A Characterization of Quadrics Among Affine Hyperspheres by Section-Centroid Location

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July 2025

Abstract

A theorem of Meyer and Reisner characterizes ellipsoids by the collinearity of centroids of parallel sections: if $K \subset \mathbb{R}^n$ is a convex body such that for every $(n-1)$–dimensional subspace $M \subset \mathbb{R}^n$ the centroids of the sections $(x+M)\cap K$ are collinear (as x ranges over those translates with $(x+M)\cap \operatorname{int} K \neq \varnothing$), then K is an ellipsoid.

The present paper investigates how far this *section–centroid collinearity* property extends to *unbounded* convex sets and to hypersurfaces in affine differential geometry. In particular, we show that among affine hyperspheres, precisely the ellipsoids, paraboloids and one sheet of a two-sheeted hyperboloid satisfy the section–centroid collinearity property. We also explore conditions under which a convex hypersurface with this property must be a quadric.

1 Introduction and main result

Let $K \subset \mathbb{R}^n$ be a convex body. A classical theorem of Blaschke asserts that if, for every direction, the midpoints of parallel chords of K lie in a common hyperplane, then K is an ellipsoid. Meyer and Reisner later recast and strengthened this statement as follows: if for every $(n-1)$–dimensional subspace $M \subset \mathbb{R}^n$, the centroids of the affine sections $(x+M)\cap K$ are collinear (as x ranges over all translates for which the section meets $\operatorname{int} K$), then—and only then—K is an ellipsoid.

The present paper investigates to what extent this *section–centroid collinearity* property (SCCP) characterizes *unbounded* convex sets. In particular, we show that among affine hyperspheres, precisely the ellipsoids, paraboloids and the one sheet of a two-sheeted hyperboloid satisfy the SCCP. Finally we explore some constructions that aim to better understand the conditions under which surfaces with SCCP are affine hyperspheres. Affine spheres were introduced by Țițeica [10–12] and were later studied by Blaschke, Calabi, and Cheng–Yau. A basic classification according to affine mean curvature may be summarized as follows: affine hyperspheres with positive affine mean curvature are ellipsoids; those with zero affine mean curvature are elliptic paraboloids; and the class with negative affine mean curvature is much richer, since for any proper convex cone there exists an affine hypersphere asymptotic to it. Identifying quadrics within this last class has been of interest; for example, Pick and Berwald proved that vanishing cubic form implies that the hypersphere is locally an open subset of a quadric; see [9].

A recurring theme is the classical relationship between centroids of parallel sections and the *affine normal*. Infinitesimally, the affine normal line at a point of a smooth strictly convex hypersurface is tangent to the locus of centroids of nearby parallel sections; thus, when the family of section centroids in a fixed normal direction is collinear, its direction agrees with the affine normal direction. This provides the bridge between centroid geometry and equi-affine surface theory that we exploit below.

Our main result is the following classification among affine hyperspheres.

Theorem 1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a strictly convex domain with C^3 boundary $\partial\Omega$, equipped with its Blaschke (equi-affine) normalization, so that $\partial\Omega$ is an affine hypersphere. Assume that for every $(n-1)$–dimensional linear subspace $M \subset \mathbb{R}^{n+1}$ such that the section $(x+M)\cap\Omega$ is bounded, the centroids of the sections $(x+M)\cap\Omega$ lie on a single affine line as x varies over all translates for which the section meets* $\operatorname{int}\Omega$*. Then $\partial\Omega$ is an open subset of one of the following quadrics:*

- *an ellipsoid,*
- *a paraboloid, or*
- *one sheet of a two-sheeted hyperboloid.*

This result is a direct consequence of the work of Meyer–Reisner [8] and Kim [4, 5]. We prove a small extension of this theorem in which we replace the affine hypersphere condition by the assumption of asymptotic convergence to a cone. Theorem 1 should be compared with the Meyer–Reisner theorem: in their setting the convex set is bounded, and so $(x+M)\cap\Omega$ is always bounded. Conveniently, for convex unbounded sets, if $(x+M)\cap\Omega$ is bounded and nonempty, then for every translate $y+M$ that meets Ω the section $(y+M)\cap\Omega$ is also bounded.

It is natural to ask to what extent the affine hypersphere assumption can be removed.

Question 2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a strictly convex domain with smooth boundary $\partial\Omega$, and suppose that for every $(n-1)$–dimensional subspace $M \subset \mathbb{R}^{n+1}$ such that $(x+M)\cap\Omega$ is bounded, the centroids of the sections $(x+M)\cap\Omega$ are collinear as x varies over all translates with $(x+M)\cap\operatorname{int}\Omega \neq \varnothing$. Must $\partial\Omega$ be a connected component of an ellipsoid, paraboloid, or one sheet of a two-sheeted hyperboloid?*

Organization. In the rest of this section we fix notation and recall basic material on recession cones, Gauss maps and equi-affine geometry. We then introduce the SCCP. In Section 2 we adapt several lemmas of Meyer and Reisner to the unbounded case and derive a volume–cut functional whose level sets determine our initial set under some homothety. We connect this to work in [4] and [5] to prove Theorem 1 and its extension Lemma 7. Finally, in the last section we discuss the remaining open cases needed for a complete classification of convex hypersurfaces with SCCP.

Notation and basic objects

Throughout, $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$, $\|\cdot\|$ the Euclidean norm, and $\mathcal{H}^k$ the k–dimensional Hausdorff measure. The unit sphere in $\mathbb{R}^n$ is denoted S^{n-1}. The interior and boundary of a set A are denoted $\operatorname{int} A$ and ∂A, respectively.

