P(T)VZZLE 15

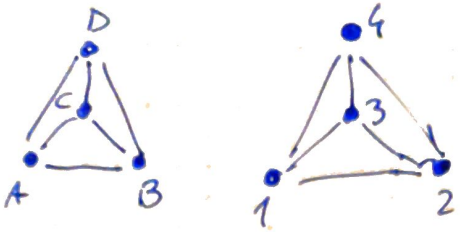
THERE ARE N PEOPLE AND EACH OF THEM CAN SPEAK WITH 3 AND ONLY 3 OTHERS. WHAT WILL BE THE MAX NUMBERS OF CHATS TO GO FROM ONE PERSON TO THE FAR AWAY ONE IN ORDER TO COMMUNICATE A MESSAGE?

SOLUTION

NOTE THAT THE STATEMENT OF THE PROBLEM IS SUCH THAT IT DOES NOT ALLOW A UNIQUE SOLUTION !!

THE N PEOPLE CAN BE THOUGHT TO BE ARRANGED INTO A GRAPH OF DEGREE 3, BUT IT WAS NOT SPECIFIED THAT SUCH A GRAPH SHOULD BE CONNECTED. INDEED IF NOT CONNECTED IT COULD BE IMPOSSIBLE TO COMMUNICATE BETWEEN DISCONNECTED COMPONENTS.

FOR EXAMPLE THE FOLLOWING SITUATION IS OF SUCH TYPE



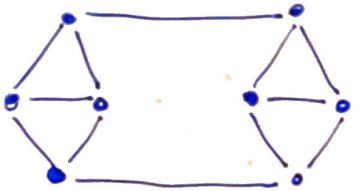
$\{A, B, C, D\}$ ARE DISCONNECTED FROM $\{1, 2, 3, 4\}$

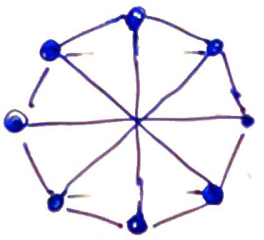
SO WE ARE ASSUMING TO HAVE A CONNECTED GRAPH.

UNFORTUNATELY THIS IS NOT ENOUGH TO MAKE UNIQUE THE SOLUTION, BECAUSE IT IS POSSIBLE TO CREATE DIFFERENT GRAPHS WITH SAME N AND DEGREE d BUT WITH DIFFERENT DIAMETER!

THE DIAMETER OF A GRAPH IS EXACTLY WHAT WE ARE LOOKING FOR, I.E. THE LONGEST POSSIBLE DISTANCE BETWEEN ANY PAIR OF VERTICES.

FOR EXAMPLE LET'S CONSIDER THE FOLLOWING TWO GRAPHS WITH $N=8$ AND $d=3$

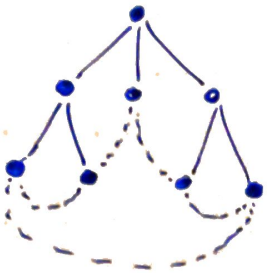
A)  AND

B) 

YOU CAN CHECK EASILY THAT FOR THE GRAPH A) THE DIAMETER IS 3 WHILE FOR THE GRAPH B) IT IS 2.

IN ORDER TO FOCUS BETTER THE PROBLEM WE CAN OBSERVE THAT THE GRAPH WE ARE CONSIDERING ARE 3-REGULAR GRAPHS, ALSO CALLED CUBIC GRAPHS, AND ONE WAY TO BUILD THEM IS TO CREATE A TREE OF DEGREE 3 AND CONNECT PROPERLY THE LEAVES. FOR EXAMPLE THE GRAPH WITH N=8 CAN BE

DRAWN AS



IF WE ASK OURSELVES TO CONSIDER ONLY THE GRAPHS THAT FOR THIS TREE REPRESENTATION HAVE THE MAXIMUM NUMBER OF NODES AT EVERY DEPTH, WE END UP WITH THE SO CALLED MOORE GRAPHS

FOR WHICH, AS YOU CAN EASILY COUNT, THE NUMBER OF NODES IS

$$N = 1 + d \sum_{i=0}^{k-1} (d-1)^i$$

AND IN OUR CASE $d = 3$, AND K IS EXACTLY THE ASKED DIAMETER !!