Truncated PCA project 2

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Abstract

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0.1 Stability

Now we can find an expression for the backward Euler scheme. The sparse matrix takes the form

$$U_m^n = -k(\frac{\sigma^2}{2h^2}x_m^2 + \frac{r}{2h}x_m)U_m^{n+1} + k(\frac{\sigma^2}{2h^2}x_m^2 + c)U_m^{n+1} - k(\frac{\sigma^2}{2h^2}x_m^2 - \frac{r}{2h}x_m)U_{m-1}^{n+1} \tag{0.1}$$

We can rewrite our problem as following

$$AU^{n+1} = U^n \tag{0.2}$$

Assume that the inverse matrix of A exists, We can rewrite our equation to an iterative form.

$$A^{-1}U^n = U^{n+1} \tag{0.3}$$

To show stability we need to show that our sparse matrix is continuous or in other words upper-bounded in some norm. We need show the existence of the inverse matrix, simply knowing that our sparse matrix is weakly chained, monotone and diagonal dominant for any h, where h is the size of the discretization. we have that the matrix is diagonal dominant if

$$\frac{2h}{2} > 0 \tag{0.4}$$

Not to show consistency, we need to who that our matrix is upper bound in some norm. Sadly our matrix is not symmetric thus not hermitian, meaning it can have complex eigenvalues. We can show that the inverse function is upper bounded in the infinity norm. There is proof that the matrix is WCDD.

$$\left\|A^{-1}\right\|_\infty \le \max_{1\le j\le n} \frac{1}{\left\|\Delta J_i\right\|_\infty} \tag{0.5}$$

Where δJ_i is defined as

$$\Delta J_i = a_{ii} - \sum_{j\neq i}^{n} a_{ij} \tag{0.6}$$

But we can rather use a super solution to show upper bound to our matrix, since is WCDD. Given the row elements of our sparse matrix, we can replace our U_i^n with the following super solution. Our scheme or in other words, the backward Euler is indeed monotone form b. the monotonously will imply maximal discrete principle and we now that the max is now at the boundary. Thus we can use a super solution, which must satisfy the boundary conditions.

$$\begin{aligned} &u(x,0) \leq K \\ &u(0,0) = K \\ &u(0,t) \leq K \\ &u(X,T) = 0 \end{aligned} \tag{0.7}$$

Thus we can choose the following function as our super solution. This function will satisfy our boundary condition.

$$\phi_m^n = K(n+1) \tag{0.8}$$

We know that there are no other values that are greater than our boundary and we can see clearly that if I look at the maximum if $W_n^m = V_m^n - \phi_m^n \leq 0$ So we can simply find the max of this function. Now we need to use this to show stability, we can make a new variable name W_m which is given by

$$\begin{aligned} \mathcal{L}\phi_m^n &= -\phi_m^{n+1} + \phi_m^n(ck+1) \\ \mathcal{L}\phi_m^n &= -(n+2) * K + K(n+1)(ck+1) \\ \mathcal{L}\phi_m^n &= -nK + 2K - Knck - Kck - kn - k \\ \mathcal{L}\phi_m^n &= -K + Knck - Kck \\ \mathcal{L}\phi_m^n &= -K + Kck(n+1) \end{aligned} \tag{0.9}$$

Now we can evaluate the actual max which might be less or equal to some negative number

$$\begin{aligned} &W_m = V_n - \phi_m^n \\ &L(W_m) = \mathcal{L}V_n - \mathcal{L}\phi(x,t) \\ &L(W_m) = -f_n - (-K + Kck(n+1)) \end{aligned} \tag{0.10}$$

Now our W_k must satisfy the DMP, which gives us that

$$\begin{aligned} &\max_m V_m^n \leq \max_m W_m^n + \max_m \phi_m^n \\ &\max_m V_m^n \leq K - K(n+1) \\ &= Kn \end{aligned} \tag{0.11}$$

and we have shown stability.

0.2 Error analysis

For the truncation error one can show that the truncation error for the forward Euler method is

$$t_h^t = O(t + h^2) \tag{0.12}$$

Now we want to show this. we have simply used that central difference in space, and a backward difference in time. This can be shown by writing out the expressions needed an taylorexpan them.

