TMA4195 Project
ATMOSPHERIC MODELS RELATED TO GLOBAL WARMING

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1 Introduction

Global warming is relevant in today's climate. The goal of this project is to develop a mathematical model to examine the effect of different factors on the global temperature. The Earth gets energy from the Sun in the form of radiation. This radiation is either absorbed or reflected by the Earth's atmosphere or the Earth itself. In the first part of this project we focus on the amount of absorbed radiation and heat transfers between the Earth and the atmosphere, and how the absorbed radiation affects temperatures. However it is also important to take into consideration polar regions. Due to the low temperatures at the poles, ice sheets form which have a high capacity to reflect solar radiation, thus lowering the temperature. Therefore in the second part of this project we examine heat fluxes from equatorial to polar regions and the Earth's capacity to reflect radiation.

2 Modeling the temperature of the Earth and atmosphere

The goal of this section is to provide a simple model for the Earth and the atmosphere temperature based on radiative heat transfers. There are several assumptions made to make it more feasible to model.

2.1 Average energy arriving to the Earth from the Sun

Assuming the rays of radiation from the Sun arrives at the Earth approximately parallel to each other, we find the mean value of the Watts per square meter. Denoting this value as the solar constant, it has a value of $q_{\text{sol}} = 1366\text{W/m}^2$.

An assumption of our upcoming model is that we will assume the radiation arriving from the sun is equally distributed across the whole surface of the earth, including the side facing away from the Sun. To do this, we calculate the total energy hitting the cross-section of the Earth, followed by dividing this by the total area of the Earth:

$$Q := q_{\text{earth}} = \frac{q_{\text{sol}} \cdot \pi \cdot r_{\text{earth}}^2}{A_{\text{earth}}} = \frac{q_{\text{sol}} \cdot \pi \cdot r_{\text{earth}}^2}{4 \cdot \pi \cdot r_{\text{earth}}^2} = \frac{q_{\text{sol}}}{4} = 341.5 \frac{\text{W}}{\text{m}^2} \tag{1}$$

2.2 General assumptions

One of the main assumptions is considering the Earth and the atmosphere as black bodies. A black body is an idealized physical body which does not reflect any radiation. This assumption allows the use of Stefan-Boltzmann's law to model the Earth and atmosphere's emitted radiation. Additionally for the atmosphere unlike the Earth, we assume that it does not absorb or emit radiation uniformly for all wavelengths due to the mixed composition of gases. Therefore the emitted power per unit area for the Earth $P_{\text{E, out}}$ and for the atmosphere $P_{\text{A, out}}$ are

$$P_{\text{E, out}} = \sigma T_E^4 \qquad P_{\text{A, out}} = \varepsilon_A \sigma T_A^4$$

where σ is the Stefan-Boltzmann constant, T_E is the Earth's temperature, T_A is the atmosphere temperature and $0 < \varepsilon_A < 1$ is the fraction representing the atmosphere's emissivity accounting for its partial absorption and emission.

Another major assumption is dividing the continuous electromagnetic spectrum into two different frequency ranges, low energy long-wave and high energy short-wave emissions. The long-waves (LW) are emitted by the Earth and atmosphere and mainly fall into the infrared range, while the short-waves (SW) consists of ultraviolet and visible light emitted from the Sun. With this assumption of two frequency ranges, we consider constant effective absorption and reflection coefficients for each range.

An incoming emission power P_{in} will be modeled with a transfer coefficient approach. The incoming emission power is decomposed into three parts,

$$P_{\text{in}} = P_{\text{abs}} + P_{\text{refl}} + P_{\text{trans}} \tag{2}$$

where

$$P_{\text{abs}} = a(1-r)P_{in}, \quad P_{\text{refl}} = rP_{\text{in}}, \quad P_{\text{trans}} = (1-a)(1-r)P_{\text{in}}.$$

Here, a and r represent absorptivity and reflectivity respectively. The constants related to SW and LW are provided in Table 1 in Appendix A. It is important to note that there is no reflection coefficient for the O_3 layer, and therefore no reflection here.

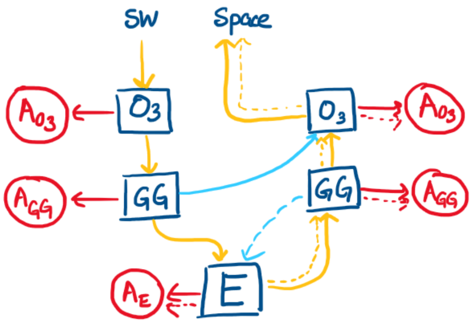
In addition to radiative transfers, there are other forms of heat transfer between the Earth's surface and the atmosphere. In this model, the temperature differences accounting for the overall heat transfer from warm to cold regions, called the sensible heat flux $P_{E\to A}$ is modeled with a linear relationship. This is very much simplified, as the energy transfers are much more complex in reality. Additionally there is an amount of heat transported through vapor from the Earth to the atmosphere. The heat stored in water molecules are released as the molecules change from gas to liquid form in the atmosphere. This release process, called latent heat P_{latent} is also modeled linearly. We combine these two processes together as non-radiative transfers $P_{\text{non-rad}}$, and get the following terms

$$P_{E\to A} = -\alpha(T_A - T_E), \quad P_{\text{latent}} = -\beta(T_A - T_E), \quad P_{\text{non-rad}} = P_{E\to A} + P_{\text{latent}} = (\alpha+\beta)(T_E - T_A) \tag{3}$$

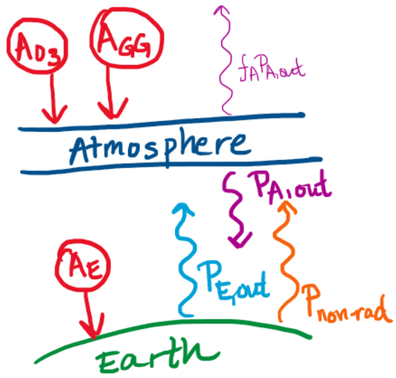
where the values of α and β are given in Table 1 in the Appendix A.

2.3 Modeling with no clouds

Assuming a cloudless atmosphere, there are two main layers of reflection and absorption, which are the atmosphere and the Earth. The atmosphere is divided into two parts, where the upper part is the ozone layer (O_3) and the lower part are the greenhouse gases (GG). That means the SW from the Sun are first absorbed in the ozone layer and then by the atmosphere.



(a) Sketch of how the SW are emitted and absorbed in the different layers following the transfer coefficient approach. Red shows absorption, blue shows reflection, and yellow shows transmission. The dotted lines show the loop of the SW reflected back to Earth.



(b) Sketch of all absorbed and emitted energy for the atmosphere and the Earth.

Figure 1: Visual representation of the how SW travel in the transfer coefficient approach and the energy balance for the Earth and the atmosphere.

We model the interactions between the Earth and the atmosphere as a series of reflections. A fraction of the SW reflected by Earth's surface reflects from the atmosphere back to Earth's surface. This loop can be considered a geometric series. However after two loops, the amount of energy being reflected is small, being less than 10^{-1} W m^{-2}. Therefore we neglect the remainder to simplify the calculations required in the model, and only calculate for the first two loops.

