Advances in Applied Mathematics **27,** 17–40 (2001)
doi:10.1006/aama.2001.0722, available online at http://www.idealibrary.com on IDEAL®

The Dynamics of Schelling-Type Segregation Models and a Nonlinear Graph Laplacian Variational Problem

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October 15, 2000; accepted December 16, 2000; published online May 18, 2001

In this paper we analyze a variant of the famous Schelling segregation model in economics as a dynamical system. This model exhibits, what appears to be, a new clustering mechanism. In particular, we explain why the limiting behavior of the non-locally determined lattice system exhibits a number of pronounced geometric characteristics. Part of our analysis uses a geometrically defined Lyapunov function which we show is essentially the total Laplacian for the associated graph Laplacian. The limit states are minimizers of a natural nonlinear, nonhomogeneous variational problem for the Laplacian, which can also be interpreted as ground state configurations for the lattice gas whose Hamiltonian essentially coincides with our Lyapunov function. Thus we use dynamics to explicitly solve this problem for which there is no known analytic solution. We prove an isoperimetric characterization of the global minimizers on the torus which enables us to explicitly obtain the global minimizers for the graph variational problem. We also provide a geometric characterization of the plethora of local minimizers. © 2001 Academic Press

[1] The work of the second author was partially supported by a National Science Foundation Grant DMS-9704913. This work began during the second author's sabbatical visit at IPST, University of Maryland, and he thanks IPST for their gracious hospitality. The second author thanks Todd Young for some interesting discussions during the genesis of this project.

0196-8858/01 $35.00

Key Words: Schelling segregation model; discrete Laplacian; nonlinear variational problem; nonhomogeneous variational problem; isoperimetric inequality.

INTRODUCTION

In this paper we introduce and study an interesting new class of deterministic and stochastic lattice dynamical systems.[2] Our motivation to study this class of models arose from trying to explain the striking limiting behavior of the seminal segregation model of Schelling in economics. This model can also be viewed as a new type of two-spin exchange kinetics for the lattice gas in statistical physics.

In a fundamental series of papers and in a book [S1–S4] (see also [Klu]), the eminent economist Thomas Schelling proposed a remarkable model which exhibits self-forming neighbourhoods based on the desire of people to live with those *of their own type*, with whom they empathize. In these models the individual's micro-level preferences (about their nearest neighbours) manifest themselves in a striking way at the macro level. This prescient model from 1971 exhibits many themes encountered in contemporary literature on agent-based modeling, social complexity, and economic evolution. By the term *own kind*, Schelling refers to membership in one of two homogeneous groups—men or women, blacks and whites, French-speaking and English speaking, officers and enlisted men, students and faculty, surfers and swimmers, the well dressed and the poorly dressed, and any other dichotomy that is exhaustive and recognizable.

The *phase space* for the Schelling model is a finite square subset of the standard lattice in $\mathbb{R}^2$. To each of the lattice sites one can associate labels for individuals of one of two kinds (and in the more sophisticated versions of the model, more (than two) kinds, along with allowing a number of unlabeled, or empty, states). Schelling's model allows pairs of individuals who are both not *happy* with the number of compatible nearest neighbours, to switch sites. The two sites may be far away, and thus the **dynamics is not local**, in the sense that a cellular automata is determined by local rules. In fact, these models can be thought of as cellular automata with *migration*, and it appears such models have not been rigorously studied before. In fact, surprisingly little is rigorously known about cellular automata in more than one dimension.

Schelling devised his model 30 years ago and studied it using nickels and pennies on a chess board (an eight-by-eight lattice). In more recent years there have been extensive computer studies of this model [EA, GD], and it

[2] The systems we study are not always dynamical systems in the strict sense of a $\mathbb{Z}^+$ action, but are discrete systems which evolve in time.

has become perhaps the most famous model of self-organizing behavior. To social scientists, this model demonstrates that spatial segregation, or ghettoization, can occur spontaneously, without being imposed by a central authority, based on *relatively modest* desires of people to live around those with whom they empathize. This can result in the clustering of people by gender, age at a social gathering, or in the clustering of people by ethnicity or race in society at large.

In this paper we present a mathematical explanation of the observed limiting behavior for a variant of Schelling's model. Schelling originally assumed that no individual would move if a certain percentage of his neighbours were from the same group. In our variant of his model we suspend the *tolerance levels*, thus an individual will move whenever he can increase his happiness, regardless of his current happiness. However, our model is similar in spirit to Schelling's model and exhibits qualitatively similar limiting behavior. We explain why the limit configurations have striking geometric features and our analysis of this variant model provides insights that cast light on Schelling's original model. The two authors are currently writing a manuscript analyzing the original model. The techniques in the second paper use ideas from ergodic theory and probability theory and are quite different than the methods we use in this paper.

We make use of basic ideas in the study of dynamical systems. We construct a Lyapunov function for the dynamics which has striking geometric, spectral, physical, and sociological interpretations. The geometric interpretation is that the Lyapunov function of a state is *essentially* the total perimeter of the boundary contour separating the two groups. We prove an isoperimetric result for *grid domains* on the flat torus to characterize the special geometry of the global minimizing states.

These models are related to the nearest neighbour lattice gas model introduced by Lee and Yang [LY], which is a close cousin of the (nearest neighbour) Ising model. For the lattice gas in the regular $N \times N$ square lattice, if there is a molecule occupying lattice site v, we put $\eta_v = 1$; otherwise $\eta_v = 0$. The Hamiltonian (or total energy) of this configuration is

$$H(\{\eta_v\}) = -\frac{1}{N^2} \sum_{v,w:|v-w|=1} \eta_v \eta_w,$$

where the sum is over nearest neighbours. By changing to spin coordinates $\eta_v = (1/2)(1 + s_v)$, the function s_v attains the values -1 and $+1$, one sees that the Lyapunov function L we construct in Section 2 is *essentially* the Hamiltonian for this lattice gas on a torus, where essentially means the two functions differ by a first integral of motion. Thus the limit states for our variant of the Schelling dynamics correspond to the ground state (minimal energy) configurations for the lattice gas. The lattice gas is closely related to the two dimensional Ising model on torus, the difference being that for the lattice gas the total number of gas molecules does not change.

In the physics literature there are several popular *dynamics* or kinetics related with the Ising and lattice gas-type models, e.g., the Glauber and Kawasaki dynamics. In these models the usual spin-interaction is replaced by certain **local** temperature-dependent transition probabilities of spin-exchange. Our non-local model seems to be a new type of lattice gas kinetics, which is different from both Glauber and Kawasaki dynamics.

