1Y Algebra

2012/13

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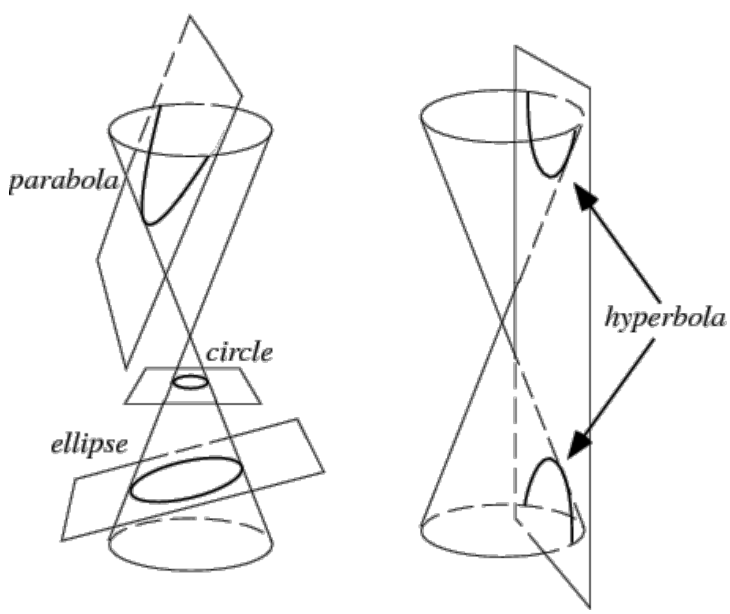
2 Conics

Conics or *conic sections* can be obtained by intersecting a (double, infinite) cone in $\mathbb{R}^3$ with a plane, but can also be defined as a plane algebraic curve of degree 2 (in plain terminology, a second degree polymomial in two variables). The latter definition is simpler since it only requires $\mathbb{R}^2$ and is based on the distance between a point P on the curve and a fixed point, called *focus*, and the distance between P and a line, called the *directrix*, being in a constant ratio. This ratio is called the *eccentricity* and denoted e and this way of defining conics is known as the *focus-directrix approach.*

There are three types of conics:

- parabola ($e = 1$);
- hyperbola ($e > 1$);
- ellipse ($e < 1$).

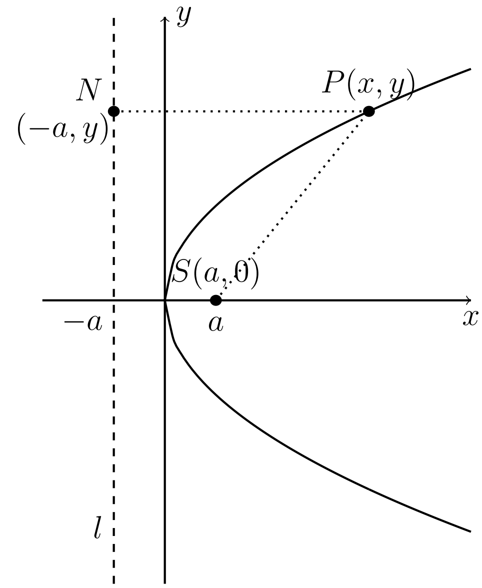
The circle is a special case of an ellipse. In the focus-directrix definition, the circle is a limiting case with $e = 0$.



2.1 Parabola

I. Equation of a parabola from the focus-directrix approach

Definition 2.1. A *parabola* in $\mathbb{R}^2$ consists of all points P that are equidistant from a fixed focus point S and a fixed directrix.



When the focus is at $(a,0)$ and the directrix is the line $x=-a$, $a>0$, we have

$$\begin{aligned} |PS| &= |PN| \\ (x-a)^2+y^2 &= (x+a)^2 \end{aligned}$$

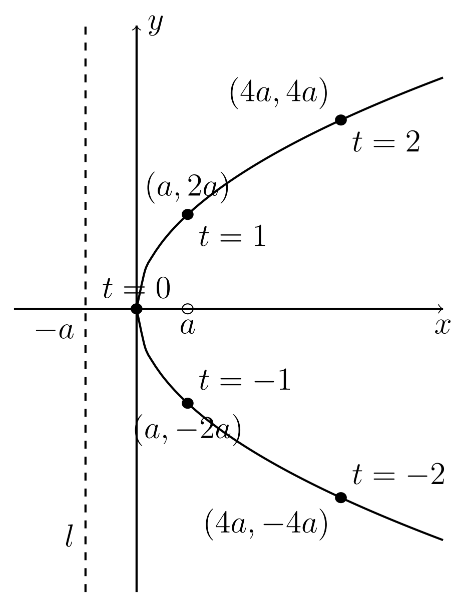
which gives the equation, describing the parabola (with focus and directrix as specified above) in the form

$$y^2 = 4ax. \tag{2.1}$$

Remarks.

1. The x-axis is the symmetry line for the curve (2.1) and is also called the *axis* of parabola.
2. Note that this parabola cans also be described as the set of points P so that $e := \dfrac{|PS|}{|PN|} = 1$.

II. Parametric equation of a parabola



The implicit equation

$$y^2 = 4ax$$

is equivalent to parametric equations

$$x = at^2, \quad y = 2at, \quad t \in \mathbb{R} \tag{2.2}$$

Derivative $y'(x)$: $\quad \dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2a}{2at} = \dfrac{1}{t}$

Equation of the tangent at $(at^2, 2at)$ is $y - 2at = \frac{1}{t}(x - at^2)$, i.e. $y = \frac{x}{t} + at$.

