

Strong and weak chaos in nonlinear networks with time-delayed couplings

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We study chaotic synchronization in networks with time-delayed coupling. We introduce the notion of strong and weak chaos, distinguished by the scaling properties of the maximum Lyapunov exponent within the synchronization manifold for large delay times, and relate this to the condition for stable or unstable chaotic synchronization, respectively. In simulations of laser models and experiments with electronic circuits, we identify transitions from weak to strong and back to weak chaos upon monotonically increasing the coupling strength.

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The cooperative behavior of a system of interacting units is of fundamental interest in nonlinear dynamics. Such complex networks have a wide range of interdisciplinary applications ranging from neural networks to coupled lasers [1]. Typically, these units interact by transmitting information about their state to their partners, and in many applications the transmission time is larger than the time scales of the individual units. Thus, networks with time-delayed couplings are a focus of active research [2].

Time-delayed feedback can produce dynamical instabilities which may lead to deterministic chaos [3–5]. Even a scalar differential equation with time-delayed feedback has an infinite-dimensional phase space which favors chaotic solutions. In physics, a single semiconductor laser produces a chaotic signal when its laser beam is reflected back into its cavity by an external mirror. Networks of nonlinear units may, similarly, become chaotic due to time-delayed coupling of the nodes. For networks of identical units, one often observes chaos synchronization. Even if the delay times are very long, the units may synchronize onto a common chaotic trajectory without time shift (zero-lag synchronization) [6, 7]. Other kinds of synchronization are possible, as well, like phase, achronal, anticipated and generalized synchronization, but here we only consider zero-lag synchronization. Chaos synchronization is being discussed in the context of secure communication [8].

In this letter we investigate networks with time-delayed couplings in the limit of large delay times [7, 9], and show that transitions between two kinds of chaos, namely *strong* and *weak* chaos, can be induced by changing the coupling strength. For strong chaos the largest Lyapunov exponent (LE) is of the order of the inverse time scales of the individual units and independent of the delay time, hence two nearby trajectories separate very quickly. For weak chaos, however, the LE is of the order of the inverse delay time, hence nearby trajectories separate very

slowly. We show that these two types of chaos possess very different synchronization properties: Networks with strong chaos cannot synchronize, whereas for weak chaos, networks can synchronize if the product of the LE and the delay time is sufficiently small compared to the eigenvalue gap of the coupling matrix, i. e. the difference between the row sum and its largest transversal eigenvalue.

We illustrate our general findings by the example of a semiconductor laser network modeled by the Lang-Kobayashi (LK) rate equations, and by experiments on chaotic electronic circuits.

We consider networks of N identical units with variables $\mathbf{x}_i(t) \in \mathbb{R}^n$, $i = 1, \dots, N$, which obey the equations

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}[\mathbf{x}_i(t)] + \sigma \sum_j G_{ij} \mathbf{H}[\mathbf{x}_j(t - \tau)] . \quad (1)$$

The nonlinear function \mathbf{F} describes the local dynamics of the individual units. The units are connected by the coupling matrix $G = \{G_{ij} \in \mathbb{R}\}$, which describes the coupling topology and the weight of each link. The coupling itself is characterized by the coupling function \mathbf{H} , the delay time τ and the strength σ . We consider coupling matrices G with normalized row sum ($\sum_j G_{ij} = 1$), such that complete synchronization $\mathbf{x}_i(t) = \mathbf{s}(t)$ is a solution of Eq. (1),

$$\dot{\mathbf{s}}(t) = \mathbf{F}[\mathbf{s}(t)] + \sigma \mathbf{H}[\mathbf{s}(t - \tau)] . \quad (2)$$

The dynamics $\mathbf{s}(t)$ within the synchronization manifold (SM) is identical to the dynamics of a single unit with time-delayed feedback. The LEs of a single unit are obtained from linearizing Eq. (2) which gives

$$\dot{\boldsymbol{\xi}}(t) = D\mathbf{F}[\mathbf{s}(t)] \boldsymbol{\xi}(t) + \sigma D\mathbf{H}[\mathbf{s}(t - \tau)] \boldsymbol{\xi}(t - \tau) . \quad (3)$$