Hyperplanes and sections. For $u \in S^{n-1}$ and $t \in \mathbb{R}$, define the hyperplane

$$\Pi(u,t) := \{x \in \mathbb{R}^n : \ \langle u, x\rangle = t\},$$

and, for any $X \subset \mathbb{R}^n$,

$$\Sigma(u,t;X) := \Pi(u,t) \cap X.$$

When $0 < \mathcal{H}^{n-1}(\Sigma(u,t;X)) < \infty$, the centroid of the section is

$$\mathrm{cen}(u,t;X) := \frac{1}{\mathcal{H}^{n-1}(\Sigma(u,t;X))} \int_{\Sigma(u,t;X)} x \, \mathrm{d}\mathcal{H}^{n-1}(x) \in \Pi(u,t).$$

Support and polar functionals. For a convex body $K \subset \mathbb{R}^n$ with $0 \in \operatorname{int} K$, the Minkowski functional and support function are

$$\|x\|_K := \inf\{\lambda > 0 : \ x \in \lambda K\}, \qquad h_K(u) := \sup_{y \in K} \langle u, y \rangle \quad (u \in \mathbb{R}^n).$$

The polar body is $K^\circ := \{x \in \mathbb{R}^n : \ \langle x, y \rangle \le 1 \ \forall\, y \in K\}$.

Recession cone. For a convex (not necessarily bounded) set $\Omega \subset \mathbb{R}^n$, the recession cone is

$$\mathrm{rec}(\Omega) := \{d \in \mathbb{R}^n : \ x + td \in \Omega \ \ \forall x \in \Omega, \ t \ge 0\}.$$

It is a closed convex cone, and we denote its linear dimension by $\dim \mathrm{rec}(\Omega)$.

Convergence to cones

We now use two different notions of convergence of an unbounded convex hypersurface to a cone. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed, strictly convex set with smooth boundary $\partial\Sigma$ an n–dimensional hypersurface, and let $C \subset \mathbb{R}^{n+1}$ be the boundary of a closed convex cone with apex at the origin, i.e., $\lambda C = C$ for all $\lambda > 0$.

For $R > 0$, write $S_R := \{x \in \mathbb{R}^{n+1} : \|x\| = R\}$ and let S^n be the unit sphere in $\mathbb{R}^{n+1}$ with its intrinsic metric d_{S^n}.

Definition 1 (Blow-down convergence). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a nonempty closed convex set with nonempty interior, and fix some $x_0 \in \Sigma$. The* asymptotic cone *(or* blow-down cone*) of Σ is the Painlevé–Kuratowski limit*

$$\Sigma_\infty \ := \ \lim_{t \to \infty} t^{-1}(\Sigma - x_0),$$

whenever this limit exists. It is well known that for closed convex sets this limit always exists, is independent of the choice of x_0, and coincides with the classical recession cone $\mathrm{rec}(\Sigma)$*, see for instance [1, Chap. 2] or [3].*

Definition 2 (Asymptotic convergence). *For each $R > 0$, equip S_R with its induced geodesic metric d_{S_R}. We say that $\partial\Sigma$* is asymptotic to C *if*

$$d_H^{(S_R)}\big(\partial\Sigma \cap S_R, \ C \cap S_R\big) \xrightarrow[R \to \infty]{} 0.$$

where $d_H^{(S_R)}$ denotes the Hausdorff distance with respect to the metric d_{S_R} on S_R.

This definition is the one used in the Calabi conjecture [2] and [6] and is not to be confused with the asymptotic cone which is equivalent to the recession cone.

If Σ is asymptotically convergent to a cone $\mathcal{C}$ then this cone must be $\mathrm{rec}(\Sigma)$. So the asymptotic and blow-down limits when they exist are equal.

It is possible however for a set with a blow-down limit to not have any asymptotic limit; see Figure 1 where the recession cone of the epigraph of f is a translate of the second quadrant, but the graph of f is not asymptotic to any translate of this cone.

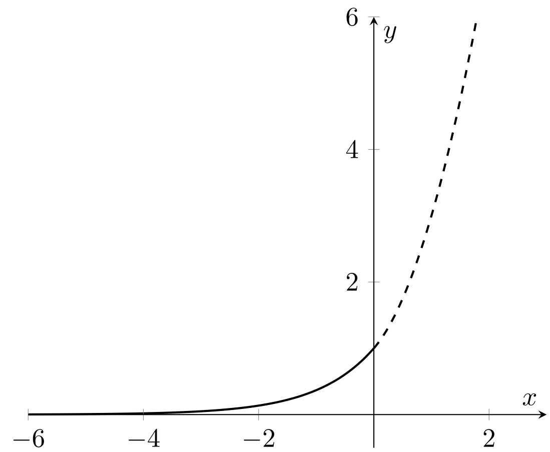

Figure 1: The function $f(x)$ defined by $\text{f(x)}=\begin{cases} e^x, & x \le 0, \\ x^2+x+1, & x > 0. \end{cases}$

Gauss map and centroid loci

Let $\Omega \subset \mathbb{R}^n$ have C^2 strictly convex boundary $\partial\Omega$. The (outer) Euclidean Gauss map

$$N : \partial\Omega \to S^{n-1}, \qquad N(x) = \text{outer unit normal at } x,$$

is well defined and a homeomorphism onto its image. The image $U := N(\partial\Omega) \subset S^{n-1}$ will be the set of admissible normals.

Section–centroid loci. Fix $u \in U$. Define

$$I(u) := \left\{ t \in \mathbb{R} : \ 0 < \mathcal{H}^{n-1}(\Sigma(u,t;\Omega)) < \infty \right\},$$

and the *centroid curve*

$$\gamma_u \ := \ \{\text{cen}(u,t;\Omega) : \ t \in I(u)\} \ \subset \mathbb{R}^n.$$

When γ_u is contained in some affine line ℓ_u, we call ℓ_u the *centroid line in the direction* u. When the underlying set is important we write ℓ_u^Ω.

Cut–volume functional. For $a \in \mathbb{R}^n \setminus \{0\}$, write

$$H(a) := \{x \in \mathbb{R}^n : \ \langle a, x\rangle = 1\}, \qquad C(a)^{\pm} := \{x \in \mathbb{R}^n : \ \pm\langle a, x\rangle \ge 1\},$$

and, for a measurable $X \subset \mathbb{R}^n$,

$$V_X(a) := \mathcal{H}^n\big(X \cap C(a)^-\big).$$

When the set is clear from context we simply write $V(a)$.

