Incomplete Markets: A Pure Credit Model, Huggett (1993)

Lecture 4

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Setup

- Main Reference: Huggett, M. 1993. "The risk-free rate in heterogeneous-agent incomplete-insurance economies", Journal of Economic Dynamics and Control.
- Huggett studies a pure consumption/loans economy.
- Households receive *persistent (not i.i.d. as in the previous model)* idiosyncratic shocks to their earnings.
- Households can only insure through **non-contingent** assets subject to borrowing constraints.
- Each household can borrow or lend at a constant **risk-free** interest rate r.
- Total borrowing cannot exceed $\underline{a}$, where $\underline{a}$ is either
    - the natural borrowing limit, or
    - a more stringent, "ad hoc" borrowing limit.

Questions

- Using Huggett's framework, we can address the following questions:
    1. If financial markets are incomplete (e.g. the only available asset is a non-contingent bond with a borrowing limit), how much does earnings inequality account for wealth inequality?
    2. If borrowing constraints become tighter (as they have in the Great Recession), how much does wealth inequality change? What are the welfare effects of tighter constraints?
    3. How much does a doubling of unemployment duration affect wealth inequality and welfare?

Methodology

- How do we tackle these questions?
- Basic methodology: feed an exogenous earnings shock process into a heterogeneous agent model to obtain endogenous asset decision rules and associated wealth distribution.
- Heterogeneous agent model: Huggett (1993).
- We can then compute summary statistics of the wealth distribution like the Gini coefficient to analyze positive and normative questions about inequality.

Outline

- Huggett's model
  - Calibration
  - Existence of equilibrium
  - Computational aspects
- Experiments
  - Tighter Financial Constraints
  - Longer Duration of Unemployment

Environment: Preferences

- Population: unit measure of households.
- Preferences:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \right]$$

- Assume $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ is ctsly differentiable, strictly increasing, strictly concave.
- We also need "bounded" to prove that Bellman operator is a contraction.

Environment: Endowments

- In any period t, households receive earning/labor market shock $s_t \in \mathcal{S} = \{e, u\}$ with interpretation:
  - e household is employed
  - u household is unemployed.
- The shock follows a first-order Markov process with transition matrix

$$\pi\left(s'|s\right) = prob\left(s_{t+1} = s'|s_t = s\right).$$

- If employed, he earns $y\left(e\right) = 1$ (a normalization).
- If unemployed he gets $y\left(u\right) = b < 1$ (unemployment benefits).

Environment: Endowments

- Transition matrix for "employment" shock:

$$\Pi = \begin{bmatrix} t \backslash t+1 & e' & u' \\ e & \pi\left(e|e\right) & \pi\left(u|e\right) \\ u & \pi\left(e|u\right) & \pi\left(u|u\right) \end{bmatrix}$$

- Restrictions:

$$\begin{aligned} \pi\left(e|e\right) + \pi\left(u|e\right) =& 1, \\ \pi\left(e|u\right) + \pi\left(u|u\right) =& 1. \end{aligned}$$

- How many independent transition rates?
- Interpretation: $\pi\left(u|e\right)$ prob. of losing a job (or employment-to-unemployment transition rate), $\pi\left(e|u\right)$ prob. of finding a job (or unemployment-to-employment transition rate), etc.

A Digression: Stationary Distribution of Markov Chain

- Let $\overline{E} = P(s_t = e)$ and $\overline{U} = P(s_t = u)$, with $\overline{U} + \overline{E} = 1$.
- $\overline{U}$ can be interpreted both as the (unconditional) prob. of being unemployed for a single agent or as the fraction of unemployed agents in the economy (i.e. unemployment rate).
- $\overline{E}$ and $\overline{U}$ evolve over time according to:

$$\overline{E}' = \pi(e|e)\overline{E} + \pi(e|u)\overline{U}$$
$$\overline{U}' = \pi(u|e)\overline{E} + \pi(u|u)\overline{U}$$

- **Stationary Distribution**. After a sufficiently long period of time, the system settles down to a statistical equilibrium where $\overline{E}$ and $\overline{U}$ are constant.