Of independent interest, along the way we find the explicit solution to a natural nonlinear nonhomogeneous variational problem for a graph Laplacian on the torus. We could find no analytic solution of this problem in the literature. The limit states for our variant of Schelling's model are minimizers for this variational problem. Thus we obtain an explicit solution as an immediate consequence of our study of the dynamics of the model.

1. DESCRIPTION OF OUR VARIANT OF SCHELLING'S MODEL

The *phase space* for the family of models consists of a $N \times N$ square sub-lattice Λ_N on the standard two-torus T^2. The torus arises because we consider periodic functions on the lattice. This is probably not an essential assumption, but it helps to simplify the exposition.[3]

We first discuss the model with two distinct populations, say 1's and -1's, which together fill all available N^2 sites. Each possible (global) configuration of 1's and -1's is specified by a function or *state* or *configuration* x: $\Lambda_N \to \{-1, 1\}$, where one associates a label 1 or -1 to each site. Let

$$\mathscr{H}_N = \{x\colon \Lambda_N \to \{-1, 1\}\}$$

denote the collection of all states.

To describe the *time evolution* of the system, we explain how the system evolves from state $x_n \in \mathscr{H}_N$ to state $x_{n+1} \in \mathscr{H}_N$. Using some mechanism we choose two sites with different labels, i.e., site v is labeled 1 and the other site w is labeled -1. There are many possible ways of selecting these sites and thus one obtains families of related models. For example, one can

[3] We say probably because we do not consider a thermodynamic limit with $N \to \infty$; we fix N which need not be very large. Computer simulations show that for moderate sized N, the qualitative features of the limit set of the models are essentially independent of the boundary conditions.

randomly choose sites until one obtains two sites with different labels. Or we can assume that the $N^2(N^2-1)$ pairs of distinct sites are ordered in some way and systematically run through these pairs until different labelings are found. Finally, one can select pairs from among the *unhappiest* label 1 and *unhappiest* label -1 at step n, meaning the sites containing 1 with the maximal number of nearest neighbours labeled -1 and the site containing -1 with the maximal number of nearest neighbours labeled 1.[4] Or one can specify an ordering of the sites and use this ordering to obtain two sites with different labels. There are many possible selection rules and our analysis applies to all such rules, so let us now assume that we have fixed one such rule.

The Notation. Let $v \in \Lambda_N$ denote a site labeled 1 and $w \in \Lambda_N$ denote a site labeled -1. Thus $x_n(v) = 1$ and $x_n(w) = -1$. Since our lattice is *periodic* we can associate to each site its four nearest neighbours to the north, east, south, and west. We note that one can also consider *eight point neighbourhoods* consisting of the eight neighbours to the north, northeast, east, southeast, south, southwest, west, and northwest. There are few qualitative differences for the dynamics with the two choices of neighbourhoods, and we will work with four point neighbourhoods (see also Section 6).

Let $\#1_v$ denote the number of nearest neighbour sites to v containing 1, $\#(-1)_v$ the number of nearest neighbour sites to v containing (-1), $\#1_w$ the number of nearest neighbour sites to w containing 1, and $\#(-1)_w$ the number of nearest neighbour sites to w containing the label (-1). Clearly,

$$\#1_v, \#(-1)_v, \#1_w, \#(-1)_w \in \{0, 1, \dots, 4\},$$

$$\#1_v + \#(-1)_v = 4,$$

and

$$\#1_w + \#(-1)_w = 4.$$

The Basic Algorithm. Suppose we are given the state of the system x_n at time n; we now describe how to obtain the state x_{n+1}. Schelling's idea is to measure each site's *happiness* by counting the number of nearest neighbour sites with the same label. Thus a labeling 1 at a particular site would be most happy if all four of its nearest neighbours were labeled 1, and would be most unhappy if all four of its nearest neighbours were

[4] Since there could be several 1's (or -1's) with the maximal number of nearest neighbours being -1 (or 1), one may need to require some extra conditions to make this prescription well defined, perhaps the closest nearest neighbour or closest nearest neighbour at smallest angle. With such a deterministic selection rule the system we will define is a dynamical system, in the usual sense, on a collection of states.

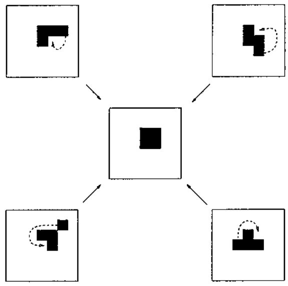

FIG. 1. Four different configurations reaching the same equilibrium configuration.

labeled -1. If the 1 at site v and the -1 at site w would increase their happiness if they *switched labels*, then we switch them. Observe that this is equivalent to either site increasing its happiness.

More precisely, if $\#1_w > \#1_v$ (or equivalently $\#(-1)_v > \#(-1)_w$), then x_{n+1} is prescribed by

$$\begin{aligned} x_{n+1}(v) &= -1 \\ x_{n+1}(w) &= 1 \\ x_{n+1}(z) &= x_n(z) \qquad \text{for } z \neq v, w. \end{aligned}$$

If $\#1_w \leq \#1_v$ (or equivalently $\#(-1)_v \leq \#(-1)_w$) then the state does not change, i.e., $x_{n+1} = x_n$.

Limiting Configurations. We define the **limiting configuration** for a system or the **equilibrium state** beginning from state x_0 of the system to be the pre-fixed points for the system; i.e., there exists $N > 0$ such that $x_n = x_N$ for $n \geq N$ (see Fig. 1).

This procedure generates a well-defined time evolution $\{x_n\}$, $n \geq 0$, of an initial state x_0, which of course depends on the pair selections at each step. The less patient readers may prefer studying a slightly different process $\{x'_k\}$ which is obtained from $\{x_n\}$ by disregarding consecutive strings of the same state. In other words, if the two chosen sites from x_n don't switch labels, then we continue to choose pairs until a switch does occur, and denote the resulting configuration x'_{n+1}. We call this finite process the **accelerated process**. If the selection process is deterministic, the system must eventually reach a limiting configuration.[5]

[5] In a modified construction where changes occur even when there is no net gain to the two individuals concerned, then there is the additional possibility of periodic limiting behaviour.

