months.

1) Analyse the close encounter conditions reading the SPK kernel and plotting in the time window [2029-01-01; 2029-07-31] the following quantities:
   a) The distance between Apophis and the Sun, the Moon and the Earth respectively.
   b) The evolution of the angle Earth-Apophis-Sun
   c) The ground-track of Apophis for a time-window of 12 hours centered around the time of closest approach (TCA).
2) Formalize an unambiguous statement of the problem specifying the considered optimization variables, objective function, the linear and non-linear equality and inequality constraints, starting from the description provided above. Consider a multiple-shooting problem with $N = 3$ points (or equivalently 2 segments) from t_0 to t_{imp}.
3) Solve the problem with multiple shooting. Propagate the dynamics of the spacecraft considering only the gravitational attraction of the Sun; propagate the post-impact orbit of Apophis using a full n-body integrator. Use an event function to stop the integration at TCA to compute the objective function; read the position of the Earth at t_0 and that of Apophis at t_{imp} from the SPK kernels. Provide the optimization solution, that is, the optimized departure date, DSM execution epoch and the corresponding $\Delta\mathbf{v}$'s, the spacecraft impact epoch, and time and Distance of Closest Approach (DCA) in Earth radii. Suggestion: try different initial conditions.

(11 points)

The closest encounter between Apophis and the Earth will take place on 2029-04-13. Evidence of this event can be grasped considering a time window of 8 months around this date and observing the relative positions between Apophis and different celestial bodies such as Earth, Moon and Sun: as expected, for Earth and Moon, the relative distance decreases monotonically until the closest approach date, after which it starts increasing.
The evolution of the Earth-Apophis-Sun angle is also shown: as anticipated, at the time of the closest approach, since Apophis and the Earth are almost overlapping, the angle suddenly spikes towards 180°, before decreasing again.

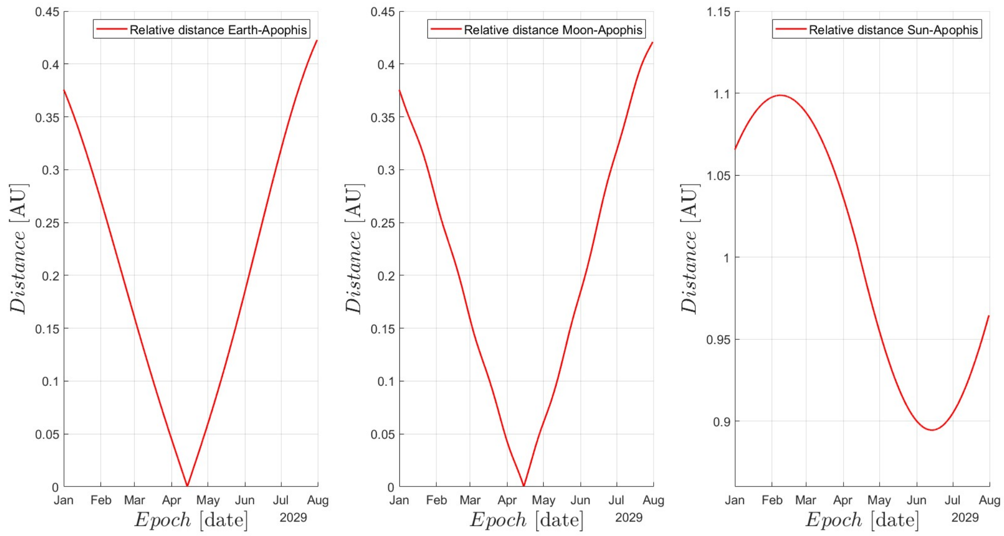


Figure 5: Relative distances between Earth-Apophis, Moon-Apophis, Sun-Apophis @ECLIPJ2000