Environment: Asset market structure

- Households can trade one-period, non-contingent discount bonds with borrowing constraint $\underline{a} \leq 0$. Define $A = \{a_t \in \mathbb{R} : a_t \geq \underline{a}\}$.
- Households enter period with assets a_t and purchase next period assets a_{t+1} at price q_t.
- Since there is no aggregate uncertainty and we will be looking for a steady state equilibrium, we will assume $q_t = q$.
- Assume that $\beta < q$ (something that must be verified in equilibrium).
  - Since $q = \frac{1}{1+r}$, this is equivalent to $\beta (1 + r) < 1$ (see lecture on Aiyagari).
- Note: for simplicity, the borrowing constraint $a_t \geq \underline{a}$ is taken as exogenous.

Environment: Strong Assumptions

- For simplicity we have made the following strong assumptions:
  - Exogenous Earnings
  - Exogenous Borrowing Constraints
  - No Redistribution
- Relax these assumptions later (if time permits).

Calibration

- Parameters to calibrate: $\beta, u(\cdot), b, \pi(e|e), \pi(u|u), \underline{a}$
- Huggett assumes the model period is one quarter.
- Preference parameters are taken from outside the model:
  - The utility function is

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}$$

    where $\alpha = 1.5$
  - Annual discount factor 0.96 $\implies \beta = 0.994$ on a quarterly basis.
- Employed earnings normalized to 1 and data on replacement rates pins down $b = 0.5$.

Calibration

- Markov chain for employment shock: need to calibrate **two** transitions: $\pi(u|u)$ and $\pi(e|e)$.
- Data on duration of unemployment pins down $\pi(u|u)$.
- Duration of unemployment D can be computed as Math

$$D = \frac{1}{1 - \pi(u|u)} \implies \pi(u|u) = 1 - \frac{1}{D}.$$

- If $D = 2$ quarters (U.S. postwar data), then $\pi(u|u) = 0.5$.

Calibration

- Data on average unemployment pin down $\pi\left(e|e\right)$.
- Transition equation for $\overline{U}$ is:

$$\overline{U}' = \underbrace{\pi\left(u|e\right)\left(1-\overline{U}\right)}_{E\rightarrow U} + \underbrace{\pi\left(u|u\right)\overline{U}}_{U\rightarrow U}.$$

- In the long-run, $\overline{U}' = \overline{U}$, hence

$$\pi\left(u|e\right) = \frac{\left(1-\pi\left(u|u\right)\right)\overline{U}}{1-\overline{U}}$$

- Thus, if $\overline{U} = 5.66\%$ (U.S. postwar data), then

$$\pi\left(u|e\right) = \frac{\left(1-0.5\right)*0.0566}{1-0.0566} = 0.03 \implies \pi\left(e|e\right) = 0.97$$

Calibration

- Given the above calculations, the Markov chain for the (exogenous) state $s \in \{e, u\}$ is given by:

$$\Pi = \begin{bmatrix} \pi(e|e) & \pi(u|e) \\ \pi(e|u) & \pi(u|u) \end{bmatrix} = \begin{bmatrix} 0.97 & 0.03 \\ 0.5 & 0.5 \end{bmatrix}$$

Calibration

- Finally $\underline{a}$ is targeted to match 2% real interest rate in U.S. postwar data.
- Sensitivity analysis:

$$\underline{a} \in \{-2, -4, -6, -8\}$$

Household's Problem

- The household's problem can be written in recursive form. What are the (individual) state variables?
- Bellman equation:

$$V(s, a; q) = \max_{a' \geq \underline{a}} u(c) + \beta \mathbb{E}\left[V(s', a'; q) | s\right]$$

subject to

$$c + qa' \leq a + y(s), \; s = e, u$$

where $y(e) = 1 > b = y(u)$.

