Overlapping generations models with Money

Lecture 2

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Pure–exchange OLG model with money (Samuelson 1958)

- In the pure-exchange OLG model w/out outside assets, the only competitive equilibrium is autarky.
  - Inter-generational lending is not possible (due to the 2-periods demographic structure).
- In order to make inter-generational trade possible, let us introduce *fiat* money into the model.
- Inside money vs outside money:
  - *Inside money* (such as bank deposits, private loans, etc.) is both an asset as well as a liability of the private sector.
  - Money that is, on net, an asset of the private economy, is *outside money*. This includes fiat currency issued by the government.

Markets

- There is a stock $M > 0$ of an infinitely–lived zero–dividend asset, owned by generation 0.
- Interpretation: fiat money, zero–interest government bonds, land without intrinsic value, chocolate paper,...
- Households trade the consumption good against the asset.
- Normalize the price of the asset to unity in all periods.
- Is this a restriction?
- Write p_t, $t \geq 1$, for the price of the consumption good in units of the asset. That is, we take money to be the numeraire (i.e. money is the unit of account).
- Define inflation rate $\pi_t \equiv \frac{p_{t+1}}{p_t}$. Note that $\pi_t = 1/R_t$.

Equilibrium

A competitive equilibrium is $(p_t)_{t\geq 1}$, $(M_t, c_t^t, c_{t+1}^t)_{t\geq 1}$, c_1^0 such that

(i) For all $t \geq 1$, (c_t^t, c_{t+1}^t, M_t) maximizes $u(c_t) + \beta u(c_{t+1})$ s.t.

$$p_t c_t + M_t \leq p_t y_t^t$$
$$p_{t+1} c_{t+1} \leq M_t + p_{t+1} y_{t+1}^t.$$

(ii) For $i = 0$, c_1^0 maximizes $u(c_1)$ subject to $p_1 c_1 \leq p_1 y_1^0 + M$.

(iii) For all $t \geq 1$, (goods market clearing or resource balance)

$$c_t^{t-1} + c_t^t = y_t^{t-1} + y_t^t \ ,$$

and (money market clearing)

$$M_t = M.$$

Exercise: Show that one of the two market clearing conditions is redundant (Walras' law).

Demand functions

- The budget constraints of generation $t \geq 1$ can be consolidated by substituting out M_t to yield the "lifetime" constraint:

$$c_t^t + \pi_t c_{t+1}^t \leq y^Y + \pi_t y^O. \tag{1}$$

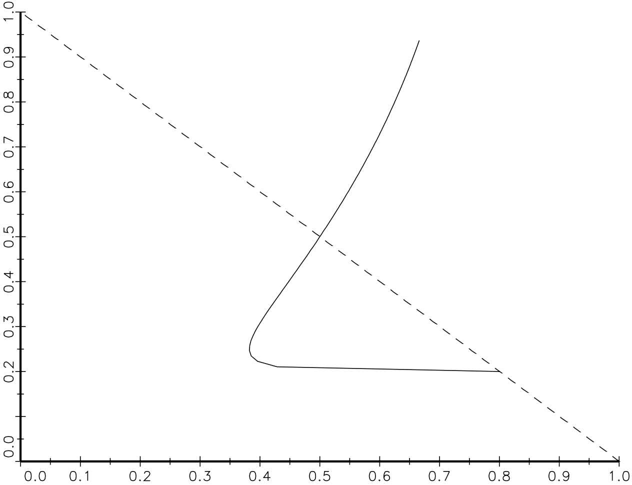
- It is important to note that if there was a further constraint that $M_t \geq 0$, then it would not be possible to characterize the constraint set by (1) only without reference to M_t.
- The decision of generation $t \geq 1$ depends only on $\pi_t \equiv p_{t+1}/p_t$, and not on p_t, p_{t+1} separately (homogeneity degree zero).
- Write $c^Y(\pi_t)$ and $c^O(\pi_t)$ for consumption demand of "young" and "old" household t.
  - First-order condition: $u'(c_t^t)\pi_t = \beta u'(c_{t+1}^t)$.
  - Budget restriction: $c_t^t + \pi_t c_{t+1}^t = y^Y + \pi_t y^O$.