Equation of the normal at $(at^2, 2at)$ is $y - 2at = -t(x - at^2)$.

Concavity: $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{1}{t}}{\dot{x}} = \frac{-\frac{1}{t^2}}{2at} = -\frac{1}{2at^3}$$

so the curve is:
- concave up when $t < 0$ (bottom half) and
- concave down when $t > 0$ (upper half).

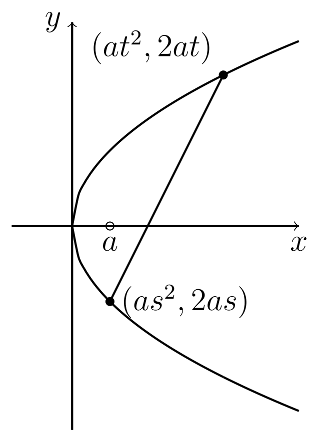
Gradient of a *chord* joining points $(at^2, 2at)$ and $(as^2, 2as)$:

$$\frac{2at - 2as}{at^2 - as^2} = \frac{2(t-s)}{t^2 - s^2} = \frac{2}{t+s}$$

Notice that

$$\text{grad. of chord} \underset{s \to t}{\longrightarrow} \frac{2}{2t} = \frac{1}{t} = \text{grad. of tangent}$$

(when the two points collide).



Mirror property:

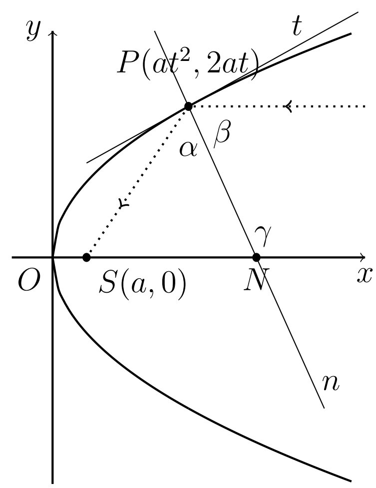
A ray of light parallel to the axis of a parabola is reflected to the focus.

Proof. We claim that $\alpha = \beta$ (see figure).
At the point $P(at^2, 2at)$ the gradient of the normal is $-t = \tan\gamma$. Notice that $\gamma + \beta = \pi$ (angles with parallel lines). Therefore,

$$\tan\beta = \tan(\pi - \gamma) = \tan(-\gamma) = -\tan\gamma = t$$

Also,

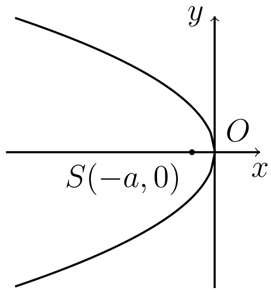
$$\begin{aligned} \tan(\alpha + \beta) &= \tan(\angle OSP) = \tan(\pi - \angle NSP) \\ &= -\text{grad}\,\overline{SP} = -\frac{2at - 0}{at^2 - a} = \frac{2t}{1 - t^2} \\ &= \tan(2\beta) \\ &\Rightarrow \alpha = \beta \end{aligned}$$ $\square$



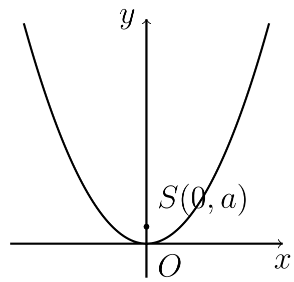
The mirror property is put to work in (parabolic) satellite dishes!

III. Other equations of parabolas

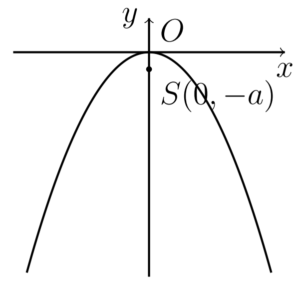
• One of the coordinate axes is also axis of the parabola



$y^2 = -4ax, \quad a > 0$



$x^2 = 4ay, \quad a > 0$



$x^2 = -4ay, \quad a > 0$

• Axis of the parabola is a shifted coordinate axis

For example, taking a parabola

$$y^2 = -4ax, \quad a > 0$$

and shifting the vertex of the parabola

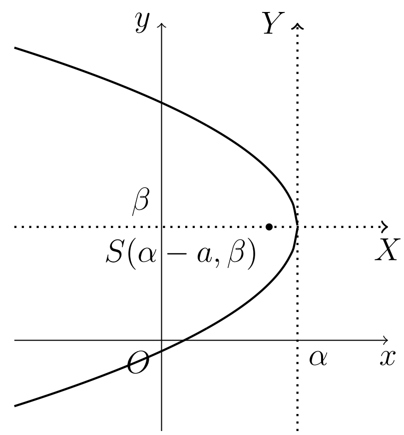
$$(0,0) \longrightarrow (\alpha, \beta)$$

gives the parabola with equation

$$Y^2 = -4aX, \quad a > 0,$$

with $X = x - \alpha, \quad Y = y - \beta$. Thus the **standard form** of the equation is

$$(y-\beta)^2 = -4a(x-\alpha), \quad a > 0\,.$$



In general, if the origin of the new coordinate system (X, Y) is at (α, β), then

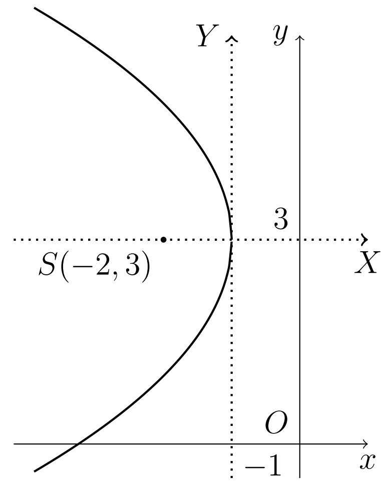
$$X = x - \alpha, \quad Y = y - \beta\,.$$