To determine the temperatures of the Earth and the atmosphere, we apply the principle of energy conservation. The amount of incoming power to a layer should equal the amount of outcoming power from the layer. See Figure 1b for a visualization. Let A_E, A_{GG}, A_{O_3} be the absorbed SW by the Earth, the GG and O_3 layer, shown in Figure 1a. The calculations are found in the Appendix B. Then, the energy balance equations for the Earth and the cloudless atmosphere are

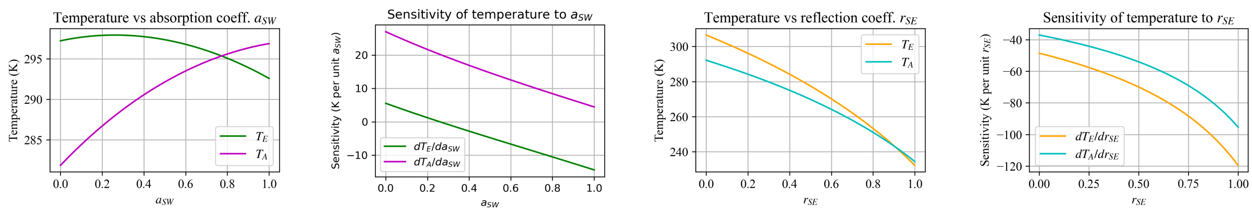
$$\begin{cases} A_E + P_{\text{A, out}} = P_{\text{non-rad}} + P_{\text{E, out}} \\ A_{GG} + A_{O_3} + a_{LW} P_{\text{E, out}} + P_{\text{non-rad}} = (1 + f_A) P_{\text{A, out}} \end{cases} \tag{4}$$

The equations are solved numerically using the function `fsolve` from the package `scipy.optimize` in Python. The method converges to the temperatures

$$T_E = 297.79 \text{ K} = 24.64 \text{ C}^o \quad T_A = 285.46 \text{ K} = 12.31 \text{ C}^o.$$

These temperatures seem reasonable considering the assumptions applied to the problem. In particular, we do not consider cloud coverage nor the ice-albedo from the Earth's poles here.

We are interested in the sensitivity of the temperatures with respect to some parameters. The parameters we choose to investigate are a_{SW} and r_{SE}. The derivative of the Earth and the atmosphere temperature with respect to these chosen parameters are calculated numerically. This is done by solving the set of equations in (5) for the range of values between 0 and 1 of a_{SW} and r_{SE}, obtain values for T_E and T_A, and then solving numerically for the gradients dT_E/da_{SW}, dT_A/da_{SW}, dT_E/dr_{SE} and dT_E/dr_{SE} using the function `numpy.gradient` in Python. The results are plotted in Figure 2.



(a) Results for a_{SW}

(b) Results for r_{SE}

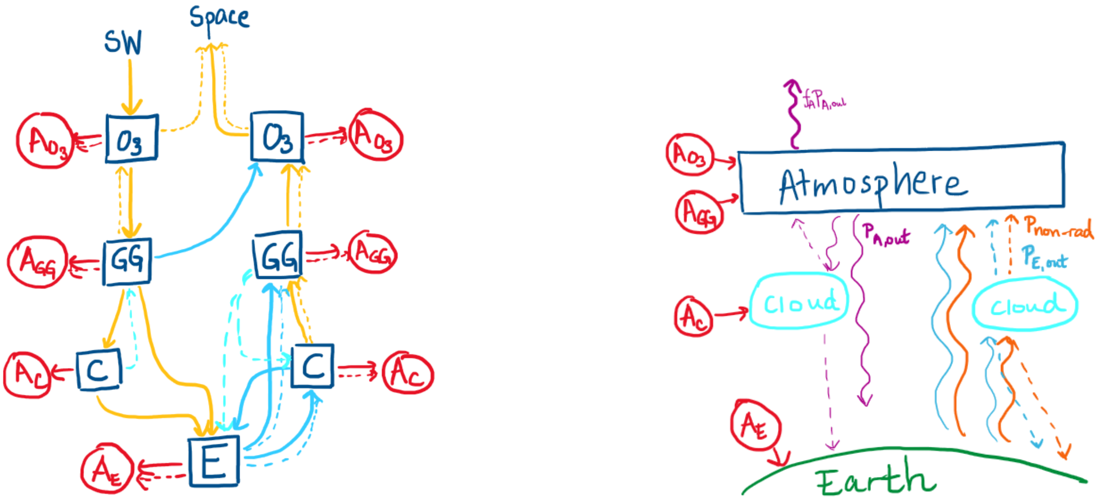
Figure 2: Temperature and sensitivity change with respect to chosen parameters

For Figure 2a, the temperature of the atmosphere rises as the absorption coefficient a_{SW} increases. The Earth's temperature starts decreasing around $a_{\text{SW}} = 0.25$. This aligns with the model's assumption of the transfer coefficient approach. With more power being absorbed in the atmosphere, less power is transmitted down to Earth. For Figure 2b, an increase of the Earth's reflection coefficient r_{SE} causes more of the SW to reflect off the Earth's surface. This means less of the SW are absorbed and thus decreasing the Earth's temperature. The temperature decreases more slowly as r_{SE} increases in the model. This also aligns with the model's assumptions, where the waves reflected off Earth's surface go back to the atmosphere and are partially absorbed.

2.4 Modeling with clouds

Now we add an additional layer of clouds. The clouds are placed right below the GG layer and above the Earth. The clouds cover a C_C fraction of the Earth's surface. The clouds absorb some of the transmitted SW from the atmosphere and the SW reflected off Earth's surface, shown in Figure 3a. Again, the amount of energy after a few reflections is quite small and therefore neglected.

Again we set up the energy balance equations for the Earth and the atmosphere, but now including the layer of clouds. Now of the LW are reflected and absorbed by the clouds. The temperature of the atmosphere takes the absorbed energy by the clouds into consideration. Let A_C be the absorbed SW by the clouds. The calculations of absorbed SW of all A are found in the Appendix C. The set of equations



(a) Sketch of how the SW are emitted and absorbed in the different layers. Red shows absorption, blue shows reflection, and yellow shows transmission. The dotted lines show the loop of the waves reflected back to where the SW were transmitted from.

(b) Sketch of all absorbed and emitted energy for the atmosphere, the clouds and the Earth. The dotted lines show the reflected LW and transmitted LW for the clouds.

Figure 3: Visual representation of shortwave interactions and energy balance.

considering clouds are

$$\begin{cases} A_E + (1-C_C)P_{\text{A, out}} + C_C(1-a_{LC})(1-r_{LC})(P_{\text{A, out}}) + C_C r_{LC}(P_{\text{non-rad}} + P_{\text{E, out}}) = P_{\text{non-rad}} + P_{\text{E, out}} \\ \\ A_{GG} + A_{O_3} + A_C + a_{LW}\big(C_C(1-a_{LC})(1-r_{LC})P_{\text{E, out}} + (1-C_C)P_{\text{E, out}}\big) + \\ \quad C_C(1-a_{LC})(1-r_{LC})P_{\text{non-rad}} + (1-C_C)P_{\text{non-rad}} + C_C r_{LC} P_{\text{A, out}} + \\ \quad C_C(1-r_{LC})a_{LC}(P_{\text{non-rad}} + P_{\text{E, out}} + P_{\text{A, out}}) = (1+f_A)P_{\text{A, out}} \end{cases} \tag{5}$$

The temperature with the additional layer of clouds, where clouds are part of the atmosphere for temperature calculations, we get

$$T_E = 293.10 \text{ K} = 19.95 \text{ C}^o \quad T_A = 293.22 \text{ K} = 20.07 \text{ C}^o. \tag{6}$$

The clouds increase the temperature of the atmosphere. This fits the model as some of the SW from the Sun, and some of the LW from the Earth and the atmosphere are absorbed by the clouds. These temperatures are still a bit high compared to realistic numbers, but reasonable given our assumptions. The ice-albedo nor the continuous spectrum of electromagnetic waves are considered here.

3 Zonal Model

In this section we will use the heat equation to derive an equilibrium equation which we will use to study heat transfer and temperature in polar- and non-polar regions.

