Exposition on Higher-Dimensional Hermite-Hadamard Inequality

Abstract

The Hermite-Hadamard inequality, originally formulated for convex functions on 1D intervals, has been extended to higher dimensions. This paper provides a summarized exposition of the proofs presented by Stefan Steinerberger [Steinerberg, 2020]. The discussion includes the necessary measure-theoretic foundations and geometric considerations used to derive the higher-dimensional inequality. Finally, I present my proof of the inequality for elliptical domains in R^2

1 Introduction

The Hermite-Hadamard inequality in its classical form establishes a bound on the average value of a convex function over an interval using its values at the endpoints. Extending this inequality to higher-dimensional domains introduces challenges in accounting for the geometry of the domain and the role of boundary integrals.

2 Measure-Theoretic Preliminaries

2.1 Hausdorff Measure

For $k \in \mathbb{N}$, the k-dimensional Hausdorff measure, denoted H^k, generalizes the notion of length (H^1), area (H^2), and volume (H^3) to arbitrary dimensions. For a set $A \subseteq \mathbb{R}^n$, the k-dimensional Hausdorff measure is defined as:

$$H^k(A) = \lim_{\delta \to 0} \inf \left\{ \sum_i (\mathrm{diam}(U_i))^k : A \subseteq \bigcup_i U_i,\ \mathrm{diam}(U_i) < \delta \right\}.$$

2.2 Radon-Nikodym Derivative

The Radon-Nikodym theorem provides a way to express one measure as being "absolutely continuous" with respect to another. For two measures μ and ν on a measurable space $(\Omega, \mathcal{F})$ with $\mu \ll \nu$ (absolute continuity), the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ satisfies:

$$\mu(A) = \int_A \frac{d\mu}{d\nu}\, d\nu, \quad \text{for all measurable } A \subseteq \Omega.$$

2.3 Push-Forward Measures

For a measurable map $\phi : \Omega \to \Omega'$, the push-forward measure $\phi_* \mu$ is defined on Ω' as:

$$\phi_* \mu(A') = \mu(\phi^{-1}(A')), \quad \text{for measurable } A' \subseteq \Omega'.$$

3 Proof of Lemma 1

Lemma 1 provides the backbone for the extension of the Hermite-Hadamard inequality by interpreting convexity geometrically and establishing a relationship between measures on the domain Ω and its boundary $\partial\Omega$.

3.1 Outline of the Proof

Suppose $\phi : \Omega \to S^{n-1}$ is a continuous map. For every point $x \in \Omega$, consider the intersection of the line $x + t\phi(x)$ with $\partial\Omega$. If x lies strictly inside the convex domain, then there are exactly two intersection points $y_1, y_2 \in \partial\Omega$. There exists a unique $0 < t < 1$ such that:

$$ty_1 + (1-t)y_2 = x.$$

Using the convexity of f, we immediately get:

$$f(x) \leq tf(y_1) + (1-t)f(y_2).$$

The proof leverages this geometric fact by interpreting x as a weighted mapping to y_1 and y_2 on the boundary, where the weights are t and $1-t$, respectively. By integrating in a neighborhood of x, this can be understood as transporting the Lebesgue measure from the domain Ω to its boundary $\partial\Omega$ through the mapping ϕ.

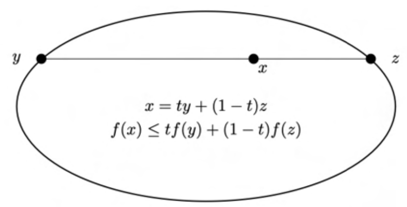


Figure 1: Associating a direction to every point x results in two points on the boundary, y_1 and y_2, whose weights are determined by x.

3.2 Measure Transport and Push-Forward

Define a map $\phi : \Omega \to \partial\Omega$ that sends x to a weighted combination of y_1 and y_2. The push-forward of the Lebesgue measure $\mathcal{L}^n$ under ϕ induces a measure μ on $\partial\Omega$.

The key result is that if the push-forward measure μ is absolutely continuous with respect to the surface measure H^{n-1}, and the Radon-Nikodym derivative $\frac{d\mu}{dH^{n-1}}$ is bounded by a constant c, then for all convex functions $f : \Omega \to \mathbb{R}$ satisfying $f|_{\partial\Omega} \geq 0$, we have:

$$\int_{\Omega} f \, dH^n \leq c \int_{\partial\Omega} f \, dH^{n-1}.$$

—

4 Proof of Theorem 1

Theorem 1 applies Lemma 1 to establish an explicit bound for the higher-dimensional Hermite-Hadamard inequality.