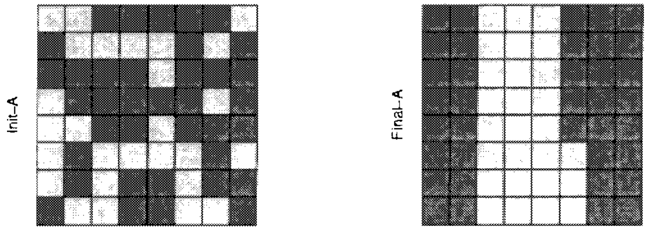

FIGURE 2

First Integrals. Clearly the time evolution preserves both the total number and the average number of 1's and -1's at each step. With an eye towards applications to variational problems for the Laplacian, we see that the average of the difference between the total number of 1's and the total number of -1's is also conserved, i.e.,

$$I(x) \equiv \frac{1}{N^2} \sum_{v \in \Lambda_N} x(v)$$

is a *first integral* of the time evolution. For instance, if for x_0 one has that $\#$ 1's $= \# -1$'s, then $I(x_n) = 0$ for all $n \geq 0$.

Plots. Figures 2–11 contain Matlab plots of the initial and final states for two parameter values with $N = 8$. In Figs. 2–5 the initial configuration contains (almost) an equal number of 1's and -1's, while in Figs. 6 and 7, the initial configuration contains 70% 1's and 30% -1's. Figures 10 and 11 are plots with $N = 30$ where the initial configuration contains 60% 1's and 40% -1's. Note that one label in the limit set in Fig. 5 is not connected. Each label in the limit set depicted in Fig. 10 has eight connected components.

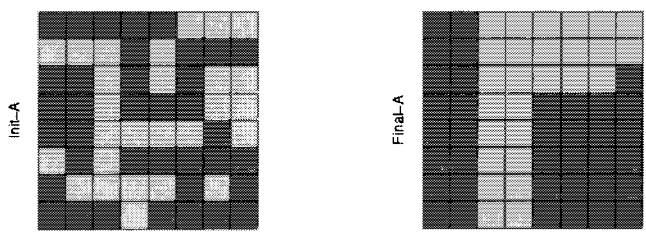

FIGURE 3

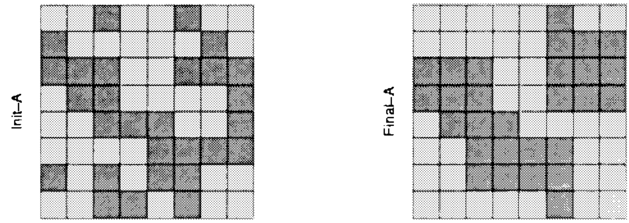

FIGURE 4

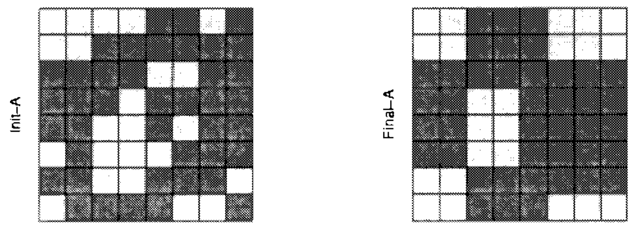

FIGURE 5

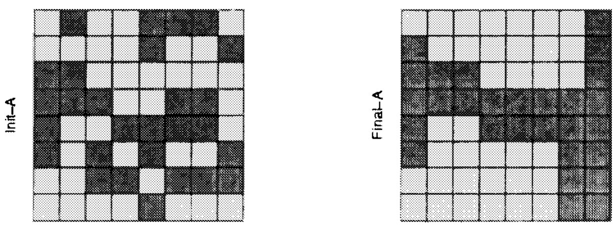

FIGURE 6

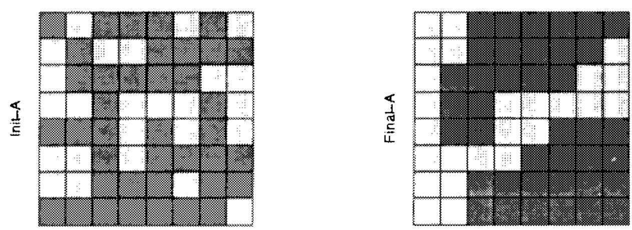

FIGURE 7

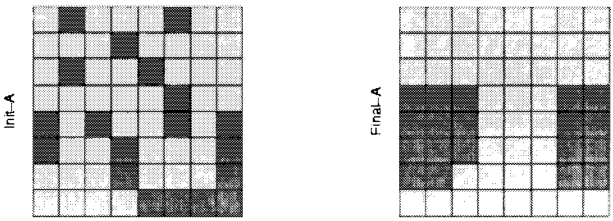

FIGURE 8

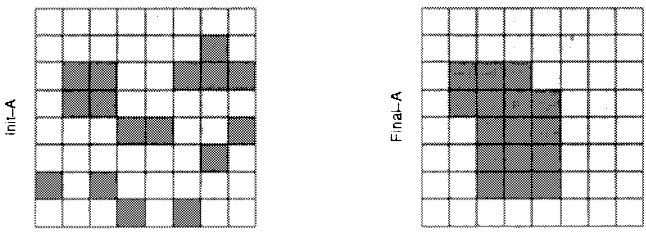

FIGURE 9

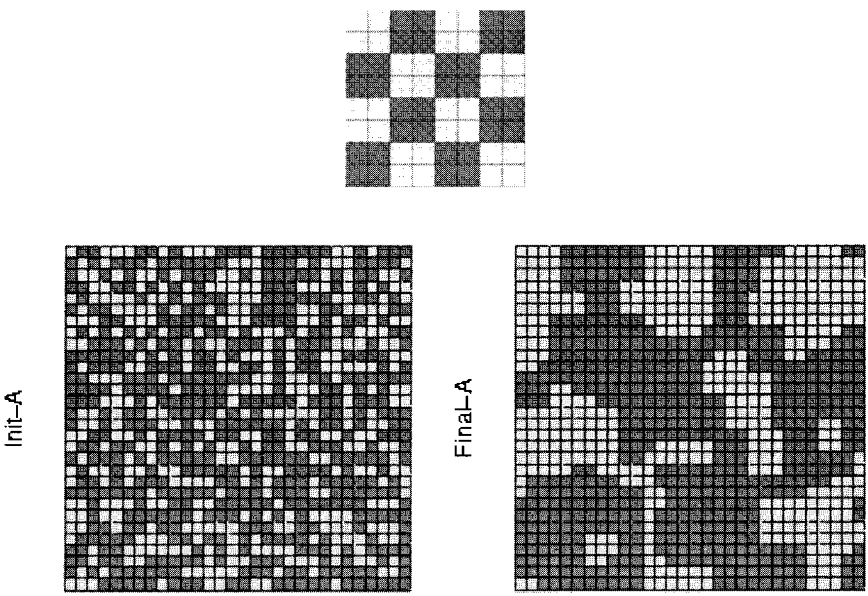

FIG. 10. Limit set where one label forms a connected set on $\mathbb{T}^2$.