3.1 Deriving the Heat Equation from Conservation Laws

The total amount of heat energy in a domain Ω is $\iiint_\Omega c\rho T \, \mathrm{d}V$, where c is the specific heat capacity with $[c] = \mathrm{JK^{-1}kg^{-1}}$, ρ is the density with $[\rho] = \mathrm{kgm^{-3}}$, and T is the temperature with $[T] = \mathrm{K}$. By Fourier's law, the heat flux contribution per time is $\iint_{\partial\Omega} -\mathbf{n} \cdot k\nabla T \, \mathrm{d}S$, where $\mathbf{n}$ is the normal vector pointing out of the domain and k is the thermal conductivity with $[k] = \mathrm{Wm^{-1}K^{-1}}$. We also have the energy per time contributed by the heat source q: $\iiint_\Omega q \, \mathrm{d}V$, where $[q] = \mathrm{Wm^{-2}}$. The total amount of energy must be conserved:

$$\frac{\partial}{\partial t}\iiint_\Omega c\rho T \, \mathrm{d}V = \iint_{\partial\Omega} -\mathbf{n}\cdot k\nabla T \, \mathrm{d}S + \iiint_\Omega q \, \mathrm{d}V,$$

where we can apply the divergence theorem to get

$$\frac{\partial}{\partial t}\iiint_\Omega c\rho T \, \mathrm{d}V = \iiint_\Omega \nabla\cdot(k\nabla T) \, \mathrm{d}V + \iiint_\Omega q \, \mathrm{d}V.$$

Since Ω is arbitrary, this can be written equivalently in differential form:

$$\frac{\partial}{\partial t}(c\rho T) = \nabla\cdot(k\nabla T) + q. \tag{7}$$

3.2 Heat Equation Dependant Only on Latitude

The heat equation for the atmosphere can be derived using a control volume. We assume that there is only a temperature gradient along the latitude, and get

$$E_\phi - E_{\phi+\Delta\phi} + q = \frac{\partial Q_{CV}}{\partial t}, \tag{8}$$

where ϕ is a given latitude, E_ϕ is the energy at a given latitude with $[E] = W$, q is a source term with $[q] = W$ and $\frac{\partial Q_{CV}}{\partial t}$ is the generated heat with units W. The energy is found by

$$AQ_\phi - AQ_{\phi+\Delta\phi} + q = A\rho c_p \frac{\partial T}{\partial t}, \tag{9}$$

where A is the area in m^2. By accounting for the area the flux is passing through using the Earth's spherical coordinates, and using Fourier's law for the heat, the equation can be written as

$$r\cos(\phi)\Delta\theta\left(\frac{k}{r}\frac{\partial T}{\partial\phi}\right) - r\cos(\phi+\Delta\phi)\Delta\theta\left(\frac{k}{r}\frac{\partial T}{\partial\phi}\right) + q = A\rho c_p\frac{\partial T}{\partial t}, \tag{10}$$

where r is the radius in meters. By dividing the equation on the area, and applying algebraic techniques, we obtain that

$$\frac{1}{r\cos(\phi)}\left(\frac{\cos(\phi)-\cos(\phi+\Delta\phi)}{\Delta\phi}\right)\left(\frac{k}{r}\frac{\partial T}{\partial\phi}\right) + Q = \rho c_p\frac{\partial T}{\partial t}, \tag{11}$$

which can be evaluated when $\Delta\phi \to 0$, which yields

$$\frac{1}{r\cos(\phi)}\frac{\partial}{\partial\phi}\left(\cos(\phi)\left(K_a\frac{\partial T}{\partial\phi}\right)\right) + Q = C_a\frac{\partial T}{\partial t}, \tag{12}$$

where $K_a = \frac{k}{r}$ and $C_a = \rho c_p$. The differential operator D_ϕ is defined to be equal to $\frac{1}{r\cos(\phi)}\frac{\partial}{\partial\phi}\cos(\phi)$, and we get

$$D_\phi\left(K_a\frac{\partial T}{\partial\phi}\right) + Q = C_a\frac{\partial T}{\partial t}. \tag{13}$$

3.3 Justifying a simplification of the incoming solar flux distribution

We now want to derive an expression for a given latitude of the solar radiation. We begin by deriving an expression for the solar flux density for an arbitrary point on the earth. Then we compute the average solar flux density over a day, and finally use this to compute the yearly average solar radiation for a given latitude. This expression will then be compared to the approximation,

$$S(\phi) = 1 + S_2\frac{1}{2}(3\sin^2\phi - 1), \tag{14}$$

where $S_2 = -0.477$, and ϕ denotes the latitudinal angle.

We begin by choosing a coordinate frame. There are two apparent choices: fixing the coordinates by the Earth's axis of rotation around the sun, or fixing the coordinates to the Earth's axis of rotation. We proceed with the latter. Define an arbitrary point on the sphere parametrized as the normalized vector

$$\mathbf{p} = \sin(\phi)\mathbf{e}_3 + \cos(\phi)\big(\cos(\lambda)\mathbf{e}_1 + \cos(\lambda)\mathbf{e}_2\big),$$

where $\mathbf{e}_3$ is the unit vector pointing from the Earth's centre to the north pole. ϕ denotes the latitudinal angle, where $\phi = 0$ denotes the angle of the equator. λ denotes the longitudinal angle.

We now derive an expression for where the Sun is in the sky for an arbitrary point on the Earth, θ. θ denotes, in a sense, the local latitudinal angle of where the Sun is in the sky, being the angle between the normal vector at the point, and the vector pointing towards the Sun. Denote the point of interest as $\mathbf{p}_0$, and the vector pointing directly towards the Sun from the centre of the Earth as $\mathbf{p}_s$. Computing the dot product:

$$\mathbf{p}_s \cdot \mathbf{p}_0 = |\mathbf{p}_s| \cdot |\mathbf{p}_0| \cos\theta = \sin\phi_s \sin\phi_0 + \cos\phi_s \cos\phi_0 \cos(\lambda_s - \lambda_0), \tag{15}$$

where $\phi_s = \delta$ is called the declination of the sun, $\lambda_s - \lambda_0 =: t$ is the local solar time, $|\mathbf{p}_s| = |\mathbf{p}_0| = 1$, and $\phi_0 =: \phi$ is the local latitude. Equation (15) becomes

$$\cos\theta = \sin\delta \sin\phi + \cos\delta\cos\phi\cos t. \tag{16}$$

We can now define the (local) solar flux density onto the tangent plane to the earth as a function of θ:

$$s(\theta) = \begin{cases} \cos\theta & \cos\theta > 0 \\ 0 & \cos\theta \leq 0 \end{cases} \tag{17}$$

To find the average daily solar exposure for a given latitude, we want to integrate over the 24 hours ($t \in [-\pi, \pi]$). To do this we first need to find when the Sun rises and sets. This happens when $\theta = \pm\frac{\pi}{2} \Leftrightarrow \cos\theta = 0$. Inserting (17) into equation (16) we get the expression

$$\cos t_0 = -\frac{\sin\delta\sin\phi}{\cos\delta\cos\phi} = -\tan\delta\tan\phi =: -T, \tag{18}$$

where we are denoting sunrise/sunset as $\mp t_0$. Depending on the value of T, we have 3 different scenarios: As $T \to 1, t_0 \to \pi$, meaning the days get longer and longer until $T > 1$, where the Sun never sets. In this range $t_0 = \pi$. As $T \to -1$, $t_0 \to 0$, meaning the days get shorter and shorter until $T < 1$, where the Sun never rises. In this range $t_0 = 0$. For $T \in [-1, 1]$, $\tan\delta\tan\phi \leq 1$ which implies that $\delta + \phi \leq \frac{\pi}{2}$. Summing this up into one function for t_0,

$$t_0 = \begin{cases} 0 & T \in [\leftarrow, -1\rangle \\ \arccos(-\tan\delta\tan\phi) & T \in [-1, 1] \\ \pi & T \in \langle 1, \rightarrow] \end{cases} \tag{19}$$

Integrating around the latitudinal circle, and dividing by 2π to get the average daily solar exposure as a function of latitude,

$$S_d(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} s(\theta)\mathrm{d}t = \frac{1}{2\pi}\big[t\sin\delta\sin\phi + \cos\delta\cos\phi\sin t\big]_{-t_0}^{t_0} \tag{20}$$

$$= \frac{1}{\pi}\big[t_0\sin\delta\sin\phi + \cos\delta\cos\phi\sin t_0\big]. \tag{21}$$