4.1 Tools and Assumptions

- **John's Ellipsoid Theorem:** Any convex domain $\Omega \subseteq \mathbb{R}^n$ contains and is contained in ellipsoids:
$$E \subseteq \Omega \subseteq nE.$$

- **Cauchy's Formula for Surface Area:**
$$H^{n-1}(\partial\Omega) = \frac{1}{|B^{n-1}|} \int_{S^{n-1}} H^{n-1}(\pi_v \Omega)\, dv,$$
where π_v projects Ω onto a hyperplane orthogonal to v.

4.2 Estimation of Constants

By bounding the surface area of ellipsoids and leveraging their symmetry, the constant c in Lemma 1 is estimated. For any convex Ω:

$$c \leq 2\sqrt{\pi}\, n^{n+1}.$$

4.3 Integration and Inequality

Combining the results:

$$\frac{1}{|\Omega|} \int_{\Omega} f\, dH^n \leq \frac{2}{\sqrt{\pi}\, n^{n+1}} \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f\, dH^{n-1}.$$

5 Discussion and Generalizations

- **Open Questions:** Optimization of constants and characterization of extremal domains remain open.

6 Findings about ellipses

The remainder of this paper focuses on presenting my questions and finding.

6.1 First in R^2

- Assume we have a well-defined transport plan T dependent on the map ϕ. Define T^{-1} as the set of points $x \in \Omega$ that map to a given $y \in \partial\Omega$, along with the associated weight t of x mapped to y.

- minimizing:
$$\max \left[\frac{d\mu}{dH^{n-1}} \right],$$
is *similar* to minimizing:
$$\max \left| T^{-1}(y_n) - T^{-1}(y_m) \right| \quad \forall\, m, n,$$

- The meaning of similar is that T is optimal if that expression equals 0

In R^2 the geometric intuition seems clear: $T^{-1}(y_n)$ is the portion t of the length of a curve in R^2. Therefore we are looking for a ϕ which balances length and weight such that if $T^{-1}(y_n)$ is longer than $T^{-1}(y_m)$ then the proportion of weight sent to y_n should be smaller than y_m

If Ω is an ellipse of major and minor axes a and b, I want to introduce the following mapping:
$\phi :=$ the tangent to the only ellipse concentric to Ω with major and minor axes $c * a$ and $c * b$ that passes through x.

This mapping has some fascinating properties. First if y_1 and y_2 are the intersection of $x + t\phi(x)$ and Ω then x is always equidistant to y_1 and y_2 and therefore sends half of its mass to each.

Lastly, maybe the most interesting property of all: $T^{-1}(y_n)$ forms an ellipse of center exactly between the center of Ω and y_n and of major and minor semi axes $a/2$ and $b/2$. (see figure 2)

These two properties reveal the incredible geometry of this mapping: The points mapping to a certain y form an ellipse which is the same for all $y \in \Omega$, simply moved around the center of Ω and tight against its border. Additionally, since all the points on that smaller ellipse send half of their weight to y and half to a unique other y_n, not only are the ellipses the same but so are the portions sent to each y

Given these properties, it seems like there is not a more optimal mapping possible.

6.2 What breaks down in R^3

Instead of ellipses, the pre-image of each y is an ellipsoid with same dimension across all y. However each x sends its weight to the intersection of a plane and an ellipsoid which in general is an ellipse. Therefore its weight would be unevenly distributed. Does it matter? Meaning, does the symmetry of the ellipse somehow make up for that? This question makes this construction of ϕ even more interesting. If it does break down in R^3, is there a reason behind it? If it doesn't, then do all ellipsoidal domains satisfy the Hermite-Hadamard inequality with constant 1?

Construction and Analysis of Tangency Properties Between Ellipses

6.3 Poof: The point of tangency of a scaled down concentric ellipse is the midpoint of the line segment formed by the tangent line and the larger ellipse

Consider two ellipses:

- **Larger ellipse:**
$$\frac{x^2}{a} + \frac{y^2}{b} = 1,$$
where a and b are the semi-major and semi-minor axes, respectively.

- **Smaller ellipse:**
$$\frac{x^2}{sa} + \frac{y^2}{sb} = 1,$$
where $s \in (0, 1)$ scales the axes of the larger ellipse.

Let (x_1, y_1) be a point on the smaller ellipse. By definition, it satisfies:

$$\frac{x_1^2}{sa} + \frac{y_1^2}{sb} = 1. \tag{1}$$

The tangent line to the smaller ellipse at (x_1, y_1) has slope:

$$m = -\frac{sbx_1}{say_1} = -\frac{bx_1}{ay_1}. \tag{2}$$

Using the point-slope form, the equation of the tangent line is:

$$y = m(x - x_1) + y_1. \tag{3}$$

Substituting this tangent line into the equation of the larger ellipse:

$$\frac{x^2}{a} + \frac{y^2}{b} = 1, \tag{4}$$

and solving for the intersection points yields two solutions (x_2, y_2) and (x_3, y_3).

