TMA4180 Optimization I
A Tensegrity Form-Finding Optimization Model
NTNU

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Abstract

From art to engineering, tensegrity structures have revolutionized the world of construction. Tensegrity structures, also called "smart structures", are mechanical structures constructed of straight elastic members (called bars) and elastic cables that are connected at joints (or nodes), and together guarantee the stability of the structure. These bars and cables are only compressed and tensioned, and that is why the structure receives the name tensegrity, defying other kind of structures.

The proposed optimization model provides a systematic and efficient way to determine the shape of tensegrity structures and can have potential applications in the design and analysis of various tensegrity-based systems in engineering and space exploration.

Contents

1 Introduction, notation & definitions

In this project, we wish to implement a form-finding model for tensegrity structures using optimisation. We will set up the initial positions of nodes connected by cables and bars and use optimization algorithms to find what form the system will take. To do this, we must introduce notation and energy functions and choose an appropriate algorithm. The project is divided into three parts, where the system increases in complexity:

- In the first part, the structure only has cables, and some nodes are fixed.
- In the second part, bars are added to the system.
- In the third part, with cables and bars, a ground is set.

These three parts will consist of an introduction to the mathematical theorems needed to understand the system's behavior in each part, followed by the graphical results obtained after the numerical implementation of the chosen algorithms.

1.1 Notation & definitions

To model tensegrity structures, we will use a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a vertex set $\mathcal{V} = \{1, \dots, N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The vertices represent nodes, and an edge e_{ij} (for $i < j$) represents a connection between node i and node j. Furthermore, we denote the set of all cable connections as $\mathcal{C}$ and the set of all bar connections as $\mathcal{B}$. The position of node i is denoted as $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \in \mathbb{R}^3$, and the collection of all nodes is in a large vector $X \in \mathbb{R}^{3N}$.

We will consider several energy components, so in the following, we will introduce all the components needed to express the system's total energy. Suppose each node has an external load. We can then model the gravitational potential energy as

$$E_{\text{ext}}(X) = \sum_{i=1}^{N} m_i g x_3^{(i)}, \tag{1.1}$$

where m_i is the mass of node i, and g is the gravitational acceleration. Now, consider a bar that connects the nodes $x^{(i)}$ and $x^{(j)}$, and denote the natural resting length of the bar as $l_{ij} > 0$. We model the gravitational potential energy of such a bar as

$$E_{\text{grav}}^{\text{bar}}(e_{ij}) = \frac{\rho g l_{ij}}{2}(x_3^{(i)} + x_3^{(j)}), \tag{1.2}$$

where $\rho > 0$ is the line density of the bar. Further, if the distance between the nodes is anything other than the natural resting length, the bar will have elastic potential energy. We model this with the quadratic model

$$E_{\text{elast}}^{\text{bar}}(e_{ij}) = \frac{c}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2, \tag{1.3}$$

where $c > 0$ is a material parameter. Cables, however, can not be compressed. We shall therefore model the elastic energy of the cables as

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2 & \text{for } \|x^{(i)} - x^{(j)}\| > l_{ij}, \\ 0 & \text{else,} \end{cases} \tag{1.4}$$

where $k > 0$ is also a material parameter. We can now model the total energy of a given system as

$$E(X) = \sum_{e_{ij} \in \mathcal{B}} (E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})) + \sum_{e_{ij} \in \mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X). \tag{1.5}$$

Minimising such a system is problematic since the functions E_{ext} and $E_{\text{grav}}^{\text{bar}}$ are unbounded below. We must therefore introduce additional constraints. In this report, we explore two possibilities. The first possibility is fixing the position of the first M nodes,

$$x^{(i)} = p^{(i)}, \quad \text{for } i = 1, \dots, M, \tag{1.6}$$

which results in an unconstrained problem in $\mathbb{R}^{3(N-M)}$. The second possibility is constraining the system to be above ground, resulting in the inequality constraints

$$x_3^{(i)} \geq 0, \quad \text{for } i = 1, \ldots, N. \tag{1.7}$$

1.2 Existence of a solution

These constraints seem logical, but do they guarantee that a global solution exists? To guarantee this, we introduce Theorem 1.1.

Theorem 1.1. *The problem of minimising* (1.5) *with constraints given either by* (1.6) *or by* (1.7) *admits a solution, provided that the graph* $\mathcal{G}$ *is connected.*

Proof. We must show that our function is lower semi-continuous, coercive, and the space we operate in must be closed and non-empty.

The cables and bars can only *continuously* shrink/stretch around their respective tether points. This means $E_{\text{elast}}^{\text{bar}}(e_{ij})$ and $E_{\text{elast}}^{\text{cable}}(e_{ij})$ are continuous. We trivially see that the two other terms in (1.5) are also continuous. Since all terms are continuous, the function is also lower semi-continuous.

First, we look at type (1.6) constraints and how they produce coercivity. In the x- and y direction, the only terms they apply are $E_{\text{elast}}^{\text{bar}}(e_{ij})$ and $E_{\text{elast}}^{\text{cable}}(e_{ij})$. Increasing x or y to $\pm\infty$ will increase $E(X)$ to $+\infty$. Now for the z-direction: for $z \to +\infty$, all terms grow to $+\infty$. The negative z-direction is a little more work. The only terms that apply here are:

$$E_{\text{elast}}^{\text{bar}}, E_{\text{elast}}^{\text{cable}} \sim z^2, \text{ and}$$
$$E_{\text{grav}}^{\text{bar}}, E_{\text{ext}} \sim z.$$

This means that when $z \to -\infty$, $E_{\text{grav}}^{\text{bar}}$ and E_{ext} become negligible and $E(X) \to +\infty$.