Example 2.1. Reduce the equation of the parabola

$$y^2 - 6y + 4x + 13 = 0$$

to the standard form and sketch the parabola.

$$
\begin{aligned}
(y-3)^2 - 9 + 4x + 13 &= 0 \\
(y-3)^2 &= -4(x+1)
\end{aligned}
$$

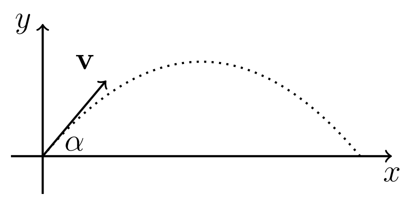


This is the standard equation of the form

$Y^2 = -4X$ where $Y = y-3,\ X = x+1$

(the centre of the new coordinate system XY is locates at $(-1, 3)$).

Path of a projectile



Suppose we throw a ball into the air with velocity v and at an angle α to the horizontal.

Ignoring air resistance,

$$
\begin{aligned}
\dot{x} &= v\cos\alpha &&\Rightarrow\quad x = v\cos\alpha\cdot t \\
\ddot{y} &= -g &&\Rightarrow\quad \dot{y} = -gt + C, \qquad C = v\sin\alpha \\
& &&\Rightarrow\quad \dot{y} = -gt + v\sin\alpha \\
& &&\Rightarrow\quad y = -\frac{1}{2}gt^2 + (v\sin\alpha)t
\end{aligned}
$$

Here $\dot{x}$ and $\dot{y}$ are the horizontal and the vertical component of the velocity. The second derivative gives the acceleration - in this case the vertical component, $\ddot{y}$, is the gravity.

Since $x = v\cos\alpha\cdot t$ and thus $t = \dfrac{x}{v\cos\alpha}$ we have that

$$
\begin{aligned}
y &= -\frac{1}{2}g\left(\frac{x}{v\cos\alpha}\right)^2 + v\sin\alpha\,\frac{x}{v\cos\alpha} \\
y &= -\frac{g}{2v^2\cos^2\alpha}\,x^2 + x\tan\alpha
\end{aligned}
$$

Which is an equation of a parabola.

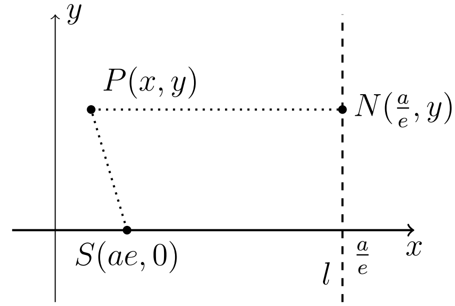
2.2 Ellipse

I. Equation of an ellipse from the focus-directrix approach

Definition 2.2. An *ellipse* in $\mathbb{R}^2$ consists of all points P s.t.

$$d(P,S) = e \cdot d(P,l), \quad 0 < e < 1$$

where S is a fixed focus point and l is a a fixed directrix.

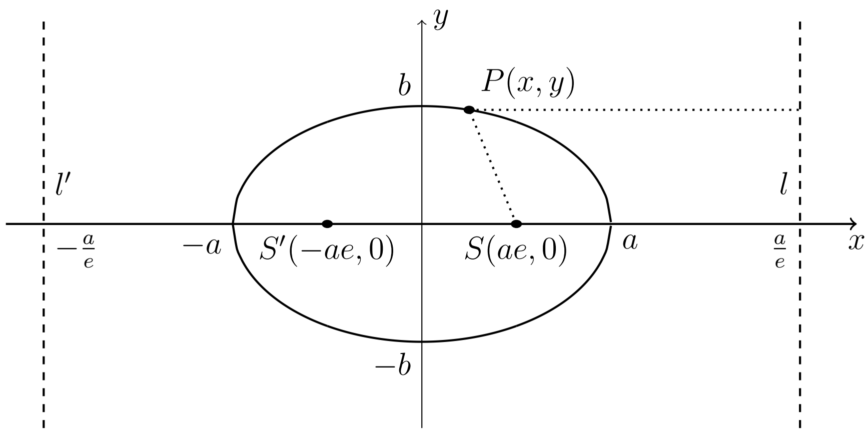


Choose the focus at $(ae, 0)$ and let the directrix be the line $x = \frac{a}{e}$, for some constant $a > 0$. Then $P(x,y)$ is on the ellipse if and only if

$$\begin{aligned} |PS| &= e \cdot |PN| \\ |PS|^2 &= e^2 \cdot |PN|^2 \\ (x-ae)^2 + y^2 &= e^2\left(\frac{a}{e} - x\right)^2 \\ x^2 - 2aex + a^2e^2 + y^2 &= a^2 - 2aex + e^2x^2 \\ x^2(1-e^2) + y^2 &= a^2(1-e^2) \\ \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} &= 1 \end{aligned}$$

Defining $b := a\sqrt{1-e^2}$ now gives the equation of the ellipse (with focus and directrix as specified above) in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{2.3}$$



a and b are called semi-axes of the ellipse.