Section–centroid collinearity and affine geometry

We now formalize the collinearity property and recall the equi-affine setup.

Definition 3 (Section–centroid collinearity property (SCCP)). *A closed convex set $\Omega \subset \mathbb{R}^n$ has the* SCCP *if for every $(n-1)$–dimensional linear subspace $M \subset \mathbb{R}^n$ the following holds: for all x with $0 < \mathcal{H}^{n-1}((x+M)\cap\Omega) < \infty$, the centroids of the sections $(x+M)\cap\Omega$ lie on a single affine line. Equivalently, for each $u \in U := N(\partial\Omega)$ the centroid curve γ_u is contained in some affine line ℓ_u.*

Definition 4 (Affine differential setup). *Let M^n be a C^3 strictly convex hypersurface in $\mathbb{R}^{n+1}$ equipped with its equi-affine (Blaschke) normalization. Denote by ξ the Blaschke normal field, by h the Blaschke metric, by ∇ the Blaschke connection, and by S the affine shape operator, so that the equi-affine Gauss–Weingarten equations hold:*

$$D_X(Y_*) = (\nabla_X Y)_* + h(X,Y)\,\xi, \qquad D_X\xi = -\,(SX)_*,$$

for all $X, Y \in \Gamma(TM)$, where D is the ambient flat connection and $(\cdot)_*$ denotes the tangent push-forward along the immersion. The affine normal line *at a point is the line spanned by ξ at that point.*

Definition 5 (Affine hypersphere). *A strictly convex Blaschke immersion is an* affine hypersphere *if its affine normals are concurrent or all parallel; equivalently, its affine shape operator satisfies*

$$S = \lambda\,\mathrm{Id}$$

for some constant $\lambda \in \mathbb{R}$.

2 Auxiliary lemmas and volume cut functionals

In this section we adapt two lemmas from [8] to unbounded convex sets and derive a key consequence: under SCCP, certain cut volumes determined by support hyperplanes are constant. This will be the main tool in the classification of affine hyperspheres with SCCP. The adated proofs are nearly identical to the originals but we include them for completeness.

2.1 Differentiability of the cut-volume functional

The following lemma is a variant of Lemma 5 in [8], adapted to cuts by half-spaces of the form $C(a)^-$.

Lemma 3. *Let $K \subset \mathbb{R}^n$ be a convex unbounded set with $0 \notin K$. Consider the cut-volume functional*

$$V(a) := \mathcal{H}^n\big(K \cap C(a)^-\big), \qquad a \in \mathbb{R}^n \setminus \{0\},\ \textit{such that } V(a) \textit{ is finite}$$

where $C(a)^- = \{x :\ \langle a, x\rangle \le 1\}$. Then V is C^1 at every a for which $0 < V(a) < \infty$. For such a, denote by $x(a)$ the centroid of the (bounded) section $H(a)\cap K$, where $H(a) = \{x : \langle a, x\rangle = 1\}$. Then

$$x(a) = \frac{\nabla V(a)}{\langle a, \nabla V(a)\rangle}.$$

Proof. Since $K \cap C(a)^-$ is convex with nonempty interior and $0 < V(a) < \infty$, it must be bounded; otherwise a convex unbounded set with nonempty interior would have infinite volume. In particular, $H(a) \cap K$ is bounded and has finite $(n-1)$–dimensional measure.

Let $\mathrm{rec}(K)$ denote the recession cone of K. If $v \in \mathrm{rec}(K) \setminus \{0\}$ and $\langle a, v\rangle \le 0$, then for any $x_0 \in K \cap C(a)^-$ we would have $x_0 + tv \in K \cap C(a)^-$ for all $t \ge 0$, so $K \cap C(a)^-$ would be unbounded, a contradiction. Hence

$$\langle a, v\rangle > 0 \qquad \forall\, v \in \mathrm{rec}(K) \setminus \{0\}. \tag{1}$$

By compactness of $\mathrm{rec}(K) \cap S^{n-1}$ there exists $\alpha > 0$ such that

$$\langle a, v\rangle \ge \alpha > 0 \qquad \forall\, v \in \mathrm{rec}(K) \cap S^{n-1}. \tag{2}$$

It follows that there exist $R > 0$ and $\varepsilon_0 > 0$ such that for all ε with $|\varepsilon| \le \varepsilon_0$ and every coordinate direction e_j,

$$K \cap C(a + \varepsilon e_j)^- \subset B_R(0). \tag{3}$$

Indeed, if (3) failed, there would be a sequence $x_k \in K \cap C(a + \varepsilon_k e_j)^-$ with $\|x_k\| \to \infty$ and $\varepsilon_k \to 0$, and after normalizing we would obtain a direction $d \in \mathrm{rec}(K) \cap S^{n-1}$ with $\langle a, d\rangle \le 0$, contradicting (1). Thus all cuts $K \cap C(a + \varepsilon e_j)^-$ for small ε lie in a fixed ball $B_R(0)$.

Step 1: One-sided derivative in a coordinate direction. Fix $j \in \{1, \dots, n\}$ and consider

$$D_j(\varepsilon) := V(a + \varepsilon e_j) - V(a), \qquad \varepsilon \in \mathbb{R}.$$

We compute the derivative at $\varepsilon = 0$ for $j = 1$; the other coordinates are analogous, and the argument for $\varepsilon < 0$ is the same as for $\varepsilon > 0$.

Write

$$\begin{aligned} K_+(\varepsilon) &:= \{x \in K :\ \langle a, x\rangle > 1,\ \langle a + \varepsilon e_1, x\rangle \le 1\}, \\ K_-(\varepsilon) &:= \{x \in K :\ \langle a, x\rangle \le 1,\ \langle a + \varepsilon e_1, x\rangle > 1\}, \end{aligned}$$

so that

$$D_1(\varepsilon) = |K_+(\varepsilon)| - |K_-(\varepsilon)|.$$

Both $K_+(\varepsilon)$ and $K_-(\varepsilon)$ lie in the slab between the hyperplanes $H(a)$ and $H(a + \varepsilon e_1)$, hence in the bounded set $K \cap \big(C(a)^- \cup C(a + \varepsilon e_1)^-\big) \subset B_R(0)$ for $|\varepsilon| \le \varepsilon_0$ by (3).