Now for the (1.7) constraint type. The tricky part is that if we move all the points to $\pm\infty$ in the x- or y direction, $E(X)$ does not tend to infinity but remains constant. The problem is invariant under horizontal shifts of the nodes[1]. To fix this problem, we can introduce the constraints $x_1^{(1)} = x_2^{(1)} = 0$, i.e., fixing the first node's x and y values to the origin. Coercivity in the x- and y direction is introduced. Both $\mathbb{R}^3$ and $\{\mathbb{R}^3 : z \geq 0\}$ are closed and nonempty, so our problem admits a global solution. □

2 Cable-nets

We begin by looking at a simpler case than the complete model. We do this by constructing a structure only consisting of cables and nodes. Moreover, we consider the constraint (1.6), where some of the nodes are fixed. The resulting optimisation problem can be stated as

$$\min_X E(X) = \sum_{e_{ij} \in \mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \quad \text{s.t.} \quad x^{(i)} = p^{(i)}, \quad i = 1, ..., M. \tag{2.1}$$

Let us explore some interesting properties of this system.

2.1 Mathematical framework

First, we need to know the smoothness of the system to choose a suitable algorithm.

Theorem 2.1. *The function* E *defined in* (2.1) *is* C^1*, but typically not* C^2*.*

Proof. Remark that the sum of C^1 functions is C^1, so we can evaluate each term separately. For $E_{\text{elast}}^{\text{cable}}(e_{ij})$, we define the term $u := \frac{k}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2$.

By symmetry, we have $\partial_{x_n^{(i)}} u = -\partial_{x_n^{(j)}} u$ for $n = 1, 2, 3$, and

$$\partial_{x_n^{(i)}} u = \frac{k\left(x_n^{(i)} - x_n^{(j)}\right)}{l_{ij}^2 \|x^{(i)} - x^{(j)}\|} \left(\|x^{(i)} - x^{(j)}\| - l_{ij}\right), \quad \text{for } n = 1, 2, 3.$$

[1]The problem is actually invariant under rotation, but the point here is that the problem is coercive.

If $\partial_{x^{(i)}} E_{\text{elast}}^{\text{cable}}$ is continuous if and only if $\partial_{x^{(i)}} u \overset{\|x^{(i)}-x^{(j)}\|\to l_{ij}}{\to} 0$, which is the case.

A potential issue with every $\partial_{x^{(i)}} u$ would be when $\|x^{(i)} - x^{(j)}\| = 0$ (and hence a 0 in the denominator). However, recall that $E_{\text{elast}}^{\text{cable}} = 0$ for $\|x^{(i)} - x^{(j)}\| < l_{ij}$, and that $l_{ij} > 0$, such that in this case, the function and its derivative is equal to 0. We conclude that all $\partial_{x^{(i)}} u$ are continuous, and $E_{\text{elast}}^{\text{cable}}(e_{ij})$ is, therefore, C^1.

Let us now compute $\partial^2_{x_n^{(i)}} u$. We get

$$\partial^2_{x_n^{(i)}} u = \frac{k}{l_{ij}^2}\left(1 - \frac{l_{ij}}{\|x^{(i)} - x^{(j)}\|} + \frac{l_{ij}(x_n^{(i)} - x_n^{(j)})^2}{\|x^{(i)} - x^{(j)}\|^3}\right), \quad \text{for } n = 1, 2, 3.$$

If $\partial^2_{x^{(i)}} E_{\text{elast}}^{\text{cable}}$ is continuous, then $\partial^2_{x^{(i)}} u \overset{\|x^{(i)}-x^{(j)}\|\to l_{ij}}{\to} 0$ holds only if $x_n^{(i)} = x_n^{(j)}$, which is not necessarily the case. Hence it is typically not C^2.

$E_{\text{ext}}(X)$ is clearly C^1 as $\partial_{x_3^{(i)}} E_{\text{ext}} = m_i g$ is constant and, therefore, continuous. □

A rather important property we need to explore is the convexity of our unconstrained problem since a convex problem means that we converge to a global minimum regardless of the initialisation.

Theorem 2.2. *The problem (2.1) is convex.*

Proof. First, we explore $E_{\text{ext}}(X)$. With our constraints, we have

$$E_{\text{ext}}(X) = \sum_{i=1}^{M} m_i g p_3^{(i)} + \sum_{i=M+1}^{N} m_i g x_3^{(i)}.$$

$E_{\text{ext}}(X)$ is convex if and only if $E_{\text{ext}}(tX + (1-t)\tilde{X}) \leq tE_{\text{ext}}(X) + (1-t)E_{\text{ext}}(\tilde{X})$ for all $0 \leq t \leq 1$. Inserting our definition gives

$$E_{\text{ext}}(tX + (1-t)\tilde{X}) = \sum_{i=1}^{M} m_i g p_3^{(i)} + \sum_{i=M+1}^{N} m_i g(t x_3^{(i)} + (1-t)\tilde{x}_3^{(i)}) =$$

$$= (1-t+t)\sum_{i=1}^{M} m_i g p_3^{(i)} + t\sum_{i=M+1}^{N} m_i g x_3^{(i)} + (1-t)\sum_{i=M+1}^{N} m_i g \tilde{x}_3^{(i)} = tE_{\text{ext}}(X) + (1-t)E_{\text{ext}}(\tilde{X}).$$

We see, from the calculations above, that $E_{\text{ext}}(X)$ is convex. Now let's explore if

$$\sum_{e_{ij}\in\xi} E_{\text{elast}}^{\text{cable}}(e_{ij}) = \sum_{e_{ij}\in\xi} \begin{cases} \frac{k}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2, & \text{for } \|x^{(i)} - x^{(j)}\| > l_{ij}, \\ 0, & \text{else}, \end{cases}$$

is convex.