Remarks.

1. $S'(-ae, 0)$ and $l' : x = -\frac{a}{e}$ is another focus-directrix pair determining the same ellipse.

2. Since $b = a\sqrt{1-e^2}$, we can express eccentricity in terms of semi-axes as follows: $e = \dfrac{\sqrt{a^2 - b^2}}{a}$.

Example 2.2. Consider the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The semi-axes are $a = 3$ and $b = 2$. Eccentricity: $e = \dfrac{\sqrt{9-4}}{3} = \dfrac{\sqrt{5}}{3}$.

The foci are at $(\pm ae, 0)$, so $S(\sqrt{5}, 0)$ and $S'(-\sqrt{5}, 0)$.

The directrices are at $\pm\dfrac{a}{e}$, so $l : x = \dfrac{9}{\sqrt{5}}$ and $l' : x = -\dfrac{9}{\sqrt{5}}$.

The meaning of eccentricity.

For an ellipse, as eccentricity varies from 0 to 1, it indicates the departure from circularity. Since $b^2 = a^2 - a^2e^2$, the closer e is to 0, the closer ellipse is to a circle. The closer e is to 1, the more elongated the ellipse.

- The eccentricity of the orbit of planet Earth is $e = 0.02$.
- The eccentricity of the orbit of planet Mercury is $e = 0.21$.
- The eccentricity of asteroid Icarus is $e = 0.83$.

Example 2.3. The orbit of comet Kohoutek is approximately 44 Au wide and 3600 Au long. (Au, the astronomical unit, is Earth's mean distance from the Sun). Its eccentricity, therefore, is

$$e = \frac{\sqrt{1800^2 - 22^2}}{1800} \approx 0.9999\,.$$

II. Parametric equation of an ellipse

The implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is equivalent to parametric equations

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi\,. \tag{2.4}$$

<u>Derivative $y'(x)$</u>: At a point $(a\cos t, b\sin t)$ the gradient of the tangent is

$$\frac{dy}{dx} = \frac{\dot y}{\dot x} = \frac{b\cos t}{-a\sin t}\,.$$

<u>Equation of the tangent at $(a\cos t, b\sin t)$</u>:

$$\begin{aligned}
y - b\sin t &= -\frac{b\cos t}{a\sin t}(x - a\cos t)\\
ay\sin t - ab\sin^2 t &= -bx\cos t + ab\cos^2 t\\
ay\sin t + bx\cos t &= ab\\
\frac{x\cos t}{a} + \frac{y\sin t}{b} &= 1
\end{aligned}$$

Replacing $(a\cos t, b\sin t)$ by (x_0, y_0) the equation of tangent becomes

$$\frac{x\,x_0}{a^2} + \frac{y\,y_0}{b^2} = 1\,.$$

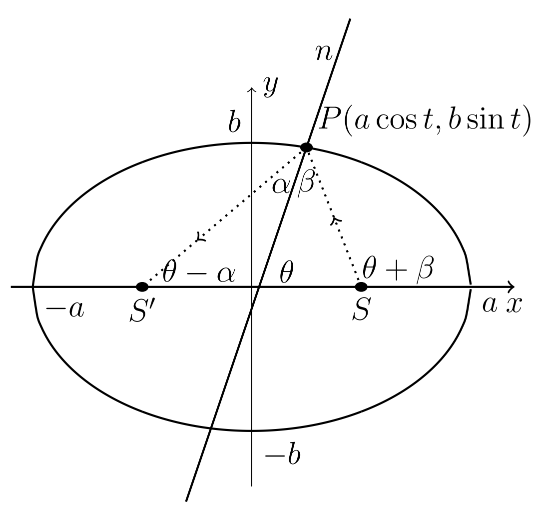
<u>Equation of the normal at $(a\cos t, b\sin t)$</u>:

$$\begin{aligned}
y - b\sin t &= \frac{a\sin t}{b\cos t}(x - a\cos t)\\
by\cos t - ax\sin t &= (b^2 - a^2)\sin t\cos t
\end{aligned}$$

<u>Mirror property:</u>

A ray of light originating at one focus is reflected to the other.

Proof: tutorial exercise.
Sketch of the proof: We claim that $\alpha = \beta$ (see figure).



Let the normal have gradient $\tan\theta = \dfrac{a\tan t}{b}$.

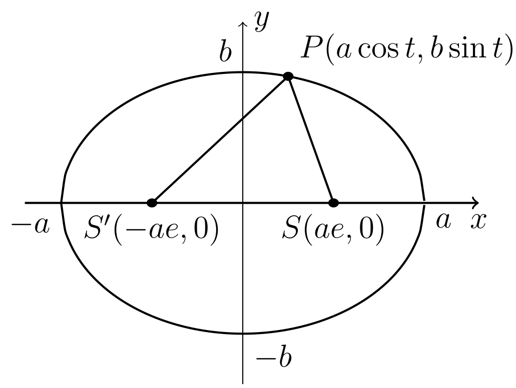
Line SP has gradient $\tan(\theta+\beta)$.