Let P denote orthogonal projection onto $H(a)$ and, for a point x, let $Q_\varepsilon x$ be the intersection of the line $x + \mathbb{R}a$ with $H(a + \varepsilon e_1)$. A computation in the 2–plane spanned by a and e_1 shows that for $x \in H(a)$,

$$\|x - Q_\varepsilon x\| = \frac{|\varepsilon|}{\|a\|}\,|x_1| + O(\varepsilon^2) \qquad (\varepsilon \to 0), \tag{4}$$

and the segment $[x, Q_\varepsilon x]$ is orthogonal to $H(a)$ up to an error of order $O(\varepsilon)$.

Define

$$U(\varepsilon) := H(a) \cap K \cap P\big(H(a + \varepsilon e_1) \cap K\big), \qquad V(\varepsilon) := \{x \in H(a) :\ [x, Q_\varepsilon x] \cap K \neq \emptyset\}.$$

Each point of $K_+(\varepsilon)$ lies on a segment $[x, Q_\varepsilon x]$ with $x \in U(\varepsilon)$, and conversely each such segment contributes a sliver of $K_+(\varepsilon)$ between the two hyperplanes. By Fubini along these segments, together with (4), yields (for $\varepsilon > 0$)

$$\frac{\varepsilon}{\|a\| + c_1 \varepsilon} \int_{U(\varepsilon) \cap \{x_1 \ge 0\}} x_1 \, d\sigma(x) \ \le\ |K_+(\varepsilon)| \ \le\ \frac{\varepsilon}{\|a\| + c_2 \varepsilon} \int_{V(\varepsilon) \cap \{x_1 \ge 0\}} x_1 \, d\sigma(x), \tag{5}$$

for some constants c_1, c_2 independent of small ε. An analogous estimate holds for $K_-(\varepsilon)$ with $x_1 \le 0$.

Because $K \cap C(a)^-$ is compact, the sets $H(a) \cap K$ and $H(a+\varepsilon e_1) \cap K$ converge in the Hausdorff metric as $\varepsilon \to 0$, and hence

$$U(\varepsilon), V(\varepsilon) \xrightarrow[\varepsilon \to 0]{d_H} H(a) \cap K.$$

Since $H(a) \cap K$ is compact, the function $x \mapsto x_1$ is bounded on all these sets. Letting $\varepsilon \downarrow 0$ in (5) and using dominated convergence, we obtain

$$\lim_{\varepsilon \downarrow 0} \frac{|K_+(\varepsilon)|}{\varepsilon} = \frac{1}{\|a\|} \int_{H(a)\cap K,\ x_1 \ge 0} x_1 \, d\sigma(x).$$

A completely similar argument for $K_-(\varepsilon)$ (considering $x_1 \le 0$) gives

$$\lim_{\varepsilon \downarrow 0} \frac{|K_-(\varepsilon)|}{\varepsilon} = \frac{1}{\|a\|} \int_{H(a)\cap K,\ x_1 \le 0} x_1 \, d\sigma(x).$$

Subtracting the two limits, we obtain

$$\frac{\partial V}{\partial a_1}(a) = \lim_{\varepsilon \to 0} \frac{D_1(\varepsilon)}{\varepsilon} = \frac{1}{\|a\|} \int_{H(a)\cap K} x_1 \, d\sigma(x).$$

Repeating the same computation for all coordinate directions e_j shows that

$$\nabla V(a) = \frac{1}{\|a\|} \int_{H(a)\cap K} x \, d\sigma(x).$$

Thus $\nabla V(a)$ is a nonzero scalar multiple of the first moment of the section $H(a) \cap K$:

$$\nabla V(a) = \gamma(a) \int_{H(a)\cap K} x \, d\sigma(x), \qquad \gamma(a) = \frac{1}{\|a\|} \neq 0. \tag{6}$$

In particular, V is differentiable at a, and the above formula shows that V is actually C^1 in a neighborhood of a (the dependence on a is continuous by the same argument).

Step 2: Centroid formula. By definition,

$$x(a) = \frac{\displaystyle\int_{H(a)\cap K} x \, d\sigma(x)}{\displaystyle\int_{H(a)\cap K} 1 \, d\sigma(x)}.$$

Taking the inner product of (6) with a and using $\langle a, x\rangle = 1$ on $H(a)$, we obtain

$$\langle a, \nabla V(a)\rangle = \gamma(a) \int_{H(a)\cap K} \langle a, x\rangle \, d\sigma(x) = \gamma(a) \int_{H(a)\cap K} 1 \, d\sigma(x).$$

Hence

$$\int_{H(a)\cap K} x \, d\sigma(x) = \frac{1}{\gamma(a)} \nabla V(a), \qquad \int_{H(a)\cap K} 1 \, d\sigma(x) = \frac{1}{\gamma(a)} \langle a, \nabla V(a)\rangle .$$

Dividing these two identities gives

$$x(a) = \frac{\nabla V(a)}{\langle a, \nabla V(a)\rangle},$$

which is the desired formula.

$\square$

2.2 Floating body characterization

We next adapt Lemma 7 of [8].

Lemma 4. *Let $K \subset \mathbb{R}^n$ and $B \subset K$ both be unbounded strictly convex sets with $0 \notin \operatorname{int} B$. Assume that for every support hyperplane H of B, the centroid of $H \cap K$ exists and belongs to B. Suppose further that for every $a \in \mathbb{R}^n \setminus \{0\}$ such that $H(a)$ is a support hyperplane of B and $B \subset C(a)^-$, we have*

$$0 < V(a) < \infty, \qquad V(a) := \mathcal{H}^n(K \cap C(a)^-).$$

Then there exists a constant $c > 0$ such that

$$V(a) = c \qquad \textit{for all such } a.$$

Geometric preliminaries. Let a be as in the statement, so that $H(a)$ supports B and $B \subset C(a)^-$. We record two consequences of strict convexity of B.