The sum of convex functions multiplied by a positive constant is convex, so we need only to show that each term (without the constant) is convex. The squared of a non-negative convex function is also convex, and

$$g(X) := \begin{cases} \|x^{(i)} - x^{(j)}\| - l_{ij} & \text{for } \|x^{(i)} - x^{(j)}\| > l_{ij}, \\ 0 & \text{else} \end{cases} \tag{2.2}$$

is non-negative, so we only need to show that $g(X)$ is convex. We have

$$g(tX + (1-t)\tilde{X}) = \|t(x^{(i)} - x^{(j)}) + (1-t)(y^{(i)} - y^{(j)})\| - l_{ij}$$

$$\overset{\text{Triang. ineq.}}{\leq} t\|x^{(i)} - x^{(j)}\| + (1-t)\|y^{(i)} - y^{(j)}\| - (1-t+t)l_{ij} = tg(X) + (1-t)g(\tilde{X}).$$

So g is convex, and by extension $\sum_{e_{ij}\in\xi} E_{\text{elast}}^{\text{cable}}(e_{ij})$ is convex. Again, the sum of convex functions is also convex, so $\min_X E(X) = \sum_{e_{ij}\in\xi} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X)$ is convex.

□

Note that since $E_{\text{ext}}(tX + (1-t)\tilde{X}) = tE_{\text{ext}}(X) + (1-t)E_{\text{ext}}(\tilde{X})$, then $E_{\text{ext}}(X)$ is not strictly convex. If we imagine a case where X and $\tilde{X}$ are such that all cables are slack, i.e., $\|x^{(i)} - x^{(j)}\| < l_{ij}$

for all e_{ij}, then the elastic energy is zero. It follows that E is not strictly convex. Therefore, we cannot confidently say that (2.1) admits a unique solution.

Now that we have seen the main properties of Problem (2.1), we can now state the conditions needed to find a solution.

Theorem 2.3. *The necessary and sufficient optimality conditions for (2.1) are*

$$\nabla E(Y) = (\partial_{x^{(M+1)}}E,\ \partial_{x^{(M+2)}}E, \dots,\ \partial_{x^{(N)}}E)^T = O_{N-M,3},$$

where O is the zero matrix and $Y = \{x^{(i)}\}_{i=M+1}^{N}$ is the collection of variable nodes.

Proof. The theorem holds in our case since the problem we are looking at is a free optimization problem, and the function E is convex and C^1. □

When $\|x^{(i)} - x^{(j)}\| > l_{ij}$, the optimality conditions reads

$$\partial_{x^{(i)}}E = \begin{bmatrix} \frac{k\left(x_1^{(i)}-x_1^{(j)}\right)}{l_{ij}^2\|x^{(i)}-x^{(j)}\|}\left(\|x^{(i)}-x^{(j)}\|-l_{ij}\right), \\ \frac{k\left(x_2^{(i)}-x_2^{(j)}\right)}{l_{ij}^2\|x^{(i)}-x^{(j)}\|}\left(\|x^{(i)}-x^{(j)}\|-l_{ij}\right), \\ \frac{k\left(x_3^{(i)}-x_3^{(j)}\right)}{l_{ij}^2\|x^{(i)}-x^{(j)}\|}\left(\|x^{(i)}-x^{(j)}\|-l_{ij}\right)+m_ig \end{bmatrix}^T = [\ 0,\ \ 0,\ \ 0\ ],$$

where $i, j = M+1, ..., N$ for $i < j$.

When $\|x^{(i)} - x^{(j)}\| \leq l_{ij}$, the optimality conditions reads

$$\partial_{x^{(i)}}E = [\ 0,\ 0,\ m_ig\ ] = [\ 0,\ 0,\ 0\ ] \quad \Longrightarrow \quad m_ig = 0,$$

and so $m_i = 0$ or $g = 0$, $\forall i = M+1, ..., N$, which is often not the case. The interpretation of this condition (the difference between the norm being greater or smaller than l_{ij}) is that when gravity is present, the cables must be in tension for a point to be a global minimum.

2.2 Numerical implementation

We are now ready to implement this system numerically, but we must first choose which algorithm to implement. As shown in Theorem (2.1), E is typically not in C^2, so Newton's method is out of the question. Therefore, we will choose the Quasi-Newton method BFGS with line search using strong Wolfe conditions. Under certain conditions, BFGS convergences super-linearly, while other methods, such as Gradient Descent, only converge linearly. Further, we are guaranteed convergence to a global solution for convex problems. For the line search, we will use the backtracking parameters $c_1 = 0.02$ and $c_2 = 0.2$.