Line $S'P$ has gradient $\tan(\theta-\alpha)$.

Expand $\tan(\theta+\beta)$ and solve for β. Similarly, expand $\tan(\theta-\alpha)$ and solve for α. Find that $\alpha = \beta$.

Focal distance property:

The sum of the distances from any point on an ellipse to the foci is constant.



$$
\begin{aligned}
|SP|^2 &= a^2(\cos t - e)^2 + b^2 \sin^2 t \\
&= a^2 \cos^2 t - 2a^2 e \cos t \\
&\quad + a^2 e^2 + b^2 \sin^2 t
\end{aligned}
$$

Now, since $b^2 = a^2 - a^2e^2$,

$$
\begin{aligned}
|SP|^2 &= a^2 \cos^2 t - 2a^2 e \cos t \\
&\quad + a^2 e^2 + (a^2 - a^2e^2) \sin^2 t \\
&= a^2 - 2a^2 e \cos t + a^2 e^2 \cos^2 t \\
&= (a - ae \cos t)^2
\end{aligned}
$$

Thus $|SP| = a - ae\cos t$. (Note that $a - ae\cos t > 0$.)

Similarly, $|S'P| = a + ae\cos t$. (*Show this as an exercise!*)

Therefore, $|SP| + |S'P| = 2a$ (i.e., constant).

III. Other equations of ellipses

Consider shifting the centre of the coordinate system

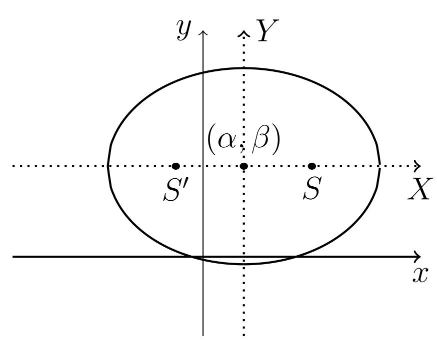
$$(0,0) \longrightarrow (\alpha, \beta)$$

and call the new coordinate axes X and Y. Then an ellipse with the equation (w.r.t. X, Y)

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,$$

where $X = x - \alpha$, $Y = y - \beta$, has the **standard form** of the equation (w.r.t. x, y)

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1\,.$$



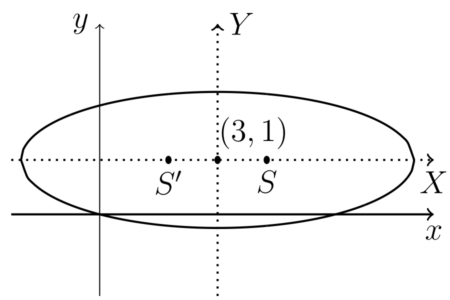
In this respect we can also define the *centre* of the ellipse to be the point halfway between the foci. For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the centre is at the origin, for $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$ it is at (α, β).

Example 2.4. Reduce the equation $x^2 - 6x + 4y^2 - 8y - 3 = 0$ to standard form. Find eccentricity, positions of centre, foci and directrices. Sketch the curve.

$$\begin{aligned}
(x-3)^2 - 9 + 4(y-1)^2 - 4 - 3 &= 0 \\
(x-3)^2 + 4(y-1)^2 &= 16 \\
\text{Standard form:} \qquad \frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} &= 1 \\
\text{or} \qquad \frac{X^2}{16} + \frac{Y^2}{4} &= 1\,; \quad X = x-3, \quad Y = y-1
\end{aligned}$$

The semi-axes are $a = 4, b = 2$, the centre is at $C(3,1)$ and the eccentricity is $e = \dfrac{\sqrt{16-4}}{4} = \dfrac{\sqrt{3}}{2}$.

The directrices are $X = \pm\dfrac{a}{e} = \pm\dfrac{8\sqrt{3}}{3}$, which is $x = 3 \pm \dfrac{8\sqrt{3}}{3}$ in original coordinates.

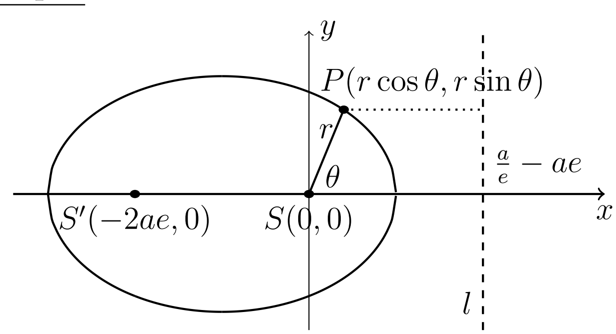


The foci are at $X = \pm ae = \pm 2\sqrt{3}$. Therefore, in original coordinates, the coordinates of the foci are $S(3 + 2\sqrt{3}, 1)$ and $S'(3 - 2\sqrt{3}, 1)$.

