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Turing pattern formation in coupled reaction-diffusion system with distributed delays

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Turing pattern formation in coupled two-layer system with distributed delayed is investigated. Numerical simulations prove that, when the coupling is weak, it can apparently accelerate the formation process and enhance the spatial amplitude of the pattern. When it is strong, it will prolong the formation process or even inhibit the pattern and turn the whole system into bulk oscillatory state by its influence on the transient oscillatory state. If the coupling covers only part of the system, Turing pattern can be prominently oriented according to the shape of the coupling area at tiny coupling strength. However, if the coupling is too strong, the Turing pattern may also be destroyed. This means that in coupled systems, the delay effect in the cross-layer signal transfer may significantly influence the spatial character and/or the evolution dynamics in Turing pattern formation, even to destroy the pattern. This work is of practical significance in the study of Turing pattern in biosystems, where bilayer membranes or multilayer tissues are often found. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2041427]

I. INTRODUCTION

The effect of delay is nontrivial in many nonlinear systems whose information processing is based on the spatiotemporal dynamics, such as electric circuits, biological neural network, etc. Related researches prove that in coupled oscillator systems, delay can not only significantly change the stability boundaries of the system,[1] but also induce phenomena such as amplitude death,[2,3] multirhythmicity,[4–6] wave,[7] spatiotemporal pattern and chaos.[8] Many of these phenomena have been experimentally tested.[9–11] However, the real systems are usually so complex that the delay may change with time and its value can only be approximately known. The discrete or constant delay often fails to adequately cover the essential character of the system. Moreover, the delay effect in chemical reaction systems cannot be easily described by constant delay, since it might bring the serious problem by violating microscopic reversibility.[12] Therefore, distributed delay have been proposed[13] and employed to investigate various systems,[14–16] especially neural network systems.[17–19] However, to our knowledge, few results have been reported concerning the effect of distributed delay on the pattern formation in the coupled reaction-diffusion systems, even though it might be more appropriate in describing their evolution dynamics.

Due to the potential connection with the biological morphogenesis, Turing pattern has been intensively studied both theoretically and experimentally,[20–25] especially after it was realized in chemical systems.[26] Since bilayer membranes or multilayer tissues are often found in biological system, Turing pattern formation in coupled system is of great significance. Barrio *et al.*[27] has proved that the coupling of Turing systems can produce complex patterns that resemble those in biological systems. Recently, Yang *et al.* have found various new Turing patterns in reaction-diffusion system with two coupled layers, like superposition patterns;[28] oscillatory Turing patterns;[29,30] symmetric, asymmetric, and antiphase Turing patterns.[31] In this work, we investigated Turing pattern formation in a coupled system with distributed delay. Simulation results prove that the delayed coupling can not only obviously shorten the formation time (the evolution time after which the pattern amplitude is stable at a high value) and improve the spatial amplitude of Turing pattern when it is weak, but also be capable of switching the system into bulk oscillatory state if it is strong enough. In the case of local coupling, delayed coupling can orient the Turing pattern according to the shape of the coupling area at tiny coupling strength.

II. MODEL

To demonstrate our results, we employ a two-layer-coupled system whose kinetics is given by the well-known two variable Lengyel–Epstein model[31] for the photosensitive chlorine dioxide-iodine-malonic acid (CDIMA) reaction,

$$\frac{\partial u_1}{\partial t} = a - u_1 - 4\frac{u_1 v_1}{1+u_1^2} - \phi + \nabla^2 u_1 + D\left[\int_0^\infty f(\tau')u_2(t-\tau') - u_1(t)\right],$$

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$$\frac{\partial v_1}{\partial t} = \sigma\left[b\left(u_1 - \frac{u_1 v_1}{1+u_1^2} + \phi \right) + d\nabla^2 v_1 + D\left[\int_0^\infty f(\tau')v_2(t-\tau') - v_1(t) \right] \right],$$

$$\frac{\partial u_2}{\partial t} = a - u_2 - 4\frac{u_2 v_2}{1+u_2^2} - \phi + \nabla^2 u_2 + D\left[\int_0^\infty f(\tau')u_1(t-\tau') - u_2(t) \right],$$

$$\frac{\partial v_2}{\partial t} = \sigma\left[b\left(u_2 - \frac{u_2 v_2}{1+u_2^2} + \phi \right) + d\nabla^2 v_2 + D\left[\int_0^\infty f(\tau')v_1(t-\tau') - v_2(t) \right] \right], \tag{1}$$