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + h^2\frac{1}{2}f''(x) + h^3\frac{1}{6}f^{(3)}(x) + h^4\frac{1}{24}f^{(4)}(x) \\ f(x-h) &= f(x) - hf'(x) + h^2\frac{1}{2}f''(x) - h^3\frac{1}{6}f^{(3)}(x) + h^4\frac{1}{24}f^{(4)}(x) \end{aligned} \tag{0.13}$$

we can write them as a linear combination, thus we get that

$$\begin{aligned} &f(x) - f(x-k) = kf'(x) + k^2\frac{1}{2}f^{(2)}(x) \\ &f(x+h) - f(x+h) = 2 * f'(x)h + h^3\frac{1}{3}f^{(3)}(x) \\ &f(x-h) - 2f(x) + f(x+h) = h^2f''(x) + h^4\frac{1}{12}f^{(4)}(x) \end{aligned} \tag{0.14}$$

collect the last terms and we get that the following truncation error, and since we are dealing with a vector, we can find an upper bound to the truncation in L^∞.

$$\begin{aligned} \tau_h^k &= k\frac{1}{2}f^{(2)}(x) + h^2\frac{1}{3}f^{(3)}(x) + h^2\frac{1}{12}f^{(4)}(x) \\ \|\tau_h^k\|_\infty &= \left\|k\frac{1}{2}f^{(2)}(x) + h^2\frac{1}{3}f^{(3)}(x) + h^2\frac{1}{12}f^{(4)}(x)\right\|_\infty \\ &= \left\|k\frac{1}{2}f^{(2)}(x)\right\|_\infty + \left\|h^2\frac{1}{3}f^{(3)}(x)\right\|_\infty + \left\|h^2\frac{1}{12}f^{(4)}(x)\right\|_\infty \\ &= k\frac{1}{2}\left\|f^{(2)}(x)\right\|_\infty + h^2\frac{1}{3}\left\|f^{(3)}(x)\right\|_\infty + h^2\frac{1}{12}\left\|f^{(4)}(x)\right\|_\infty \\ &= k\frac{1}{2}\left\|f^{(2)}(x)\right\|_\infty + h^2\left[\frac{1}{3}\left\|f^{(3)}(x)\right\|_\infty + \frac{1}{12}\left\|f^{(4)}(x)\right\|_\infty\right] \end{aligned} \tag{0.15}$$

We can see as k and h is going to zero we get that the truncation error goes towards zero. And the complexity of the algorithm is big $O(t)$ in time and $O(h^2)$ in space. In the Crank Nicholson method, the only thing that changes is the complexity in time. We just how to find the error of the trapezoid rule, which is.

$$\int_0^k f(x)dx = \frac{1}{2}k(f(x) - f(0)) + \frac{1}{12}f^2(x)k^2 \tag{0.16}$$

Thus we collect the new truncation error in time and we get the new truncation error for the Crank Nicholson.

$$\left\|\tau_h^k\right\|_\infty = k^2\frac{1}{12}\left\|f^{(2)}(x)\right\|_\infty + h^2\left[\frac{2}{3}\left\|f^{(3)}(x)\right\|_\infty + \frac{1}{6}\left\|f^{(4)}(x)\right\|_\infty\right] \tag{0.17}$$

Observe that the truncation error goes to zero at h and t goes approaches zero, and the complexity of the method is $O(k^2)$ is time and $O(h^2)$ in space. We get an additional factor of two due to symmetry. Thus both method are stable. Now we can. To evaluate the error, we have the following relation between the truncation error and the error. Lest A denote the sparse matrix, u the approximate solution and U the exact solution.

$$\begin{aligned} e &= Au - AU \\ &= A(u-U) \\ &= A\tau \end{aligned} \tag{0.18}$$

Notice that both methods are stability and consistent this implies that both methods convergence accordion to the Lax Equivalence Theorem. Now we would like to find an upper bound to the numerical error.

$$\begin{aligned} \|e\|_\infty &= \left\|A^{-1}\tau\right\|_\infty \\ &\leq \left\|A^{-1}\right\|_\infty\|\tau\|_\infty \end{aligned} \tag{0.19}$$

Now from b, we have already shown an upper bound to the $\left\|A^{-1}\right\|_\infty$. We have shown that an upper bound exists for the backward Euler scheme for Black Sholes equation. A different approach would be.

$$e^{n+1} - Ae^n = (u^{n+1} - Au^n - b^n) - (U^{n+1} - AU^n - b^n) = -k\tau^{n+1} \tag{0.20}$$

and we have show it goes to zero. Now for the Crank Nicholson scheme this is not so easy. We can do a similar approach

$$e^n + 1 - Ae^n = (u^{n+1} - Au^n - b^n) - (U^{n+1} - AU^n - b^n) = -k\tau^{n+1} \tag{0.21}$$