The maximum LE λ_m of Eq. (3) is a measure of the chaoticity within the SM. It turns out that it is useful to consider the maximum LE λ_0 from an integration of the reduced linear system [4]

$$\dot{\boldsymbol{\xi}}(t) = D\mathbf{F}[\mathbf{s}(t)] \boldsymbol{\xi}(t) . \quad (4)$$

We call this LE λ_0 the *instantaneous Lyapunov exponent* of the system, since there is no delayed term in the corresponding variational equation. Note, however, that this should not be confused with a *finite-time* LE. Furthermore, λ_0 still depends on the coupling strength σ , since Eq. (4) contains the trajectory $\mathbf{s}(t)$. The following results hold in the limit of large delay times τ : *Weak chaos* occurs if $\lambda_0 < 0$; in this case $\lambda_m = \eta/\tau$ in the leading order, where η is independent of τ . *Strong chaos* is encountered if $\lambda_0 > 0$; here $\lambda_m \approx \lambda_0$ up to a correction which is exponentially small with respect to τ . At first, we give a sketch of the proof.

Weak chaos ($\lambda_0 < 0$) — Let us denote by $X(t, s)$ the fundamental matrix solution [10] of the instantaneous linear system Eq. (4). In the case of weak chaos, it satisfies $\|X(t, s)\| \leq M e^{\lambda_0(t-s)}$ with negative λ_0 . Let us split the solution of Eq. (3) into pieces of the length τ as follows $\xi_j(\theta) := \xi(\theta + \tau j)$ with $0 \leq \theta \leq \tau$. Then, ξ_j can be expressed using the variation of constants formula [10] as follows

$$\begin{aligned} \xi_j(\theta) = & X_j(\theta, 0) \xi_{j-1}(\tau) \\ & + \sigma \int_0^\theta X_j(\theta, t') DH[\mathbf{s}(t' - \tau)] \xi_{j-1}(t') dt' \end{aligned} \quad (5)$$

where $X_j(\theta, t') = X(\theta + \tau j, t' + \tau j)$. Using the exponential decrease of $X_j(\theta, t')$, it is straightforward to obtain from Eq. (5) the estimate $\max_\theta \|\xi_j(\theta)\| \leq L \max_\theta \|\xi_{j-1}(\theta)\| \leq L^j \max_\theta \|\xi_0(\theta)\|$, where L is some constant. This immediately implies that the exponential growth of the solutions is possible only with respect to the slow time $j = t/\tau$ [4, 5], and, hence, the maximum LE is scaled by $1/\tau$ in the case when $\lambda_0 < 0$. Strictly speaking, the constant L depends on τ , since the properties of the chaotic attractor change with τ . However, we argue that for large τ this dependence disappears: A chaotic attractor is characterized by its “skeleton” of periodic orbits and in particular by the orbits of low period. The set of periodic orbits, which exists for low values of τ , reappears generically also for larger delays [11]. Thus, we conjecture that in the limit of large τ generically all characteristics of the attractor converge to a limit and in particular L becomes independent of τ . All experimental and theoretical results about chaos in delayed systems, for instance in lasers with delayed self-feedback and optoelectronic oscillators [12], support this argument [13].

Strong chaos ($\lambda_0 > 0$) — Let us make the coordinate transformation to the frame diverging exponentially with rate λ_0 , i. e. $\xi(t) = e^{\lambda_0 t} \mathbf{u}(t)$. In the new coordinates, the variational Eq. (3) has the form

$$\begin{aligned} \dot{\mathbf{u}}(t) = & (DF[\mathbf{s}(t)] - \lambda_0 I) \mathbf{u}(t) \\ & + \sigma e^{-\lambda_0 \tau} DH[\mathbf{s}(t - \tau)] \mathbf{u}(t - \tau), \end{aligned} \quad (6)$$

where the largest LE of the instantaneous vector field is zero. Applying the same arguments to the rescaled system (6) as in the case of weak chaos, we conclude that

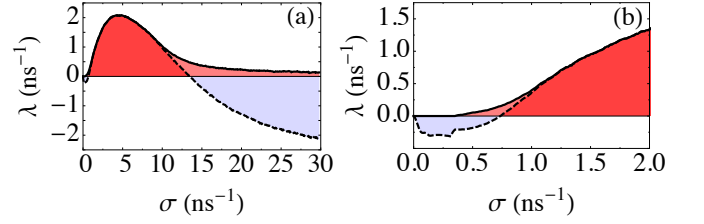


Figure 1. (Color online) (a) Maximum Lyapunov exponents λ_m (solid line) and λ_0 (dashed line) of the synchronization manifold (SM) for $\tau = 10$ ns vs. coupling strength σ . (b) Enlarged view for small coupling strengths σ .