Our implementation uses two neighbor matrices to denote connections between nodes: one for the cables and one for the bars. If the index $[i, j]$ is non-zero, there is a connection between node i and j, and l_{ij} is given by the value of index $[i, j]$ in the matrix.

In the code, we will be splitting the node matrix, $X \in \mathbb{R}^{3N}$, into the fixed nodes, $P \in \mathbb{R}^{3M}$, and the variable nodes, $Y \in \mathbb{R}^{3(N-M)}$. We then define the functions $f : \mathbb{R}^{3(N-M)} \to \mathbb{R}$ and $\nabla f : \mathbb{R}^{3(N-M)} \to \mathbb{R}^{3(N-M)}$ such that $f(Y) = E([P, Y])$ and $\nabla f(Y) = \{\partial_{x_k}E([P, Y])\}_{k=3M+1}^{3N}$, meaning that we only return the last $3(N - M)$ indices of the gradient. We then use BFGS to minimize $f(Y)$, using $\nabla f(Y)$ as the gradient. Our stopping criteria is when the 2-norm of the gradient is smaller than some tolerance (in most test cases, we have chosen tolerance 10^{-16}). In Figure 2.1 below, we plot a test case using the algorithm described here.

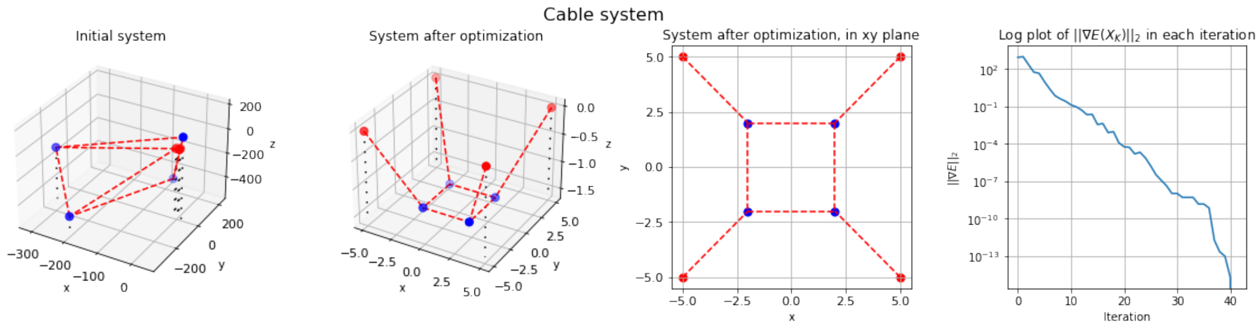

Figure 2.1: Plots of the system - before and after optimization, as well as a plot of the gradient norm in each iteration. The following parameter values were used: $g = 9.81$, $k = 3$, $N = 8$, $M = 4$, and $l_{ij} = 3$ for all cables. The mass of the nodes are $1/6g$. Red dots represent the fixed nodes, while blue dots represent the variable nodes. Dashed red lines represent the cables. The position of the fixed nodes and the difference between the exact and numerical solution can be found in the supplied code.

From Figure 2.1 we see that even when placing the free nodes at random, far away from the fixed nodes, the algorithm converges superlinearly to the global minimum, as expected. As mentioned, this is due to the convexity of E.

3 Tensegrity-domes

We have now optimized a structure only consisting of cables and nodes. Now we will add cables to the system as well. The resulting optimisation problem can be stated as

$$\min_X E(X) \quad \text{s.t.} \quad x^{(i)} = p^{(i)}, \text{ for } i = 1, \dots, M. \tag{3.1}$$

where $E(X)$ is the same function as (1.5). By adding the bars to the system, new terms are added to the energy function, and so we need to study again some important properties of this new system to make sure our code works as expected.

3.1 Mathematical framework

First, we need to know the smoothness of the new system in order to decide if a new algorithm is required to solve the problem.

Theorem 3.1. *The function E in (3.1) is typically not differentiable, but this poses no problem in practical situations.*

Proof. The main issue we face in (3.1) lies in having the bars. What happens is that by the definition of $E^{\text{bar}}_{\text{elast}}$, there is no restriction on the values $x^{(i)}$ and $x^{(j)}$ can take, so the case where $x^{(i)} = x^{(j)}$ is theoretically possible. For the cable system, we found that by the definition of their energy, there was no problem with having a hypothetical case where $x^{(i)} = x^{(j)}$, because their energy would be 0, and so would it's derivative. Taking the gradient for $E^{\text{bar}}_{\text{elast}}$ gives

$$\nabla E^{\text{bar}}_{\text{elast}}(Y) = \left[\ \partial_{x^{(M+1)}} E^{\text{bar}}_{\text{elast}},\ \ \partial_{x^{(M+2)}} E^{\text{bar}}_{\text{elast}},\ \ \dots,\ \ \partial_{x^{(N)}} E^{\text{bar}}_{\text{elast}}\ \right]^T,$$

where $Y = \left(x^{(i)}\right)_{i=M+1}^{N}$ and $x^{(i)} = [\ x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\ ]$, where for every i

$$\partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}} = \left[\partial_{x_1^{(i)}} E^{\text{bar}}_{\text{elast}},\ \partial_{x_2^{(i)}} E^{\text{bar}}_{\text{elast}},\ \partial_{x_3^{(i)}} E^{\text{bar}}_{\text{elast}}\right].$$