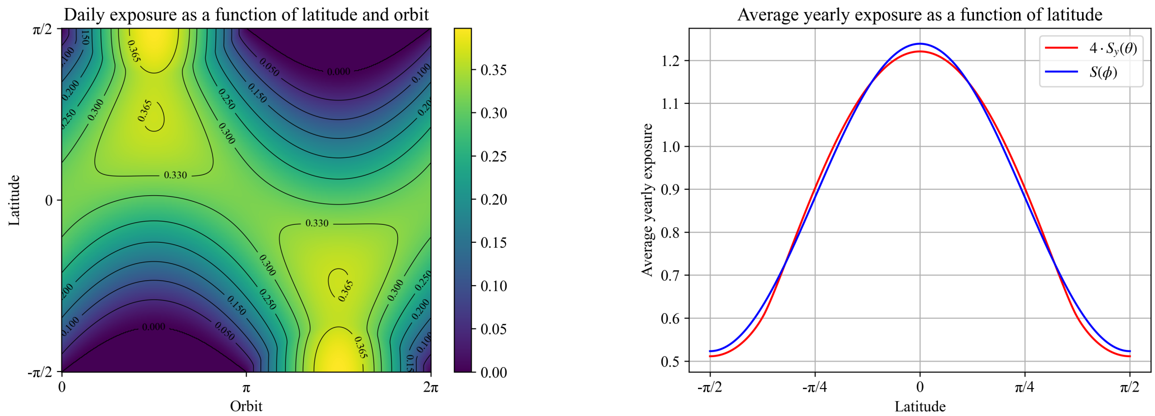
To go from a daily average to a yearly average, we want to integrate (21) over the whole year. Assuming a circular orbit, and defining 0 declination at the equator, we get an expression for the declination angle throughout the year:

$$\delta(\mathcal{O}) = 0.409\sin\mathcal{O}, \tag{22}$$

where $\mathcal{O}$ is the angle denoting the planetary orbit around the Sun, and 0.409 being the angle between the Earth's rotational axis and the orbital axis. Define $\mathcal{O} = 0$ as the day when the Sun passes the equator from the south to the north (for example, the December solstice is when $\sin\mathcal{O} = 1$ for the northern hemisphere). We end up with an integral expression of the yearly average radiation for a given latitude,

$$S_y(\phi) = \frac{1}{2\pi^2}\int_0^{2\pi} t_0(\mathcal{O})\sin\delta(\mathcal{O})\sin\phi + \cos\delta(\mathcal{O})\cos\phi\sin t_0(\mathcal{O})\mathrm{d}\mathcal{O}, \tag{23}$$

where $t_0(\mathcal{O})$ is as in (19), and δ is as in (22). See Figure 4 for (a) a plot of the daily exposure (21) as a function of latitude and the location in the orbit, as well as (b) numerical integration of equation (23) plotted alongside the approximation from (14). Note that by how $S(\phi)$ is defined, we need to multiply $S_y(\phi)$ by a factor of 4 for them to represent the same distribution in equation (6) in the problem description. Computing the root mean squared error between the two expressions, we get $\mathrm{RMSE} = (\frac{1}{n}\Sigma_i^n(f(x_i) - g(x_i))^2)^{\frac{1}{2}} = 0.0173$. To get a better intuition on what this difference means, compute $\%\mathrm{RMSE} := 100 \cdot \mathrm{RMSE}/\max(f(x_i), g(x_i)) = 1.40\%$. We conclude that the difference is small enough to justify the use of the approximation $S(\phi)$ instead of $S_y(\phi)$.



(a) (b)

Figure 4: (a) Daily exposure (21) as a function of latitude and the location in the orbit. (b) Numerical integration of equation (23) plotted alongside the approximation from (14).

3.4 Deriving the equilibrium equation

We start by dividing the Earth into segments by latitudinal and longitudinal lines. See Figure 5.

The flux in and out will respectively be

$$\Phi\Delta\lambda R\cos(\phi)$$

$$\Phi\Delta\lambda R\cos(\phi + \Delta\phi)$$

where R is the radius of the earth, and Φ is the heat flux from down to up in latitude. We differentiate T with respect to $r = R\phi$. By the chain rule and setting $x = \sin(\phi)$, we get that the heat flux Φ is

$$\Phi = -k\frac{\partial T}{\partial r} = -\frac{k}{R}\frac{\partial T}{\partial \phi} = -\frac{k}{R}\frac{\partial x}{\partial \phi}\frac{\partial T}{\partial x} = -\frac{k\cos(\phi)}{R}\frac{\partial T}{\partial x}.$$

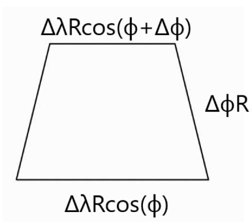


Figure 5: Sketch of a segment acquired by dividing the globe by latitudinal and longitudinal lines

We want an equilibrium, therefore the heat source on each segment $q(\phi)R^2\Delta\lambda\Delta\phi\cos(\phi)$ has to equal the difference of the flux in and flux out

$$\begin{aligned} q(\phi)R^2\Delta\lambda\Delta\phi\cos(\phi) &= \Phi\Delta\lambda R\cos(\phi+\Delta\phi) - \Phi\Delta\lambda R\cos(\phi) \\ q(\phi) &= -\frac{k}{R\cos(\phi)}\frac{T_\phi(\phi+\Delta\phi)\cos(\phi+\Delta\phi) - T_\phi(\phi)\cos(\phi)}{\Delta\phi} \\ q(\phi) &= -\frac{k}{R\cos(\phi)}\frac{\partial(T_\phi(\phi)\cos(\phi))}{\partial\phi}. \end{aligned}$$

By using the chain rule twice we get

$$q(x) = -\frac{k}{R}\frac{\partial}{\partial x}(\cos^2(\phi)\frac{\partial T}{\partial x}) = -\frac{k}{R}\frac{\partial}{\partial x}((1-x^2)\frac{\partial T}{\partial x}).$$

We assume that the heat source q is

$$q(x) = -A_{out} - B_{out}T(x) + \frac{G_{SC}}{4}S(x)(1-\alpha(x)) = -I(x) + QS(x)a(x,x_s),$$

where $G_{SC} = 1360\mathrm{Wm}^{-2}$ and $Q = G_{SC}/4$. The term $A_{\text{out}} + B_{\text{out}}T(x)$ is a linear approximation of the outgoing emitted long wave radiation with $A_{\text{out}} = 201.4\mathrm{Wm}^{-2}$ and $B_{\text{out}} = 1.45\mathrm{Wm}^{-1}\mathrm{C}$. $a(x,x_s) = 1-\alpha(x)$ is the co-albedo, which we assume is of the form $a = \begin{cases} a_u = 0.38 & x > x_s \\ a_l = 0.68 & x < x_s, \end{cases}$ where x_s is the location of the ice sheet. We get the equilibrium equation

$$-D\frac{\partial}{\partial x}((1-x^2)\frac{\partial T}{\partial x}) = -I(x) + QS(x)a(x,x_s) \tag{24}$$

with the constant $D = 0.3\mathrm{Wm}^{-2}\mathrm{K}^{-1}$ being an amalgamation of $\frac{k}{R}$ and compensating for heat transport effects other than heat conduction.

3.4.1 The boundary $[0, x_s]$ and $[x_s, 1]$

Because our temperature function T only depends on latitude ϕ and is symmetric by the equator, we get a flux and boundary conditions that are symmetric, meaning

$$\frac{\partial T}{\partial x}(0) = \frac{\partial T}{\partial x}(1) = 0$$

By x_s we denote the point between 0 and 1 where the ice cap starts. We can therefore get our boundary conditions at x_s easily with

$$T(x_s) = 0$$

because the freezing temperature of water is $0°C$. This gives us our boundary conditions. Given a solution T, we can find x_s by iterating with Newton's method

$$x_{n+1} = x_n - \frac{T(x_n)}{T_x(x_n)}$$

with for example $x_0 = 0.5$ and using some numerical derivative T_x until we get some acceptable error.

3.5 Solving the heat equation for the earth

To solve Equation (24), the solution is split into a homogeneous part and an inhomogeneous part.