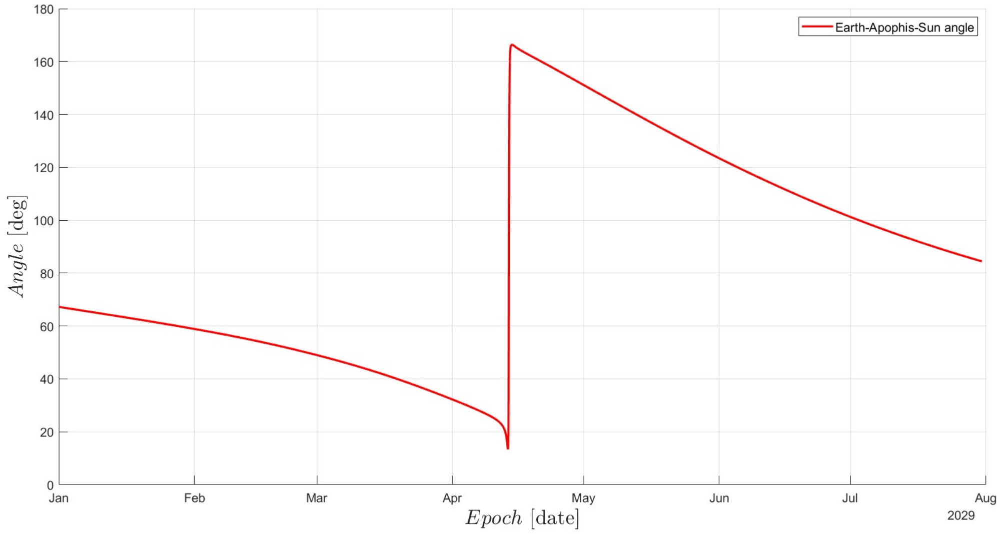


Figure 6: Earth-Apophis-Sun angle @ECLIPJ2000

Ground-tracks are the path on the surface of a planet directly below a satellite's or asteroid's trajectory. Here the ground-track of Apophis on the Earth's surface, with the starting and ending point, on a time window of 12 hours centered around the closest approach time are displayed:

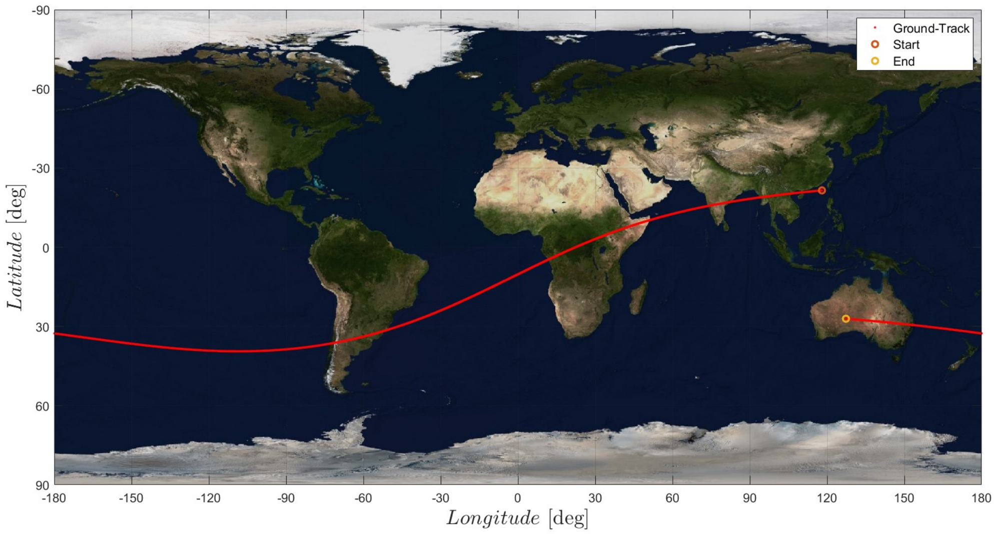


Figure 7: Apophis ground-tracks @IAUEARTH

The optimization problem is stated as a three-point transfer between two moving bodies. A total of 21 variables are considered, 3 time-variables (launch, deep space manoeuvre and impact) with the associated sets of states:

$$\mathbf{y} = (\mathbf{x}_0, \mathbf{x}_{dsm}, \mathbf{x}_{imp}, t_0, t_{dsm}, t_{imp}) \tag{4}$$

The function to be maximized is the distance between the Earth and Apophis at the time of the closest approach, along the asteroid's new trajectory generated by the impact with the spacecraft:

$$f(\mathbf{y}) = \|\boldsymbol{\rho}_A - \boldsymbol{\rho}_E\| \tag{5}$$

The optimization problem is then hereby formalized:

$$\max_{\mathbf{y}} f(\mathbf{y}) \quad \text{s.t.} \quad \begin{cases} \mathbf{c}(\mathbf{y}) = 0 \\ g(\mathbf{y}) < 0 \end{cases} \tag{6}$$

with the following equality and inequality constraints:

$$\mathbf{c}(\mathbf{y}) = 0 \implies \begin{cases} z_1 = \varphi(\mathbf{r}_0, t_0, t_{dsm}) - \mathbf{r}_{dsm} \\ z_2 = \varphi(\mathbf{x}_{dsm}, t_{dsm}, t_{imp}) - \mathbf{x}_{imp} \\ \psi_i = \mathbf{r}_0 - \mathbf{r}_E(t_0) \\ \psi_f = \mathbf{r}_{imp} - \mathbf{r}_V(t_{imp}) \end{cases} \tag{7}$$

