Mean & Variance of Sum of Two Independent Variables

Premmi

May 13, 2016

Suppose x and y are two statistically independent variables. Then their mean and variance are given by

$$\mathbb{E}[x+y] = \mathbb{E}[x] + \mathbb{E}[y] \tag{1}$$
$$\mathrm{var}[x+y] = \mathrm{var}[x] + \mathrm{var}[y] \tag{2}$$

Proof. Since the two variables x and y are independent, their joint distribution $f_{x,y}(x,y)$ factorizes into $f_x(x)f_y(y)$. Hence,

$$\begin{aligned}
\mathbb{E}[x+y] &= \iint (x+y) f_{x,y}(x,y)\,\mathrm{d}x\,\mathrm{d}y && (3)\\
&= \iint (x+y) f_x(x) f_y(y)\,\mathrm{d}x\,\mathrm{d}y && (4)\\
&= \iint \{x\, f_x(x) f_y(y)\,\mathrm{d}x\,\mathrm{d}y + y\, f_x(x) f_y(y)\,\mathrm{d}x\,\mathrm{d}y\} && (5)\\
&= \iint x\, f_x(x) f_y(y)\,\mathrm{d}x\,\mathrm{d}y\ + \iint y\, f_x(x) f_y(y)\,\mathrm{d}x\,\mathrm{d}y && (6)\\
&= \int x\, f_x(x)\,\mathrm{d}x \int f_y(y)\,\mathrm{d}y\ + \int y\, f_y(y)\,\mathrm{d}y \int f_x(x)\,\mathrm{d}x && (7)\\
&= \int x\, f_x(x)\,\mathrm{d}x\ + \int y\, f_y(y)\,\mathrm{d}y && (8)\\
&= \mathbb{E}[x] + \mathbb{E}[y] && (9)
\end{aligned}$$

as required. (8) follows from the fact that the pdf integrates to 1 and (9) follows from the definition of expectation.

$$
\begin{aligned}
\operatorname{var}[x+y] &= \mathbb{E}[\,(x+y) - \mathbb{E}[x+y]\,]^2 && (10)\\
&= \mathbb{E}[\,(x+y)^2 - 2\,(x+y)\,\mathbb{E}[x+y] + \mathbb{E}[x+y]^2\,] && (11)\\
&= \mathbb{E}[\,(x+y)^2\,] - 2\,\mathbb{E}[x+y]^2 + \mathbb{E}[x+y]^2 && (12)\\
&= \mathbb{E}[\,(x+y)^2\,] - \mathbb{E}[x+y]^2 && (13)\\
&= \mathbb{E}[x^2] + \mathbb{E}[y^2] + 2\,\mathbb{E}[xy] - (\,\mathbb{E}[x] + \mathbb{E}[y]\,)^2 && (14)\\
&= \mathbb{E}[x^2] + \mathbb{E}[y^2] + 2\,\mathbb{E}[xy] - \mathbb{E}[x]^2 - \mathbb{E}[y]^2 - 2\,\mathbb{E}[x]\,\mathbb{E}[y] && (15)\\
&= \mathbb{E}[x^2] + \mathbb{E}[y^2] + 2\,\mathbb{E}[x]\,\mathbb{E}[y] - \mathbb{E}[x]^2 - \mathbb{E}[y]^2 - 2\,\mathbb{E}[x]\,\mathbb{E}[y] && (16)\\
&= \mathbb{E}[x^2] - \mathbb{E}[x]^2 + \mathbb{E}[y^2] - \mathbb{E}[y]^2 && (17)\\
&= \operatorname{var}[x] + \operatorname{var}[y] && (18)
\end{aligned}
$$

as required. Also, $\operatorname{cov}(x, y) = \mathbb{E}[xy] - \mathbb{E}[x]\,\mathbb{E}[y]$ and $\operatorname{cov}(x, y) = 0$ since x and y are independent, it follows that $\mathbb{E}[xy] = \mathbb{E}[x]\,\mathbb{E}[y]$.

□

References

[1] Christopher M. Bishop. *Pattern Recognition and Machine Learning*. **Exercise 1.10**