where u and v are the dimensionless concentrations of I^- and ClO_2^-, a, b, and σ are parameters that are proportional to other initial conditions and rate constant in the CDIMA kinetics, d is the ratio of diffusion coefficient of inhibitor to that of activator, and ϕ denotes the effect of photochemical reaction. D is the interlayer coupling strength and f represents a distribution of delay values. If f is chosen as $\delta(\tau)$ with $\tau > 0$, it will give the situation of discrete delayed coupling; as a special case, if f is $\delta(0)$, the nondelayed system can be obtained. The subscripts specify the different layers where the reactive species are. Here, the kinetic parameters are fixed at $a=36$, $b=2.5$, $d=1.2$, $\sigma=9$, $\phi=2$ to make sure that the dynamic state of each uncoupled sublayer is in the Turing pattern region.[32] The delay distribution function f is chosen as a common seen shape of distribution, normal distribution, i.e.

$$\begin{cases} \langle f(\tau') \rangle = \tau_0 \\ \langle f(\tau')T(\tau'') \rangle = 2G\delta(\tau' - \tau''), \end{cases}$$

in which τ_0 is the averaged delay and G is the standard deviation of the distribution.

Integration of Eq. (1) is performed on a 100×100 square lattice. We employ the first order Euler algorithm with zero-flux boundary condition and random initial condition. The time step and space step are fixed at 0.005 t.u. (time unit) and 0.5 s.u. (space unit), respectively. The diffusion term is treated by five-points methods, where $u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}$, so does the $v_{i,j}$ case. In simulation, we use four three-dimension arrays $u_i(v_i)[\mathrm{Ti}][N\_x] \times [N\_y]$ to store the spatiotemporal information in the past Ti evolution steps in the ith sublayer. Ti is decided according to the value of delay distribution deviation G. A random number, t_i, governed by the normal distribution is generated every evolution step; therefore, the spatiotemporal information stored in the arrays $u_i(v_i)[t_i][N\_x][N\_y]$ are used in the coupling.

III. RESULTS AND DISCUSSION

Under the above-mentioned distributed delayed coupling, we find that the formation time and the spatial amplitude of the pattern can be significantly influenced by the delayed coupling. Figure 1 shows the Turing pattern amplitude evolution under different coupling parameters with the comparison of some special cases. Compared with the nondelayed [Fig. 1(a)] situation, it is obvious that the spatial amplitude of the pattern can be significantly enhanced by weak coupling, but the rising extent of the amplitude does not so much rely on the exact value of the averaged delay. From Fig. 1(b) it can be seen that under very weak coupling, the pattern amplitude is scarcely enhanced, but the amplitude rises earlier than the uncoupled case. That means there is a certain critical coupling strength above which the distributed coupling can influence the spatial character of Turing pattern, but even below this critical value, the coupling is still capable of impacting the evolution dynamics of the pattern and shortening the pattern formation time.

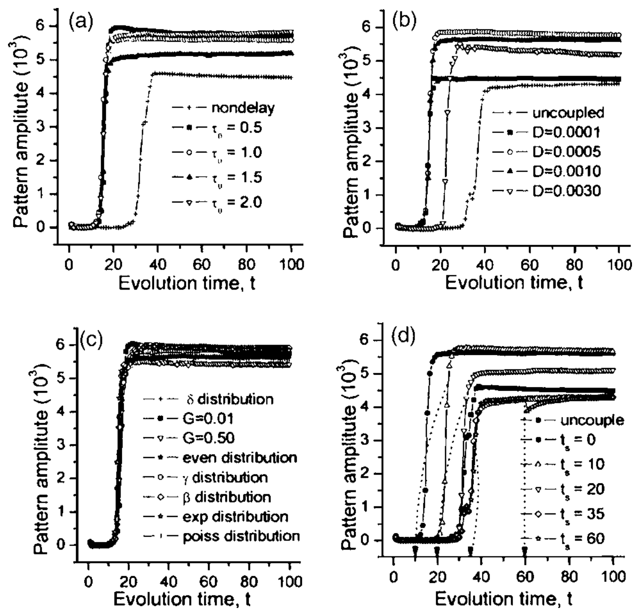

FIG. 1. The evolution of Turing pattern spatial amplitude in distributed delayed systems with different delay parameters. (a) The influence of the averaged delay, (b) the influence of the coupling strength, (c) the influence of the distribution derivation and distribution type, (d) the influence of the coupling turn on time. (Other figures are all the result of full time coupling.) (a) $D=0.001$, $G=0.1$; (b) $\tau_0=1.0$, $G=0.1$; (c) $D=0.001$, $\tau_0=1.0$; (d) $D=0.001$, $G=0.1$, $\tau_0=1.0$.