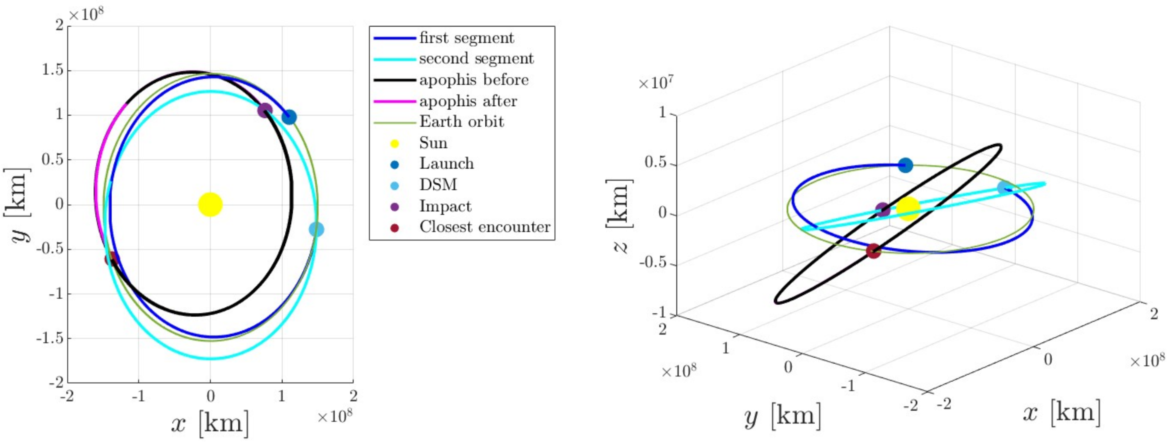
$$g(\mathbf{y}) < 0 \implies g = \Delta v_0 + \Delta v_{dsm} - \Delta v_{tot} \tag{8}$$

The optimized solution for the planetary protection guidance mission is provided using the *GlobalSearch* algorithm. It initially looks for a local minimum near the provided initial conditions, then a set of trial points, based on how good the initial result was, are generated using the *Scatter Search Algorithm*. The set of retrieved minima are then ranked based on their quality and so the global one is found.
Upper and lower boundaries for the time variables are defined from the available windows for launch, deep space manoeuvre and arrival, while boundaries for the states are relaxed and taken as sufficiently small and high values. Initial conditions are taken randomly at each iteration inside those domains.

Due to the function being highly non-convex, the algorithm cannot track the global minimum and provides different results based on the initial conditions.
Since the spacecraft is not able to impart a significant Δv to the asteroid, the trajectory of Apophis after the collision is only slightly different from the natural one; therefore the closest approach will still take place on the same day but the distance is greatly increased, at least tripled with respect to the one expected.
Different simulations are provided below; the factors that determine the final results are, along with the selected dates for the different phases, the magnitudes of the manoeuvres: more costly launch manoeuvres provide greater close approach distances.

Launch	`2024-11-03-21:38:24.3 UTC`		
DSM	`2025-08-28-02:13:38.2 UTC`		
Impact	`2028-10-25-03:45:43.6 UTC`		
TCA	`2029-04-13-23:14:26.7 UTC`		
$\Delta\mathbf{v}_L$ [km/s]	-0.3606	-0.5715	0.1817
$\Delta\mathbf{v}_{DSM}$ [km/s]	-4.1046	0.8285	-0.9789
DCA [Re]	16.3558		

Table 1: First solution for the impactor mission.



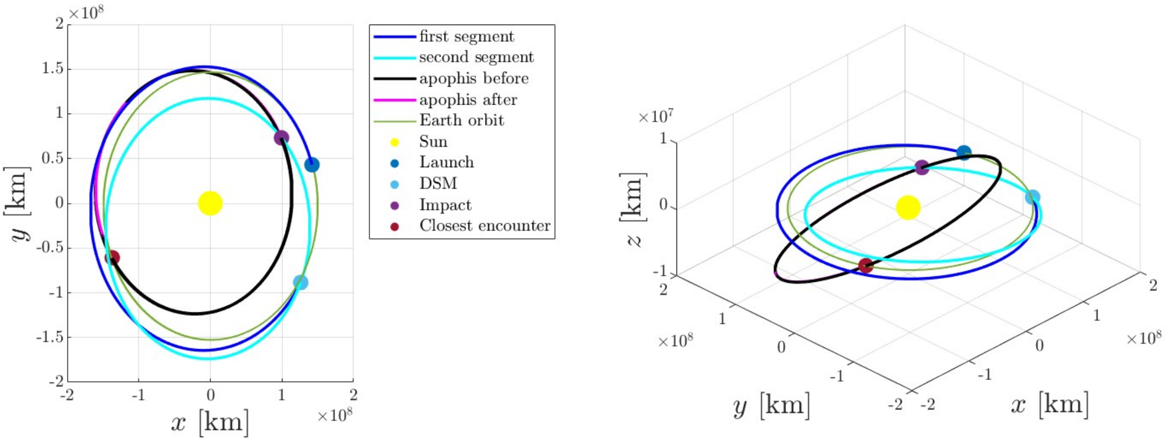
(a) x-y view

(b) x-y-z view

Figure 8: First asteroid impacting mission @Sun-ECLIPTIC-J2000

Launch	`2024-10-10-05:03:52.0 UTC`		
DSM	`2025-09-22-10:54:46.9 UTC`		
Impact	`2028-10-11-12:04:36.1 UTC`		
TCA	`2029-04-13-23:17:19.2 UTC`		
$\Delta\mathbf{v}_L$ [km/s]	-0.2870	1.0309	0.0696
$\Delta\mathbf{v}_{DSM}$ [km/s]	-3.8646	0.5755	-0.4001
DCA [Re]	19.4126		

Table 2: Second solution for the impactor mission.

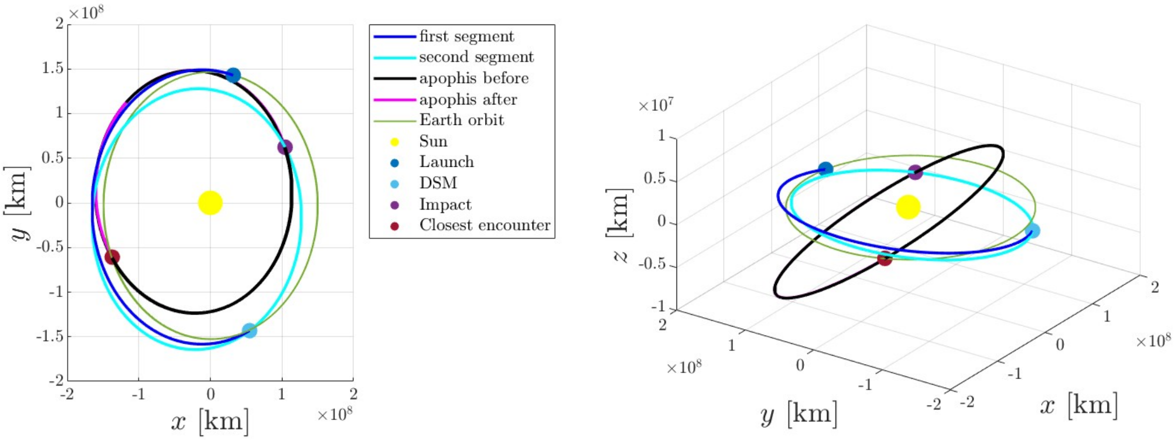


(a) x-y view **(b)** x-y-z view

Figure 9: Second asteroid impacting mission @Sun-ECLIPTIC-J2000

Launch	`2024-12-09-01:56:11.4 UTC`		
DSM	`2025-08-10-03:10:34.7 UTC`		
Impact	`2028-10-12-12:21:08.9 UTC`		
TCA	`2029-04-13-23:18:10.1 UTC`		
$\Delta \mathbf{v}_L$ [km/s]	-0.0962	2.1129	0.4557
$\Delta \mathbf{v}_{DSM}$ [km/s]	-1.8986	2.1031	0.1322
DCA [Re]	21.2739		

Table 3: Third solution for the impactor mission.



(a) x-y view **(b)** x-y-z view

Figure 10: Third asteroid impacting mission @Sun-ECLIPTIC-J2000

3 Continuous guidance

Exercise 3

A low-thrust option is being considered for an Earth-Venus transfer*. Provide a *time-optimal* solution under the following assumptions: the spacecraft moves in the heliocentric two-body problem, Venus instantaneous acceleration is determined only by the Sun gravitational attraction, the departure date is fixed, and the spacecraft initial and final states are coincident with those of the Earth and Venus, respectively.

1) Using the PMP, write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition.

2) Adimensionalize the problem using as reference length $LU = 1$ AU† and reference mass $MU = m_0$, imposing that $\mu = 1$. Report all the adimensionalized parameters.