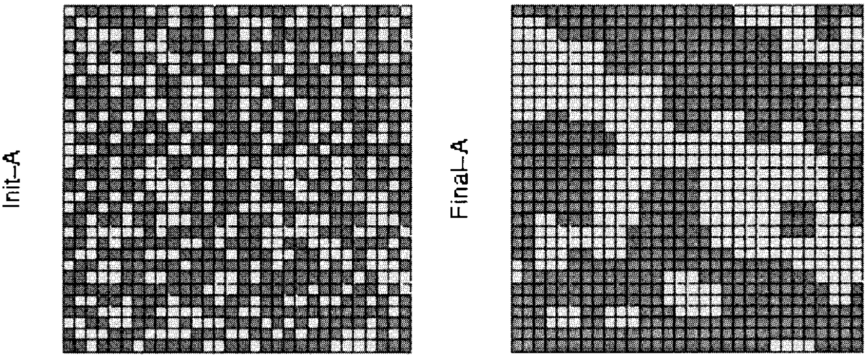

FIG. 11. Limit state where neither label forms a connected set $\mathbb{T}^2$.

2. LYAPUNOV FUNCTIONS FOR THE BASIC MODEL

We now construct a **Lyapunov function** for states of this dynamical system, i.e., construct a non-negative function $L : \{-1, 1\}^{\Lambda_N} \to \mathbb{R}$ such that $L(x_{n+1}) \leq L(x_n)$ for all $n \geq 0$. We actually construct a **strict Lyapunov function** for the accelerated algorithm such that $L(x'_{n+1}) < L(x'_n)$ for all $n \geq 0$, provided that the state x'_n is not a limit state, in which case $L(x'_k) = L(x'_n)$ for $k \geq n$.

We first describe a measure of the *average happiness of a state* and show that this function is a Lyapunov function. Given a state x, we define the **average happiness** $S(x; v)$ **of a site v** $\in \Lambda_N$ by counting the number of immediate neighbours having label $x(v)$, subtracting the number of immediate neighbours which are not labeled $x(v)$, and normalizing by dividing by the number of nearest neighbours (i.e., four). We can quantify this by defining the **happiness of the site** v **given the configuration** x to be

$$S(x; v) = x(v) \cdot \frac{1}{4} \sum_{\substack{(i,j)=(-1,0),(1,0)\\(0,1),(0,-1)}} x(v + (i, j)).$$

(We use the convention that we consider the indices modulo N.) Observe that $-1 \leq S(x; v) \leq 1$. The maximum value 1 is achieved when v takes the same value as all of its four neighbours and the minimum value -1 is attained then v takes the opposite value as all of its four neighbours.

We then define the **average happiness** $S(x)$ **of a state** x by averaging

$$S(x) = \frac{1}{N^2} \sum_{v \in \Lambda_N} S(x; v)$$
$$= \frac{1}{N^2} \sum_{v \in \Lambda_N} \frac{1}{4} \sum_{\substack{(i,j)=(-1,0),(1,0)\\(0,1),(0,-1)}} x(v) \cdot x(v + (i,j)),$$

where we average the function x over the four nearest neighbours of v. Clearly the function S attains values in $[-1, 1]$, and thus, in particular, the function $S + 1$ is non-negative.

THEOREM 1. *The average happiness function S is increasing, and strictly increasing (until it reaches an equilibrium configuration) for the accelerated system. In particular,* $L \equiv -S$ *is a Lyapunov function.*

Proof. This follows from the definition of the algorithm. The system evolves from a configuration x_n to a configuration x_{n+1} if two prescribed states $v, w \in \Lambda_N$ with different labels switch and both benefit by increasing their proportion of similar nearest neighbours. In terms of the average happiness function $S(x)$, observe that the switch only influences the, at most, total of four nearest neighbours of these two states. However, if the switch benefits labeling at the state v, say, then there must be a greater number of like signed nearest neighbours to the new site w then at the original site and each of these like signed nearest neighbours to w increases its own local happiness by $1/4$.

If $x_n(v) = +1$ and $x_n(w) = -1$, then a simple calculation shows that

$$L(x_n) - L(x_{n+1}) = (\#(+1)_w - \#(+1)_v) + (\#(-1)_v - \#(-1)_w).$$

Since a switch requires that both $(\#(+1)_w - \#(+1)_v) > 0$ and $(\#(-1)_v - \#(-1)_w) > 0$, it follows that $L(x_{n+1}) > L(x_n)$. ∎

The next proposition gives a striking geometric characterization of the Lyapunov function L.

PROPOSITION 1. *For a state x, the Lyapunov function*

$$L(x) = \frac{P(x)}{2N^2} - 1,$$

where $P(x)$ *denotes the perimeter of the boundary contour (see Fig. 12) that separates inhomogeneous neighbours of x.*

Proof. For a given state x, the collection of all 1's comprises finitely many connected regions, whose boundary consists of finitely many closed contours. The boundary contours are composed of horizontal and vertical

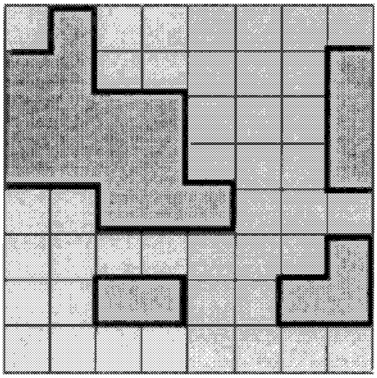

FIG. 12. Boundary contour of state x.

line segments and separate sites containing 1 from the neighbouring sites containing -1. Any site containing a 1 can be surrounded by one, two, three, or four sites containing -1, and thus can be surrounded by one, two, three, or four segments of the boundary contour.

From the definition of $S(x)$ it follows that

$$\begin{aligned} 4N^2(1 + L(x)) &= 4N^2(1 - S(x)) \\ &= \sum_{v \in \Lambda_N} \sum_{\substack{(i,j)=(-1,0),(1,0) \\ (0,1),(0,-1)}} (1 - x(v) \cdot (v + (i,j))). \end{aligned}$$

For each site v, the inner sum is precisely the number of segments of the boundary contour surrounding the site v. Thus when summing this quantity over all sites v, one obtains twice the total number of boundary segments, twice because each boundary segment is counted by each of the two inhomogeneous sites which the segment separate. The formula easily follows. ∎

The next proposition shows that the maximum number of steps for the accelerated system to reach equilibrium is at most quadratic in N.