The partial derivative can be written as

$$\partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}} = \begin{bmatrix} \frac{c}{l_{ij}^2}\left[\left(x_1^{(i)} - x_1^{(j)}\right) - \frac{l_{ij}}{\|x^{(i)}-x^{(j)}\|}\right], \\ \frac{c\left(x_2^{(i)}-x_2^{(j)}\right)}{l_{ij}^2\|x^{(i)}-x^{(j)}\|}\left(\|x^{(i)} - x^{(j)}\| - l_{ij}\right), \\ \frac{c\left(x_3^{(i)}-x_3^{(j)}\right)}{l_{ij}^2\|x^{(i)}-x^{(j)}\|}\left(\|x^{(i)} - x^{(j)}\| - l_{ij}\right) \end{bmatrix}^T ,$$

where $i, j = M+1, ..., N$, and $i < j$. If $x^{(i)} = x^{(j)}$ for some j, then every term goes to minus infinity, and therefore our problem is not differentiable.

Using any reasonable algorithm and initialization, the nodes connected by bars will not coincide since the energy would grow fast as the points grow closer. □

The next property that we will explore is convexity. Again, a convex problem guarantees convergence to a global solution regardless of initialisation. If the problem is non-convex, the system may admit local (non-global) minimisers, and we must be careful with the initial conditions.

Recall our strategy to prove convexity for cable nets, specifically the argument that the squared of a non-negative convex function is also convex. Since (2.2) was non-negative and convex, then Problem (2.1) was convex. However, this same line of reasoning is not the case for (3.1) since the corresponding function for (3.1) is $\tilde{g}(X) = \|x^{(i)} - x^{(j)}\| - l_{ij}$, which is not non-negative. In fact, we propose the following theorem:

Theorem 3.2. *The problem* (3.1) *is not convex if* $\mathcal{B} \neq \emptyset$*.*

Proof. We will prove this non-convexity with a counter-example. Take the system $X \in \mathbb{R}^{15}$ such that $x^{(i)} = p^{(i)}$ for $i = 1, 2, 3, 4$, where $p^{(1)} = [1, 1, 0]^T$, $p^{(2)} = [-1, 1, 0]^T$, $p^{(3)} = [-1, -1, 0]^T$, $p^{(4)} = [1, -1, 0]^T$. We set the bar lengths as $l_{15} = l_{25} = l_{35} = l_{45} = \sqrt{3}$. Denote $b_{\text{elast}}(X) := \sum_{e_{ij}\in\mathcal{B}} E^{\text{bar}}_{\text{elast}}(e_{ij})$ and $b_{\text{grav}}(X) := \sum_{e_{ij}\in\mathcal{B}} E^{\text{bar}}_{\text{grav}}(e_{ij})$. This system admits two solutions where $b_{\text{elast}}(X) = 0$, namely $X_1 = [0, \ldots, 0, 1]$, and $X_2 = [0, \ldots, 0, -1]$.

$E(X)$ is convex iff $E(tX_1 + (1-t)X_2) \leq tE(X_1) + (1-t)E(X_2)$ for all $t \in [0, 1]$. Taking $t = \frac{1}{2}$, denoting $X_3 = [0, \ldots, 0]$, and using $b_{\text{elast}}(X_1) = b_{\text{elast}}(X_2) = 0$, we obtain

$$E(X_3) \leq \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2),$$

$$\sum_{i=1}^{4} m_i g p_3^{(i)} + b_{\text{elast}}(X_3) + b_{\text{grav}}(X_3) \leq \frac{2}{2}\sum_{i=1}^{4} m_i g p_3^{(i)} + 1\cdot m_5 g - 1\cdot m_5 g + \frac{1}{2}b_{\text{grav}}(X_1) + \frac{1}{2}b_{\text{grav}}(X_2).$$

Using $b_{\text{grav}}(X_1) = -b_{\text{grav}}(X_2)$, we get that

$$b_{\text{elast}}(X_3) \leq 0 \overset{(1.3)}{\Longrightarrow} \sum_{i=1}^{4} \frac{c}{2l_{ij}^2}(\|p^{(i)} - [0,\ \ 0,\ \ 0]\| - l_{ij})^2 \leq 0 \implies 4\cdot\frac{c}{2\sqrt{3}^2}(\sqrt{2}-\sqrt{3})^2 \leq 0,$$

which is a contradiction. (3.1) is not convex if $\mathcal{B} \neq \emptyset$. □

Moreover, selecting "small enough"[2] values for $g, \rho, m_i > 0$ for $i = 1, \ldots, N$, it becomes evident that our example admits a (non-global) local solution close to X_1 and a global solution close to X_2. We have now seen the main properties of Problem (3.1). We can now state the optimality conditions needed to find a solution.