3) Solve the problem considering the following data:

   - Launch date: `2023-05-28-14:13:09.000 UTC`
   - Spacecraft mass: $m_0 = 1000$ kg
   - Electric propulsion properties: $T_{\max} = 800$ mN, $I_{sp} = 3120$ s

   To obtain an initial guess for the costate, generate random numbers such that $\lambda_{0,i} \in [-20; +20]$, while $t_f < 2\pi$. Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Assess your results exploiting the properties of the Hamiltonian in problems that are not time-dependent and time-optimal solutions.

4) Solve the problem for a lower thrust level $T_{\max} = [500]$ mN. Tip: exploit numerical continuation.

(11 points)

A heliocentric two-body problem is considered for the motion of the spacecraft between the Earth and Venus. When low-thrust is considered, it is convenient to write the equations of motion as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\alpha}, \mathbf{u}) \implies \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{r^3}\mathbf{r} - u^*(m, \lambda_v, \lambda_m)\frac{T_{max}}{m}\frac{\boldsymbol{\lambda_v}}{\lambda_v} \\ -u^*(m, \lambda_v, \lambda_m)\frac{T_{max}}{I_{sp}g_0} \end{bmatrix} \tag{9}$$

where $\mathbf{r} = [x, y, z]^T$ and $\mathbf{v} = [v_x, v_y, v_z]^T$ are the position and velocity vectors of the spacecraft; m denotes the spacecraft's mass, T_{max} is the maximum thrust magnitude, I_{sp} is the thruster specific impulse and g_0 is the gravitational acceleration at sea level.
Considering the optimal thrust-pointing direction $\boldsymbol{\alpha} = -\frac{\boldsymbol{\lambda_v}}{\lambda_v}$, the Hamiltonian of the problem is:

$$H = 1 + \boldsymbol{\lambda_r} \cdot \mathbf{v} - \frac{\mu}{r^3}\mathbf{r} \cdot \boldsymbol{\lambda_v} + \frac{T_{max}}{I_{sp}g_0}u^*\left(-\frac{\lambda_v}{m}I_{sp}g_0 - \lambda_m\right) \tag{10}$$

where u^*, being the problem time-optimal, is considered constant and unitary and $\boldsymbol{\lambda} = [\boldsymbol{\lambda_r}, \boldsymbol{\lambda_v}, \lambda_m]$ is the vector of costates, for which the dynamics are

$$\dot{\boldsymbol{\lambda}} = \mathbf{g}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) \implies \begin{bmatrix} \dot{\boldsymbol{\lambda}}_{\boldsymbol{r}} \\ \dot{\boldsymbol{\lambda}}_{\boldsymbol{v}} \\ \dot{\lambda}_m \end{bmatrix} = \begin{bmatrix} -\frac{3\mu}{r^5}(\mathbf{r} \cdot \boldsymbol{\lambda_v})\mathbf{r} + \frac{\mu}{r^3}\boldsymbol{\lambda_v} \\ -\boldsymbol{\lambda_r} \\ -u^*(m, \lambda_v, \lambda_m)\frac{\lambda_v T_{max}}{m^2} \end{bmatrix} \tag{11}$$

*Read the necessary gravitational constants and planets positions from SPICE. Use the kernels provided on WeBeep for this assignment.

†Read the value from SPICE

The problem is to find $\{\boldsymbol{\lambda}_0, t_f\}$ such that $\varphi(\begin{bmatrix}\mathbf{x_0}\\ \boldsymbol{\lambda_0}\end{bmatrix}, t_0, t_f)$ produces a solution $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$ that satisfies:

$$\begin{cases} \mathbf{r}(t_f) - \mathbf{r}_V(t_f) = 0 \\ \mathbf{v}(t_f) - \mathbf{v}_V(t_f) = 0 \\ \lambda_m(t_f) = 0 \end{cases} \tag{12}$$

where $\boldsymbol{\Psi}(t_f) = (\mathbf{r}_V(t_f), \mathbf{v}_V(t_f))$ is the state of Venus at the final time. The following transversality condition is considered:

$$H(t_f) - \boldsymbol{\lambda}(t_f) \cdot \dot{\boldsymbol{\Psi}}(t_f) = 0 \tag{13}$$

Before looking for the time-optimal solution, the problem has been completely adimensionalized using length LU, mass MU and time TU units. The obtained parameters are reported in the table below:

$\mathbf{r}_0$	-0.4009	-0.9306	4.8795×10^{-5}
$\mathbf{v}_0$	0.9023	-0.3992	4.2073×10^{-5}
m_0		1.0000	
I_{sp}		6.2119×10^{-4}	
T_{max}		0.1349	
g_0		1.6537×10^{3}	
GM		1.0000	

Table 4: Adimensionalized quantities ($T_{max} = 800$ mN).

Starting from random initial conditions for $\{\boldsymbol{\lambda}_0, t_f\}$ and solving the zero-finding problem until a certain degree of precision is reached, a time-optimal solution is provided.