- The exogenous shock s follows a discrete Markov chain $\pi(s'|s) \rightarrow$ the continuation value/expected future value function can be written as

$$\mathbb{E}\left[V(s', a'; q) | s\right] = \sum_{s' \in S} \pi(s'|s) \, V(s', a'; q) \, .$$

Household's Problem

- The Bellman operator associated to the above problem is $T : C(S \times A) \to C(S \times A)$ such that:

$$TV(s, a; q) = \max_{a' \in \Gamma(s,a;q)} u\left(a + y(s) - qa'\right) + \beta \sum_{s' \in S} \pi\left(s'|s\right) V\left(s', a'; q\right) \tag{1}$$

  where

$$\Gamma(s, a; q) = \left\{ a' \in \mathbb{R} : \underline{a} \leq a' \leq \frac{a + y(s)}{q} \right\}$$

  and $C(S \times A)$ denotes the space of continous and bounded functions on $S \times A$.

- Solution to this problem: policy function $a' = g(s, a; q)$.

Wealth Distribution

- Since households differ in their employment histories, and they cannot perfectly insure employment risk, they will in general differ in their asset holdings. No representative agent.
  - Households who have received a long sequence of bad shocks are wealth-poor (they have decumulated their stock of savings).
  - Households who have received a long sequence of good shocks are wealth-rich (they have increased their stock of savings).
- Describe the economy-wide distribution of assets and employment with the help of a probability measure μ.
- Think of $\mu(S_0, A_0; q)$ as the fraction of people with shocks in the set S_0 and asset holdings in the set A_0, when the price is q (cross-sectional interpretation).

Wealth Distribution

- To avoid measure-theoretic notation, we assume the asset space A is discrete, i.e. $A = \{a_1, a_2, \ldots, a_n\}$, with $a_1 = \underline{a}$.
- The space of the shocks is discrete by construction, i.e. $S = \{e, u\}$ or, more generally, $S = \{s_1, s_2, \ldots, s_m\}$.
- Define the probability measure as

$$\mu(a, s) = \Pr(a_t = a, s_t = s).$$

Wealth Distribution, cont.

- Since μ is a probability,

$$\mu(s,a) \geq 0 \text{ for all } s, a \in S \times A$$

and

$$\sum_{s\in S}\sum_{a\in A}\mu(s,a) = 1$$

- Moreover,

$$\sum_{a\in A}\mu(s,a) = \mu^*(s)$$

where $\mu^*(s)$ is the invariant distrib. of the Markov chain for the shock, $\pi(s'|s)$.

Wealth Distribution

- Using the discretized distribution μ, several statistics can be easily computed, e.g.
  - Aggregate (=average) asset holdings:

$$E(a) = \sum_{s \in S} \sum_{a \in A} a \cdot \mu(s, a)$$

  - Aggregate (=average) labor earnings:

$$E(y) = \sum_{s \in S} \sum_{a \in A} y(s) \cdot \mu(s, a)$$

- Note: in this model $E(y)$ is exogenous, i.e. it can be computed without the need to solve for the equilibrium μ. Indeed,

$$\begin{aligned} E(y) &= \sum_{s \in S} y(s) \sum_{a \in A} \mu(s, a) \\ &= \sum_{s \in S} y(s) \mu^*(s) \end{aligned}$$

  where $\mu^*(s)$ depends only on the exogenous Markov chain $\pi(s'|s)$.