Demand functions, cont'd

- Alternatively, write $s(R_t) = 1 - \varepsilon - c^Y(1/R_t)$ for the *savings function*, with $R_t = 1/\pi_t$.
- Write $\bar{\pi} = \frac{\beta u'(\varepsilon)}{u'(1-\varepsilon)}$ and $\bar{R} = 1/\bar{\pi}$.
- If $\pi < \bar{\pi}$ ($R > \bar{R}$), the household saves: $c^Y(\pi) < 1 - \varepsilon$ and $c^O(\pi) > \varepsilon$.
- For varying π, $(c^Y(\pi), c^O(\pi))$ describes the household's "offer curve".
- $c^Y(.)$ (and savings $s(.)$) may not be monotonic. Why?
- If "gross substitutability" condition holds, then $c^Y(.)$ (and savings $s(.)$) are monotonic.

Offer curve with CRRA utility

$\varepsilon = 0.2$, $\gamma = 5$, $\beta = 1$.



Demand functions: an example

- $u(c) = \ln(c)$ and $\beta = 1$.
- Demand functions are:

$$c^Y(\pi) = \frac{1}{2}\Big(1 - \varepsilon + \varepsilon\pi\Big)\ ,\ c^O(\pi) = \frac{1}{2}\Big(\frac{1-\varepsilon}{\pi} + \varepsilon\Big)\ .$$

- Solving for c^O as a function of c^Y by eliminating π_t, yields

$$c^O = \frac{\varepsilon(1-\varepsilon)}{2[2c^Y - (1-\varepsilon)]} + \frac{\varepsilon}{2},$$

for $c^Y \in \left(\frac{1-\varepsilon}{2}, \infty\right)$. This is the offer curve in the (c^Y, c^O) plane.

Equilibrium

- The initial old generation consumes $c_1^0 = y^O + M/p_1$.
- An equilibrium is a sequence π_t, satisfying

$$c^O(\pi_t) + c^Y(\pi_{t+1}) = 1 \quad \text{for all } t \geq 2 \ ,$$
$$c^O(p_1, M) + c^Y(\pi_1) = 1.$$

  Note: $c^Y(\pi_1) \leq 1 - \varepsilon$ (i.e. initial young must be savers).
- An alternative characterization of equilibrium are the money market equilibrium conditions:

$$s(R_t) = \frac{M}{p_t} \ , \ t \geq 1 \ ,$$

- i.e. the demand for assets (saving) equals the outside supply of assets, M. Note that the demanders of the assets are the currently young whereas the suppliers are the currently old people.
- What are stationary and non-stationary equilibria?

Stationary equilibrium

- Only candidate solutions are given by the intersections of the offer curve and the feasibility line:
  - Autarkic equilibrium, $\pi = \bar{\pi}$
  - "Monetary" equilibrium $\pi^* = 1$.
- $\pi = \bar{\pi}$ is a "limiting equilibrium" at which the price of the asset is zero (or $p_t = \infty$ in all periods). There is no intergenerational trade $\Longrightarrow$ Autarky
- $\pi^* = 1$ is an equilibrium iff $c^Y(1) < y^Y = 1 - \varepsilon$ which is the case iff $\bar{\pi} > 1$ ($\bar{R} < 1$). Recall: this is Samuelson case.
- The stationary asset price p^* is positive and finite and determined from $1 - \varepsilon - c^Y(1) = M/p^*$. The asset is traded between generations in all periods ("monetary equilibrium").
- If $\bar{\pi} \leq 1$, the unique stationary equilibrium is autarky.

Non–stationary equilibria in the stationary economy

- Graphical characterization: $(c^Y(\pi_t), c^O(\pi_t))$ must be on the offer curve of generation t, and the market-clearing condition holds in all periods $t \geq 2$:

$$c^O(\pi_t) + c^Y(\pi_{t+1}) = 1 \ .$$

Further, in initial period

$$c^O(p_1, M) + c^Y(\pi_1) = 1 \ .$$

- If $\bar{\pi} > 1$ ($\bar{R} < 1$), there are many non–stationary equilibria converging to the autarkic equilibrium.
- (Local) indeterminacy.
- Otherwise, autarky is the unique equilibrium.