$\boldsymbol{\lambda}_{0,r}$	0.5093	-13.3116	0.1158
$\boldsymbol{\lambda}_{0,v}$	5.0464	-10.0181	1.4568
$\lambda_{0,m}$		1.7096	
t_f		`2023-10-17-00:20:47.1 UTC`	
TOF [days]		141.4220	

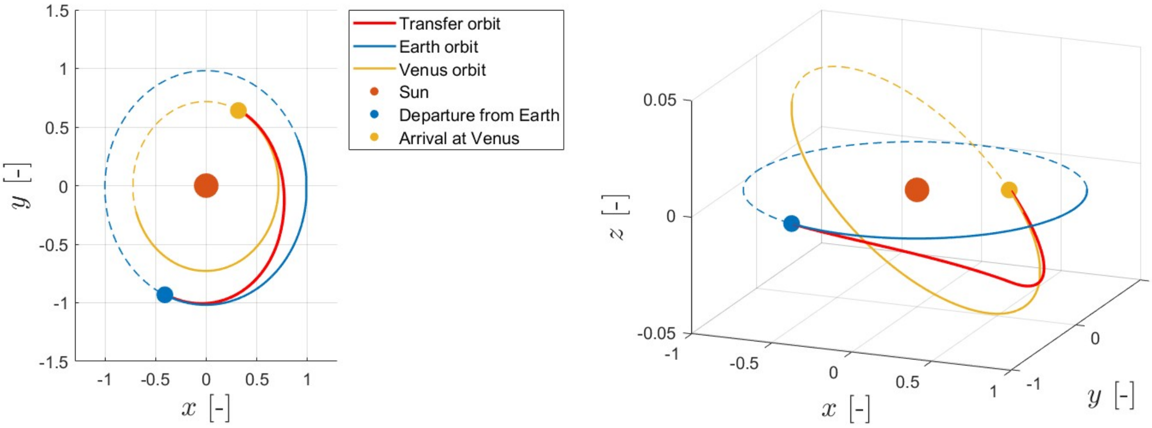
Table 5: Time-optimal Earth-Venus transfer solution ($T_{max} = 800$ mN).

Arrival at Venus is achieved with a certain degree of accuracy, as both the position and velocity errors are very restrained.

$\lvert\lvert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\rvert\rvert$	[km]	6.930×10^{-4}
$\lvert\lvert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\rvert\rvert$	[m/s]	3.144×10^{-7}

Table 6: Final state error with respect to Venus' center ($T_{max} = 800$ mN).

The transfer is here graphically represented, with the starting point at the Earth, arrival point at Venus and the in between optimized trajectory; the thrust pointing unit vector along the voyage is also visually emphasized.



(a) x-y view **(b)** x-y-z view

Figure 11: Time optimal Earth-Venus transfer orbit for $T_{\max} = 800$ mN (@Sun-ECLIPTIC-J2000)

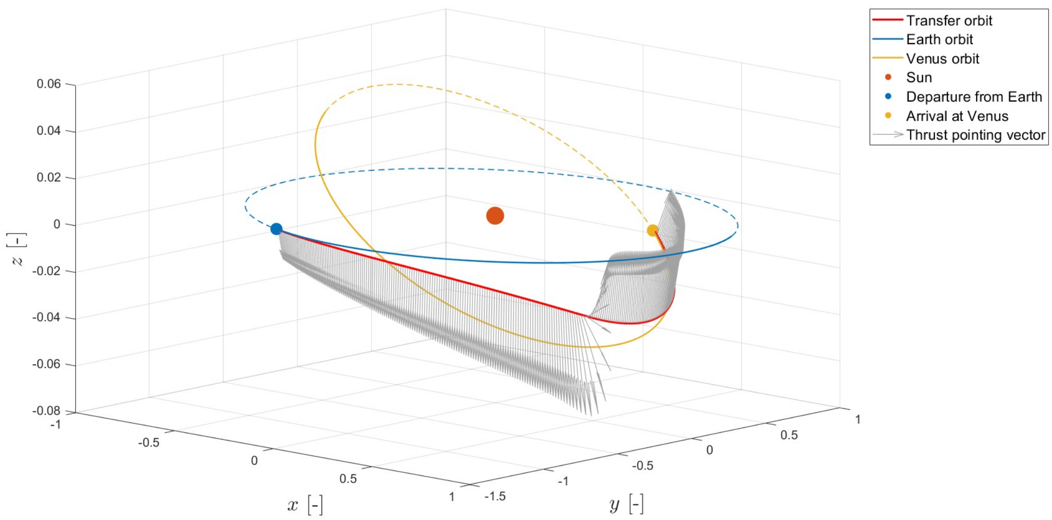
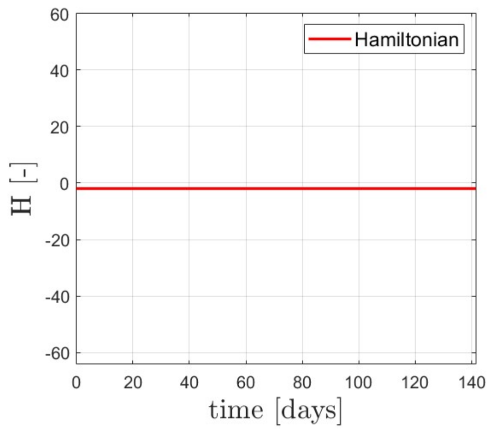
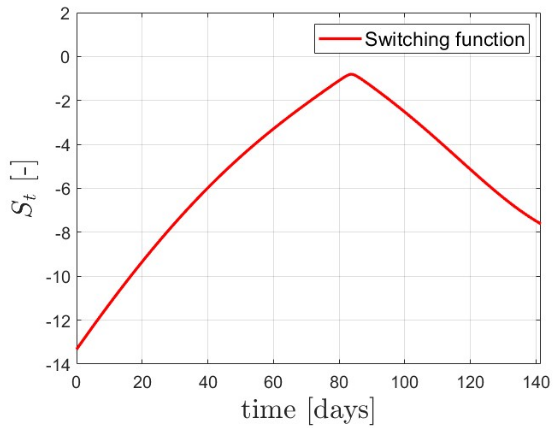


