Chaining Garbled Circuits

Alex Ledger

Reed College

April 3, 2016

Overview

1. Secure Computation
2. Garbled Circuits
3. Better Garbled Circuits
4. Chaining Garbled Circuits
5. Better Chaining of Garbled Circuits

Goals of this talk

1. Understand the high level idea of secure computation
   - Maybe you'll run into a situtation where it's useful, and it can help solve an otherwise unsolvable, real world problem.
2. Understand garbled circuits - the most basic construction
3. Understand chaining - how we (my thesis) made secure computation faster
4. We will not be focusing on security (different from most crypto talks)

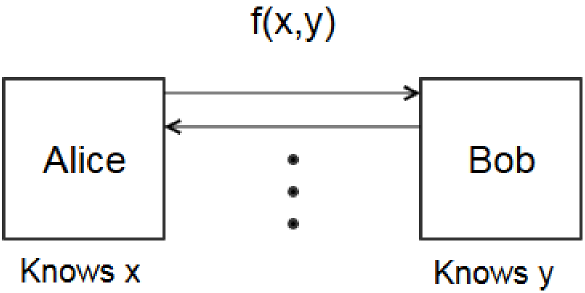
Before we start:

1. Lots of notation: ask me if I brush over something, or you forget what something means.
2. Lots of moving parts.

The Millionaire Problem

- Alice and Bob want to determine who is wealthier.
- They do want to disclose their wealth to each other.
  - Alice has \$$x$, Bob has \$$y$.
  - Alice should not learn anything about y.
  - Bob should not learn anything about x.

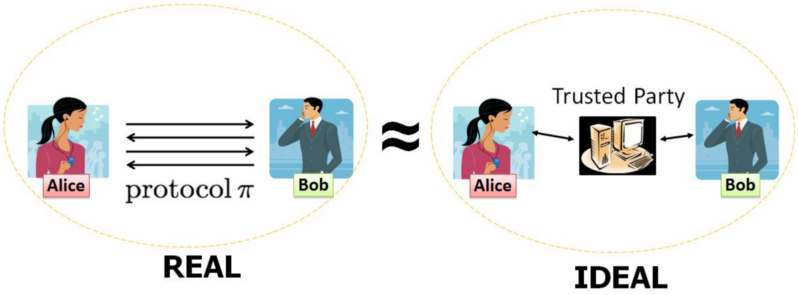
$$f(x,y) = \begin{cases} Alice, & y \leq x; \\ Bob, & y > x. \end{cases} \tag{1}$$



Security Properties

- Privacy of inputs
  - Alice and Bob do not learn anything about the other's input.
  - Except for info that is inferable from x and $f(x, y)$.
  - Bob should not learn that $1,000,000 < x \leq 2,000,000$.
  - But if $y < x$ and $y = 2,000$, then he learns $x < 2,000$.
- Correctness
  - Alice and Bob receive $f(x, y)$.
  - As opposed to some value near $f(x, y)$.
  - Or not receiving a value at all
- Semi-honest
  - We assume that each party obeys the protocol, but attemps to learn extra information from its interactions

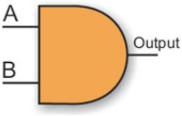
Security



A secure computation protocol is secure if Alice and Bob learn the same informatoin in the real world as the ideal world.

Boolean Circuits

- We encode a function f into a circuit C.
- Circuit C is made of AND, XOR and NOT gates.
- Each gate has two input wires and a single output wire

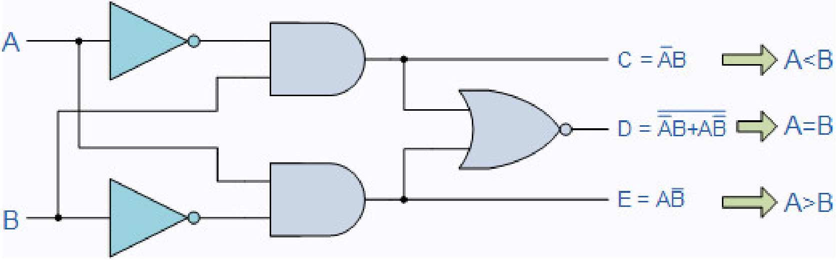


A	B	Output
0	0	0
0	1	0
1	0	0
1	1	1

A	B	O
0	0	0
0	1	1
1	0	1
1	1	0

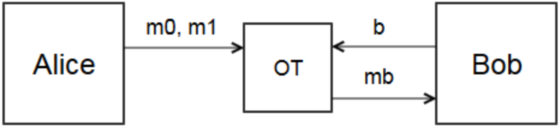
Boolean Circuits

- Any function can be encoded into a circuit.
- Here is the less than circuit.