PROPOSITION 2. *The strict Lyapunov function L provides an estimate for the time T required for the accelerated system to reach an equilibrium point. More precisely, there exists $T \le N^2(1 + L(x)) \le 2N^2$ and an equilibrium state x'_* such that the accelerated system reaches its equilibrium state in T steps, i.e., $x'_T \equiv x'_*$.*

Proof. We need only observe that $1 + L(x')$ is strictly decreasing at each step, and that the function attains values in $(1/N^2)\mathbb{N}$. ∎

We can define an L^1-metric on $\mathbb{R}^{\Lambda_N}$ by

$$d(x, y) = \sum_{v \in \Lambda_N} |x(v) - y(v)|, \qquad x, y \in \mathbb{R}^{\Lambda_N}.$$

This induces a metric on $\mathscr{H}_N$ taking values in $2\mathbb{Z}^+$. A natural small neighbourhood around a configuration x takes the form $U(x) = \{y \in \Lambda_N : d(x, y) \leq 2\}$. This corresponds to all configurations which can be achieved by a single interchange of two labels. This will quantify our notion of **nearby** configurations.

DEFINITION. Consider a state $x \in \mathscr{H}_N$.

The state ix is a **local minimum** for the Lyapunov function L if for any $y \in U(x)$, we have $L(y) \geq L(x)$.

The state x is a **global minimum** for the Lyapunov function L if $L(y) \geq L(x)$ for all $y \in \mathscr{H}_N$.

The state x is a **local maximum** for the Lyapunov function L if for any $y \in U(x)$, we have $L(y) \leq L(x)$.

The state x is a **global maximum** for the Lyapunov function L if $L(y) \leq L(x)$ for all $y \in \mathscr{H}_N$.

The following corollary and proposition are immediate consequences of Proposition 1.

COROLLARY 1. *If a state x is a local (global) minimum for L, then the state x is a local (global) minimum for P.*

PROPOSITION 3 (Contour Minimizing Principle). *The limit states are local minima of the Lyapunov function L and are thus locally boundary contour perimeter minimizing configurations.*

Let $R \subset \Lambda_N$ be a collection of sites. We say that R is an **island** if all the sites in R have the same label, the boundary contour of R has one component and is a simply connected curve on the torus, and all sites along the boundary contour of R have a different label. We say that R is a **strip** if all the sites in R have the same label, the boundary contour of R has two components, each boundary component is a closed curve which winds once around the torus, and all sites along the boundary contour of R have a different label.

PROPOSITION 4. *The global minimizers of L contain a single island or a single strip.*

Proof. If we assume for a contradiction there were more than one island, or more than one strip, or one island and one strip for a configuration, then we could always consider a new configuration in which one of the domains were translated until it overlaps another along at least one segment of the boundary contour. This would result in both a strict decrease in P and a smaller number of components. Proceeding inductively, we see that minimizers for L are configurations which contain at most a single island or a single strip. ∎

The limit state illustrated in Fig. 5 is a local minimizer of L where one label has two connected components. Thus the claim of Proposition 4 need not hold for local minimizing states.

3. GEOMETRY OF THE LIMIT SET

We will first characterize those configurations which are global minima for L. The following isoperimetric result states that the global minimal are essentially squares, complements of squares, or strips which traverse the torus.

Case A deals with global minima for L where there are a comparable number of $+1$'s and -1's. Case B describes the global minima of L in the case of an abundance of $+1$'s (or -1's). By a **strip** we mean a $k \times 1$ block of sites with the same labeling.

THEOREM 2 (Isoperimetric Characterization of Global Minima).

Case A. *Denote by N_1 the total number of sites with sign $+1$, $N_{-1} = N^2 - N_1$ the total number of sites with sign -1, and suppose that*

$$\frac{N}{2} + \frac{1}{3} \le N_1^{1/2} \le \frac{N}{\sqrt{2}} \qquad \text{or} \qquad \frac{N}{2} + \frac{1}{3} \le N_{-1}^{1/2} \le \frac{N}{\sqrt{2}}.$$

Then a global minimum x_* for L is a horizontal or vertical band traversing the torus with the addition of possibly at most one strip attached to the band. More precisely,

(1) *If $N_1 = kN$, $k \in \mathbb{N}$, then the limit state has the $+1$ (or alternatively -1's) aligned in a horizontal or vertical band of width k, and $P(x_*) = 2N$.*

(2) *Assume that $N_1 = kN + i$, $k \in \mathbb{N}$ and $0 < i < N$. Then the limit state has the $+1$ (or alternatively -1) aligned in a horizontal or vertical band of width k plus a strip of length i attached at the side, and $P(x_*) = 2N + 2$.*

Case B. *Suppose that*

$$N_1^{1/2} \le \frac{N}{2} + \frac{1}{3} \qquad \text{or} \qquad N_{-1}^{1/2} \le \frac{N}{2} + \frac{1}{3}.$$

The limit states x'_* which globally minimize the function L have islands R of -1's (or alternatively $+1$'s) in the form of a square with at most two additional strips attached. Define $l \in \mathbb{N}$ by $l^2 \leq N_{\pm 1} < (l+1)^2$. Then

(1) *if $N_{\pm 1} = l^2$, then R is a $l \times l$ square and $P(x'_*) = 4l$;*

(2) *if $l^2 < N_{\pm 1} = l^2 + i < l(l+1)$, then R is a $l \times l$ square with the additional sites attached as a $i \times 1$ strip to the sides, where $P(x'_*) = 4l + 2$;*

(3) *if $N_{\pm 1} = l(l+1)$, then R is a $l \times (l+1)$ rectangle and $P(x'_*) = 4l + 2$;*

(4) *if $l(l+1) < N_{\pm 1} = l^2 + i < (l+1)^2$, then R is a $l \times l$ square with the additional sites attached to the sides as two strips and $P(x'_*) = 4l + 4$.*

Proof. By Proposition 4, the global minimizers are connected and consist of a single island or strip.

Consider first those limiting configurations x which consist of a strip. We can consider the torus as a square with edge identifications, and this provides a notion of horizontal and vertical axes. Without loss of generality let us assume that x contains a vertical strip. Let W denote the sum total of the lengths of the projections of the perimeter curves onto the horizontal axis. Clearly, $P(x) \geq 2(N + W)$ and is minimized when $W = 0$, in (1) of Case A, or when $W = 1$, in case (2) of Case A. Within this latter case $P(x)$ is minimized where the extra boundary sites occur on just one boundary component and are contiguous.