Theorem 3.3. *The necessary optimality conditions for* (3.1) *are*

$$\nabla E(Y) = [\ \partial_{x^{(M+1)}}E, \ldots, \partial_{x^{(N)}}E\ ]^T = O_{N-M,3},$$

where O *is the zero matrix and* $Y = \{x^{(i)}\}_{i=M+1}^{N}$ *is the collection of variable nodes.*

Proof. The theorem holds since Problem (3.1) can be treated as differentiable because this will be the case in practical situations, and as the function is not convex (see Theorem 3.2), these conditions

[2]Small enough here means that $E_{\text{ext}} + b_{\text{grav}}$ do not overpower b_{elast}.

are only necessary and not sufficient. We can write

$$\nabla E(Y) = \nabla E_{\text{bar}}(Y) + \nabla E_{\text{cable}}(Y) + \nabla E_{\text{ext}}(Y) = [\ \partial_{x^{(M+1)}} E, \ldots, \partial_{x^{(N)}} E\ ]^T$$

where writing down the optimality conditions, we can express every partial derivative as

$$\partial_{x^{(i)}} E = \partial_{x^{(i)}} E_{\text{bar}} + \partial_{x^{(i)}} E_{\text{cable}} + \partial_{x^{(i)}} E_{\text{ext}} = [\ 0,\ \ 0,\ \ 0\ ]$$

Now, each of these terms can be written as follows:

$$\partial_{x^{(i)}} E_{\text{ext}} = [0, 0, m_i g] \quad \text{where} \quad i = M+1, ..., N, \text{and}$$

$$\partial_{x^{(i)}} E_{\text{bar}} = \frac{c(\|x^{(i)} - x^{(j)}\| - l_{ij})}{l_{ij}^2 \|x^{(i)} - x^{(j)}\|} \left[x_1^{(i)} - x_1^{(j)},\quad x_2^{(i)} - x_2^{(j)},\quad x_3^{(i)} - x_3^{(j)}\right] + \left[0,\quad 0,\quad \frac{\rho g l_{ij}}{2}\right]$$

where $e_{ij} \in \mathcal{B}$, $i, j = M+1, ..., N$ for $i < j$.
When $\|x^{(i)} - x^{(j)}\| > l_{ij}$:

$$\partial_{x^{(i)}} E_{\text{cable}} = \frac{k(\|x^{(i)} - x^{(j)}\| - l_{ij})}{l_{ij}^2 \|x^{(i)} - x^{(j)}\|} \left[x_1^{(i)} - x_1^{(j)},\quad x_2^{(i)} - x_2^{(j)},\quad x_3^{(i)} - x_3^{(j)}\right]$$

where $e_{ij} \in C$, $i, j = M+1, ..., N$ for $i < j$.
When $\|x^{(i)} - x^{(j)}\| \leq l_{ij}$:

$$\partial_{x^{(i)}} E_{\text{cable}} = [0, 0, 0]$$

□

3.2 Numerical implementation

One drawback of BFGS is that for non-convex functions, it may converge to a non-global minima. However, in our case, we *want* to converge to non-global minimas since the configurations with variable nodes above the fixed ones are the most interesting. Since the problem is not convex, any solution we find depends on the initial conditions, so we must choose these carefully. Figure 3.1 shows a tensegrity dome. The fixed nodes are set at $z = 0$, and the four variable nodes are initialized randomly with $x, y \in [-1, 1]$ and their height $z = 4$.

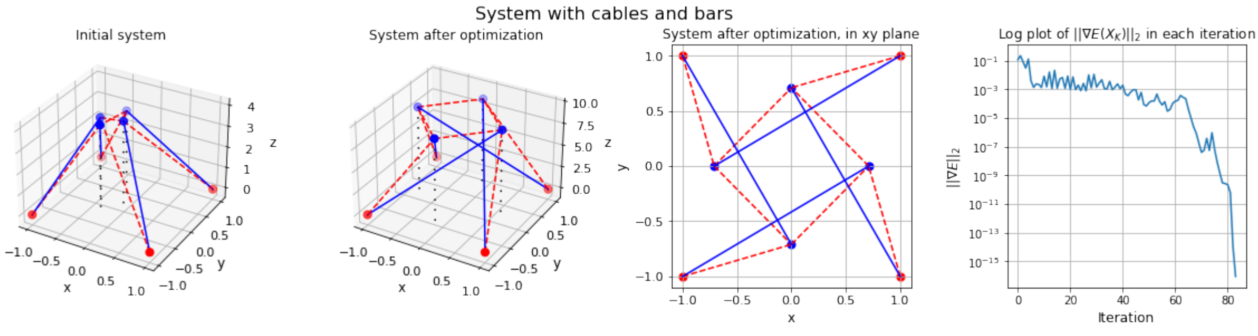

Figure 3.1: Plots of the system - before and after optimization, as well as a plot of the norm of the gradient of the energy function for each iteration. The following parameter values were used: $g = 0$, $c = 1$, $k = 0.1$, $N = 8$, $M = 4$. All the bars have natural length 10, the cables connecting fixed- to variable nodes are 8, and the cables connecting the variable nodes together have length 1.

The solution the method converges to is a local minimum, but the global minimum would be the same system with the variable nodes below the fixed ones. We double-check that the structure has found a local solution[3] by seeing that its $\|\nabla E\|_2$ approach zero in the log plot.

[3] We do this to ensure that we have not stopped the iterations too early so that we can ensure that we, in fact, have actually arrived at a minimum.

4 Free-standing structures

Now we consider the case of a free-standing structure, where none of the nodes are fixed and the only constraint we have is that the whole structure is above the ground. The resulting optimisation problem can be stated as

$$\min_X E(X), \quad \text{s.t.} \quad x_3^{(i)} \geq 0 \ \text{ for } \ i = 1, ..., M, \tag{4.1}$$

where $E(X)$ is the same function as (1.5). The energy function is the same as in Problem (3.1), as we still use cables and bars. We, therefore, see that the same important properties are found, and Theorem (3.1) and Theorem (3.2) apply here. For the problem to admit a global minimum, we will also use the discussion about type (1.7) constraints in Theorem 1.1. This means that we apply the equality constraints $x_1^{(1)} = x_2^{(1)} = 0$. Now that there are no fixed nodes, the problem grows in dimension, and having a ground changes the optimality conditions needed to find a solution.