Recursive Competitive Equilibrium

Definition

A recursive (stationary) competitive equilibrium is an allocation (c, a'), a price q^*, and an invariant distribution $\mu^* = \mu(s, a; q^*)$ such that

1. For given q^*, (c, a') solves the household optimization $V = TV$.
2. Given μ^*, goods and asset markets clear:

$$\sum_{a \in A} \sum_{s \in S} [c(s, a; q^*) - y(s)] \mu^*(s, a) = 0 \tag{2}$$

$$\sum_{a \in A} \sum_{s \in S} a'(s, a; q^*) \mu^*(s, a) = 0 \tag{3}$$

3. μ^* is a stationary probability measure consistent with $a' = g(s, a; q^*)$ and $\pi(s'|s)$.

Existence: Policy function for savings

- (**Theorem 1**) Huggett (1993) proves that if $q > 0$ and other technical conditions are satisfied, then
  - $T^n V_0 \to V^*$, where $V^* = TV^*$ is a fixed point of the Bellman operator (1).
  - $V^*(s, a; q)$ is strictly increasing, strictly concave, and continuously differentiable in a.
  - Policy function $a' = g^*(s, a; q)$ is continuous, nondecreasing in a; and strictly increasing in a for $g(s, a; q) > \underline{a}$.
- Remarks: concavity of $u \implies$ concavity of $V \implies g^*(s, a; q)$ is increasing in a.

Existence: Stationary Distribution

- **Theorem 2 (Huggett)**. If the conditions of Theorem 1 hold, $\beta < q$ (i.e. people are impatient and borrowing rates are not too high), and $\pi(e|e) > \pi(e|u)$ (i.e. the probability of staying employed is higher than becoming employed), then there exists a unique invariant measure given q.

Existence: Stationary Distribution

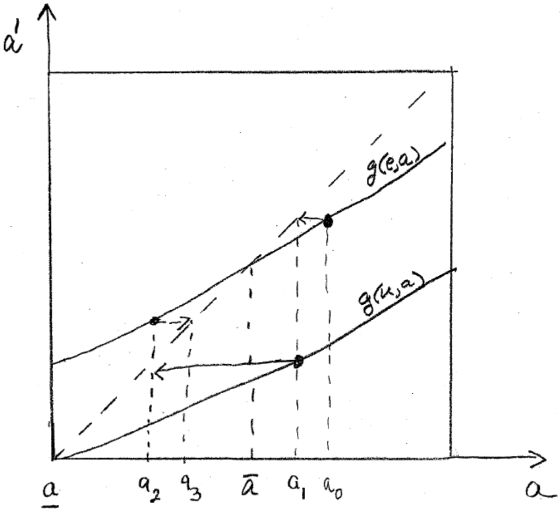
- One of the big issues is that for μ to be invariant, the distribution cannot fan out.
- While asset holdings are bounded below by $\underline{a}$, they are not necessarily bounded above.
- It is impossible for μ to be invariant if there is mass being put on higher and higher a.
- One of the important parts of Huggett's proofs is to show that agents never accumulate savings beyond an endogenously determined upper bound $\overline{a}$.

Existence: Stationary Distribution, cont.

- **Result 1**. Under the conditions of Theorem 2, $g(u, a) < a$ for $a > \underline{a}$
  - Households dissave when unemployed.
- **Result 2**. There exists an $\overline{a}$ such that $g(s, a) < a$ for $a \geq \overline{a}$ and for $s = e, u$.
  - Eventually every person (even the employed) dissave if they have enough assets.

Existence: Stationary Distribution, cont.

Graphical intuition



Existence: Stationary Distribution, cont.

- Ergodic set is $[\underline{a}, \overline{a}]$.
- Formal definition of ergodic set:

$$(i)\ a_t \in [\underline{a}, \overline{a}] \implies a_{t+1} \in [\underline{a}, \overline{a}], \forall t$$

and

$$(ii)\ a_t \notin [\underline{a}, \overline{a}] \text{ only for finitely many } t\text{'s}$$

- "Informal" definition: *(i)* Once you are inside the set, you never go out, *(ii)* If you start outside the set, you always come back in, and never go out again.

Existence: Stationary Distribution, cont.