the maximum LE for $\mathbf{u}(t)$ is at most of the order $1/\tau$ and converges to zero for large delays. Hence, λ_0 approximates λ_m for large τ . Numerical calculations (see below) show that, in fact, the largest LE converges to λ_0 with an error $e^{-\mu\tau}$, $\mu > 0$. The convergence rate, however, is much slower than in the case of steady states [14]. Note that the existence of LEs that are independent of τ has also been reported in [4] for time-discrete maps with delay. Such exponents have been called anomalous there. They can be also computed by regarding Eq. (3) as a nonautonomous differential equation with the delay term acting as “stochastic contribution”. Note further that in most chaotic delayed systems that have been studied, such as Ikeda and Mackey-Glass oscillators, the local dynamics is a constant damping, such that these systems only exhibit weak chaos. We thus propose to investigate delayed systems with strong chaos, such as lasers with delayed feedback in certain parameter regimes, further, since these systems may have important applications for instance as random number generators [15].

Consequences — We now discuss the consequences of these results for systems with large delay. For strong chaos, the maximum LE of the system is already given by the instantaneous term, Eq. (4). The coupling strength σ contributes only indirectly through the orbit $\mathbf{s}(t)$. For weak chaos, however, chaos is generated by the delayed term with strength σ in the variational equation, and the maximum LE is of order $1/\tau$. It is important to note here that both types of chaos are delay-induced. In fact, the laser system that we consider exhibits stable continuous wave output without delayed feedback.

Our theoretical predictions are compared to numerical simulations of the LK equations modeling a semiconductor laser with optical feedback. Now the variables $\mathbf{x}_i(t) \in \mathbb{R}^3$ contain the real and imaginary parts of the electromagnetic field and the charge carrier inversion. Details of the equations and parameters can be found in [9]. Fig. 1 shows the two maximum LEs λ_m and λ_0 as a function of the coupling strength σ . Without coupling, $\sigma = 0$, the laser relaxes to a constant intensity, both LEs are zero and correspond to the Goldstone mode. For a small coupling, the laser becomes chaotic but the instantaneous LE is negative, i. e., the chaos is weak. With

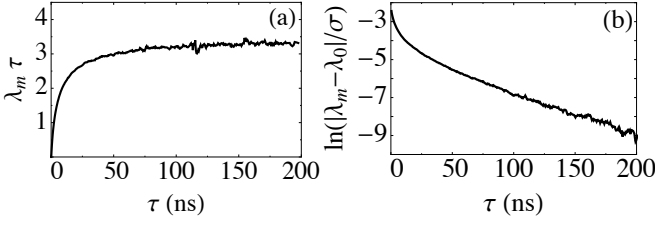


Figure 2. (a) $\lambda_m \tau$ for the SM vs. delay time τ in the regime of weak chaos ($\sigma = 21 \text{ ns}^{-1}$). (b) $\ln(|\lambda_m - \lambda_0|/\sigma)$ for the SM vs. delay time τ in the regime of strong chaos ($\sigma = 12 \text{ ns}^{-1}$). Parameters as in Fig. 1.

increasing coupling strength, λ_0 increases to positive values. Hence, the laser is strongly chaotic in some interval of σ . For higher values of σ , the laser is weakly chaotic again.

Fig. 2(a) shows $\lambda_m \tau$ as a function of the delay time τ in the regime of weak chaos. We observe that this product saturates at a constant value for large delay times. Note that for our parameters a delay time of 200 ns is much larger than the internal time scale (1 ns). For the regime of strong chaos, Fig. 2(b) depicts $\ln(|\lambda_m - \lambda_0|/\sigma)$ as a function of the delay time τ . We observe that it decreases linearly with τ in agreement with the analysis of Eq. (6).