3.5.1 Homogeneous equation

To solve the homogeneous part, let $y_l(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ for $x < x_s$. Then, the homogeneous equation becomes

$$
\begin{aligned}
-D(1-x^2)y''(x) + 2Dxy'(x) + B_{\text{out}}y(x) = 0 \\
-(1-x^2)\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\alpha_n x^n\right) + 2x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\alpha_n x^n\right) + \frac{B_{\text{out}}}{D}\sum_{n=0}^{\infty}\alpha_n x^n = 0 \\
-(1-x^2)\sum_{n=2}^{\infty}\alpha_n n(n-1)x^{n-2} + 2x\sum_{n=1}^{\infty}\alpha_n n x^{n-1} + \frac{B_{\text{out}}}{D}\sum_{n=0}^{\infty}\alpha_n x^n = 0.
\end{aligned}
$$

After some calculations, the final solution becomes

$$
y_l(x) = \alpha_0 y_{l,1}(x), \tag{25}
$$

where

$$
y_{l,1}(x) = 1 + \sum_{n=1}^{\infty}\frac{\frac{B_{\text{out}}}{D}(\frac{B_{\text{out}}}{D}+2\cdot 3)(\frac{B_{\text{out}}}{D}+4\cdot 5)\cdots[\frac{B_{\text{out}}}{D}+(2n-1)(2n-2)]}{(2n)!}x^{2n}. \tag{26}
$$

The calculations are given in Appendix D.
For $x > x_s$ we do a variable substitution, $x = 1-u$, so that the function is well behaved at $x = 1$. We then get the equation

$$
\begin{aligned}
-(1-(1-u)^2)\frac{\partial^2 y(u)}{\partial u^2} - 2(1-u)\frac{\partial y(u)}{\partial u} + \frac{B_{\text{out}}}{D}y(u) = 0 \\
-2u\frac{\partial^2 y(u)}{\partial u^2} + u^2\frac{\partial y(u)}{\partial u} - 2\frac{\partial y(u)}{\partial u} + 2u\frac{\partial y(u)}{\partial u} + \frac{B_{\text{out}}}{D}y(u) = 0
\end{aligned}
$$

Let $y_u(u) = \sum_{n=0}^{\infty}\alpha_n u^n$ be a solution to the homogeneous equation for $x > x_s$. By solving this equation the same way we did above, we end up with a series consisting of one linearly independent solution. By substituting back to the x variable, we get that

$$
y_u(x) = \beta_0 y_{u,1}(x), \tag{27}
$$

where

$$
y_{u,1}(x) = 1 + \sum_{n=1}^{\infty}\frac{B(B+1\cdot 2)(B+2\cdot 3)\cdots[B+N(n-1)]}{2\cdot 4\cdot 6\cdots(2n)}(1-x)^n \tag{28}
$$

3.5.2 Inhomogeneous equation

The inhomogeneous part of the equation is solved by using the ansatz $T_p = \sum_{n=0}^{\infty} T_n P_n(x)$, where P_n are Legendre polynomials. The equation then becomes

$$
-D\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_n P_n(x)}{\partial x}\right) + B_{\text{out}}T_n P_n(x) = -A_{\text{out}} + QS(x)a(x,x_s)
$$

Using $S(x) = 1 + S_2\frac{1}{2}(3x^2-1) = P_0 + S_2P_2(x)$, the equation becomes

$$
-D\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_n P_n(x)}{\partial x}\right) + B_{\text{out}}T_n P_n(x) = -A_{\text{out}} + Q(1+S_2P_2(x))a(x,x_s). \tag{29}
$$

The Legendre polynomials satisfy

$$
-\frac{\partial}{\partial x}((1-x)^2\frac{\partial}{\partial x}P_n) = n(n+1)P_n, \tag{30}
$$

which means that the equation becomes

$$
\begin{aligned}
D\sum_{n=0}^{\infty} n(n+1)T_nP_n(x) + B_{\text{out}}T_nP_n(x) &= -A_{\text{out}} + Q[P_0 + S_2P_2(x)]a(x,x_s) \\
D\sum_{n=0}^{\infty}(n(n+1)T_n + B_{\text{out}}T_n)P_n(x) &= -A_{\text{out}} + Qa(x,x_s)P_0 + QS_2a(x,x_s)P_2(x) \\
D\sum_{n=0}^{\infty}(n(n+1)T_n + B_{\text{out}}T_n)P_n(x) &= [-A_{\text{out}} + Qa(x,x_s)]P_0(x) + QS_2a(x,x_s)P_2(x)
\end{aligned}
$$

Then, due to the orthogonality of Legendre polynomials, we have that

$$
\begin{aligned}
[D\cdot 0(0+1)T_0 + B_{\text{out}}T_0]P_0(x) &= [-A_{\text{out}} + Qa(x,x_s)]P_0(x), \\
[D\cdot 2(2+1)T_2 + B_{\text{out}}T_2]P_2(x) &= QS_2a(x,x_s)P_2(x),
\end{aligned}
$$

where the final particular solution becomes

$$
T_p(x) = T_0P_0(x) + T_2P_2(x) = \frac{Qa - A_{\text{out}}}{\frac{B_{\text{out}}}{D}}P_0(x) + \frac{QaS_2}{6\frac{B_{\text{out}}}{D}}P_2(x) \tag{31}
$$

where

$$
T_{p,l} = \frac{Qa_l - A_{\text{out}}}{\frac{B_{\text{out}}}{D}}P_0(x) + \frac{Qa_lS_2}{6\frac{B_{\text{out}}}{D}}P_2(x), \qquad T_{p,u} = \frac{Qa_u - A_{\text{out}}}{\frac{B_{\text{out}}}{D}}P_0(x) + \frac{Qa_uS_2}{6\frac{B_{\text{out}}}{D}}P_2(x). \tag{32}
$$

3.5.3 The total temperature equation

The final equation for temperature are

$$
T(x) = y_u(x) + y_l(x) = \alpha_0 y_{l,1}(x) + \beta_0 y_{u,1}(x), \tag{33}
$$

where the coefficients α_0 and β_0 need to be determined. These coefficients are found by evaluating the function at the boundary condition, as we know that the temperature and flux to the upper and lower function should be equal to each other at $x = x_s$. We therefore get two equations,

$$
\begin{aligned}
y_l(x_s) &= y_u(x_s) \\
\alpha_0 y_{l,1} + T_{p,l} &= \beta_0 y_{u,1} + T_{p,u}
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
y_l'(x_s) &= y_u'(x_s) \\
\alpha_0 y_{l,1}' + T_{p,l}' &= \beta_0 y_{u,1}' + T_{p,u}'.
\end{aligned} \tag{35}
$$

This can be written as the matrix equation

$$
\begin{bmatrix} y_{l,1}(x_s) & -y_{u,1}(x_s) \\ y_{l,1}'(x_s) & -y_{u,1}'(x_s) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} T_{p,u}(x_s) - T_{p,l}(x_s) \\ T_{p,u}'(x_s) - T_{p,l}'(x_s). \end{bmatrix} \tag{36}
$$

Since Q is proportional to α_0 and β_0, we can define two new coefficients,

$$
\alpha_0 = \alpha_0' \cdot Q, \qquad \beta = \beta_0' \cdot Q \tag{37}
$$

These coefficients will be used in in the next part, as Q will be treated as a function of x_s.

3.5.4 Solving for $Q(x_S)$

In order to determine x_s for a given Q in Equation 24, we first have to find an expression for Q, which results in a ice cap at $x = x_s$. We can then numerically find $x_s(Q)$. This problem is easier, as Q enters the equation linearly, which x_s does not. The relation between Q and x_s is found by knowing that the temperature is zero at the ice cap. We get that

$$
\begin{aligned}
T(x_s) = 0 \Leftrightarrow y_u(x_s) = 0 \Leftrightarrow \beta_0 y_{u,1}(x_s) + T_{p,u}(x_s) = 0 \Leftrightarrow Q\beta_0' y_{u,1}(x_s) + T_{p,u}(x_s) = 0 \\
Q\beta_0' y_{u,1}(x_s) + \left[\frac{Qa_u - A_{\text{out}}}{B_{\text{out}}/D} P_0(x_s) + \frac{Qa_u S_2}{6B_{\text{out}}/D}\right] = 0 \\
\frac{Qa_u}{B_{\text{out}}/D} + \frac{Qa_u S_2}{6B_{\text{out}}/D} + Q\beta_0' y_u(x_s) = \frac{A_{\text{out}}}{B_{\text{out}}/D}
\end{aligned}
$$

which can be solved for Q. We then get that

$$
Q(x_s) = \frac{\frac{A_{\text{out}}}{B_{\text{out}}/D}}{\frac{a_u}{B_{\text{out}}/D} + \frac{a_u S_2}{6B_{\text{out}}/D} + \beta_0' y_u(x_s)} \tag{38}
$$

3.6 Numerically solving for the Earth's temperature distribution

Following from the analytical calculations in subsection 3.5 we first need to compute where x_s is. To that end, we will first numerically solve the expression for $Q(x_s)$ as defined in 38, then compute its inverse through an optimization scheme as $x_s(G_{\text{SC}}/4)$. Finally we will use the obtained x_s in Equation 33 to compute $T(x)$ to find a solution of 24. The code for this part of the project can be found in the file *"Question10.ipynb"* in [1]

We begin by solving the system 36 numerically. We then have all the components necessary to calculate $Q(x_s)$. See Figure 6 (a) for plot of $Q(x_s)$. To calculate the inverse numerically we begin by defining the difference function from $Q(x) = Q$:

$$
f(x) := Q(x) - Q = 0 \tag{39}
$$

To compute the inverse numerically we only need to find the root(s) of $f(x)$. Denoting the inverse as $f^{-1}(x) := x_s(Q)$, we find that $x_s(G_{\text{SC}}/4) \approx 0.843$.

Now, to compute $T(x)$ we utilize the infrastructure defined for finding $Q(x_s)$, as well as defining 32 in the code. In Figure 6 (b) we see Earth's yearly average temperature as a function of x, where $x = 0$ is the equator, while $x = 1$ is the north pole. We get $T(0) \approx 31.4°\text{C}$, and $T(1) \approx -16.6°\text{C}$ which seem reasonable. We also notice, as expected, that the temperature seem to decrease at a faster rate after crossing x_s.

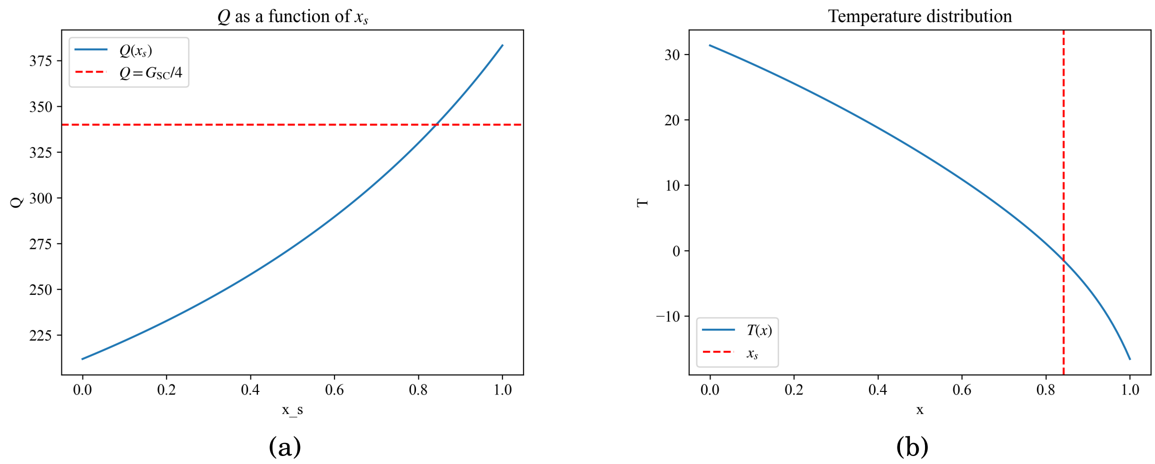


Figure 6: (a) $Q(x_s)$ over the possible values of x_s. (b) Earth's temperature distribution as a function of x, where $x = 0$ is the equator, while $x = 1$ is the north pole.

3.7 Stability analysis

We have now computed an equilibrium solution T, and are interested in analyzing the stability. Consider a small perturbation δT around the equilibrium solution. Substituting $T + \delta T$ in the non-equilibrium version of 24 yields

$$\begin{aligned}C_a\frac{\partial}{\partial t}(T+\delta T) &= D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial}{\partial x}(T+\delta T)\right) - B_{\text{out}}(T+\delta T) - A_{\text{out}} + QS(x)a(x,x_s)\\ C_a\frac{\partial T}{\partial t} + C_a\frac{\partial \delta T}{\partial t} &= D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T}{\partial x}\right) - B_{\text{out}}T - A_{\text{out}} + QS(x)a(x,x_s) + D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial \delta T}{\partial x}\right) - B_{\text{out}}\delta T.\end{aligned} \tag{40}$$

Since T is the equilibrium solution, this simplifies to

$$C_a\frac{\partial \delta T}{\partial t} = D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial \delta T}{\partial x}\right) - B_{\text{out}}\delta T = D(1-x^2)\frac{\partial^2 \delta T}{\partial x^2} - 2Dx\frac{\partial \delta T}{\partial x} - B_{\text{out}}\delta T. \tag{41}$$

Let's use the ansatz $\delta T = e^{\lambda t}u(x)$. We want to investigate the value of λ for a $u \neq 0$, where a negative value of λ means that the perturbation sinks as a function of time, and that the equilibrium solution is stable. Using this ansatz leads to the equation

$$\begin{aligned}C_a\frac{\partial}{\partial t}(e^{\lambda t}u(x)) &= D(1-x^2)\frac{\partial^2}{\partial x^2}(e^{\lambda t}u(x)) - 2Dx\frac{\partial}{\partial x}(e^{\lambda t}u(x)) - B_{\text{out}}(e^{\lambda t}u(x))\\ \lambda C_a e^{\lambda t}u(x) &= D(1-x^2)e^{\lambda t}u''(x) - 2Dxe^{\lambda t}u'(x) - B_{\text{out}}e^{\lambda t}u(x)\\ \lambda u(x) &= \frac{D}{C_a}\left[(1-x^2)u''(x) - 2xu'(x) - \frac{B_{\text{out}}}{D}u(x)\right].\end{aligned} \tag{42}$$

Take the spatial discretization $x_i = ih$ for $i = 0, \dots, n$, and denote $u_i = u(x_i)$. We utilize second-order central difference approximations such that the discretized equation for the interior nodes becomes

$$\begin{aligned}\lambda u_i &= \frac{D}{C_a}\left[(1-x_i^2)\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} - 2x_i\frac{u_{i+1}-u_{i-1}}{2h} - \frac{B_{\text{out}}}{D}u_i\right]\\ \lambda C_a u_i &= \frac{D}{C_a}\left[\left(\frac{1-(ih)^2}{h^2} - \frac{ih}{h}\right)u_{i+1} + \left(-\frac{2(1-(ih)^2)}{h^2} - \frac{B_{\text{out}}}{D}\right)u_i + \left(\frac{1-(ih)^2}{h^2} + \frac{ih}{h}\right)u_{i+1}\right]\\ \lambda C_a u_i &= \frac{D}{C_a}\left[\left(\frac{1}{h^2} - i^2 - i\right)u_{i+1} + \left(-\frac{2}{h^2} + 2i^2 - \frac{B_{\text{out}}}{D}\right)u_i + \left(\frac{1}{h^2} - i^2 + i\right)u_{i+1}\right].\end{aligned} \tag{43}$$

For $i = 0$, we use forward differences to get

$$\begin{aligned}\lambda u_0 &= \frac{D}{C_a}\left[\frac{\partial^2 u_0}{\partial x^2} - \frac{B_{\text{out}}}{D}u_0\right]\\ \lambda u_0 &= \frac{D}{C_a}\left[\frac{u_2 - 2u_1 + u_0}{h^2} - \frac{B_{\text{out}}}{D}u_0\right]\\ \lambda u_0 &= \frac{D}{C_a}\left[\frac{1}{h^2}u_2 - \frac{2}{h^2}u_1 + \left(\frac{1}{h^2} - \frac{B_{\text{out}}}{D}\right)u_0\right].\end{aligned} \tag{44}$$