- The result for why agents don't increase savings beyond $\bar{a}$ relies on concavity.
- The marginal benefit of savings is $\partial V\left(s', a'\right) / \partial a'$ while the marginal cost is $q \cdot u'\left(y + a - qa'\right)$.
- Hence increasing savings increases expected future marginal benefits at a decreasing rate (i.e. $\partial^2 V\left(s', a'\right) / \partial a'^2 < 0$) while it increases marginal costs at an increasing rate (i.e. $-q \cdot u''\left(y + a - qa'\right) > 0$).

Computation

- Now that we have established existence and uniqueness of the equilibrium and some nice theoretical properties of the objects of interest, we need to *actually solve* for the equilibrium on a computer.

Computation: Algorithm

- The algorithm is (see ILIAS for Matlab code):
  1. Taking $q \in (0,1]$ as given, solve the agent's dynamic programming problem for his decision rule $a' = g(a; s; q)$ (i.e. solve for a fixed point of the Bellman operator);
  2. Given $g(a; s; q)$ and the stochastic process for earnings, solve for the invariant wealth distribution $\mu^*(s, a; q)$;
  3. Given μ^*, check whether the asset market clears at q, i.e. if

     $$ED(q) \equiv \sum_{s \in S} \sum_{a \in A} a'(s, a; q)\, \mu^*(s, a; q) = 0$$

     If it does, we are done. If not, then change q in the direction that clears the market and go back to step 1.
- Note to step 3: If e.g. $ED(q) > 0$ (excess demand of assets) we need to increase the price, hence $q' > q$. In practice we can use a bisection routine.

Solving the model

- Three parts of solving the model:
  1. Solving the household problem for a given price q.
  2. Solving for the invariant distribution $\mu(a, s)$.
  3. Updating the price q to clear the assets market.

- See each step in detail.

Solving the household problem

- The stochastic process for the employment shock is described by a 2×2 discrete Markov chain: $\pi(s'|s)$, with $s \in \mathcal{S} = \{e, u\}$.
- What about asset holdings a?
- The simplest way to go is to discretize also the asset space:

$$\mathcal{A} = \{a_1 < a_2 < \ldots < a_n\}$$

- How to choose a_1 and a_n? How to choose n (i.e. number of grid points)? How to choose the spacing b/w grid points?

Solving the household problem

How to choose n (i.e. number of grid points)?

- The more the better, but: computing power is limited.
- Trade-off b/w accuracy and speed.

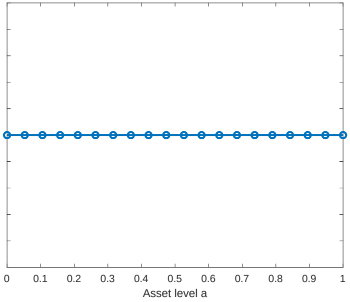
How to choose the spacing b/w grid points?

- Linear spacing: use Matlab command `linspace(amin,amax,n)`.
- Geometric spacing, to cluster points around certain areas.
- For $i = 2, \ldots, n$, set

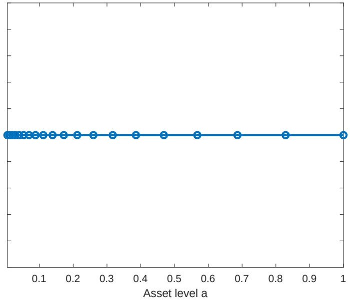
$$(a_{i+1} - a_i) = \gamma \left(a_i - a_{i-1} \right)$$

  for a user-chosen γ. This will allocate more points closer to the lower bound.

Solving the household problem



(a) Linear spacing, $\gamma = 1$



(b) Geometric spacing, $\gamma > 1$

The figures above are drawn for $n = 20$, $a_1 = 0$, $a_n = 1$. In panel (b), $\gamma = 1.2$.

If γ is higher, the grid will have more points around the lower bound a_1.