Finding equilibria using offer curve

- In equilibrium, allocations and prices must satisfy:

$$OC = \Big(c^Y(\pi_t), c^O(\pi_t)\Big) \tag{2}$$

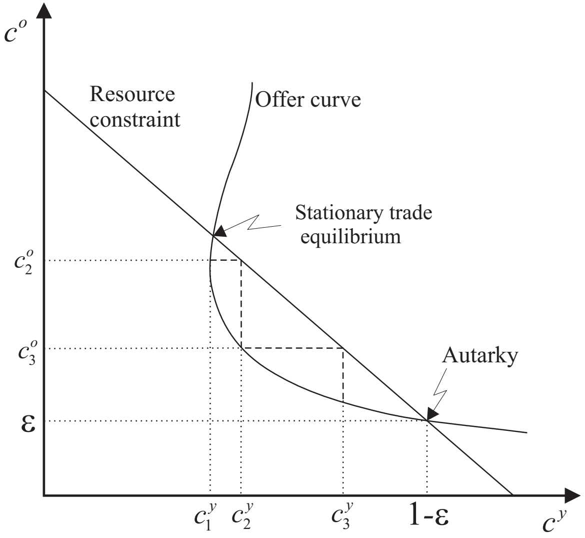
$$c_1^0 = y_1^0 + \frac{M}{p_1} \tag{3}$$

$$c_1^0(p_1, M) + c^Y(\pi_1) = 1 \tag{4}$$

$$c^O(\pi_t) + c^Y(\pi_{t+1}) = 1 \tag{5}$$

  for all $t \geq 1$.
- Choose p_1. This will determine the value of money held by the initial old and hence their consumption c_1^0 by (3).
- Using feasibility condition for $t = 1$, eq. (4), find $c^Y(\pi_1)$.
- Read $c^O(\pi_1)$ on the offer curve (2).
- Use feasibility condition (5) to determine $c^Y(\pi_2)$, and so forth.
- In this way we determine entire equilibrium allocation.
- Any initial p_1 that induces sequences c_1^0, $c^Y(\pi_t), c^O(\pi_t)$ such that the consumption values are positive is an equilibrium.

Equilibria in the OLG model



Example, cont'd

- $u(c) = \ln(c)$ and $\beta = 1$.
- Then
$$c^Y(\pi) = \frac{1}{2}\left(1 - \varepsilon + \varepsilon\pi\right)\,,\ c^0(\pi) = \frac{1}{2}\left(\frac{1-\varepsilon}{\pi} + \varepsilon\right)\,.$$
- From market clearing conditions $c^O(\pi_t) + c^Y(\pi_{t+1}) = 1$,
$$\frac{1}{2}\left(\frac{1-\varepsilon}{\pi_t} + \varepsilon\right) + \frac{1}{2}\left(1 - \varepsilon + \varepsilon\pi_{t+1}\right) = 1.$$
- All equilibria satisfy
$$\pi_{t+1} = \frac{1}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon\pi_t}\,.$$
- First-order nonlinear difference equation. Two steady-states, solutions of
$$\pi^2 - \frac{1}{\varepsilon}\pi + \frac{1-\varepsilon}{\varepsilon} = 0.$$

Example, cont'd

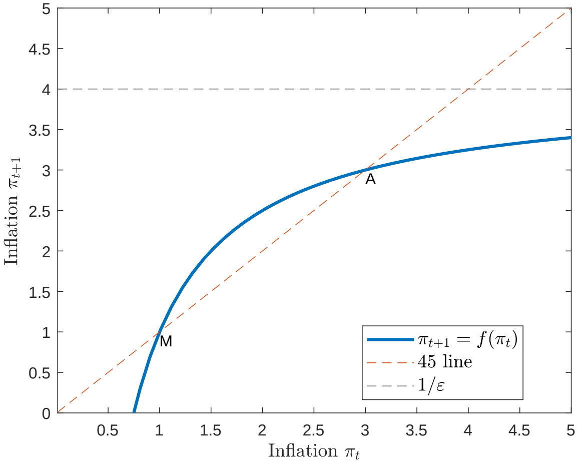


Figure: Equilibrium difference equation for π_t, given $\varepsilon = 0.25$. M: monetary steady-state, A: autarky steady-state.