For $i = n$, we use backward differences to get

$$\begin{aligned}\lambda u_n &= \frac{D}{C_a}\left[-2x_n\frac{\partial u_n}{\partial x} - \frac{B_{\text{out}}}{D}u_n\right]\\ \lambda u_n &= \frac{D}{C_a}\left[-2\frac{u_n - u_{n-1}}{h} - \frac{B_{\text{out}}}{D}u_n\right]\\ \lambda u_n &= \frac{D}{C_a}\left[\left(-\frac{2}{h} - \frac{B_{\text{out}}}{D}\right)u_n + \frac{2}{h}u_{n-1}\right].\end{aligned} \tag{45}$$

Then we have an eigenvalue problem of the form $\lambda \mathbf{u} = \mathbf{M}\mathbf{u}$, which we will compute numerically for different choices of the number of spatial nodes n. We tested a range of values for n between 10 and 1000 and confirmed that all eigenvalues were, in fact, negative. The equilibrium solution is therefore stable. See Question12.ipynb in [1] for implementation details. We used the values $C_a = \rho \cdot c = 1.293\,\text{kgm}^{-3} \cdot 1000\,\text{JK}^{-1}\text{kg}^{-1}$.

4 Conclusion

In the first part of the project, the modeling of the Earth and the atmosphere temperatures were higher than real values. However the modeled temperatures aligned with the assumptions made. Further on the sensitivity of some parameters was calculated to examine the effect on the temperatures. In particular, increasing the absorption coefficient a_{SW} related to greenhouse gases increased the atmosphere temperature. The model was further extended to include clouds. This led to an increase of the atmosphere temperature and a slight decrease of the Earth's temperature compared to the model without clouds. The inclusion of clouds allowed for more energy to be absorbed in the atmosphere, and thus more heat was retained. At the same time, the clouds allowed for more reflection of the short-waves from the Sun. As a result, less of these waves reached the Earth and therefore contributing to the slight temperature decrease.

In the second part of the project, we assumed a simple zonal model considering the polar regions and the earth's capacity to reflect radiation. It seems that the temperature decreases noticeably faster above the polar circle, and the ice cap considerably affects the temperature. The resulting temperatures seem to align reasonably with observed values.

Appendix

A Constants

Parameter	Symbol	Value	Unit
Averaged Solar Flux	P_0^S	341.3	W m^{-2}
Cloud Cover	CC	66.0	%
SW Molecular Scattering Coefficient	r_{SM}	10.65	%
SW Cloud Scattering Coefficient	r_{SC}	22.0	%
SW Earth Reflectivity	r_{SE}	17.0	%
SW Absorptivity: O_3	a_{O3}	8.0	%
SW Cloud Absorptivity	a_{SC}	12.39	%
SW Absorptivity: H_2O-CO_2-CH_4	a_{SW}	14.51	%
LW Cloud Scattering Coefficient	r_{LC}	19.5	%
LW Earth Reflectivity	r_{LE}	0.0	%
LW Cloud Absorptivity	a_{LC}	62.2	%
LW Absorptivity: H_2O-CO_2-CH_4-O_3	a_{LW}	82.58	%
Earth Emissivity	$\varepsilon_{\mathrm{E}} = 1 - r_{\mathrm{LE}}$	100.0	%
Atmospheric Emissivity	ε_{A}	87.5	%
Asymmetry Factor	f_{A}	61.8	%
Sensible Heat Flux	α	3	W m^{-2} K^{-1}
Latent Heat Flux	β	4	W m^{-2} K^{-1}

Table 1: Summary of parameters and their values.

B Calculations of A for cloudless atmosphere

First O_3 Absorption	$P_0^S = 341.3\,\mathrm{W/m^2}, \quad P_{O_3a1} = a_{O_3} P_0^S, \quad P_1 = P_0^S - P_{O_3a1}$
Atmosphere (First Pass)	$P_{1r} = r_{\mathrm{SW}} P_1, \quad P_{1a} = a_{\mathrm{SW}}(1 - r_{\mathrm{SW}}) P_1, \quad P_{1t} = (1 - a_{\mathrm{SW}})(1 - r_{\mathrm{SW}}) P_1$
Transmitted to Earth	$P_{\mathrm{Er}} = r_{\mathrm{SE}} P_{1t}, \quad P_{\mathrm{Ea}} = (1 - r_{\mathrm{SE}}) P_{1t}$
Atmosphere (Second Pass)	$P_{2r} = r_{\mathrm{SW}} P_{\mathrm{Er}}, \quad P_{2a} = a_{\mathrm{SW}}(1 - r_{\mathrm{SW}}) P_{\mathrm{Er}}, \quad P_{2t} = (1 - a_{\mathrm{SW}})(1 - r_{\mathrm{SW}}) P_{\mathrm{Er}}$
Second O_3 Absorption	$P_{O_3a2} = a_{O_3}(P_{2t} + P_{1r})$
Reflections Back to Earth	$P_{\mathrm{Er2}} = r_{\mathrm{SE}} P_{2r}, \quad P_{\mathrm{Ea2}} = (1 - r_{\mathrm{SE}}) P_{2r}$
Atmosphere (Third Pass)	$P_{3r} = r_{\mathrm{SW}} P_{\mathrm{Er2}}, \quad P_{3a} = a_{\mathrm{SW}}(1 - r_{\mathrm{SW}}) P_{\mathrm{Er2}}, \quad P_{3t} = (1 - a_{\mathrm{SW}})(1 - r_{\mathrm{SW}}) P_{\mathrm{Er2}}$
Third O_3 Absorption	$P_{O_3a3} = a_{O_3} P_{3t}$

Total absorbed SW Radiation components:

$$A_E = P_{\mathrm{Ea}} + P_{\mathrm{Ea2}} \quad A_{O_3} = P_{O_3a1} + P_{O_3a2} + P_{O_3a3} \quad A_{GG} = P_{1a} + P_{2a} + P_{3a}$$