IV. Polar equation of an ellipse

Consider an ellipse having the right focus at the origin. Its equation is

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$



By the focus-directrix definition of an ellipse,

$$|PO| = e \cdot d(P, l)$$

which translates into polar coordinates as

$$\begin{aligned}
r &= e \cdot \left(\frac{a}{e} - ae - r\cos\theta\right) \\
r &= a - ae^2 - re\cos\theta \\
r(1 + e\cos\theta) &= a(1-e^2) \qquad \Rightarrow \qquad \boxed{r = \frac{a(1-e^2)}{1 + e\cos\theta}}
\end{aligned}$$

2.3 Hyperbola

I. Equation of a hyperbola from the focus-directrix approach

Definition 2.3. A *hyperbola* in $\mathbb{R}^2$ consists of all points P s.t.

$$d(P,S) = e \cdot d(P,l), \quad e > 1$$

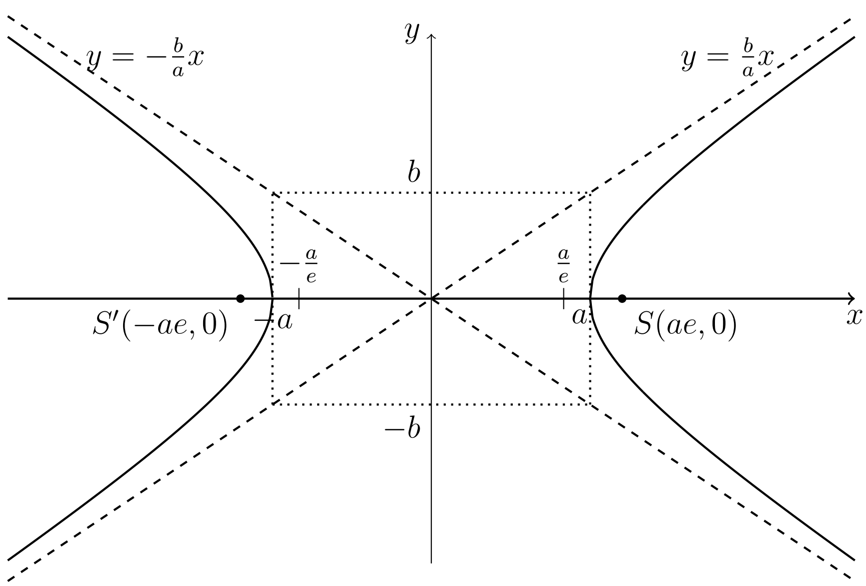
where S is a fixed focus point and l is a a fixed directrix.

Choose the focus $S(ae,0)$ and let the directrix be the line $x = \frac{a}{e}$, for some constant $a > 0$. Then $P(x,y)$ is on the hyperbola if and only if

$$\begin{aligned}
(x-ae)^2 + y^2 &= e^2\left(\frac{a}{e} - x\right)^2 \\
x^2 - 2aex + a^2e^2 + y^2 &= a^2 - 2aex + e^2x^2 \\
x^2(1-e^2) + y^2 &= a^2(1-e^2) \qquad \text{now} \quad 1 - e^2 < 0 \\
x^2(e^2-1) - y^2 &= a^2(e^2-1)
\end{aligned}$$

Defining $b := a\sqrt{e^2-1}$ now gives the equation of the hyperbola (with focus and directrix as specified above) in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,. \tag{2.5}$$



Remarks.

1. $S'(-ae,0)$ and $l' : x = -\frac{a}{e}$ is another focus-directrix pair determining the same hyperbola.
2. Eccentricity in terms of semi-axes: $e = \dfrac{\sqrt{a^2+b^2}}{a}$.
3. The x-axis is the axis of the hyperbola (2.5).

Example 2.5. For the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

determine its eccentricity, the positions of foci and the equations of asymptotes and directrices.

$$a = 3,\ b = 2 \quad \Rightarrow \quad e \;=\; \frac{\sqrt{a^2+b^2}}{a} \;=\; \frac{\sqrt{13}}{3}\,;\;\; \text{Foci at } \;(\pm ae, 0) \;=\; \big(\pm\sqrt{13}, 0\big)\,;$$

$$\text{Asymptotes: } \; y \;=\; \pm\frac{b}{a}\,x \;=\; \pm\frac{2}{3}\,x\,;\;\; \text{Directrices } \; x \;=\; \pm\frac{a}{e} \;=\; \pm\frac{9}{\sqrt{13}}\,.$$

II. Other equations of hyperbolas

Consider shifting the centre of the coordinate system $(0,0) \longrightarrow (\alpha, \beta)$ and call the new coordinate axes X and Y. Then a hyperbola with the equation (w.r.t. X, Y)

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} \;=\; 1,$$

where $X = x - \alpha$, $Y = y - \beta$, has the **standard form** of the equation (w.r.t. x, y)

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} \;=\; 1\,.$$

Example 2.6. Reduce the equation

$$x^2 - 4y^2 - 2x + 8y - 7 \;=\; 0$$

to standard form and hence show that it defines a hyperbola. Find its eccentricity, asymptotes and foci.

$$\begin{aligned} x^2 - 2x - 4y^2 + 8y - 7 &= 0 \\ (x-1)^2 - 1 - 4(y-1)^2 + 4 - 7 &= 0 \\ (x-1)^2 - 4(y-1)^2 &= 4 \\ \frac{(x-1)^2}{4} - \frac{(y-1)^2}{1} &= 1 \end{aligned}$$

So we have a standard equation $\dfrac{X^2}{4} - \dfrac{Y^2}{1} = 1$ with $X = x - 1$, $Y = y - 1$.