Example, cont'd

- Steady states are:
  - $\pi^* = 1$, monetary equilibrium. $c^Y(1) = c^O(1) = 1/2$. Here $p_t = p_1$ for all $t \geq 1$. Q: What is p_1?
  - $\bar{\pi} = (1-\varepsilon)/\varepsilon$, autarkic equilibrium. $c^Y(\bar{\pi}) = 1-\varepsilon$, $c^O(\bar{\pi}) = \varepsilon$. Q: What is p_1?
- Autarky with $\pi_t = \bar{\pi} = (1-\varepsilon)/\varepsilon$ is the unique equilibrium if $\bar{\pi} \leq 1$, i.e. Classical case, with $\varepsilon > 0.5$.
- Otherwise there is a continuum of equilibria (case depicted in figure).
  - (A) The non–monetary autarky equilibrium $\pi_t = \bar{\pi} > 1$.
  - (M) The monetary equilibrium at $\pi_t = \pi^* = 1$.
  - Inflationary equilibria $\pi_t \to \bar{\pi}$.

Discussion

- Are all equilibria Pareto optimal (first welfare theorem)?
- What about example on previous slide?
- Another example: $u(c) = c$, $\beta = 1$, $\varepsilon = 1/2$.
- What would happen if the economy ended in finite time?

Overlapping generations and economic growth

OLG model with production (Diamond 1965)

- The same generational pattern as before, with labor endowment profiles $(L_t^t, L_{t+1}^t) = (1, 0)$. (Samuelson case).
- The population grows at rate n: there are $N_t = N_0(1+n)^t$ members of generation t.
- The old generation in period $t = 1$ is endowed with K_1 units of capital. Physical capital is the only asset.
- Output is produced from capital and labor inputs with aggregate production function $F(K_t, L_t)$.
- F satisfies the standard assumptions of neoclassical growth theory.
- In each period, perfectly competitive firms rent capital at rate r_t. Capital depreciates at rate δ. Hence,
  $r_t = F_1(K_t, L_t) - \delta = f'(k_t) - \delta$ with $k_t = K_t/L_t$ and
  $f(k) = F(k, 1)$. Also, $R_t \equiv 1 + r_t$.

Competitive equilibrium in the Diamond model

Given K_1, a competitive equilibrium is an allocation $(c_t^t, c_{t+1}^t, s_t^t)_{t\geq 1}$, c_1^0, $(K_t, L_t)_{t\geq 1}$, and factor prices $(r_t, w_t)_{t\geq 1}$ such that

1. For all $t \geq 1$, $(c_t^t, c_{t+1}^t, s_t^t)$ maximizes $u(c_t) + \beta u(c_{t+1})$ s.t.
$$c_t + s_t \leq w_t,$$
$$c_{t+1} \leq (1 + r_{t+1}) s_t.$$
2. c_1^0 maximizes $u(c_1^0)$ s.t. $c_1^0 \leq (1 + r_1) K_1 / N_0$.
3. For all $t \geq 1$, (K_t, L_t) maximizes $F(K_t, L_t) - (r_t + \delta) K_t - w_t L_t$.
4. Factor markets are in equilibrium, i.e. for all $t \geq 1$
$$L_t = N_t \text{ and } N_t s_t^t = K_{t+1} \ .$$

Note: Goods market equilibrium follows:
$$N_t c_t^t + N_{t-1} c_t^{t-1} + K_{t+1} - (1 - \delta) K_t = F(K_t, L_t) \ .$$

Saving Function

- Again write $1 + r_t = R(k_t) = 1 + f'(k_t) - \delta$ and $w_t = w(k_t) = f(k_t) - f'(k_t)k_t$.
- Optimal savings of generation t, $s(w_t, R_{t+1})$, satisfy:

$$u'(w_t - s(w_t, R_{t+1})) = \beta R_{t+1} u'(R_{t+1} s(w_t, R_{t+1})).$$

- Saving is an increasing function of the wage by concavity of u.
    - Using the Implicit Function Theorem, one can show that $s_w \in (0, 1)$.
- The effect of an increase in the interest rate is ambiguous, depending on income and substitution effects.
    - If the coefficient of relative risk aversion is less than 1, then SE dominates and saving function is monotonically increasing in R_{t+1}.