The influences of distribution deviation and distribution type on the pattern formation are also investigated. In Fig. 1(c), it is clear that when the coupling strength and the averaged delay time are fixed, the formation time and amplitude of the pattern are generally kept at the same level for different values of G. When other distributions are selected [here, we keep the max delay less than double of the averaged delay time (τ_0) in the normal distribution case], the general effect of the coupling does not show great changes when other coupling parameters are the same. These indicate that it is the distributed character that contributes to the general effect of distributed delay coupling, the shape of the distribution is of less importance. Fatihcan[16] also had similar results when investigating the distributed delay facilitated amplitude death of coupled oscillators.

Combining all the above-mentioned results, it seems to suggest that as long as there is some weak delayed coupling effect whose strength is above a certain value present in the system, the formation time and/or the spatial amplitude of the Turing pattern will be significantly influenced. These phenomena can be explained in the transient oscillation process in the pattern evolution. The intrinsic reaction-diffusion mechanism is prone to stabilize the system in some spatial inhomogeneous stable state, i.e., Turing pattern state, by the transient oscillation. The presence of the delayed coupling effect means each layer will receive the cross-layer signal from the other after a certain delay. When the coupling is weak, probably because the delayed coupling signals increase the spatial heterogeneity, the delayed coupling signals can help the system to stabilize at Turing pattern state, the pattern is formed in some relatively early stage. When the coupling strength is relatively bigger, the coupling becomes so strong that its effect on the dynamic evolution dominates.

We also study the situations when the coupling is turned on after the pattern evolution begins. The corresponding results are plotted in Fig. 1(d). Here, we use the symbol t_s to represent the evolution time when the coupling is turned on. It seemed that only when the coupling is turned on before the pattern formation, it can help to accelerate pattern formation and increase the pattern amplitude after some period of adjustment period. This might be because the reaction diffusion mechanism is relatively weak before the formation of Turing pattern; therefore, the coupling signal can accelerate pattern formation and increase the spatial amplitude of the pattern (Fig. 1(d), t_s=0, 10, 20 case). However, the reaction diffusion mechanism becomes very strong in keeping the subsystems in their original state after the formation of Turing pattern. The delayed coupling signal added now is relatively too weak to influence the pattern (Fig. 1(d), t_s=60 case). Besides, since the system needs a period of adjusting time to respond to the coupling signal, even if the coupling is turned on shortly before the pattern formation [Fig. 1(d), t_s=35 case], it still cannot influence the pattern. In our previous investigation,[33] we also find similar phenomenon that weak spatial perturbation can only effectively influence the Turing pattern when it is added before the Turing pattern formation.

Since it is the distributed delayed coupling, it will generally produce certain limited phase difference between the concentration evolution signal in each subsystem and the cross-layer concentration signal. Thus, when the coupling is strong, it will prevent the oscillation from being stabilized and prolong the pattern formation process. We can also figure this tendency out from Fig. 1(b): when the coupling strength is 0.003, the pattern formation time is relatively longer than other coupled cases. When the coupling strength is further increased, the pattern formation time will keep on increasing, even to the extent that is far longer than the uncoupled situation. However, when the coupling strength is above a certain value, the coupling signal becomes so strong that it overcomes the intrinsic reaction-diffusion mechanism, and the Turing pattern is inhibited and turned into bulk oscillatory state.

In Fig. 2, we plot the biparameter phase diagram of Turing pattern inhibition under different distribution deviation. It can be seen that when the averaged delay is very small,

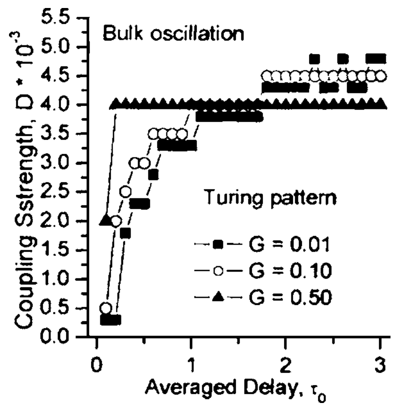

FIG. 2. The biparameter phase diagram of Turing pattern inhibition in the coupled system with distributed delays.