The intersection points are:

$$x_2 = \frac{\sqrt{a}\, y_1 \sqrt{ab - ay_1^2 - bx_1^2}}{\sqrt{b}\sqrt{ay_1^2 + bx_1^2}} + x_1, \tag{5}$$

$$y_2 = y_1 - \frac{\sqrt{b}\, x_1 \sqrt{ab - ay_1^2 - bx_1^2}}{\sqrt{a}\sqrt{ay_1^2 + bx_1^2}}, \tag{6}$$

$$x_3 = -\frac{\sqrt{a}\, y_1 \sqrt{ab - ay_1^2 - bx_1^2}}{\sqrt{b}\sqrt{ay_1^2 + bx_1^2}} + x_1, \tag{7}$$

$$y_3 = y_1 + \frac{\sqrt{b}\, x_1 \sqrt{ab - ay_1^2 - bx_1^2}}{\sqrt{a}\sqrt{ay_1^2 + bx_1^2}}. \tag{8}$$

The midpoint of the segment connecting (x_2, y_2) and (x_3, y_3) is given by:

$$x_{\text{mid}} = \frac{x_2 + x_3}{2}, \tag{9}$$

$$y_{\text{mid}} = \frac{y_2 + y_3}{2}. \tag{10}$$

After simplification, it can be shown that:

$$x_{\text{mid}} = x_1, \tag{11}$$

$$y_{\text{mid}} = y_1. \tag{12}$$

Thus, the point of tangency (x_1, y_1) is the midpoint of the segment formed by the intersection points (x_2, y_2) and (x_3, y_3). This property demonstrates the symmetry of the tangent line with respect to the two ellipses.

6.4 Proof: The Locus of Tangency Points Forms an Ellipse

We aim to show that the locus of tangency points of tangent lines to scaled ellipses passing through a point (x_1, y_1) on the boundary of a larger ellipse satisfies the equation:

$$\frac{\left(x - \frac{x_1}{2}\right)^2}{\frac{a}{2}} + \frac{\left(y - \frac{y_1}{2}\right)^2}{\frac{b}{2}} = \frac{1}{2}.$$

1. Setup

The larger ellipse is given by:

$$\frac{x^2}{a} + \frac{y^2}{b} = 1,$$

where (x_1, y_1) lies on this ellipse:

$$\frac{x_1^2}{a} + \frac{y_1^2}{b} = 1.$$

The family of scaled ellipses is parameterized by a scaling factor $s \in (0, 1]$, with each scaled ellipse defined as:

$$\frac{x^2}{sa} + \frac{y^2}{sb} = 1.$$

2. Tangent Line

The slope m of the tangent line to a scaled ellipse at a tangency point (x, y) is derived from the implicit differentiation of the ellipse equation:

$$m = -\frac{x(sb)}{y(sa)}.$$

The tangent line passing through (x_1, y_1) is:

$$y - y_1 = m(x - x_1).$$

Substituting $m = -\frac{x(sb)}{y(sa)}$, we obtain:

$$y - y_1 = -\frac{x(sb)}{y(sa)}(x - x_1).$$

3. Tangency Points

The tangency points (x_t, y_t) are determined by solving the system of equations:

$$\frac{x^2}{sa} + \frac{y^2}{sb} = 1,$$
$$y - y_1 = -\frac{x(sb)}{y(sa)}(x - x_1).$$

The solutions are:

$$x_t = \frac{a(bsx_1 \pm y_1\sqrt{s(-abs + ay_1^2 + bx_1^2)})}{ay_1^2 + bx_1^2},$$

$$y_t = \frac{b(asy_1 \mp x_1\sqrt{s(-abs + ay_1^2 + bx_1^2)})}{ay_1^2 + bx_1^2}.$$

4. Verification of the Locus

To verify the locus equation:

$$\frac{\left(x - \frac{x_1}{2}\right)^2}{\frac{a}{2}} + \frac{\left(y - \frac{y_1}{2}\right)^2}{\frac{b}{2}} = \frac{1}{2},$$

we substitute (x_t, y_t) into the left-hand side:

$$\text{LHS} = \frac{\left(x_t - \frac{x_1}{2}\right)^2}{\frac{a}{2}} + \frac{\left(y_t - \frac{y_1}{2}\right)^2}{\frac{b}{2}}.$$

Simplifying, we find:

$$\text{LHS} = 1,$$

which holds true if:

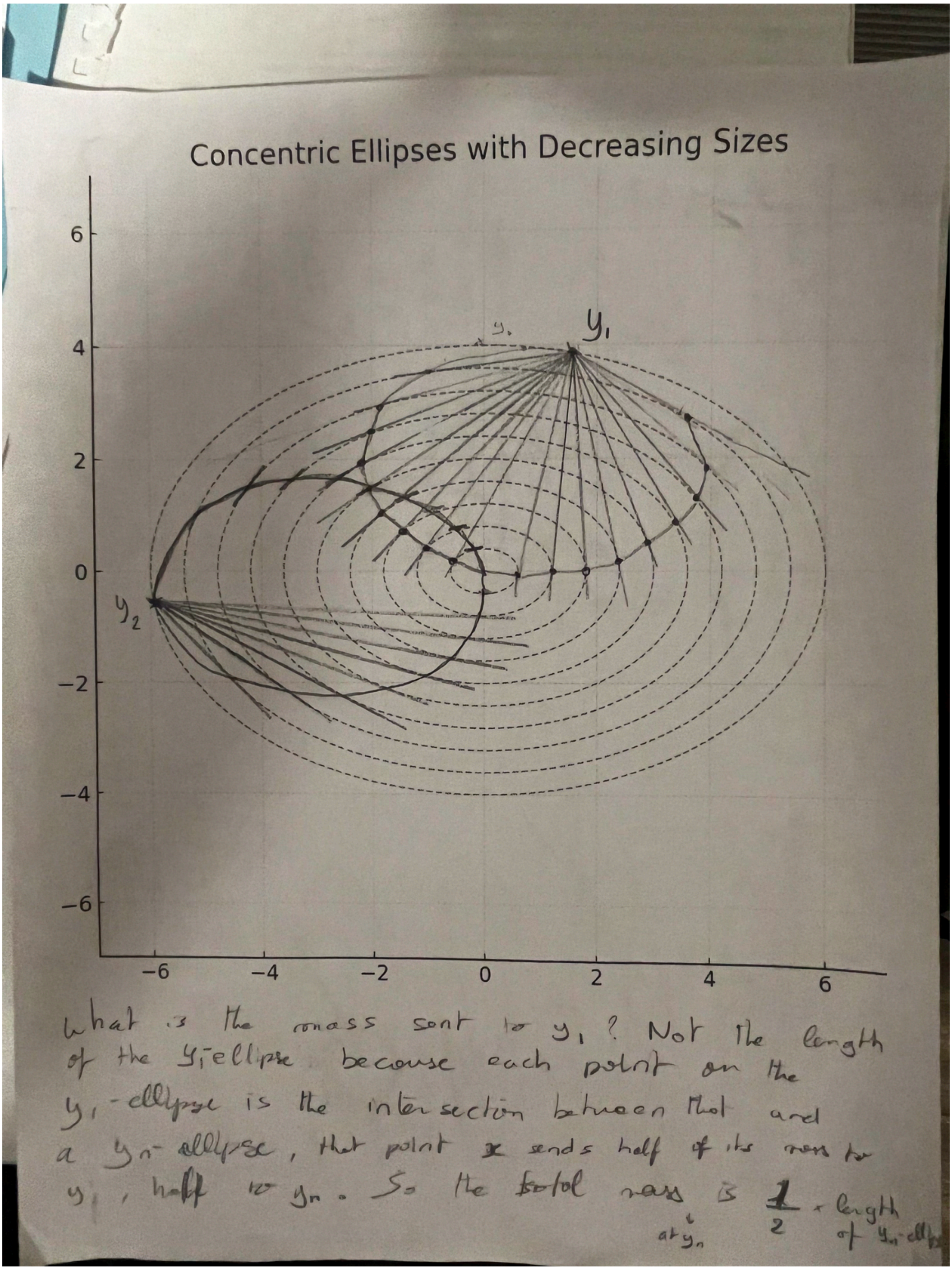
$$\frac{x_1^2}{a} + \frac{y_1^2}{b} = 1.$$

Since this condition is satisfied by the definition of (x_1, y_1) on the larger ellipse, the locus of tangency points indeed satisfies the target ellipse equation.

5. Conclusion

The locus of tangency points as s varies from 0 to 1 forms the ellipse:

$$\frac{\left(x - \frac{x_1}{2}\right)^2}{\frac{a}{2}} + \frac{\left(y - \frac{y_1}{2}\right)^2}{\frac{b}{2}} = \frac{1}{2}.$$



Concentric Ellipses with Decreasing Sizes
6
4
2
0
−2
−4
−6
−6
−4
−2
0
2
4
6
y_1
y_2
what is the mass sent to y_1? Not the length of the y_1-ellipse because each point on the y_1-ellipse is the intersection between that and a y_n-ellipse, that point x sends half of its mass to y_1, half to y_n. So the total mass is $\frac{1}{2}$ × length of y_n-ellipse
at y_n

Figure 2: