Univariate Gaussian Distribution is Normalized

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To prove that the univariate Gaussian distribution is normalized, we will first show that it is normalized for a zero-mean Gaussian and extend that result to show that $\mathcal{N}(x|\mu,\, \sigma^2)$ is normalized.

The pdf of the zero-mean Gaussian distribution is given by:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}x^2\right) \qquad -\infty < x < \infty. \tag{1}$$

To prove that the above expression is normalized, we have to show that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx = \sqrt{2\pi\sigma^2} \tag{2}$$

Proof. Let

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx \tag{3}$$

Squaring the above expression,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2\right) dx\, dy \tag{4}$$

To integrate this expression we make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by

$$x = r\, cos\, \theta \tag{5}$$

$$y = r\, sin\, \theta \tag{6}$$

and using the trigonometric identity $cos^2\theta + sin^2\theta = 1$, we have $x^2 + y^2 = r^2$. Also the Jacobian of the change of variables is given by,

$$
\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \dfrac{\partial(x)}{\partial(r)} & \dfrac{\partial(x)}{\partial(\theta)} \\ \\ \dfrac{\partial(y)}{\partial(r)} & \dfrac{\partial(y)}{\partial(\theta)} \end{vmatrix} \\
&= \begin{vmatrix} cos\,\theta & -r\,sin\,\theta \\ sin\,\theta & r\,cos\,\theta \end{vmatrix} \\
&= r\,cos^2\theta + r\,sin^2\theta \\
&= r
\end{aligned}
$$

using the same trigonometric identity $cos^2\theta + sin^2\theta = 1$. Thus equation (4) can be rewritten as

$$
\begin{aligned}
I^2 &= \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr\,d\theta && (7) \\
&= 2\pi \int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr && (8) \\
&= 2\pi \int_0^{\infty} \exp\left(-\frac{u}{2\sigma^2}\right) \frac{1}{2}\,du && (9) \\
&= \pi\left[\exp\left(-\frac{u}{2\sigma^2}\right)\left(-2\sigma^2\right)\right]_0^{\infty} && (10) \\
&= 2\pi\sigma^2 && (11)
\end{aligned}
$$

where we have used the change of variables $r^2 = u$. Thus

$$
I = \left(2\pi\sigma^2\right)^{1/2}.
$$

Finally to prove that $\mathcal{N}(x|\mu,\,\sigma^2)$ is normalized, we make the tranformation $y = x - \mu$ so that,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{N}(x \mid \mu, \sigma^2)\,dx &= \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy \\
&= \frac{I}{(2\pi\sigma^2)^{1/2}} \\
&= 1
\end{aligned}
$$

as required. $\square$

References

[1] Kevin P. Murphy. *Machine Learning: A Probabilistic Perspective.* **Exercise 2.11** Normalization constant for a 1D Gaussian.

[2] Christopher M. Bishop. *Pattern Recognition and Machine Learning.* **Exercise 1.7**