(i) *Unique contact point.* Strict convexity of B implies that the support hyperplane $H(a)$ meets B at a unique point, denoted $b(a) \in \partial B$. Thus $H(a) \cap B = \{b(a)\}$ and $\langle a, b(a)\rangle = 1$.

(ii) *Tangent space to the support parameter set.* Let

$$\mathcal{S} := \{a \in \mathbb{R}^n \setminus \{0\} : \ B \subset C(a)^-, \ H(a) \text{ supports } B\}.$$

Since $B = \bigcap_{a \in \mathcal{S}} C(a)^-$ and the active inequality at a is $\langle a, b(a)\rangle \geq 1$ with equality at the unique contact point, the tangent space of $\mathcal{S}$ at a satisfies

$$T_a\mathcal{S} \subset \{v \in \mathbb{R}^n : \ \langle v, b(a)\rangle = 0\}. \tag{7}$$

In other words, $b(a)$ is normal to the hypersurface of parameters realizing the same support point.

Proof of Lemma 4. Fix $a \in \mathcal{S}$ with $0 < V(a) < \infty$. By hypothesis, the centroid of $H(a) \cap K$ lies in B, and since it also lies on $H(a)$ we must have

$$x(a) = b(a). \tag{8}$$

Indeed, by strict convexity $H(a) \cap B = \{b(a)\}$, so the only point of B lying on $H(a)$ is $b(a)$.

By Lemma 3 and (8), the gradient of V at a is parallel to $b(a)$:

$$\nabla V(a) = \lambda(a)\, b(a) \qquad \text{with} \quad \lambda(a) := \langle a, \nabla V(a)\rangle \neq 0. \tag{9}$$

Let $v \in T_a\mathcal{S}$. Using (7) and (9),

$$D_v V(a) = \langle \nabla V(a), v\rangle = \lambda(a)\ \langle b(a), v\rangle = 0.$$

Thus all directional derivatives of V along $T_a\mathcal{S}$ vanish at a. Since a was arbitrary, V is locally constant on $\mathcal{S}$.

The set $\mathcal{S}$ of support parameters for a strictly convex body is connected (because the Gauss map of ∂B is continuous and surjective onto the set of outer unit normals). Hence local constancy implies global constancy: there exists $\alpha > 0$ such that

$$V(a) = \alpha \qquad \text{for all } a \in \mathcal{S} \text{ with } 0 < V(a) < \infty.$$

This is exactly the desired conclusion. $\square$

Proof of Theorem 1. An affine hypersphere has all affine normal lines either

(i) parallel to some vector e_n; or

(ii) concurrent through a single point.

In both cases (i) and (ii), we may construct B in the following way:

$$B = \begin{cases} \Omega + \lambda e_n & \text{in case (i),} \\ \lambda\,\Omega & \text{in case (ii),} \end{cases}$$

For each support hyperplane P_u of B with normal $u \in U$, the contact point x_0 between P_u and B can be written as

$$x_0 = \begin{cases} N^{-1}(u) + \lambda e_n & \text{in case (i),} \\ \lambda\, N^{-1}(u) & \text{in case (ii),} \end{cases}$$

for some $\lambda > 0$, where $N^{-1}(u)$ denotes the point of $\partial\Omega$ with outer normal u. In particular, the centroid of the section $P_u \cap \Omega$ lies on the centroid line ℓ_u^{Ω}, hence lies in B by construction of B as a suitable translate or scaling of Ω.

Thus the hypotheses of Lemma 4 are satisfied with $K = \Omega$ and B as above. We conclude that there exists a constant $\alpha > 0$ such that $V(a) = \alpha$ for all support parameters a corresponding to $u \in N(\partial\Omega)$.

We recall a theorem which summarizes results from Proposition 1 of [4] and Theorem 5 of [5]

Theorem 5. *Let M be the epigraph of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and let $\Phi(x)$ be the plane tangent to $f(x)$. For $k \geq 0$ denote by $V_x(k)$ the volume of the region bounded between $\Phi(x) + k$ and M. For $k \geq 1$ denote by $V_x^*(k)$ the volume of the region bounded between $k\ \Phi(x)$ and M. If $V_x(k)$ is constant for every $x \in \mathbb{R}^n$ then M is an elliptic paraboloid. If $V_x^*(k)$ is constant for every $x \in \mathbb{R}^n$ then M is one sheet of a two-sheeted hyperboloid.*

The first case (concurrent centroid lines) corresponds to the situation described in Proposition 1 of [4], ensuring $\partial\Omega$ is one sheet of a two-sheeted hyperboloid.

In the second case (parallel centroid lines), by Theorem 5 of [5], $\partial\Omega$ must be an elliptic paraboloid.

Finally, if $\partial\Omega$ is closed and bounded, the conclusion that $\partial\Omega$ is an ellipsoid follows directly from the bounded Meyer–Reisner theorem [8]. $\square$

Notice that the proof relies on the centroid lines being either parallel or concurrent, and hence on $\partial\Omega$ being an affine hypersphere. In fact, since a compact $\partial\Omega$ has all centroid lines intersecting at its centroid, the classification of compact affine hyperspheres implies that $\partial\Omega$ must be an ellipsoid. Similarly, when all centroid lines are parallel, $\partial\Omega$ must be a paraboloid. This result is most interesting in the hyperbolic case.

A resolution to question 2 follows from whether a convex hypersurface with SCCP is necessarily an affine hypersphere; this reduction is the motivation for the next sections.