Characterization of competitive equilibrium

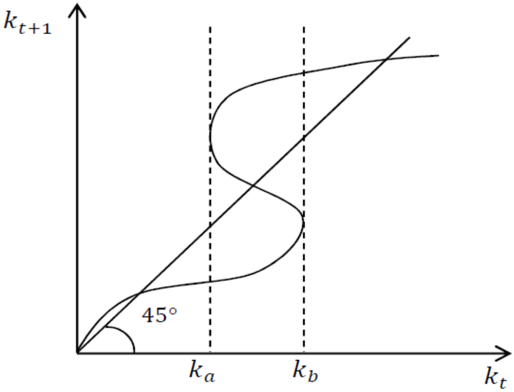
- Optimal savings of generation t depends on w_t and $1 + r_{t+1}$. In equilibrium, wage and interest rate depend on capital. Hence write $s_t = s(w(k_t), R(k_{t+1}))$.
- From capital market equilibrium:

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{K_{t+1}}{N_t(1+n)} = \frac{s(w(k_t), R(k_{t+1}))}{1+n} \ . \qquad (6)$$

- This equation defines $(k_t)_{t \geq 1}$ in any competitive equilibrium starting from a given k_1.
- Does a competitive equilibrium exist for any k_1? Is it unique? Does it converge?

Equilibrium dynamics I

- If $s_R < 0$, then eq (6) may not give a unique solution for k_{t+1}, given k_t.
- Figure: k_{t+1} is not uniquely determined if $k_t \in (k_a, k_b)$.

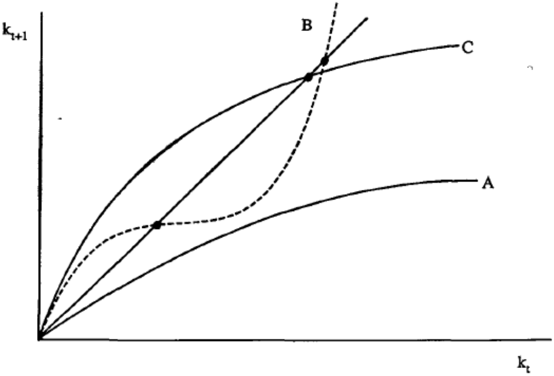


Equilibrium dynamics II

- If $s_R \geq 0$, then we can solve eq (6) for k_{t+1} as a function of k_t, $k_{t+1} = G(k_t)$.
- Even if $s_R \geq 0$, existence and uniqueness of a positive steady state are not ensured.

Equilibrium dynamics II

- If $G'(0) > 1$, there exists at least one steady state. But uniqueness cannot be guaranteed. (Case B in diagram has two positive steady-states)
- And if $G'(0) \leq 1$, no steady state may exist (case A in diagram).

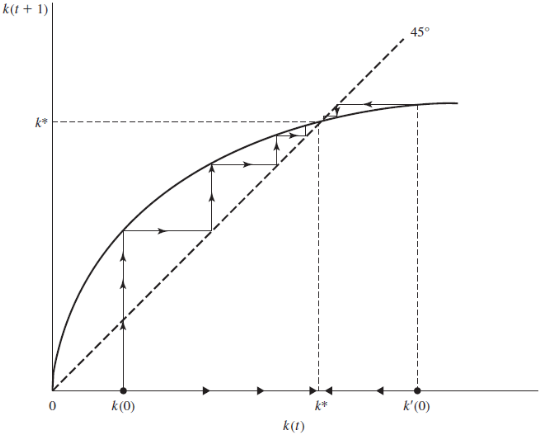


Canonical example

- $u(c) = \ln(c)$, $f(k) = k^{\alpha}$, with $\alpha \in (0,1)$.
- Saving function
$$s(w_t, R_{t+1}) = \frac{\beta}{1+\beta} w_t.$$
- Note: $s_R(w_t, R_{t+1}) = 0$. Why?
- Since $w_t = (1-\alpha)k_t^{\alpha}$, $s(w_t, R_{t+1}) = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$. The difference equation characterizing the equilibrium is
$$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha} \equiv G(k_t). \tag{7}$$
- G is increasing (indeed $s_R(w_t, R_{t+1}) \geq 0$), there is a unique positive steady state k^* given by
$$k^* = \left(\frac{\beta(1-\alpha)}{(1+\beta)(1+n)} \right)^{\frac{1}{1-\alpha}}.$$
- k^* is globally stable on $(0, \infty)$.