Oblivious Transfer (OT) in brief

- Alice potentially sends either m_0 or m_1 to Bob.
- Bob receives m_b.
- Property 1: Alice does not know which message Bob recieved.
- Property 2: Bob doesn't anything about m_{1-b}.

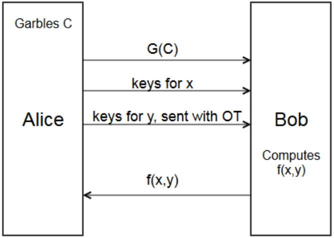


Hash Function in brief

- Hash function $H : \{0,1\}^* \to \{0,1\}^{128}$
- For our purposes, H maps any string to a uniform, random 128-bit string.
- A.k.a. H is a random oracle.

Roadmap of Garbled Circuits

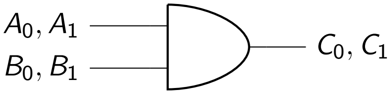
1. Suppose Alice and Bob are computing a circuit that is only the AND gate.
2. Alice *garbles* the AND gate.
3. Alice then sends *garbled table* of the gate and some auxiliary information to Bob.
4. Bob *evaluates* the gate.



Garbling a gate 1

Step 1. Assign *wire labels* to each wire.

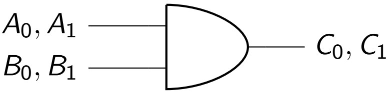
- For each wire in the circuit, assign two random labels to each wire
- Wire A has wire labels A_0 and A_1.
- We say A_0 *semantically represents* 0, and A_1 *semantically represents* 1.
- And A_0 and A_1 are sampled uniform randomly from $\{0, 1\}^n$.
  - Generally use AES-128 for encryption, so $n = 128$.



Garbling a gate 2

Step 2. Construct garbled table.

- Encrypt wire labels for C, C_0 and C_1, using wire labels of A and B.
- Randomly permute table



A	B	Encryption
A_0	B_0	$\mathrm{Enc}_{A_0,B_0}(C_0)$
A_1	B_0	$\mathrm{Enc}_{A_1,B_0}(C_0)$
A_0	B_1	$\mathrm{Enc}_{A_0,B_1}(C_0)$
A_1	B_1	$\mathrm{Enc}_{A_1,B_1}(C_1)$

Garbling a gate 3

Step 3. Send garbled table to Bob.

$$\boxed{\begin{array}{c} \mathsf{Enc}_{A_0,B_0}(C_0) \\ \mathsf{Enc}_{A_1,B_0}(C_0) \\ \mathsf{Enc}_{A_0,B_1}(C_0) \\ \mathsf{Enc}_{A_1,B_1}(C_1) \end{array}}$$

Garbling a gate 4

Step 4. Send input wire labels to Bob.

- Suppose $x_a, x_b \in \{0, 1\}$ are Alice's inputs.
- Alice sends Bob A_{x_a} and B_{x_b}
  - The wire labels corresponding to her inputs.
- Bob has:

Garbled Table
$\mathrm{Enc}_{A_0,B_0}(C_0)$
$\mathrm{Enc}_{A_1,B_0}(C_0)$
$\mathrm{Enc}_{A_0,B_1}(C_0)$
$\mathrm{Enc}_{A_1,B_1}(C_1)$

Input Labels
A_*
B_*

Garbling a gate 5

Step 5. Bob evaluates the circuit

- Bob has the garbled table, A_{x_a} and B_{x_b}.
- Bob trial decrypts each row of the garbled table, until an encrytion succeeds.
- Bob acquires $C_{a \wedge b}$.
- Bob has:

Garbled Table
$\mathrm{Enc}_{A_0,B_0}(C_0)$
$\mathrm{Enc}_{A_1,B_0}(C_0)$
$\mathrm{Enc}_{A_0,B_1}(C_0)$
$\mathrm{Enc}_{A_1,B_1}(C_1)$

Input Labels
A_*
B_*

Output Label
$C_{a \wedge b}$

Garbling a gate 6

Step 6. Bob gets a final answer.

- Alice sends Bob $\mathrm{Enc}_{C_0}(0)$ and $\mathrm{Enc}_{C_1}(1)$.
- Bob trial decrypts these with $C_{a \wedge b}$.
- One will succeed, and Bob will acquire $c = a \wedge b$.
- So Bob knows c, but not a and b!

Garbled Table
$\mathrm{Enc}_{A_0,B_0}(C_0)$
$\mathrm{Enc}_{A_1,B_0}(C_0)$
$\mathrm{Enc}_{A_0,B_1}(C_0)$
$\mathrm{Enc}_{A_1,B_1}(C_1)$

Input Labels
A_a
B_b

Output Label
$C_{a \wedge b}$

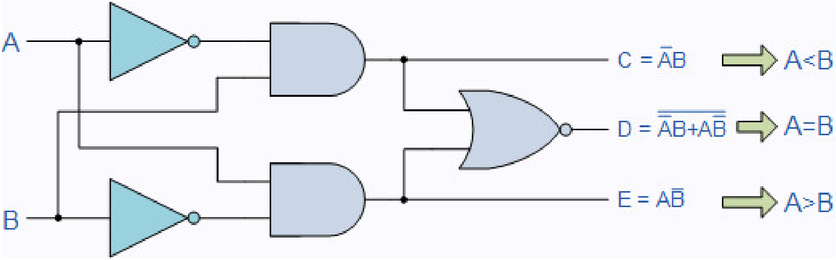
Output Map
$\mathrm{Enc}_{C_0}(0)$
$\mathrm{Enc}_{C_1}(1)$

Security Considerations

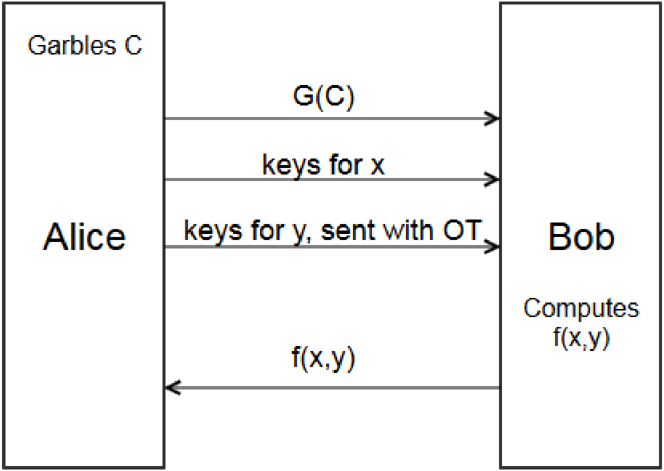
- Think about what Alice acquires:
  - Well Bob had no input in this case, so ...
  - In general, Alice generates objects and sends them to Bob
  - So she doesn't have many opportunities to learn about Bob's input.
- Think about what Bob acquires:
  1. Garbled table
  2. Input wire labels: A_a and B_b
  3. Encryptions of output: $\mathsf{Enc}_{C_0}(0)$ and $\mathsf{Enc}_{C_1}(1)$
- Can Bob learn anything about a or b?
  - If he could decrypt another row of the garbled table, then we would learn something.
    - But he can't, because he doesn't have the keys.
  - Not really, because everything is encrypted.

Extending a garbled gate into a garbled circuit

- To operate on a more complex function, the operation is recursed.
- The output wires of the first gates are used as inputs to subsequent gates.
- Alice only sends output maps for the final gates.



The Garbled Circuit Protocol



The cost of garbled circuits

- Alice sends Bob 4 ciphertexts per gate, since the garbled table has 4 rows.
- This is the bandwidth, or the amount of the communication required.
- Based on empirical work, bandwidth is the biggest bottleneck in garbled circuits.
- So reducing the size of the garbled table is of utmost priority.

Reducing the size of the garbled table with Free XOR

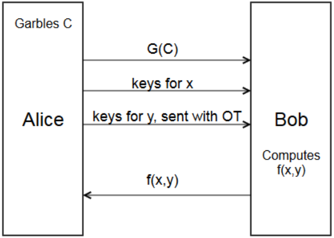
- Most improvements to garbled circuits are about choosing wire labels intelligently.
- Let $\Delta \leftarrow \{0,1\}^n$ be fixed globally in a circuit.
- Let $W_0 \oplus W_1 = \Delta$ for all wires.
  - aka $W_1 = W_0 \oplus \Delta$.
  - We now say that $W_0 = W$.
- Then an XOR gate can be computed by XORing the two input labels.
- TODO fix this notation to not use subscripts

$$
\begin{aligned}
A_0 \oplus B_0 &= C_0 \\
A_0 \oplus B_1 &= A_0 \oplus (B_0 \oplus \Delta) = (A_0 \oplus B_0) \oplus \Delta = C_1 \\
A_1 \oplus B_0 &= (A_0 \oplus \Delta) \oplus B_0 = (A_0 \oplus B_0) \oplus \Delta = C_1 \\
A_1 \oplus B_1 &= (A_0 \oplus \Delta) \oplus (B_0 \oplus \Delta) = (A_0 \oplus B_0) = C_0
\end{aligned}
$$

- So XOR gates do not require a garbled table.
- They are free
- Secure?

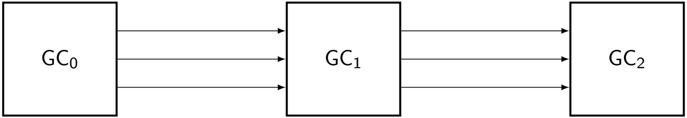
Offline/Online

- Imagine two banks use secure computation during their daily operations
- At night - the offline phase - they exchange garbled circuits (the garbled tables)
- During the day - the online phase - they exchange input wire labels and use the pre-exchanged garbled circuits.
- Problems
  - Functions are decided at night - no room for flexibility
  - Input size is fixed at night



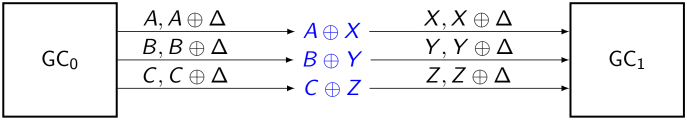
Chaining Garbled Circuits

- Goal: improve offline/online garbled circuits by adding flexibility
- Key observation: many useful functions in the real world are constructed in a modular way
  - They are composed of standard components
  - E.g. addition, subtraction, matrix multiplication
- Idea: chain garbled circuits together
  - Take the output of one garbled circuit and plug it into another garbled circuit



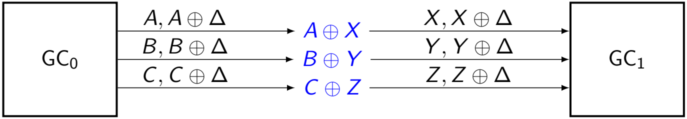
How to chain garbled circuits

- Suppose that we are chaining a garbled circuit with output wire A to garbled circuit with input wire X.
- We want $A \to B$ and $X \oplus \Delta \to X \oplus \Delta$.
- Straightforward:
  - Alice sends Bob $W_{AB} = A \oplus B$
  - Bob sets $B_* \leftarrow A_* \oplus W_{AB}$



How to chain garbled circuits

- Suppose that we are chaining a garbled circuit with output wire A to garbled circuit with input wire X.
- We want $A \to B$ and $X \oplus \Delta \to X \oplus \Delta$.
- Straightforward:
  - Alice sends Bob $W_{AB} = A \oplus B$
  - Bob sets $B_* \leftarrow A_* \oplus W_{AB}$



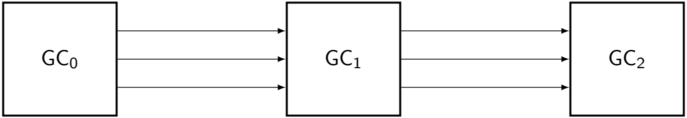
- Is this secure? What does Bob learn?
  - Not really anything.

Efficienty of chaining garbled circuits

- In online phase, chaining requires communicating a ciphertext per chain
- This can be a lot of communication
  - Suppose 100 by 100 matrix with entries of max value 2^8
  - Then requires $256,000$ ciphertexts
  - That's 4 megabytes!
- We can reduce this chaining to a single ciphertext

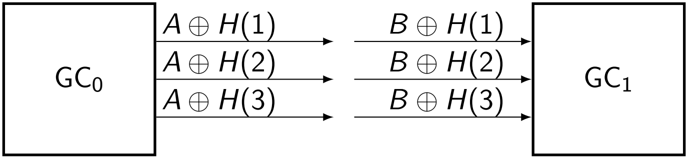
Single Communication Multiple Connections SCMC

- Problem: number of ciphertexts communicated scales linearly with the number of chains
- Key observation: the chaining is predictable, consecutive wires are chained to consecutive wires
- Solution: give consecutive wire labels a predictable pattern



Single Communcation Multiple Connections SCMC

- Set ith input wire label of circuit to $A \oplus H(i)$.
- Set jth output wire label of circuit to $X \oplus H(j)$.
    - Technically, we the labels to $A \oplus H(T \oplus (i||b))$
    - Where H is a hash function or a random oracle



My thesis work

- I implemented naive chaining and SCMC in a system called
- Written in C
- And make it fast.