3 Centroid lines for cones and recession geometry

In this section we analyze the behavior of centroid lines "at infinity". We find that for strictly convex sets with SCCP, centroid lines of Ω and of its recession cone $\mathrm{rec}(\Omega)$ have the same direction. If in addition $\partial\Omega$ is asymptotic to $\mathrm{rec}(\Omega)$ then the centroid lines of Ω and of its recession cone $\mathrm{rec}(\Omega)$ are the same. $\partial\Omega$ must then be an affine hypersphere and therefore quadric.

Lemma 6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a strictly convex, unbounded domain with C^2 boundary, and assume that Ω has the section–centroid collinearity property (SCCP). Let $\mathcal{C} := \mathrm{rec}(\Omega)$ be its (nontrivial) recession cone. Fix $u \in U := N(\partial\Omega)$, and denote by ℓ_u^Ω the centroid line associated to Ω and by $\ell_u^{\mathcal{C}}$ the centroid line associated to $\mathcal{C}$. Then*

$$\ell_u^\Omega \parallel \ell_u^{\mathcal{C}}.$$

Proof. By translating Ω if necessary, we may assume that $0 \in \Omega$. For $u \in U$ and $t \in \mathbb{R}$ such that the section

$$\Sigma(u,t;\Omega) := \Pi(u,t) \cap \Omega$$

has finite $(n-1)$–dimensional measure, let

$$\lambda_u(t) := \mathrm{cen}(u,t;\Omega)$$

denote its centroid. By SCCP, λ_u depends affinely on t:

$$\lambda_u(t) = q_u + t\,w_u, \qquad t \in I(u),$$

for some $q_u, w_u \in \mathbb{R}^n$ with $\langle u, w_u\rangle = 1$; the condition $\langle u, w_u\rangle = 1$ simply expresses that $\lambda_u(t) \in \Pi(u,t)$.

For the cone $\mathcal{C} = \mathrm{rec}(\Omega)$, consider the sections

$$\Sigma_{\mathcal{C}}(u,t) := \Pi(u,t) \cap \mathcal{C}.$$

Whenever these have finite $(n-1)$–dimensional measure, their centroids $\lambda_u^{\mathcal{C}}(t)$ lie on a line $\ell_u^{\mathcal{C}}$ passing through the apex of $\mathcal{C}$. Since $\mathcal{C}$ is a cone, the sections $\Sigma_{\mathcal{C}}(u,t)$ are homothetic as t varies, and their centroids scale linearly:

$$\lambda_u^{\mathcal{C}}(t) = t\,w_u^{\mathcal{C}},$$

for some $w_u^{\mathcal{C}} \in \mathbb{R}^n$ with $\left\langle u, w_u^{\mathcal{C}}\right\rangle = 1$.

Now, let $R > 0$ and consider the scaled sets

$$\Omega_R := \frac{1}{R}\Omega.$$

By [1], the asymptotic cone

$$\Omega_\infty := \lim_{R\to\infty} \frac{1}{R}\Omega$$

exists and coincides with the recession cone:

$$\Omega_\infty = \mathrm{rec}(\Omega) = \mathcal{C}.$$

In particular, for any fixed $t > 0$ and $u \in N(\partial\Omega)$,

$$\Omega_R \cap \Pi(u,t) \xrightarrow[R\to\infty]{d_H} \mathcal{C} \cap \Pi(u,t)$$

in Hausdorff distance inside the hyperplane $\Pi(u,t)$. Since each slice is a bounded convex subset of $\Pi(u,t)$ with positive finite $(n-1)$–measure, convergence in Hausdorff distance implies convergence of centroids:

$$\mathrm{cen}\big(u,t;\Omega_R\big) \xrightarrow[R\to\infty]{} \mathrm{cen}\big(u,t;\mathcal{C}\big) = \lambda_u^{\mathcal{C}}(t) = t\,w_u^{\mathcal{C}}.$$

On the other hand, the section of Ω_R at level t is just a rescaled section of Ω at level Rt:

$$\Omega_R \cap \Pi(u,t) = \frac{1}{R}\big(\Omega \cap \Pi(u,Rt)\big),$$

so its centroid is

$$\mathrm{cen}\big(u,t;\Omega_R\big) = \frac{1}{R}\,\lambda_u(Rt) = \frac{1}{R}\big(q_u + Rt\,w_u\big) = \frac{1}{R}q_u + t\,w_u.$$

Letting $R \to \infty$ we obtain

$$\mathrm{cen}\big(u,t;\Omega_R\big) \xrightarrow[R\to\infty]{} t\,w_u.$$

Combining the two limits, we get

$$t\,w_u = t\,w_u^{\mathcal{C}} \qquad \text{for all } t > 0,$$

hence $w_u = w_u^{\mathcal{C}}$. Therefore the direction vector of the centroid line ℓ_u^{Ω} coincides with that of $\ell_u^{\mathcal{C}}$, and

$$\ell_u^{\Omega} \parallel \ell_u^{\mathcal{C}}.$$

□

Lemma 7. *Let $\Omega \subset \mathbb{R}^n$ be a strictly convex, unbounded domain with C^2 boundary. Assume either that*

1. *Ω is asymptotic to a cone or,*
2. *$\mathrm{rec}(\Omega)$ is a single ray*

Then $\partial\Omega$ is a connected component of an ellipsoid, a paraboloid, or a two-sheeted hyperboloid.

Proof. First note that the direction of l_u must lie in $\operatorname{rec}\Omega$ or else l_u would intersect $\partial\Omega$ at two distinct points which must then have antipodal normal vectors; a contradiction with the strict convexity of $\partial\Omega$.

In the first case, the cone to which $\partial\Omega$ is asymptotic must be a translate of $\operatorname{rec}(\Omega)$ and must have full $(n+1)$–dimensional span.