Consider next those limiting configurations x which form a single island. We can consider the horizontal and vertical projections, and let H and V be their respective lengths. We observe that $P(x) \geq 2(H + V)$, with equality if the island is convex, in the natural sense. Moreover, since $HV \geq l^2$, we may write $P(x) \geq 2(l^2/V + V)$. By elementary calculus we see that $P(x) \geq 4l$, and since equality only occurs when $N_1 = l^2$ this completes the proof of (1) of Case B. For (2) and (3) we note that $P(x) \geq 4l + 2$, since $P(x) = l^2$ cannot be realized for $N_1 > l^2$. In these two subcases $P(x) = 4l + 2$ can only be realized for a island contained in a rectangle with sides of length l and $l + 1$. This is achieved precisely for the islands described in (2) and (3). Finally, in (4), where $N_1 > l(l+1)$, then we note that $P(x) \geq 4l + 4$ with equality only for an island contained in a square with sides of length $l + 1$. Among such islands, the minimum $P(x) = 4l + 4$ is achieved for the islands described in (4).

Finally, one needs to differentiate between Case A and Case B, i.e., when islands have shorter perimeters than strips. Since perimeter minimizing islands have perimeter $4l$ or $4l + 2$ and perimeter minimizing strips have perimeter $2N$ or $2N + 1$ a simple calculation completes the proof. ∎

Remark. The two chessboard states are the only global maxima for L since they have maximal perimeter. These chessboard states are not limit states (see Fig. 13).

We will call a limit state which is globally minimizing for L a **globally minimizing limit state**. According to Theorem 2, there are two types of geometries for globally minimizing limit states, corresponding to Case A and Case B.

Local Minima for L and P. Not all minima for L are global minima. Figures 2 and 8 are examples of limit states which are global minima, while Figs. 3–7, 9, 10, and 11 are examples of limit states which are local but not global minima. Experimentally, it appears that local minima occur with much greater frequency than global minima.

There are some simple criteria for a configuration not to be a local minimum. For example, if we have a site with a particular labelling, but its four neighbours are of the other labelling, then it is clearly not part of a configuration which is a local minimum.

More generally, consider a state x and denote by Γ the boundary between the two types of sites. We say that the boundary Γ has a **sharp bend** if there exists a site in the configuration for which three of the four sides are part of the boundary curve Γ.

The following result gives a simple geometric criterion to check whether a given configuration is a local minimum.

THEOREM 3 (Geometric Criterion for Local Minima). *Assume that there are at least three sites of each label. A configuration x is a local*

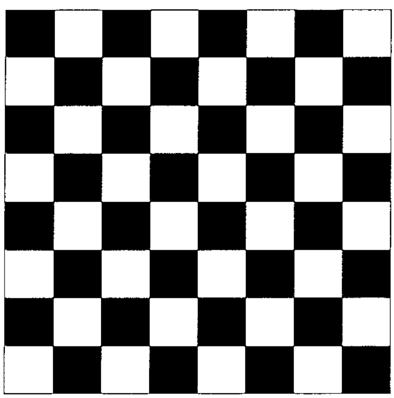

FIG. 13. One of two chessboard configurations.

minimum if and only if either

(1) *the boundary curve contains no sharp bend*, *or*

(2) *there exists a single sharp bend which is attached to a disjoint union of rectangular islands or bands*.

Proof. Let x be a limit state. A necessary and sufficient condition for x to be a local minimum is that no two sites can switch in such a way as to further decrease the perimeter. Assume first that the separating perimeter has no sharp bends. It is a simple observation that every site must have at least two neighbours of the same type. In particular, a site could only switch with another site on the boundary of one of these components which has at least three neighbours with the same label. However, the existence of such sites is forbidden by our hypothesis.

Consider next the case that the perimeter of x has sharp bends. Any component containing at least four sites can have at most one sharp bend, since if there were more, then the sites corresponding to the sharp bends could coalesce. Furthermore, if the perimeter of x has sharp bends then we see that the components must consist of rectangular islands or strips, with a single site attached to the boundary, since in all other cases the site associated to the sharp bend could move into a boundary site (corner). Thus, the only states whose perimeters contain more than one sharp bend have islands containing two or three sites.[6] (see Fig. 14.) ∎

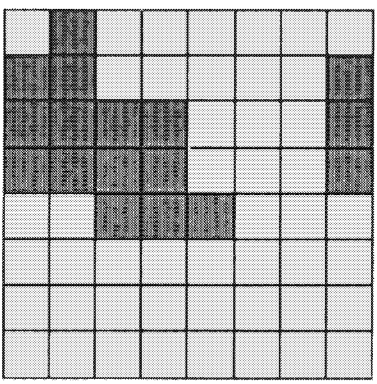

FIG. 14. State with two sharp bends (but not a limit state).

[6] Such islands are isolated, in the sense that their boundary contour is simply connected.

4. STABILITY OF LIMIT STATES

We now examine the stability of limit states. We begin with the following definition.

DEFINITION. A limit state x is **stable** if for every nearby configuration $y \in U(x)$, there exists $N \geq 0$ such that $y_n \in U(x)$ for $n \geq N$. Thus x is stable if the evolution of every nearly configuration eventually re-enters $U(x)$ and then stays in $U(x)$ for all future time. A limit state x is **unstable** if there exists $N > 0$ with $y_n \notin U(x)$ for $n \geq N$. Thus x is unstable if the evolution of some nearby configuration eventually moves out, and stays out, of the neighbourhood $U(x)$.

PROPOSITION 5. *Any globally minimizing Case A limit state is stable, while any globally minimizing Case B limit state is unstable.*

Proof. Figure 15 illustrates a mechanism which causes any globally minimizing Case B limit state, which is essentially a square box of sites labeled 1's in a sea of sites labeled -1's, to be unstable. The first picture is of a globally minimizing limit state x. The second picture shows the perturbation, i.e., the state y obtained from the original limit state by switching two (different) labels. The perturbation is chosen to introduce three new sharp bends. Then in two steps, the perturbed state evolves into a limit state y_2 which is not globally minimizing. The distance $d(x, y_2) = 6$, and thus $y_2 \notin U(x)$.