Equilibrium dynamics in canonical example

- Equilibrium dynamics for $(k_t)_{t \geq 1}$ in the log utility-Cobb Douglas case (eq. 7). If $k_1 = k(0)$, $k_t \uparrow k^*$; if $k_1 = k'(0)$, $k_t \downarrow k^*$.



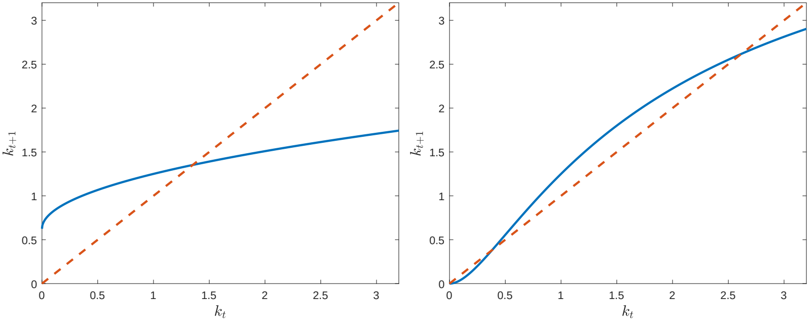
Another example

$u(c) = \ln(c)$, $n = 0$ and $\beta = 1$ so that $s(w, R) = 0.5w$, and

$$f(k) = A\left[ak^{-(1-\sigma)/\sigma} + (1-a)\right]^{-\sigma/(1-\sigma)},$$

with constant elasticity of substitution $\sigma > 0$. Set $a = 0.5$, $A = 5$. Details

$\sigma = 2$ $\sigma = 0.5$



If $\sigma = 0.5$, there are two steady states, $0 < k^* < k^{**} < \infty$. Stability?

Efficiency

- Is the competitive equilibrium Pareto optimal?
- Let (k^*, c_1^*, c_2^*) be a steady–state allocation satisfying

$$(1+n)k^* = s(w(k^*), R(k^*)) \ ,$$

$$c_1^* + \frac{c_2^*}{1+n} + (1+n)k^* - (1-\delta)k^* = f(k^*) \ .$$

- If $c^* = c_1^* + c_2^*/(1+n)$ is consumption per worker in the steady state:

$$c^* = f(k^*) - (n+\delta)k^* \ .$$

- Let k_{GR} be the golden–rule capital intensity, satisfying $f'(k_{GR}) = n + \delta$.
- If $k^* > k_{GR}$, the stationary competitive equilibrium is not Pareto optimal: *Dynamic Inefficiency*

Dynamic inefficiency

- If $k^* > k_{GR}$, then $f'(k^*) < n + \delta$. Capital is so high that its marginal product is outweighed by cost of replacement.
- Suppose economy is at the steady state c^*, k^* with $k^* > k_{GR}$. Then at date t a planner reduces the capital to be saved by marginal $\Delta k^* < 0$ to $k^{**} = k^* + \Delta k^*$ and keep it there forever.
- Consumption is given by

$$c_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}.$$

- Clearly consumption in the current period increases:

$$\Delta c_t = -(1+n)\Delta k^* > 0$$

- But also in all subsequent periods: for all $i \geq 1$,

$$\Delta c_{t+i} = \underbrace{[f'(k^* + \Delta k^*) - (n+\delta)]}_{<0} \underbrace{\Delta k^*}_{<0} > 0$$

Dynamic inefficiency, cont'd

- With this redistribution, the planner can increase total consumption per worker in every period.
- Then he can just divide the extra consumption in each period b/w the two generations alive, so that both are better off.
- Note again: for this to work, it is crucial to have an infinite horizon economy.
- If there is a last generation, it will dislike giving up some capital. So it is not possible to construct a Pareto dominating allocation.

Social Security

- Another possibility to obtain a Pareto improvement is a pay–as–you–go (i.e. unfunded) social security system that collects from every young person $\tau > 0$ (small enough) and pays pension benefit $b = (1 + n)\tau$ to every old person. Definitions
- Budget constraints of generation t become:

$$c_t + s_t = w_t - \tau$$
$$c_{t+1} = R_{t+1} s_t + (1 + n)\tau$$

- The savings function is

$$s(w_t, R_{t+1}, \tau) = \underset{s}{\operatorname{argmax}} \left\{ u(w_t - \tau - s) + \beta u(R_{t+1} s + (1 + n)\tau) \right\} .$$

- It follows $s_\tau < 0$.