Using the same notation as the previous lemma, $\partial\Omega$ converges to $\partial\mathcal{C}$ asymptotically in directions in which the recession cone is nondegenerate, and thus the sections $\Sigma(u,t;\Omega)$ converge to $\Sigma_{\mathcal{C}}(u,t)$ as $t \to \infty$. In particular, their centroids satisfy

$$\lambda_u(t) - \lambda_u^{\mathcal{C}}(t) \to 0 \text{ as } t \to \infty,$$

Therefore ℓ_u^Ω equals $\ell_u^{\mathcal{C}}$ and all lines of centroids are concurrent. If $\operatorname{rec}(\Omega)$ is a single ray then all ℓ_u^Ω must have the same direction and are therefore parallel. We then have two cases:

1. The centroid lines ℓ_u^Ω are all concurrent at a point $p \in \mathbb{R}^{n+1}$; or
2. The centroid lines ℓ_u^Ω are all parallel to a fixed direction.

Thus $\partial\Omega$ is an affine hypersphere and we may apply Theorem 1 to reach the desired conclusion. □

4 Comparison with the asymptotic affine hypersphere

After analyzing the behavior of centroid lines "at infinity" we now analyze them on the surface and compare them to the affine normals of the hypersphere asymptotic to $\operatorname{rec}\Omega$. We recall some concepts from affine differential geometry and refer to [7] for a more thorough survey.

Assume that $\partial\Omega$ is a strictly convex hypersurface with nondegenerate recession cone $\mathcal{C} := \operatorname{rec}(\Omega)$. By the solution of the Calabi conjecture for affine spheres, there exists a hyperbolic affine hypersphere that is the boundary of a convex set Y asymptotic to $\mathcal{C}$ ([2] introduces the Calabi conjecture and [6] presents an outline of the solution). In particular,

$$\operatorname{rec}(Y) = \operatorname{rec}(\Omega).$$

Moreover, the Euclidean Gauss images of $\partial\Omega$ and ∂Y coincide: the set U of normals to $\partial\Omega$ is determined solely by the recession cone, and the same holds for ∂Y. Thus $\partial\Omega$ and ∂Y share the same Gauss image U.

By Lemma 6, for each $u \in U$ the centroid line ℓ_u^Ω of Ω has the same direction as the centroid line of the cone $\mathcal{C}$, which in turn coincides with the affine normal direction of the asymptotic affine hypersphere Y at the point with normal u. Infinitesimally this implies that the affine normal fields of Y and $\partial\Omega$ are parallel at corresponding points.

To make this precise, let U be a smooth n–manifold parametrizing normal directions (e.g., $U \subset S^n$). Let $f, g : U \to \mathbb{R}^{n+1}$ be smooth, strictly convex, nondegenerate equi-affine (Blaschke) hypersurface immersions with the *same* affine normal field

$$\xi : U \to \mathbb{R}^{n+1},$$

and the *same* Gauss image U (so both f and g are given in support parametrization by the normal direction $u \in U$).

Let D denote the standard flat connection on $\mathbb{R}^{n+1}$. The equi–affine structure equations (Gauss–Weingarten) for f are

$$D_X f_* Y = f_*\big(\nabla^f_X Y\big) + h_f(X,Y)\,\xi, \tag{GW$_f$-1}$$

$$D_X \xi = -\, f_*\big(S^f X\big), \tag{GW$_f$-2}$$

for all $X, Y \in \Gamma(TU)$, where

- ∇^f is the induced torsion–free affine connection on TU,
- h_f is the Blaschke metric (a positive definite symmetric 2–tensor on TU),
- $S^f : TU \to TU$ is the affine shape operator of f.

Assume also that there is a bundle isomorphism

$$L : TU \to TU$$

such that

$$g_* = f_* \circ L. \tag{10}$$

We want to express L in terms of the difference $g - f$ and the Blaschke data of f.

Define

$$w := g - f : U \to \mathbb{R}^{n+1}.$$

At each point $u \in U$, the tangent space of the ambient space decomposes as

$$T_{f(u)}\mathbb{R}^{n+1} \;\cong\; f_*(T_u U) \oplus \mathbb{R}\,\xi(u),$$

because f is an immersion and $\xi(u)$ is transversal. Therefore there exist unique

$$a(u) \in T_u U, \qquad \varphi(u) \in \mathbb{R}$$

such that

$$w(u) = g(u) - f(u) = f_*\big(a(u)\big) + \varphi(u)\,\xi(u). \tag{11}$$

Equivalently,

$$w = f_* a + \varphi\,\xi, \tag{12}$$

for unique $a \in \Gamma(TU)$ and $\varphi \in C^\infty(U)$.

Let $X \in \Gamma(TU)$ be any vector field. Differentiating $g = f + w$ along X using D and the definition of f_*, g_* gives

$$g_* X = D_X g = D_X f + D_X w = f_* X + D_X w. \tag{13}$$

On the other hand, by hypothesis (10),

$$g_* X = f_*(LX). \tag{14}$$

Combining (13) and (14), we obtain

$$f_*(LX) = f_* X + D_X w. \tag{15}$$

We now compute $D_X w$ using (12) and the structure equations (GW$_f$-1)–(GW$_f$-2).

First, using (GW_f-1) with $Y = a$ we get

$$D_X(f_*a) = f_*\big(\nabla^f_X a\big) + h_f(X,a)\,\xi. \tag{16}$$

Next,

$$D_X(\varphi\,\xi) = X(\varphi)\,\xi + \varphi\,D_X\xi = X(\varphi)\,\xi - \varphi\,f_*(S^f X), \tag{17}$$

using (GW_f-2).

Adding (16) and (17) yields

$$\begin{aligned} D_X w &= D_X(f_*a) + D_X(\varphi\,\xi) \\ &= f_*\big(\nabla^f_X a\big) + h_f(X,a)\,\xi + X(\varphi)\,\xi - \varphi\,f_*(S^f X) \\ &= f_*\big(\nabla^f_X a - \varphi\,S^f X\big) + \big(h_f(X,a) + X(\varphi)\big)\,\xi. \end{aligned} \tag{18}$$

Substituting (18) into (15), we obtain

$$f_*(LX) = f_*X + f_*\big(\nabla^f_X a - \varphi\,S^f X\big) + \big(h_f(X,a) + X(\varphi)\big)\,\xi. \tag{19}$$

Now decompose both sides of (19) into components tangent to $f_*(TU)$ and normal to it along ξ. Since the left-hand side $f_*(LX)$ has no ξ–component, and f_* is injective on TU, we must have:

$$\text{tangent part:}\quad LX = X + \nabla^f_X a - \varphi\,S^f X, \tag{T}$$

$$\text{normal part:}\quad h_f(X,a) + X(\varphi) = 0, \tag{N}$$

for all $X \in TU$.