At the transitions from weak to strong chaos, $\lambda_m \tau$ diverges as shown in Fig. 3(a). In order to obtain the scaling for this divergence, we first consider the simple case of a scalar delay equation $\dot{s} = F(s) + b s(t - \tau)$ with constant coefficient b . The corresponding characteristic equation of the fixed point is $\lambda_m = \lambda_0 + b e^{-\lambda_m \tau}$, which can be solved using the Lambert-W function. For $\lambda_0 \rightarrow 0^-$, which corresponds to the transition from weak to strong chaos, it predicts a divergence of $\lambda_m \tau$ with $\ln(b/|\lambda_0|)$. Assuming that the coupling strength b in this simple model can be identified with σ in the chaotic LK equations, we observe a similar scaling for the divergence of $\lambda_m \tau$ at the two critical points of small and large values of σ . Fig. 3(b) reveals that $\lambda_m \tau$ indeed depends linearly on $\ln(\sigma/|\lambda_0|)$. The slope is within the same order of magnitude as for the simple case of steady states but systematically larger. This deviation is related to the chaotic time dependence of Eq. (3) and differs between the left (gray) and right (black) divergences since the degree of chaotic fluctuations is different for small and large σ .

Up to now we have considered a single unit with time-delayed feedback or, equivalently, the dynamics in the SM Eq. (2). The stability of chaos synchronization can be computed using the master stability function [16]. It is defined as the maximum LE $\lambda(r e^{i\psi})$ arising from the variational Eq. (3) where σ is replaced by the complex parameter $\sigma r e^{i\psi}$ (the input trajectory $\mathbf{s}(t)$ is still governed by Eq. (2)). For a given network the stability of the synchronized solution is determined by the eigenvalues of G .

Due to the unity row sum the coupling matrix G has

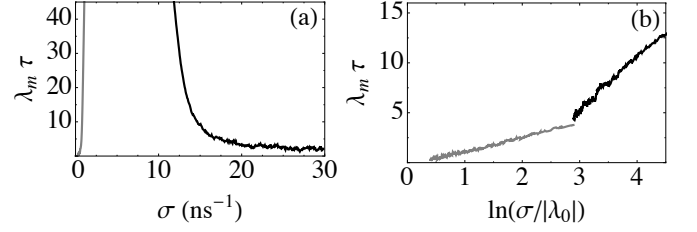


Figure 3. (a) $\lambda_m \tau$ for the SM vs. coupling strength σ for $\tau = 100 \text{ ns}$. (b) $\lambda_m \tau$ for the SM vs. $\ln(\sigma/|\lambda_0|)$ in the regime of weak chaos near the left (gray line) and right (black line) divergences.

one eigenvalue $\tilde{\gamma} = 1$ with eigenvector $(1, 1, \dots, 1)$, which corresponds to perturbations in the SM. The other $N - 1$ transversal eigenvalues $\gamma_1, \dots, \gamma_{N-1}$ correspond to perturbations transversal to the SM. Synchronization in the network is stable if $\lambda(\sigma \gamma_k) < 0$ for all transversal eigenvalues γ_k .

We now show that synchronization is stable for weak chaos if

$$|\gamma_{\max}| < e^{-\lambda_m \tau}, \quad (7)$$

where γ_{\max} is the transversal eigenvalue of G with largest magnitude [9].

As follows from [7], in the large delay case, $\lambda(r e^{i\psi})$ does not depend on the phase ψ , and there exists a critical value r_0 for the stability of the variational Eq. (3) ($\lambda(r_0) = 0$); i.e., for $r < r_0$, the perturbation $\xi(t)$ vanishes asymptotically $\xi(t) \rightarrow 0$ and grows otherwise. The maximum LE is zero for r_0 . If the maximum LE λ_m for a given σ is known, then the threshold r_0 can be expressed as

$$r_0 = \sigma e^{-\lambda_m \tau}. \quad (8)$$

This can be shown by the following arguments. Let us make the coordinate transformation $\xi(t) = \mathbf{u}(t) \exp[t \ln(\sigma/r_0)/\tau]$ in Eq. (3). Then the variational equation in the transformed coordinates reads

$$\begin{aligned} \dot{\mathbf{u}}(t) = & \left(DF[\mathbf{s}(t)] - \frac{1}{\tau} \ln\left(\frac{\sigma}{r_0}\right) I \right) \mathbf{u}(t) \\ & + r_0 DH[\mathbf{s}(t - \tau)] \mathbf{u}(t - \tau). \end{aligned} \quad (9)$$

