Mode of Univariate Gaussian Distribution

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Proof. The pdf of the univariate Gaussian distribution is given by:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} \qquad -\infty < x < \infty. \tag{1}$$

To find the mode i.e. the maximum of the Gaussian distribution, we differentiate the pdf with respect to x and equate it to 0 to find the critical point where the function is maximum or minimum and then use the second derivative test to ascertain that the function is maximized at that point.(*a function f attains its local maximum at a critical point x if it is twice differentiable at x i.e. $f'(x) = 0$ and its second derivative at x is negative i.e. $f''(x) < 0$*).

Differentiating (1) with respect to x,

$$\frac{d}{dx}\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(-\frac{1}{2\sigma^2}\right)\left\{2\left(x-\mu\right)\right\} \tag{2}$$

Equating (2) to zero, we get $x = \mu$.

To verify that the function is maximized at μ we differentiate (2) with respect to x,

$$\begin{aligned}\frac{d^2}{d^2x}\varphi(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(-\frac{1}{2\sigma^2}\right)(2) \\ &\quad + \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(-\frac{1}{2\sigma^2}\right)^2\left\{2\left(x-\mu\right)\right\}^2\end{aligned} \tag{3}$$

Substituting $x = \mu$ in (3) we get,

$$\frac{d^2}{d^2x}\,\varphi(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\exp\left\{-\frac{1}{2\sigma^2}\,(\mu-\mu)^2\right\}\left(-\frac{1}{2\sigma^2}\right)(2) \tag{4}$$

$$+\frac{1}{\sqrt{2\pi\sigma^2}}\,\exp\left\{-\frac{1}{2\sigma^2}\,(\mu-\mu)^2\right\}\left(-\frac{1}{2\sigma^2}\right)^2\{2\,(\mu-\mu)\}^2$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\,\exp\left\{-\frac{1}{2\sigma^2}\,(0)^2\right\}\left(-\frac{1}{2\sigma^2}\right)(2) \tag{5}$$

$$+\frac{1}{\sqrt{2\pi\sigma^2}}\,\exp\left\{-\frac{1}{2\sigma^2}\,(0)^2\right\}\left(-\frac{1}{2\sigma^2}\right)^2\{2\,(0)\}^2$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\left(-\frac{1}{2\sigma^2}\right)(2) + 0 \tag{6}$$

$$= -\frac{1}{\sigma^2\sqrt{2\pi\sigma^2}} < 0 \tag{7}$$

Since the second derivative of φ at μ is negative i.e < 0, the univariate Gaussian distribution is maximized at μ.

Hence the mode of the univariate Gaussian distribution is μ. $\square$

References

[1] Christopher M. Bishop. *Pattern Recognition and Machine Learning.* **Exercise 1.9**