4.1 Mathematical framework

To formulate the first order optimality conditions (KKT-conditions) for (4.1), first we have that $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ and the constraints are $C_i(X) = x_3^{(i)} \geq 0$, $i = 1, 2, .., N$. Now, using that our gradient for the i-th row is

$$\nabla C_i(X) = \left[\partial_{x^{(1)}} C_i, \partial_{x^{(2)}} C_i, \ldots, \partial_{x^{(N)}} C_i\right]^T,$$

we end up with the gradients

$$\begin{aligned}
\nabla C_1(X) &= [(0,0,1), (0,0,0), \ldots, (0,0,0)]^T, \\
\nabla C_2(X) &= [(0,0,0), (0,0,1), \ldots, (0,0,0)]^T, \\
&\vdots \\
\nabla C_N(X) &= [(0,0,0), (0,0,0), \ldots, (0,0,1)]^T.
\end{aligned}$$

In addition, the constraints to secure the first node's x and y value to the origin gives

$$\nabla C_{N+1}(X) = [(1,1,0), (0,0,0), \ldots, (0,0,0)]^T.$$

We also have that $\{\nabla C_i(X)\}_{i=1,2,\ldots,N}$ are linearly independent because

$$\lambda_1 \nabla C_1(X) + \lambda_2 \nabla C_2(X) + \cdots + \lambda_{N+1} \nabla C_{N+1}(X) = 0 \quad \Leftrightarrow \quad \lambda_i = 0, \quad i = 1, \ldots, N.$$

Then LICQ holds at every point, and we can formulate the KKT conditions as

- $\nabla_X \mathcal{L}(X, \lambda) = 0,$ where $\mathcal{L}(X, \lambda)$ is the Lagrangian.
- $C_i(X) \geq 0$ for $i = 1, \ldots, N$ and $C_i(X) = 0$ for $i = N+1$.
- $\lambda_i \geq 0$ for $i = 1, \ldots, N$.
- $C_i(X)\lambda_i = 0$ for $i = 1, \ldots, N$.

We can write $\nabla_X \mathcal{L}(X, \lambda) = \nabla E(X) - \lambda \nabla C(X) = 0$. Or in index form, $\partial_{x^{(i)}} \mathcal{L}_i(X, \lambda) = \partial_{x^{(i)}} E(X) - \lambda_i \partial_{x^{(i)}} C(X) = 0$, where $i = 1, \ldots, N$. The gradient of the energy function then takes on the form

$$\nabla E(X) = \left[\partial_{x^{(1)}} E, \ldots, \partial_{x^{(N)}} E\right]^T = [(0,0,0), \ldots, (0,0,0)]^T.$$

We can express every partial derivative as $\partial_{x^{(i)}} E = \partial_{x^{(i)}} E_{\text{bar}} + \partial_{x^{(i)}} E_{\text{cable}} + \partial_{x^{(i)}} E_{\text{ext}}$.
Now, each of these terms can be written as follows:

$$\partial_{x^{(i)}} E_{\text{ext}} = [0, 0, m_i g] \ \text{ where } \ i = 1, \ldots, N, \ \text{and}$$

$$\partial_x^{(i)} E_{\text{bar}} = \frac{c(\|x^{(i)} - x^{(j)}\| - l_{ij})}{l_{ij}^2 \|x^{(i)} - x^{(j)}\|} \left[x_1^{(i)} - x_1^{(j)}, \quad x_2^{(i)} - x_2^{(j)}, \quad x_3^{(i)} - x_3^{(j)}\right] + \left[0, \quad 0, \quad \tfrac{\rho g l_{ij}}{2}\right],$$

where $e_{ij} \in \mathcal{B}$, $i, j = 1, \ldots, N$, and $i < j$.
When $\|x^{(i)} - x^{(j)}\| > l_{ij}$:

$$\partial_{x^{(i)}} E_{\text{cable}}^{\text{elast}} = \frac{k(\|x^{(i)} - x^{(j)}\| - l_{ij})}{l_{ij}^2 \|x^{(i)} - x^{(j)}\|} \left[x_1^{(i)} - x_1^{(j)}, \quad x_2^{(i)} - x_2^{(j)}, \quad x_3^{(i)} - x_3^{(j)}\right],$$

where $e_{ij} \in C$, $i, j = 1, \dots, N$. $i < j$.
When $\|x^{(i)} - x^{(j)}\| \leq l_{ij}$:

$$\partial_{x^{(i)}} E_{\text{cable}}^{\text{elast}} = [0, 0, 0].$$

As shown in 3.2, the energy function is not convex if there are bars ($\mathcal{B} \neq \emptyset$). Therefore, we cannot say anything about the KKT points being a local solution to our problem. Hence KKT conditions are not sufficient, only necessary. If there are no bars ($\mathcal{B} = \emptyset$), then as shown in Theorem 3.2, the function is convex and so KKT conditions are sufficient optimality conditions, and every KKT point is a global solution of our problem. However, a system without bars is severely uninteresting, as all nodes would lie on the ground, at $z = 0$.