Turing pattern will be easily turned into bulk oscillatory state by weak coupling at any distribution deviation value. When the delays are narrowly distributed (e.g., G=0.01 in Fig. 2), the min coupling strength that is needed to inhibit the Turing pattern gradually increases with the averaged delay. When the delays are relatively widely distributed (e.g., G=0.50 in Fig. 2) and the averaged delay is not too small, Turing pattern will be inhibited when the coupling strength reaches a certain value no matter what the averaged delay is. This means that the impact of the coupling is weakened with the increase of averaged delay when it is narrowly distributed. When the distribution is wide, the exact value of the averaged delay shows almost no impact on the general effect of the delayed coupling. When the dynamic parameters in Eq. (1) are set at other values that can produce Turing pattern in the single layer systems, similar results are also obtained. Noticing that there are some fluctuations in the Turing boundary for G=0.01 at large values of the averaged delay in Fig. 2, we adjust the time step and the space step of the integration. Further simulation results prove that this kind of "fluctuation" will disappear when smaller integration steps (both time and space) are employed. This might be because the integration algorithm used here could show some instability under the strong coupling with narrow delay distribution.

If the coupling influences only part of the system, we find Turing patterns can be spatially oriented according to the shape of the coupling area. Figure 3 shows the Turing pattern and their corresponding Fourier spectrums at different cou-

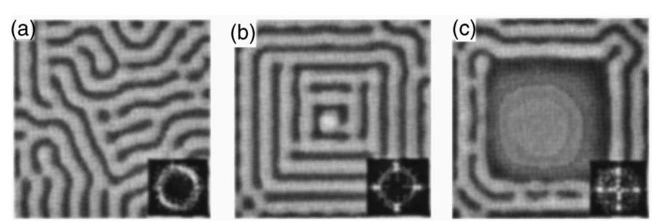

FIG. 3. Turing pattern and their corresponding Fourier spectrums in the local coupled system at different coupling strengths, τ_0=1.0. The coupling influences a square area whose center is the same as that of the system. The size of the coupling area is 36% of the whole system. (a) The classical Turing pattern under every weak local coupling, $D=10^{-12}$; (b) the oriented Turing pattern, $D=10^{-10}$; (c) the pattern under strong local coupling, $D=10^{-2}$.

pling strengths when the two subsystems are partially coupled in a square area. From Fig. 3(b) it can be seen that the pattern is apparently oriented by the coupling area shape (square), but the intrinsic wavelength still exists in its corresponding Fourier spectrum, which means the pattern is still a Turing pattern in nature. What differs with the classical Turing patterns here is that the spatial modes are gathered in some given directions (depending on the shape of the coupling area) on the intrinsic wavelength. When local coupling is above a certain level, the pattern is partially inhibited: many other spatial modes arise in its Fourier spectrum. That is to say, the Turing pattern is destroyed [Fig. 3(c)]. This kind of Turing pattern orientation phenomenon arises from the cooperation of two pattern formation mechanisms. One is the intrinsic reaction-diffusion mechanism, the other is induced by the local coupling. The local distributed delayed coupling generates spatial concentration difference between the coupled and uncoupled area, so it can produce a new pattern formation mechanism with the combination of the diffusion mechanism. When the coupling is weak, the new mechanism is far weaker than the intrinsic one, there's only classical Turing pattern in the system [Fig. 3(a)]. With the increase of the coupling strength, the new mechanism is gradually strengthened, but the intrinsic one still dominates. Therefore, the pattern is spatially oriented according to the shape of the coupling area, but it is still Turing pattern [Fig. 3(b)]. If the coupling is very strong, the new mechanism becomes overwhelming, so the original Turing pattern will be destroyed [Fig. 3(c)]. Further simulation results prove that the similar phenomena will arise when the coupling area is of other shapes. The pattern will also be oriented when the system produces other types of Turing pattern [spots, strips, or honeycomb, depending on the dynamic parameter choice in Eq. (1)], which are not shown here. The above-mentioned phenomena suggest that as long as there is some local distributed delay in the interlayer coupling of the bilayer/multilayer systems, Turing pattern will be oriented according to the shape of the coupling area. This kind of pattern orientation phenomena also appear in nondelay coupled systems, but the lowest coupling strength needed is about 10^5 times bigger than in the delay-coupled case.