Figure 12: Thrust pointing vector during the transfer for $T_{\max} = 800$ mN (@Sun-ECLIPTIC-J2000)

As expected in problems that are optimal and not time dependent, the Hamiltonian $H(t)$ is constant along the whole path; additionally, for time optimal problems, since $\dot{\lambda}_m(t) < 0$ and $\lambda_m(t_f) = 0$, which imply that $\lambda_m(t) \geq 0$, then $S_t(t) < 0$. These properties of the problem are also validated analyzing the evolution of the Hamiltonian and the switching function.



(a) hamiltonian



(b) Switching function

Figure 13: Hamiltonian and Switching function during the transfer for $T_{\max} = 800$ mN

Solving the problem with a $T_{max} = 500$ mN, a longer transfer time is certainly expected. Exploiting numerical continuation, so using the values from the previous case ($T_{max} = 800$ mN) as initial conditions to start the zero-finding problem, a lower convergence time is reached. The results for this second case are hereby reported:

$\boldsymbol{\lambda}_{0,r}$	-0.5601	-24.6682	-0.4760
$\boldsymbol{\lambda}_{0,v}$	14.2314	-19.0114	2.0082
$\lambda_{0,m}$	2.7363		
t_f	2023-12-31-22:00:34.4 UTC		
TOF [days]	217.3246		

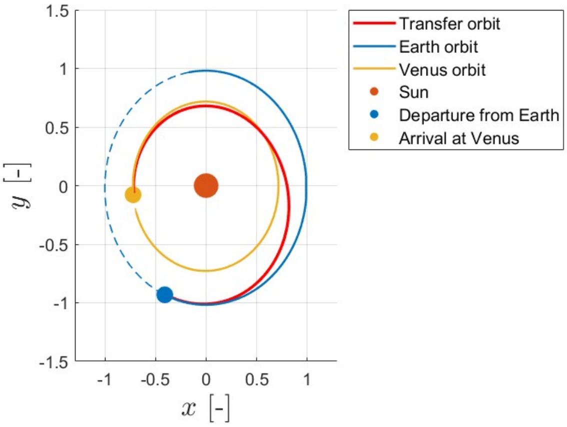
Table 7: Time-optimal Earth-Venus transfer solution ($T_{\max} = 500$ mN).

Also in this case, Venus is reached with a satisfactory error margin on position and velocity:

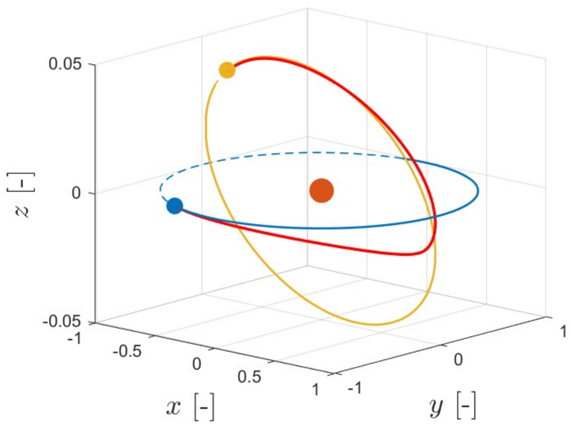
$\lvert\lvert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\rvert\rvert$	[km]	3.254×10^{-3}
$\lvert\lvert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\rvert\rvert$	[m/s]	9.427×10^{-7}

Table 8: Final state error with respect to Venus' center ($T_{\max} = 500$ mN).