On the other hand, suppose one considers a globally minimizing Case A limit state, which is essentially a strip of sites labeled 1 traversing the torus. First assume that the limit set is precisely a strip, with no extra 1's attached. In this case each site labeled 1 has at least three like neighbours. If one now switches any two (different) sites, then the new site labeled 1 will have at most one like neighbour. If it has one like neighbour, the system must return to the initial state at the next step of the iteration. If the new site labeled 1 has no like neighbours, then at the next step it could switch with a -1 to attain one like neighbour, but then the system must return to the initial state at the next step of the iteration.

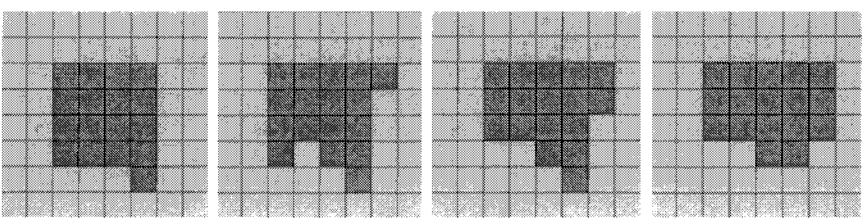

FIG. 15. A globally minimizing unstable limit state.

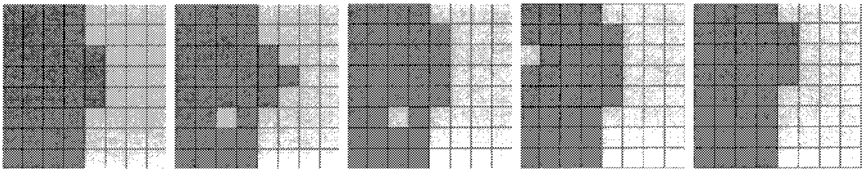

FIGURE 16

In the case that the initial limit state has some additional 1's attached to the traversing strip of 1's, one must consider a couple of additional cases. For instance, after the initial switch, the new site labeled 1 may have two like neighbours. We leave the easy enumeration of additional cases to the reader. Figure 16 illustrates a typical scenario and is an example where after one switches two sites, the system does not eventually return to the initial state, but to a nearby state. ∎

5. THE GRAPH LAPLACIAN AND A NONHOMOGENEOUS VARIATIONAL PROBLEM

We can associate to our set a finite graph with vertices Λ_N and edges joining vertices $(i,j),(i',j') \in \Lambda_N$ if $|i-i'| \leq 1$ and $|j-j'| \leq 1$. The spectrum of graph Laplacians on such graphs are the subject of intense research and practical interest [Chu].

Let us define a graph **Laplacian** operator $\Delta : \mathbb{R}^{\Lambda_N} \to \mathbb{R}^{\Lambda_N}$ acting on the space of states.

Given a function $x \in \mathbb{R}^{\Lambda_N}$, we define

$$(-\Delta x)(v) = x(v) - \frac{1}{4} \sum_{\substack{(i,j)=(-1,0),(1,0)\\(0,1),(0,-1)}} x(v+(i,j)). \tag{L}$$

Given a state x and a vertex v, the Laplacian of x at v is simply the average of x over all four nearest neighbours of v minus the value of x at v.

We follow the usual convention and work with the positive operator $-\Delta$ instead of the negative operator Δ. We denote by $\langle f,g\rangle = (1/N^2)\Sigma_{v\in\Lambda_N} f(v)\cdot g(v)$ the natural inner product on the set of functions $f,\ g : \mathbb{R}^{\Lambda_N} \to \mathbb{R}^{\Lambda_N}$. This induces a norm $\|f\|_2^2 = \langle f,f\rangle$ on such functions.

The following two relations are immediate.

PROPOSITION 6.

$$-\Delta x(v)\cdot x(v) = 1 - S(x;v)$$

$$\langle -\Delta x, x\rangle \equiv \frac{1}{N^2}\sum_{v\in\Lambda_N}(-\Delta x)(v)\cdot x(v) = 1 - S(x) = 1 + L(x).$$

By Proposition 3, the limit state x'_* is a local minimizer of the variational problem

$$\inf_{\substack{x:\Lambda_N\to\{-1,1\}\\ I(x)=a}} \langle -\Delta x, x\rangle, \qquad \text{where } a = I(x_0). \tag{V}$$

Since we constrain $x\in\mathscr{H}_N$, this variational problem is **nonlinear**. If $a\neq 0$, this variational problem is **nonhomogeneous**.

Since $\|x\|_2 = 1$ for all states $x\in\mathscr{H}_N$ (because there are no unoccupied sites and $x:\Lambda_n\to\{-1,1\}$), problem (V) is the same as the following variation problem

$$\inf_{\substack{x:\ \Lambda_N\to\{-1,1\}\\ I(x)=a}} \frac{\langle -\Delta x, x\rangle}{\|x\|_2^2}. \tag{V*}$$

By Propositions 1 and 6, $\langle -\Delta x, x\rangle = 1 + L(x) = P(x)/2N^2$, and thus the following corollary provides a geometric interpretation of (V) and (V*).

COROLLARY 2 (Equivalent Geometric Variational Problem). *The following two variational problems have the same minimizers*:

$$\inf_{\substack{x:\Lambda_N\to\{-1,1\}\\ I(x)=a}} \frac{\langle -\Delta x, x\rangle}{\|x\|_2^2} = \frac{1}{N^2}\inf_{\substack{x:\Lambda_N\to\{-1,1\}\\ I(x)=a}} P(x).$$

In the special case $a = 0$, the expression on the left is reminiscent of the famous Rayleigh–Ritz variational problem for the first eigenvalue λ_1 of Δ.[7] However, in the expression on the left, we require $x:\Lambda_N\to\{-1,1\}$

[7] To some readers, the smooth version of the Rayleigh–Ritz variational problem may be more familiar than for a graph. The solution on the flat two-torus (or any compact Riemannian manifold) is the first non-zero eigenvalue λ_1 of the Laplacian Δ, i.e.,

$$\lambda_1 = \inf_{\substack{f\in C^2(\mathbb{T}^2,\mathbb{R})\\ \int f\,dA=0}} \frac{\langle -\Delta f, f\rangle}{\|f\|_2}, \tag{RR}$$

and the infimum is attained by eigenfunction(s) of Δ with eigenvalue λ_1.

and not $x : \Lambda_N \to \mathbb{R}$. This is an essential difference, since this extra constraint makes the variational problem nonlinear!