To quantitatively study the orientation phenomenon, we define the orientation factor *Ori* as the ratio of the sum of the pattern's wave numbers in the given directions to that of other directions, the definition equation is as below:

$$Ori = \frac{\Sigma_{\mu_{k_0}} S_{k_0}}{\Sigma_{k(k \neq k_0)} \Sigma_{\mu_k} S_k}, \quad (k_0 \in k), \tag{2}$$

in which $S_k = \langle \hat{v}_k \hat{v}_{-k} \rangle_t$, and μ_k stands for the wave number in a given direction k. Here $\hat{v}_k$ is the spatial fast Fourier transform (FFT) of the concentration field v (we take the v value of one of the layers as example), and $\langle \rangle_t$, indicates averaging over the time of pattern formation process. A sharp peak in S_k, S_{k_0}, will manifest the presence of order in a given direction. k_0 in wave vector S_{k_0} tells the spatial direction property of the oriented patterns.

Figure 4 shows the change of orientation factor as a function of the coupling parameters. It is clear from Fig. 4(a) that the Turing pattern can be oriented at a tiny coupling strength. With the increase of coupling strength, the change of the orientation factor shows some stochastic resonance-like trend, i.e., the pattern is best ordered at some intermediate coupling strength. We can get the explanation directly from Fig. 3. When the coupling strength is tiny, the local concentration difference is too weak to orient the pattern. However, too strong local coupling may destroy the Turing pattern so that the orientation factor will drop instead of rise. In the case of weak coupling, when the averaged delay is at small values, its increase may also dramatically enhance the orientation factor. However, when the averaged delay is bigger than a certain value, *Ori* is stable at some high value with the change of the averaged delay [Fig. 4(b)]. This again tells us that the exact value of averaged delay will not show great influence on the general effect of delayed coupling unless it is very short.

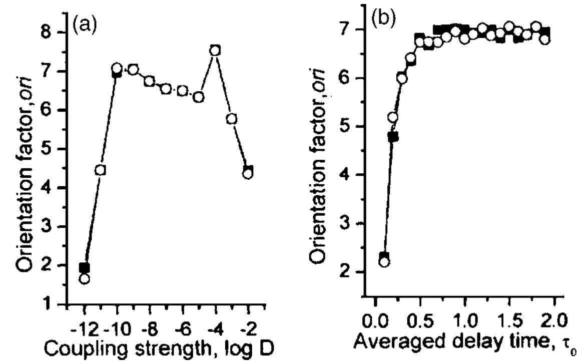

FIG. 4. The change of orientation factor as a function of the coupling parameter. Circles (◯) and filled squares (■) stand for the cases of the two layers respectively. $G=0.1$, (a) $\tau_0=1.0$, (b) $D=10^{-10}$.

All the above-mentioned results are confirmed by further simulations in other models (like Brusselor model) or when periodic boundary condition is employed. Therefore, the above results should be the general property of Turing pattern formation in the coupled multilayer systems with distributed delays.

IV. SUMMARY

In this work, we demonstrate the effect of distributed delayed coupling on Turing pattern formation in two-layer system in the two-variable Lengyel–Epstein model. Simulation results prove that the distributed delayed coupling can help to speed up the formation process and improve the spatial amplitude of the Turing pattern when it is weak. However, under strong coupling, Turing pattern formation can also be prolonged or even inhibited. The whole effect of the coupling can mainly be determined by the coupling strength, the exact value of the averaged delay and the distribution width of the delay show little contribution, unless they are very small. The general influence of the delayed coupling is independent from the shape of the distribution. When the coupling influences only part of the system, the tiny coupling can orient the Turing pattern according to the shape of the coupling area, while very strong coupling may destroy the pattern. These results are confirmed by further simulations under other simulation conditions or in other models. Since many bilayer membranes or multilayer tissues exist in biosystems and it is probable that their interlayer global/local signal transition will be delayed, this work is of practical significance to study Turing pattern.

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