Variance of Univariate Gaussian Distribution

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Proof. The normalization condition of the univariate Gaussian distribution is given by:

$$\int_{-\infty}^{\infty} \mathcal{N}\left(x \mid \mu, \sigma^2\right) dx = 1 \tag{1}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx = 1 \tag{2}$$

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx = \left(2\pi\sigma^2\right)^{1/2} \tag{3}$$

Differentiating both sides of (3) with respect to σ^2,

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left\{\frac{\left(x-\mu\right)^2}{2}\right\}\left\{\frac{1}{\left(\sigma^2\right)^2}\right\} dx = \left(\frac{1}{2}\right)\left(2\pi\sigma^2\right)^{-1/2}\left(2\pi\right) \tag{4}$$

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(x-\mu\right)^2 dx = \sigma^2\sqrt{2\pi\sigma^2} \tag{5}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(x-\mu\right)^2 dx = \sigma^2 \tag{6}$$

It follows directly from (6) that

$$\mathbb{E}\left[\left(x-\mu\right)^2\right] = var\left[x\right] = \sigma^2 \tag{7}$$

Expanding the left hand side of equation (7),

$$\begin{aligned}
\mathbb{E}\left[x^2\right] - 2\mu\,\mathbb{E}\left[x\right] + \mu^2 &= \sigma^2 \\
\mathbb{E}\left[x^2\right] - 2\,\mathbb{E}\left[x\right]^2 + \mathbb{E}\left[x\right]^2 &= \sigma^2 \\
\mathbb{E}\left[x^2\right] - \mathbb{E}\left[x\right]^2 &= \sigma^2
\end{aligned}$$

$\square$

References

[1] Christopher M. Bishop. *Pattern Recognition and Machine Learning*. **Exercise 1.8**