Social Security

- Capital market equilibrium implies

$$k_{t+1} = \frac{s(w_t, R_{t+1}, \tau)}{1+n}$$
$$k_{t+1} = \frac{s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}), \tau)}{1+n}$$

- Implicit differentiation gives:

$$\frac{dk_{t+1}}{d\tau} = \frac{s_\tau}{1 + n - s_R f''(k_{t+1})}$$

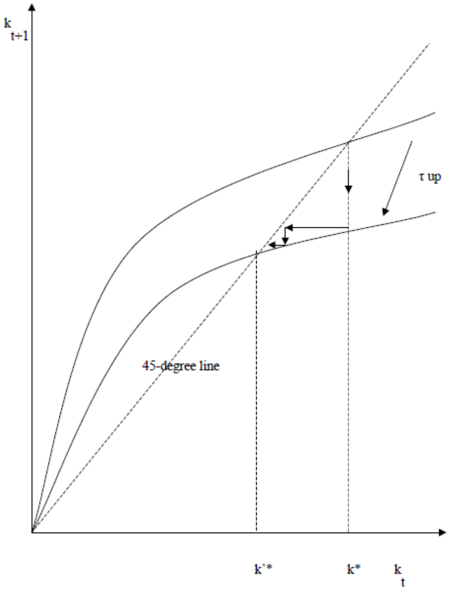
- Since $s_\tau < 0$, numerator is negative. If $s_R \geq 0$, denominator is positive. Hence $\frac{dk_{t+1}}{d\tau} < 0$ and the function $k_{t+1} = G(k_t, \tau)$ shifts downwards.

- Hence unfunded social security $\rightarrow$ smaller capital/labor ratio in the new steady-state.

Transition

- Economy is initially in old steady-state k^*.
- At period T, government announces introduction of (marginal) PAY-GO system.
- The equilibrium locus $k_{t+1} = G(k_t)$ shifts down and capital/labor declines over time until it reaches $k^{'*} < k^*$.

Transition



Transition

- Economy is initially in old steady-state k^*.
- At period T, government announces introduction of (marginal) PAY-GO system.
- The equilibrium locus $k_{t+1} = G(k_t)$ shifts down and capital/labor declines over time until it reaches $k'^* < k^*$.
- What happens to interest rate and wage during the transition?
- Is the intro of marginal PAY-GO system Pareto-improving?

Appendix

CES example: details

- Saving function: $s(w_t, R_{t+1}) = 0.5w_t$. Note that $s_R = 0$ due to log utility.
- Wage function:

$$w(k) = A(1-a)\left[ak^{\frac{\sigma-1}{\sigma}} + 1 - a\right]^{\frac{1}{\sigma-1}}.$$

- It follows that the equilibrium dynamics for k_t is given by

$$(1+n)k_{t+1} = \frac{1}{2}A(1-a)\left[ak_t^{\frac{\sigma-1}{\sigma}} + 1 - a\right]^{\frac{1}{\sigma-1}} \equiv G(k_t).$$

- $G(k)$ is monotonically increasing in k and
  1. if $\sigma > 1$, concave in k and $G'(0) > 1$
  2. if $\sigma < 1$, there exists $\bar{k} > 0$ such that G is convex for all $k < \bar{k}$, concave for all $k > \bar{k}$ and $G'(0) < 1$.
- Hence in the second case, i.e. $\sigma < 1$, multiple steady-states may arise.

Back

Social Security: Definitions

- Fully funded system: workers pay (compulsory) contributions to the government. These funds are invested in productive assets, earning the same return as capital.
- Unfunded or pay-as-you-go system: workers pay (compulsory) contributions to the government. The government distributes these funds to the current old people as per capita transfers (pensions). The implicit rate of return of a pay-as-you-go system is given by population growth rate n.
  - Not surprisingly, falling fertility rates and population aging pose a challenge to PAY-GO systems.
- The social security systems of most OECD countries are unfunded, with current pensions paid out of current social contributions. However, funded pension arrangements are becoming more and more important. Back