Since the transfer time is similar to Venus' orbit period ($\simeq 224.7$ days), the planet covers almost a whole orbital between the launch and arrival times. The transfer for this second case is hereby shown, considering also the thrust pointing unit vector.



(a) x-y view



(b) x-y-z view

Figure 14: Time optimal Earth-Venus transfer orbit for $T_{\max} = 500$ mN (@Sun-ECLIPTIC-J2000)

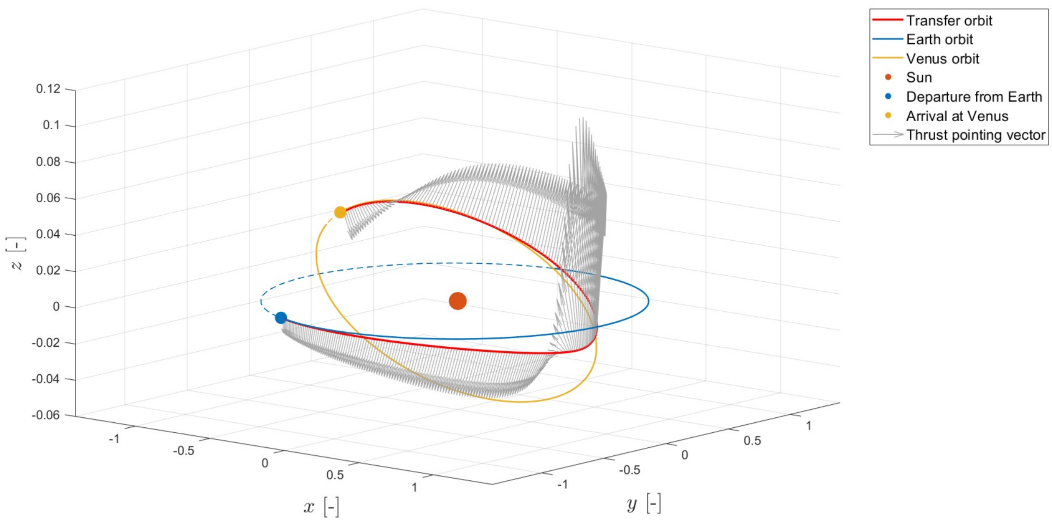
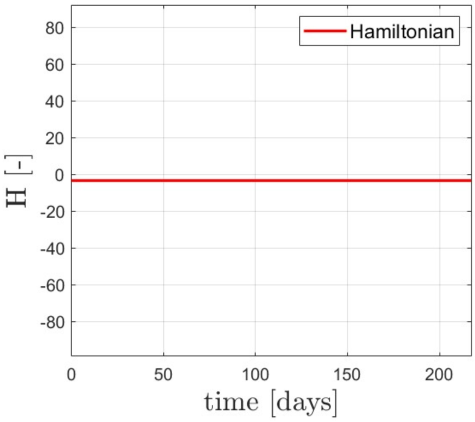
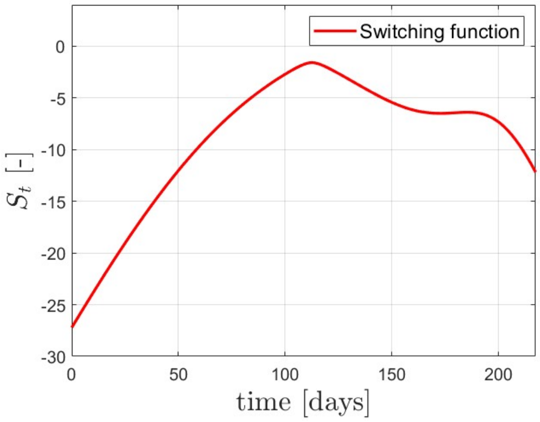


Figure 15: Thrust pointing vector during the transfer for $T_{\max} = 500$ mN (@Sun-ECLIPTIC-J2000)

Also in this case the correctness of the results is validated through the evolution of the Hamiltonian and the switching functions, that remain respectively constant and negative for the whole simulation.



(a) Hamiltonian



(b) Switching function

Figure 16: Hamiltonian and Switching function during the transfer for $T_{\max} = 500$ mN