$$a = 2,\ b = 1 \quad \Rightarrow \quad e \;=\; \frac{\sqrt{a^2+b^2}}{a} \;=\; \frac{\sqrt{5}}{2}\,;$$

$$\text{Foci at } \;(X, Y) = (x-1, y-1) = (\pm ae, 0) \quad \Rightarrow \quad (x, y) = \big(1 + \pm\sqrt{5}, 1\big)\,;$$

$$\text{Asymptotes: } \; Y \;=\; \pm\frac{b}{a}\,X \quad \Rightarrow \quad y = \pm\frac{1}{2}\,(x-1) + 1 \;=\; \pm\frac{1}{2}\,x + \frac{1}{2}\,.$$

III. Parametric equation of a hyperbola

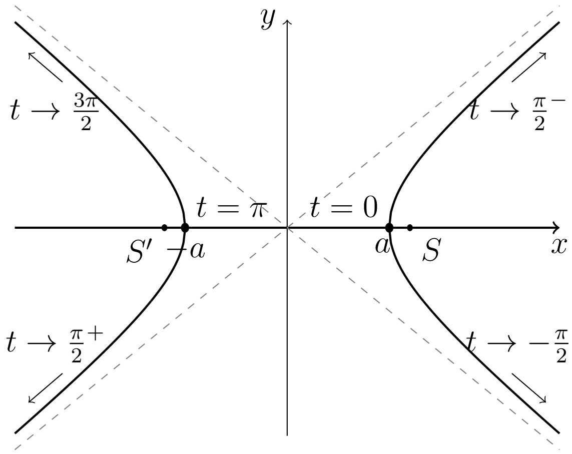
A hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be parametrized in the following way:

$$x = a\sec t,\ y = b\tan t, \quad (2.6)$$

$$-\frac{\pi}{2} < t < \frac{3\pi}{2}, \quad t \neq \frac{\pi}{2}.$$



Indeed, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2\sec^2 t}{a^2} - \frac{b^2\tan^2 t}{b^2} = \sec^2 t - \tan^2 t = 1$.

Also note that, for a fixed $y = b\tan t$, there are two possible values of $t \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $t \neq 0, \pi$, hence two different values of x.

Equation of the tangent at $(a\sec t, b\tan t)$:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b}{a\sin t}$$

so the equation of the tangent at $(a\sec t, b\tan t)$ is

$$\begin{aligned}
y - b\tan t &= \frac{b}{a\sin t}(x - a\sec t) \qquad \text{(multiply by } a\sin t\text{)}\\
a\sin t\cdot y - \frac{ab\sin^2 t}{\cos t} &= bx - \frac{ab}{\cos t}\\
a\sin t\cdot y &= bx + \frac{ab(\sin^2 t - 1)}{\cos t}\\
a\sin t\cdot y - bx &= -ab\cos t \qquad \text{(divide by } -ab\cos t\text{)}\\
\frac{x}{a\cos t} - \frac{\sin t\cdot y}{b\cos t} &= 1
\end{aligned}$$

for $(x_0, y_0) = (a\sec t, b\tan t)$:

$$\frac{x}{a^2}\cdot\frac{a}{\cos t} - \frac{y}{b^2}\cdot\frac{b\sin t}{\cos t} = 1 \qquad \Rightarrow \qquad \frac{x\cdot x_0}{a^2} - \frac{y\cdot y_0}{b^2} = 1.$$

2.4 Conics and rotation of axes

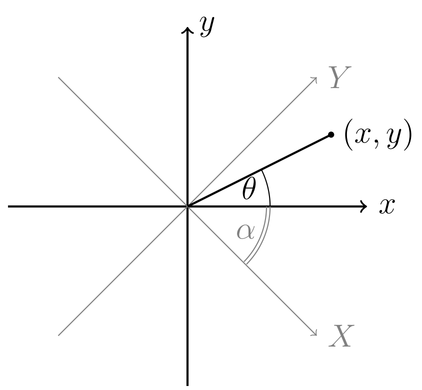
We saw in the previous sections that the quadratic equation

$$Ax^2 + By^2 + Cx + Dy + Exy + F = 0, \qquad A, B \neq 0, \quad \underline{E = 0},$$

can describe an ellipse (or a circle), a parabola or a hyperbola and we used *translation of axes* to reduce the equations to standard form

$$\left.\begin{array}{ll} \bullet \text{ ellipse:} & \dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1 \\[2ex] \bullet \text{ hyperbola:} & \dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = 1 \\[2ex] \bullet \text{ parabola:} & Y^2 = 4aX \ \text{ or } \ X^2 = 4aY \end{array}\right\} \quad X = x - \alpha, \quad Y = y - \beta$$

Sometimes, if $E \neq 0$, a *rotation of axes* is required.



Change of coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ X &= r\cos(\theta+\alpha) \\ &= r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ Y &= r\sin(\theta+\alpha) \\ &= r(\cos\theta\sin\alpha + \sin\theta\cos\alpha) \end{aligned}$$

In matrix notation,

$$\begin{aligned} \begin{bmatrix} X \\ Y \end{bmatrix} &= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\cos\theta\sin\alpha + r\sin\theta\cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{\text{rotation matrix } R_\alpha} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

The inverse matrix of the rotation matrix $R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is the matrix

$$(R_\alpha)^{-1} = \frac{1}{\cos^2\alpha + \sin^2\alpha} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = R_{-\alpha}$$

thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = R_{-\alpha} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X\cos\alpha + Y\sin\alpha \\ -X\sin\alpha + Y\cos\alpha \end{bmatrix}.$$

Example 2.7. Reduce the equation of a conic

$$3x^2 - 4xy + 3y^2 = 25$$

to standard form and sketch the conic.