Equation (N) can be rewritten as

$$X(\varphi) = -h_f(X,a) \qquad \forall X \in TU.$$

By the definition of the h_f–gradient $\operatorname{grad}_{h_f}\varphi$, characterized by

$$h_f(\operatorname{grad}_{h_f}\varphi, X) = X(\varphi) \quad \forall X,$$

we see that

$$h_f(\operatorname{grad}_{h_f}\varphi, X) = X(\varphi) = -\,h_f(X,a) = -\,h_f(a,X).$$

Since h_f is symmetric and nondegenerate, this implies

$$a = -\,\operatorname{grad}_{h_f}\varphi. \tag{20}$$

Substituting (20) into (T), we obtain the formula

$$LX = X + \nabla^f_X(-\operatorname{grad}_{h_f}\varphi) - \varphi\,S^f X \qquad \forall X \in TU. \tag{21}$$

Equations (20) and (21) express a and L in terms of the scalar function φ and the Blaschke data of f. Suppose now that f is a *proper affine hypersphere*, so that

$$S^f = H\,I$$

for some nonzero constant $H \in \mathbb{R}$. Then (21) simplifies to

$$LX = X + \nabla^f_X a - H\,\varphi\,X, \qquad a = -\operatorname{grad}_{h_f}\varphi. \tag{22}$$

If f and g are the support parametrizations of ∂Y and $\partial\Omega$, respectively, then the conditions under which $\partial\Omega$ is not asymptotic to any translate of $\operatorname{rec}(\Omega)$ should be related to Equation (22). For any affine hypersphere ∂Y centered at the vertex of the cone C it is asymptotic to, the affine mean curvature is 1, and affine normal vector to ∂Y at a point $f(u)$ is $f(u)$ after Blaschke normalization.

Therefore, for any sequence $\{y\}_n$ of points on ∂Y such that $\|y_n\| \to \infty$, the minimum distance d_n between $T_{y_n}\partial Y$ and $\xi(f^{-1}(y_n))$ goes to 0 w.r.t. the Euclidean metric.

If along some sequence $\{x\}_n \subset \partial\Omega$ with $f^{-1}(y_n) = g^{-1}(x_n)$ the minimum distance between $T_{y_n}\partial Y$ and $T_{x_n}\partial\Omega$ is bounded away from zero by a constant c, then

$$\varphi(f^{-1}(y_n)) \geq \frac{c}{d_n} \to \infty$$

hence $\varphi(f^{-1}(y_n)) \to \infty$ as $n \to \infty$.

5 Conclusion and open problems

Settling Question 2 amounts to answering whether the following dichotomy is true:

Let $\Omega \subset \mathbb{R}^{n+1}$ be connected with smooth strictly convex boundary $\partial\Omega$ and satisfying SCCP. Does necessarily

$$\bigcap_{u \in N(\partial\Omega)} \ell^\Omega_u = \{p\}$$

for some $p \in \mathbb{R}^{n+1}$, or else

$$\ell^\Omega_{u_1} \parallel \ell^\Omega_{u_2} \qquad \forall u_1, u_2 \in U\ ?$$

There are four subcases, two of which are settled, while two remain open.

Known cases

Case 1: full-dimensional recession cone, asymptotic to its cone. Assume that Ω has an $(n+1)$–dimensional recession cone $\mathcal{C} = \operatorname{rec}(\Omega)$ and that $\partial\Omega$ is asymptotic to $\mathcal{C}$. By Lemma 7, for each $u \in U$ the centroid line ℓ^Ω_u coincides with the centroid line of $\mathcal{C}$, so

$$\bigcap_{u \in N(\partial\Omega)} \ell^\Omega_u = \{p\},$$

where p is the vertex of $\operatorname{rec}(\Omega)$. In this case $\partial\Omega$ is asymptotic to a cone and shares its centroid lines.

Case 2: one-dimensional recession cone. If $\dim\operatorname{rec}(\Omega) = 1$, then $\operatorname{rec}(\Omega)$ is a ray,

$$\ell^\Omega_{u_1} \parallel \ell^\Omega_{u_2} \qquad \forall u_1, u_2 \in U,$$

and by Lemma 7 $\partial\Omega$ is a paraboloid.

Open cases

The remaining two cases are, to our knowledge, open and contain the core difficulty of classifying convex hypersurfaces with SCCP.

Case 3: full-dimensional recession cone, not asymptotic to the cone. Assume that $\mathrm{rec}(\Omega)$ has full dimension $(n+1)$ but that $\partial\Omega$ is *not* asymptotic to $\partial\,\mathrm{rec}(\Omega)$. By the Calabi conjecture there exists a hyperbolic affine hypersphere Y asymptotic to $\mathrm{rec}(\Omega)$, whose affine normal at the point with Euclidean normal u is parallel to ℓ_u^Ω. The crucial question is whether this forces $\partial\Omega$ itself to be an affine hypersphere. If so, Lemma 7 would apply and the classification would be complete in this case.

Case 4: intermediate-dimensional recession cone. Finally, suppose that $\mathrm{rec}(\Omega)$ has dimension m with $1 < m < n+1$. In this situation the SCCP implies that all centroid lines ℓ_u^Ω lie in a fixed m–dimensional subspace V, but neither concurrent nor parallel behavior is automatic. Understanding how the geometry of $\mathrm{rec}(\Omega)$ constrains the affine normal field and centroid lines of $\partial\Omega$ in this intermediate-dimensional regime is an interesting and challenging problem.

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