PROPOSITION 7. *The global minimizers attain the values*

(1) *Given*

$$\inf_{\substack{x:\Lambda_N \to \mathbb{R}\\ I(x)=0}} \frac{\langle -\Delta x, x\rangle}{\|x\|_2^2} = \lambda_1 = 1 - (1/2)(1 + \cos(2\pi/N)),$$

where λ_1 *is the first non-zero eigenvalue of the graph Laplacian.*

(2) *The spectrum of the Laplacian* $-\Delta : \mathbb{R}^{\Lambda_N} \to \mathbb{R}^{\Lambda_N}$ *consists of the set of numbers*

$$\lambda_{p,q} = 1 - \frac{1}{2}\left[\cos\left(\frac{2\pi p}{N}\right) + \cos\left(\frac{2\pi q}{N}\right)\right],$$

where $0 \le p, q \le N-1$.

(3) *Given*

$$\inf_{\substack{x:\Lambda_N \to \mathbb{R}\\ I(x)=a}} \frac{\langle -\Delta x, x\rangle}{\|x\|_2^2}$$
$$= (1-a^2)\lambda_1 = (1-a^2)(1 - (1/2)(1 + \cos(2\pi/N))),$$

where λ_1 *is the first non-zero eigenvalue of the graph Laplacian.*

Proof. A calculation shows that the characters are eigenfunctions of $-\Delta$, i.e., $-\Delta\chi_{p,q} = \lambda_{p,q}\chi_{p,q}$, where $\chi_{p,q}(x,y) = \exp 2\pi i(px+qy)/N$. Another calculation shows that for $x = \sum_{p,q} c_{p,q}\chi_{p,q}$, the total Laplacian $\langle -\Delta x, x\rangle = \sum_{p,q}\lambda_{p,q}|c_{p,q}|^2$. As in the usual Fourier series proof of the isoperimetric inequality, it follows that $\langle -\Delta x, x\rangle \ge \lambda_{0,1}|c_{0,1}|^2 + \lambda_{1,0}|c_{1,0}|^2 = \lambda_1(|c_{0,1}|^2 + |c_{1,0}|^2)$, with equality if $f(x) = a + b\chi_{0,1} + c\chi_{1,0}$. The result follows since $1 = \|x\|_2^2 = a^2 + |b|^2 + |c|^2 = a^2 + |c_{0,1}|^2 + |c_{1,0}|^2$, and thus $\lambda_1(|c_{0,1}|^2 + |c_{1,0}|^2) = \lambda_1(1-a^2)$. ∎

With the help of Corollary 2 we have geometrical tools to study the more general nonlinear nonhomogeneous variational problem (V*). Although the minimization problem is finite dimensional, this seems to be a very challenging to solve analytically, and we could find no related references in the literature. Even for $a = 0$, since we are minimizing which attain values only ± 1, the minimizer will certainly not be an eigenfunction.

The following theorem is the nonlinear analog of parts (1) and (3) of Proposition 7 for states $x : \Lambda_N \to \{-1, 1\}$ and follows immediately from Corollary 2 and Theorem 2.

THEOREM 4. *The global minimizers of* (V^*) *attain the values*

(1) *Given*

$$\inf_{\substack{x:\Lambda_N \to \{-1,1\} \\ I(x)=0}} \frac{\langle -\Delta x, x\rangle}{\|x\|_2^2} = \frac{2}{N} + \frac{k}{N^2},$$

where $k = 0$ *if* N *is even, and* $k = 1$ *if* N *is odd. If* N *is even, the global minimizers are states which attain the value* $+1$ *on a* $N \times N/2$ *strip or a* $N/2 \times N$ *strip, and attains the value* -1 *on the complementary strip. If* N *is odd the global minimizers are almost of this form.*

(2) *If* $I(x) = a \neq 0$, *explicit formulas for the minimizers can easily be obtained from Corollary* 2 *and Theorem* 3.

We again remark (see Remark 1) that there are many local minimizers to this problem which are not global minimizers.

The following theorem is the nonlinear analog of part (2) of Proposition 7 for states $x : \Lambda_N \to \{-1, 1\}$. The numbers μ_n can be thought of as the spectrum for $-\Delta$ acting on functions $x : \Lambda_N \to \{-1, 1\}$.

THEOREM 5. *Let* $k = [\log_2 N]$ *and for each* $n \in \{1, \dots, k\}$ *let* μ_n *denote the global minimizer*

$$\mu_n = \inf_{\substack{x:\Lambda_N \to \{-1,1\} \\ I(x)=0 \\ x \in Y_n}} \frac{\langle -\Delta x, x\rangle}{\|x\|_2^2},$$

where Y_n *denotes the subspace of* $\mathscr{H}_N$ *consisting of functions that are orthogonal to eigenfunctions corresponding to* $\mu_1, \dots, \mu_n$. *Then if* N *is even,*

$$\mu_n = \frac{2^n}{N}, \qquad n \in \{1, \dots, k\}.$$

Proof. To obtain μ_1 the extremal configuration is the square divided into two equal strips. To get μ_n the extremal configuration is the square divided into 2^n congruent strips which are alternating in signs. ∎

QUESTION. *Is it possible to obtain analytic expressions for the minimizers of the discrete nonlinear problem from the minimizers for the continuous problem?*

6. FINAL COMMENTS

In Schelling's original model neighbourhoods consist of eight neighbours. However, there seem to be no essential qualitative differences between using the four and eight point neighbourhoods or Laplacians in these models.

Some formulas (e.g., Proposition 1) become significantly more complicated and lose their geometric interpretations when using eight point neighbourhoods. One intriguing difference is that using four point neighbourhoods, there are no unstable equilibrium states, while the chessboard pattern (Fig. 13) is an unstable equilibrium state using eight point neighbourhoods. Another feature of the model with eight point neighbours is the existence of additional global minimizing states consisting of diagonal strips.

There is a natural higher dimensional analogue of the model we have been considering. Consider the phase space consisting of a three dimensional periodic lattice with each site occupied by one of two distinct populations. If we were to model, for example, the populations of quiet and noisy people in a large apartment building, for each site it would be natural to consider the six nearest neighbours (four on the same horizontal level, plus the neighbours directly above and below). With this notion of neighbourhood, all of our results should easily generalize to this setting.

In Schelling's original model the switching mechanism is essentially based on tolerance thresholds, where, for example, two labelings for a pair of sites do not switch if the percentage of their similar neighbours lies above some tolerance threshold. For example, Schelling considered the case where sites were happy provided that at least three out of its eight neighbours share the same labeling. The effect of this is to *dampen down* the evolution of the system before the true minimum of the Lyapunov function is attained. Our algorithm can be viewed as a high tolerance approximation to the Schelling model. Empirically, the models exhibit some similar features in their limit configurations.

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