We wish to get rid of the $xy-$term by rotation $\begin{cases} x = X\cos\alpha + Y\sin\alpha \\ y = -X\sin\alpha + Y\cos\alpha \end{cases}$:

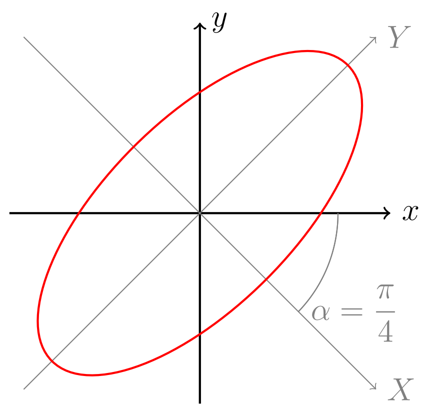
$$\begin{aligned} 3(X\cos\alpha + Y\sin\alpha)^2 - 4(X\cos\alpha + Y\sin\alpha)(-X\sin\alpha + Y\cos\alpha) & \\ +3(-X\sin\alpha + Y\cos\alpha)^2 &= 25 \\ X^2(3 + 4\cos\alpha\sin\alpha) + Y^2(3 - 4\sin\alpha\cos\alpha) + 4XY(\sin^2\alpha - \cos^2\alpha) &= 25 \end{aligned}$$

For the $XY-$term to vanish, we need $\cos(2\alpha) = 0$.

Take $\alpha = \frac{\pi}{4}$. Then $\cos(2\alpha) = 0$ and $\sin\alpha = \cos\alpha = \frac{\sqrt{2}}{2}$.

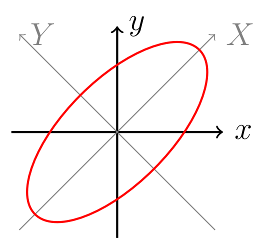
Then the equation becomes

$$\begin{aligned} X^2\left(3 + 4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\right) + Y^2\left(3 - 4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\right) + 0 &= 25 \\ \frac{X^2}{5} + \frac{Y^2}{25} &= 1 \end{aligned}$$



Notice: the choice of $\alpha = -\frac{\pi}{4}$ gives

$\sin\alpha = -\frac{\sqrt{2}}{2}$, $\cos\alpha = \frac{\sqrt{2}}{2}$ and thus $\frac{X^2}{25} + \frac{Y^2}{5} = 1$



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Note that rotation by $\frac{\pi}{2}$ swaps semi-axes of the ellipse, giving the equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b > 0.$$

Example 2.8. To see that

$$xy = c, \quad c \text{ constant}$$

describes a hyperbola, apply the rotation *clockwise* by $\frac{\pi}{4}$ (i.e., use the rotation matrix R_α with $\alpha = -\frac{\pi}{4}$ or the rotation matrix $R_{(-\alpha)}$ with $\alpha = \frac{\pi}{4}$; we shall do the former.) Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = R_{-\frac{\pi}{4}} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X\cos\left(-\frac{\pi}{4}\right) + Y\sin\left(-\frac{\pi}{4}\right) \\ -X\sin\left(-\frac{\pi}{4}\right) + Y\cos\left(-\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y \\ \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y \end{bmatrix}$$

And the equation $xy = c$ becomes

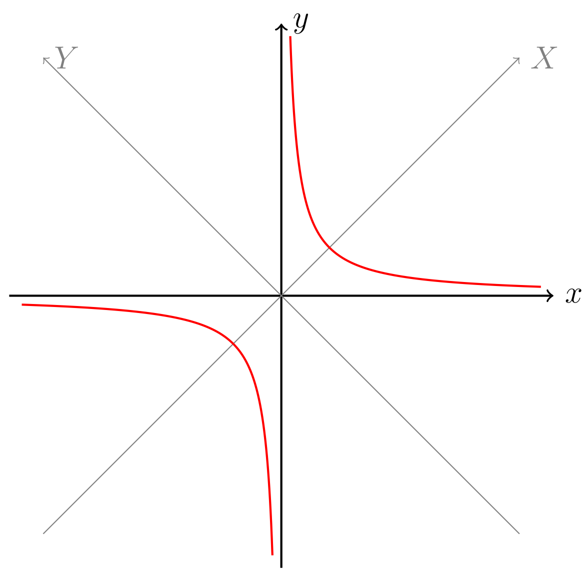
$$\begin{aligned} \frac{\sqrt{2}}{2}(X - Y) \cdot \frac{\sqrt{2}}{2}(X + Y) &= c \\ \frac{X^2}{2c} - \frac{Y^2}{2c} &= 1 \end{aligned}$$

This is the standard form (for $c > 0$). This particular type of hyperbola (with semi-axes of equal length) is called *rectangular hyperbola* and has perpendicular asymptotes.

In terms of x and y,

$$\frac{X^2}{2c} - \frac{Y^2}{2c} = 1$$

is a hyperbola, rotated by $\dfrac{\pi}{4}$ radians anti-clockwise.
Can you *show* that its asymptotes are the coordinate axes x and y?



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Note that rotation by $\frac{\pi}{2}$ swaps semi-axes of the hyperbola, giving the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \quad a, b > 0.$$