4.2 Numerical implementation

Our choice to handle these constraints is to convert all inequality constraints to equality constraints by defining $C_i^-(X) = \min\{C_i(X), 0\}$. Then, we apply the quadratic penalty method by defining the penalty function

$$Q(X, \mu_1, \mu_2) = E(X) + \frac{\mu_1}{2} \sum_{i=1}^{N} (C_i^-(X))^2 + \frac{\mu_2}{2} (C_{N+1}(X))^2,$$

where $\mu_1, \mu_2 > 0$. We can then minimise Q instead of E. The first sum represents the inequality constraints relating to the ground, and the second term represents the equality constraint for keeping the first node at the origin. We have different values for μ_1, μ_2 because we only need a small μ_2 to keep the problem invariant compared to μ_1, which prevents the structure from sinking below $z = 0$.

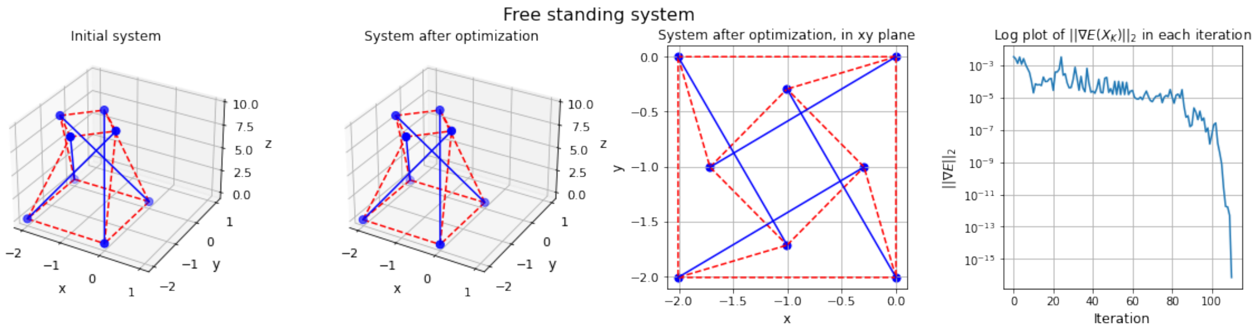

Figure 4.1: Plots of the system - before and after optimization, as well as a plot of the norm of the gradient of the energy function for each iteration. The following parameter values were used: $g = 9.81$, $\rho = 10^{-7}$, $c = 1$, $k = 0.1$, $\mu_1 = 10$, $\mu_2 = 0.001$, $N = 8$, and node masses are 0. Apart from the cable lengths at the bottom, with length 2, all of the other lengths are the same as in Fig 3.1

Here, in Figure 4.1, the resulting structure looks very similar to the one in Figure 3.1. Also, we can see (from the logplot of $\|\nabla E\|_2$) that our algorithm converges to a local minimum.

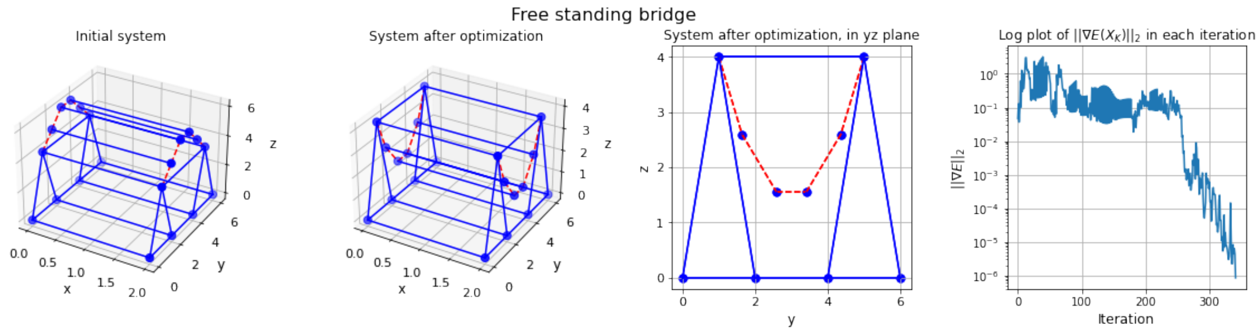

Figure 4.2: Plots of the system - before and after optimization, as well as a plot of the norm of the gradient of the energy function for each iteration. The following parameter values were used: $g = 0.1$, $\rho = 0$, $c = 200$, $k = 0.1$, $\mu_1 = 1000$, $\mu_2 = 0.1$, $N = 20$. Node masses are 0.1.

Figure 4.2 shows a model of a free-standing tower bridge. Due to the complexity and number of nodes, the code ran for about 60 seconds with the tolerance set to 10^{-6}. This result seems physically correct, as we expected the nodes connected by cables to fall and curve downwards.

5 Conclusion

In this report, we have presented a comprehensive approach to modeling tensegrity structures, by optimizing the potential energy of the system. We incorporated various constraints, to ensure that a solution existed to our posed problems. We applied mathematical theorems to understand the energy behavior in each of the three parts. Through our numerical implementations, using BFGS with Wolfe conditions, we obtained graphical results that demonstrate the viability of the algorithm, as well as the effect of increasing complexity on the behavior of the structure. Our study provides insights into the design and optimization of tensegrity structures and lays the foundation for further exploration.