Problem Set 3

Due on Monday, May 27, h 13:00. Mail box F230.

Problem 1 (15 points)

There are two agents who live for only two periods, periods 0 and 1. Both agents 1 and 2 get the same endowment of 1 in period 0. In period 1, agent 1 gets either $1+e$ or $1-e$ with probability 1/2, where $e \in (0,1)$. But agent 2 still gets 1 for sure.

Agents differ in terms of their utility: agent 1 has a utility function of $u(c) = \log(c)$, while agent 2 has a utility function of $v(c) = c$. Markets are complete, i.e. agents can trade state-contingent bonds.

Let us denote the state in period 1 in which agent 1 gets the high and low endowments as H and L, respectively, and normalize the price of period 0 consumption to 1. There is no discounting. The problem for agent 1 is

$$\max\left\{u\left(c_0^1\right) + \frac{1}{2}u\left(c_1^1(H)\right) + \frac{1}{2}u\left(c_1^1(L)\right)\right\}$$

subject to:

$$\begin{aligned} c_0^1 + q_H a^1(H) + q_L a^1(L) &= 1, \\ c_1^1(H) &= 1 + e + a^1(H), \\ c_1^1(L) &= 1 - e + a^1(L) \end{aligned}$$

and for agent 2,

$$\max\left\{c_0^2 + \frac{1}{2}u\left(c_1^2(H)\right) + \frac{1}{2}u\left(c_1^2(L)\right)\right\}$$

subject to:

$$\begin{aligned} c_0^2 + q_H a^2(H) + q_L a^2(L) &= 1, \\ c_1^2(H) &= 1 + a^2(H), \\ c_1^2(L) &= 1 + a^2(L) \end{aligned}$$

where c_0^i is agent i's consumption in period 0, and $c_1^i(S)$ is agent i's consumption in period 1 in state S. The prices of a security that insures you against state S is denoted q_S, and $a^i(S)$ denotes the quantities of those securities purchased by agent i. Assume each agent faces a natural borrowing constraint.

1. Define an equilibrium. The problems for the individuals are already stated, so you only need to be explicit about the borrowing constraints and all markets that need to clear.

2. Now solve for equilibrium allocations and the price of each security, q_H, q_L. What is the implied equilibrium interest rate, R^*, with complete markets? Is the equilibrium unique?

3. Now forget about complete markets, and assume that there is only a risk-free bond that is tradable in period 0 that promises R^f in period 1. The period 0 budget constraint becomes:

$$c_0^i + a^i = 1$$

where a^i is savings of agent i, and the period 1 budget constraint is

$$\begin{aligned} c_1^1(H) &= 1 + e + R^f a^1, \\ c_1^1(L) &= 1 - e + R^f a^1 \end{aligned}$$

for agent 1 and

$$\begin{aligned} c_1^2(H) &= 1 + R^f a^2, \\ c_1^2(L) &= 1 + R^f a^2 \end{aligned}$$

for agent 2. Reformulate a competitive equilibrium with incomplete markets. You only need to be explicit about the individual borrowing constraints and all markets that need to clear. Just state it, don't solve it.

4. What is the equilibrium gross interest rate, R^f ? Compare it with R^* from above and explain the relationship. Provide some economic intuition for your finding.

Problem 2 (15 points)

Consider a simple 2-period, partial equilibrium, model where a (single) consumer solves

$$\max \{u(c_0) + \beta [\pi u(c_1(L)) + (1-\pi) u(c_1(H))]\}$$

subject to

$$\begin{aligned} c_0 + a &= y_0, \\ c_1(L) &= y_1(L) + Ra, \\ c_1(H) &= y_1(H) + Ra \end{aligned}$$

where a denotes savings (in the form of a risk-free bond), c_0 is consumption in period 0, and $c_1(S)$ is consumption in period 1 in state $S = L, H$. Assume that the utility function satisfies $u'(\cdot) > 0$, $u''(\cdot) < 0$ (**risk aversion**) and $u'''(\cdot) > 0$ (**prudence**). Today's income y_0 is known with certainty but $y_1(S)$ is random. For concreteness, assume that $y_1(L) = \overline{y}_1 - \varepsilon$ and $y_1(H) = \overline{y}_1 + \varepsilon$, where $\overline{y}_1 > 0$ is a fixed component and $0 < \varepsilon < \overline{y}_1$. Moreover, assume that the two states of the world are equiprobable and that the discount factor is one.

Observe that the maximization problem of the household can be restated as:

$$\max_{a \in \mathcal{A}} \left\{ u(y_0 - a) + \frac{1}{2} u(Ra + \overline{y}_1 - \varepsilon) + \frac{1}{2} u(Ra + \overline{y}_1 + \varepsilon) \right\}.$$

1. What is the natural borrowing limit for the household? Specify the constraint set $\mathcal{A}$.

2. If there is no income risk (i.e. $\varepsilon = 0$), what is the optimal level of savings? What is the optimal profile of consumption over the two periods of life? Discuss the economic intuition.

3. Go back to the case of risky income (i.e. $0 < \varepsilon < \overline{y}_1$). Denote the optimal level of savings as $a^*(\varepsilon)$. Show formally that savings are increasing in income risk, i.e. that $\frac{da^*(\varepsilon)}{d\varepsilon} > 0$. What is the key assumption on preferences behind this property? *Hint*: argue that $a^*(\varepsilon)$ must satisfy the Euler equation and apply the Implicit Function Theorem.

4. Assume the following utility function

$$u(c) = -\frac{1}{\alpha} \exp(-\alpha c), \; \alpha > 0.$$

   Given this specification, compute analytically $a^*(\varepsilon)$. Specifically, show that

$$a^*(\varepsilon) = \frac{1}{2\alpha} \left[\alpha (y_0 - \overline{y}_1) - \log(2) + \log(D(\varepsilon)) \right].$$

   What is the term $D(\varepsilon)$? Discuss the economic intuition.