C Calculations of A for atmosphere including clouds

First O_3 Absorption	$P_0^S = 341.3, \mathrm{W/m^2}, \quad P_{O_3a1} = a_{O_3} P_0^S, \quad P_1 = P_0^S - P_{O_3a1}$
Atmosphere (First Pass)	$P_{1r} = r_{\mathrm{SW}} P_1, \quad P_{1a} = a_{\mathrm{SW}}(1 - r_{\mathrm{SW}}) P_1, \quad P_{1t} = (1 - a_{\mathrm{SW}})(1 - r_{\mathrm{SW}}) P_1$
Clouds (First Pass)	$P_{1t,\mathrm{cloud}} = C_C \cdot P_{1t} \qquad C_{r1} = r_{\mathrm{SC}} P_{1t,\mathrm{cloud}}$ $C_{a1} = a_{\mathrm{SC}}(1 - r_{\mathrm{SC}}) P_{1t,\mathrm{cloud}}, \quad C_{t1} = (1 - a_{\mathrm{SC}})(1 - r_{\mathrm{SC}}) P_{1t,\mathrm{cloud}}$
Transmitted to Earth (First Pass)	$P_{1t,\mathrm{earth}} = (1 - C_C) P_{1t} + C_{t1}$ $E_{r1} = r_{\mathrm{SE}} P_{1t,\mathrm{earth}}, \quad E_{a1} = (1 - r_{\mathrm{SE}}) P_{1t,\mathrm{earth}}$
Clouds (Second Pass)	$P_{2t,\mathrm{cloud}} = C_C \cdot E_{r1} \qquad C_{r2} = r_{\mathrm{SC}} P_{2t,\mathrm{cloud}}$ $C_{a2} = a_{\mathrm{SC}}(1 - r_{\mathrm{SC}}) P_{2t,\mathrm{cloud}}$ $C_{t2} = (1 - a_{\mathrm{SC}})(1 - r_{\mathrm{SC}}) P_{2t,\mathrm{cloud}}$
Atmosphere (Second Pass)	$P_{2r} = r_{\mathrm{SW}} \left[(1 - C_C) E_{r1} + C_{r1} + C_{t2}\right]$ $P_{2a} = a_{\mathrm{SW}}(1 - r_{\mathrm{SW}}) \left[(1 - C_C) E_{r1} + C_{r1} + C_{t2}\right]$ $P_{2t} = (1 - a_{\mathrm{SW}})(1 - r_{\mathrm{SW}}) \left[(1 - C_C) E_{r1} + C_{r1} + C_{t2}\right]$
Second O_3 Absorption	$P_{O_3a2} = a_{O_3} \left(P_{2t} + P_{1r}\right)$
Clouds (Third Pass)	$P_{3t,\mathrm{cloud}} = C_C \cdot P_{2r} \qquad C_{r3} = r_{\mathrm{SC}} P_{3t,\mathrm{cloud}}$ $C_{a3} = a_{\mathrm{SC}}(1 - r_{\mathrm{SC}}) P_{3t,\mathrm{cloud}}, \quad C_{t3} = (1 - a_{\mathrm{SC}})(1 - r_{\mathrm{SC}}) P_{3t,\mathrm{cloud}}$
Back to Earth (Second Pass)	$P_{2t,\mathrm{earth}} = (1 - C_C) P_{2r} + C_{t3} + C_{r2},$ $E_{r2} = r_{\mathrm{SE}} P_{2t,\mathrm{earth}}, \quad E_{a2} = (1 - r_{\mathrm{SE}}) P_{2t,\mathrm{earth}}$
Clouds (Fourth Pass)	$P_{4t,\mathrm{cloud}} = C_C \cdot E_{r2} \qquad C_{r4} = r_{\mathrm{SC}} P_{4t,\mathrm{cloud}}$ $C_{a4} = a_{\mathrm{SC}}(1 - r_{\mathrm{SC}}) P_{4t,\mathrm{cloud}}, \quad C_{t4} = (1 - a_{\mathrm{SC}})(1 - r_{\mathrm{SC}}) P_{4t,\mathrm{cloud}}$
Atmosphere (Third Pass)	$P_{3r} = r_{\mathrm{SW}} \left[(1 - C_C) E_{r2} + C_{r3} + C_{t4}\right],$ $P_{3a} = a_{\mathrm{SW}}(1 - r_{\mathrm{SW}}) \left[(1 - C_C) E_{r2} + C_{r3} + C_{t4}\right],$ $P_{3t} = (1 - a_{\mathrm{SW}})(1 - r_{\mathrm{SW}}) \left[(1 - C_C) E_{r2} + C_{r3} + C_{t4}\right]$
Third O_3 Absorption	$P_{O_3a3} = a_{O_3} P_{3t}$

Total absorbed SW Radiation components for model including clouds:

$$A_E = E_{a1} + E_{a2} \quad A_{O_3} = P_{O_3a1} + P_{O_3a2} + P_{O_3a3} \quad A_{GG} = P_{1a} + P_{2a} + P_{3a} \quad A_C = C_{a1} + C_{a2} + C_{a3} + C_{a4}$$

D Solution to the homogeneous equation for $x < x_s$

The homogeneous equation are on the form

$$-(1 - x^2) \sum_{n=2}^{\infty} \alpha_n n(n-1) x^{n-2} + 2x \sum_{n=1}^{\infty} \alpha_n n x^{n-1} + \frac{B_{\mathrm{out}}}{D} \sum_{n=0}^{\infty} \alpha_n x^n = 0 \tag{46}$$

The first term is

$$\begin{aligned}
-(1-x^2)\sum_{n=2}^{\infty}\alpha_n n(n-1)x^{n-2} &= -\sum_{n=2}^{\infty}\alpha_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty}\alpha_n n(n-1)x^n \\
&= -\sum_{n=0}^{\infty}\alpha_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty}\alpha_n n(n-1)x^n \\
&= \sum_{n=0}^{\infty}\left(-\alpha_{n+2}(n+2)(n+1)+\alpha_n n(n-1)\right)x^n,
\end{aligned}$$

where in the second step we used that $n(n-1)=0$ when $n=0$ and $n=1$. The second term is

$$2x\sum_{n=1}^{\infty}\alpha_n n x^{n-1} = 2\sum_{n=1}^{\infty}\alpha_n n x^n = 2\sum_{n=0}^{\infty}\alpha_n n x^n,$$

so we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(-\alpha_{n+2}(n+2)(n+1)+\alpha_n n(n-1)\right)x^n + 2\sum_{n=0}^{\infty}\alpha_n n x^n + \frac{B_{\text{out}}}{D}\sum_{n=0}^{\infty}\alpha_n x^n &= 0 \\
\sum_{n=0}^{\infty}\left[-\alpha_{n+2}(n+2)(n+1)+\alpha_n n(n-1)+2\alpha_n n+\frac{B_{\text{out}}}{D}\alpha_n\right]x^n &= 0
\end{aligned}$$

To find the non-trivial solutions, we put

$$\begin{aligned}
-\alpha_{n+2}(n+2)(n+1)+\alpha_n n(n-1)+2\alpha_n n+\frac{B_{\text{out}}}{D}\alpha_n &= 0 \\
-\alpha_{n+2}(n+2)(n+1)+\left[n(n-1)+2n+\frac{B_{\text{out}}}{D}\right]\alpha_n &= 0 \\
\alpha_{n+2} &= \frac{n(n-1)+2n+\frac{B_{\text{out}}}{D}}{(n+2)(n+1)}\alpha_n \\
\alpha_{n+2} &= \frac{n(n+1)+\frac{B_{\text{out}}}{D}}{(n+2)(n+1)}\alpha_n
\end{aligned}$$

We then get two linearly independent sums, one for even n and one for odd. For even n, we get

$$a_{2n} = \frac{\frac{B_{\text{out}}}{D}(\frac{B_{\text{out}}}{D}+2\cdot 3)(\frac{B_{\text{out}}}{D}+4\cdot 5)\cdots[\frac{B_{\text{out}}}{D}+(2n-1)(2n-2)]}{(2n)!}\alpha_0, \tag{47}$$

and for odd n, we get

$$a_{2n+1} = \frac{(\frac{B_{\text{out}}}{D}+1\cdot 2)(\frac{B_{\text{out}}}{D}+2\cdot 3)(\frac{B_{\text{out}}}{D}+4\cdot 5)\cdots[\frac{B_{\text{out}}}{D}+(2n)(2n-1)]}{(2n+1)!}\alpha_1. \tag{48}$$

The series for $x<x_s$ can therefore be written as

$$y_l(x) = \alpha_0 y_{l,1}(x) + \alpha_1 y_{l,2}(x), \tag{49}$$

where

$$y_{l,1}(x) = 1+\sum_{n=1}^{\infty}\frac{\frac{B_{\text{out}}}{D}(\frac{B_{\text{out}}}{D}+2\cdot 3)(\frac{B_{\text{out}}}{D}+4\cdot 5)\cdots[\frac{B_{\text{out}}}{D}+(2n-1)(2n-2)]}{(2n)!}x^{2n}, \tag{50}$$

$$y_{l,2}(x) = x+\sum_{n=1}^{\infty}\frac{(\frac{B_{\text{out}}}{D}+1\cdot 2)(\frac{B_{\text{out}}}{D}+2\cdot 3)(\frac{B_{\text{out}}}{D}+4\cdot 5)\cdots[\frac{B_{\text{out}}}{D}+(2n)(2n-1)]}{(2n+1)!}x^{2n+1} \tag{51}$$

We have a boundary condition that says that the flux is 0 at $x=0$, meaning that $\alpha_1=0$, which means that the final solution to the homogeneous equation for $x<x_s$ are

$$y_l(x) = \alpha_0 y_{l,1}(x) \tag{52}$$

References

[1] *GitHub Repository TMA4195 Project*. URL: https://github.com/aledha/TMA4195-Project.