The term $\ln(\sigma/r_0)/\tau$ in the instantaneous part of the vector field does not influence the maximum LE of Eq. (9) in leading order $1/\tau$ for weak chaos. Indeed, by substituting $\mathbf{u} \sim e^{i\omega t + \gamma/\tau}$ we see that only the terms $|\dot{\mathbf{u}}| \sim \omega$, $|DF[\mathbf{s}(t)] \mathbf{u}| \sim 1$, as well as $|r_0 DH[\mathbf{s}(t - \tau)] \mathbf{u}(t - \tau)| \sim r_0 e^{-\gamma}$ contribute to the leading order. Hence, the maximum LE of Eq. (9) is zero, as well, and we obtain $\lambda_m = \ln(\sigma/r_0)/\tau$, taking into account the relation between \mathbf{u} and ξ . This leads to the estimate (8) for the critical value r_0 . Then synchronization is stable if $\lambda(\sigma \gamma_k) < 0$, and hence if $|\sigma \gamma_k| < r_0$ for all k . With Eq. (8) this results in Eq. (7).

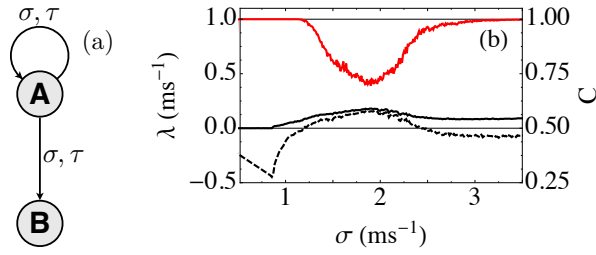


Figure 4. (Color online) (a) Experimental setup to measure the difference between strong and weak chaos. (b) Simulated λ_m (solid line) and λ_0 (dashed line) of the SM of the two electronic circuits and experimentally measured cross-correlation C (red (gray) line) between the maxima of the time series of the two electronic circuits vs. coupling strength σ .

The condition (7) rules out synchronization for networks with strong chaos since the right-hand side of Eq. (7) decreases to zero in the limit of large delay times τ . For weak chaos, however, one can always find networks for which Eq. (7) is true, i.e., chaos synchronization is stable. In addition, for weak chaos, condition (7) becomes independent of τ , as $\lambda_m \tau \rightarrow \text{const}$, in agreement with recent results [7, 9]. The network can be synchronized in this case even for arbitrarily large τ if Eq. (7) is fulfilled.

For a single laser with feedback we have found a scenario leading from weak to strong chaos and back to weak chaos with increasing feedback strength σ . But also for networks outside the regime of synchronization, we can define an instantaneous LE for each unit by the maximum LE of the equation $\dot{\xi}(t) = DF[\mathbf{x}_i(t)] \xi(t)$. Simulating this together with Eq. (1) for a triangle of bidirectionally coupled lasers, we found similar results as in Fig. 1. The network changes from weak to strong chaos and back to weak chaos with increasing coupling strength σ . The critical coupling strengths, however, have different values.

Finally, we have performed an experiment with two coupled electronic circuits [17] to measure the difference between strong and weak chaos. For general chaotic networks, one can add two identical units which are driven by one unit of the network with identical strengths, similar to the test for generalized synchronization [18]. Chaos is weak if and only if the two units synchronize. For determining the type of chaos on the SM, it is sufficient to add one unit which is driven by a single unit with delayed feedback representing the SM, as sketched in Fig. 4(a). The stability of synchronization of the two units is given by Eq. (4). Fig. 4(b) shows the simulated LEs λ_m and λ_0 of the SM in comparison with the experimentally measured cross-correlation C between the maxima of the time series of the two electronic circuits as a function of the coupling strength σ . For small σ we observe zero-lag synchronization of periodic dynamics. With increasing σ the dynamics becomes chaotic while complete synchro-

nization is maintained. With further increase of σ the cross-correlation first decreases and then increases again until synchronization is reached once more, indicating transitions from weak to strong chaos and back to weak chaos.

The notion of strong and weak chaos allows for a classification of the synchronizability for coupled chaotic nodes and, most notably, shows a significant difference of the chaotic behavior, characterized by the maximum and the instantaneous LE. Our findings are promising for applications in laser dynamics and beyond. In random number generators, where a high entropy is crucial, the regime of strong chaos will potentially lead to an increase in randomness. Similarly, in other applications like optoelectronic oscillators, devices can be deliberately constructed to operate in the regime of strong chaos.

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