Solving the household problem

How to choose a_1 and a_n?

- Set lower bound a_1 equal to the borrowing limit $\underline{a}$
- Set the upper bound equal to a large positive value s.t. it is never binding.
- Does such an upper bound on assets always exist?
  - Huggett (1993), Result 2.
  - Other models?

Solving the household problem

- The asset space is discrete.
- We have to decide if also want a discrete choice space (for next period asset holdings a').
- Suppose for now that the choice space is also discrete with $a' \in \mathcal{A}$.

Solving the household problem

- Recall the original, "true" problem:

$$V(a,s) = \max_{c,a \geq \underline{a}} \left\{ u(c) + \beta \sum_{s'} \pi(s'|s) V(a', s') \right\}$$

subject to

$$c + qa' = a + y(s).$$

- Discretized problem:

$$V(i,j) = \max_{i' \in \{1,\dots,n\}} u\left(c\left(i,j,i'\right)\right) + \beta \sum_{j'} \pi\left(j'|j\right) V\left(i',j'\right)$$

subject to

$$c\left(i,j,i'\right) = a + y(s) - qa'$$

and

$$s \in S_j,\ a \in \mathcal{A}_i,\ a' \in \mathcal{A}_{i'}$$

for $(i,j) \in \{1,\dots,n\} \times \{1,2\}$. Note: V is an $n \times 2$ matrix.

Solving the household problem

- Discretized problem $\rightarrow$ the computer can understand this.
- We know that the operator defined above is a *contraction*.
- Taking advantage of the *Contraction Mapping Theorem*, we solve the discretized problem by iterating until convergence on $t = 0, 1, \ldots$

$$V_{t+1}(i,j) = \max_{i' \in \{1,\ldots,n\}} u\left(c\left(i,j,i'\right)\right) + \beta \sum_{j'} \pi\left(j'|j\right) V_t\left(i',j'\right)$$

  given an initial guess $V_0(i',j')$.

- Convergence is achieved when

$$\max_{i,j} \left| V_{t+1}(i,j) - V_t(i,j) \right| < \varepsilon$$

  where ε is a very small positive number.

Solving the household problem

- How do we solve for $V_{t+1}(i,j)$?
- Fix an index j for the current shock (i.e. loop over j).
- Define the continuation value as $EV_j(i') \equiv \beta \sum_{j'} \pi(j'|j) V_t(i',j')$.
- Define the payoff from choosing action i', given that the current endogenous state is i, as follows:

$$F\left(i,i'\right) = u\left(c\left(i,j,i'\right)\right) + EV_j\left(i'\right)$$

- Compute

$$g_j(i) = \arg\max_{i' \in \{1,\dots,n\}} F\left(i,i'\right)$$

- Can speed up this algorithm using *matrix algebra*.

Solving the household problem

- For $j = 1, 2$, define $n \times 1$ vectors v_j whose ith rows are:

$$v_j\left(i\right) = V\left(a_i, s_j\right)$$

for $i = 1, \ldots, n$.

- For $j = 1, 2$, define two $n \times n$ matrices whose $\left(i, i'\right)$ elements are

$$R_j\left(i, i'\right) = u\left[a_i + y\left(s_j\right) - qa_{i'}\right]$$

- Define Bellman operator as $T : \left[v_1, v_2\right] \rightarrow \left[tv_1, tv_2\right]$ where

$$tv_j(i) = \max_{i'}\left\{R_j(i, i') + \beta\pi_{j1}v_1(i') + \beta\pi_{j2}v_2(i')\right\}$$

for $j = 1, 2$, or

$$\begin{aligned} tv_1 &= \max\left\{R_1 + \beta\pi_{11}\mathbf{1}v_1' + \beta\pi_{12}\mathbf{1}v_2'\right\} \\ tv_2 &= \max\left\{R_2 + \beta\pi_{21}\mathbf{1}v_1' + \beta\pi_{22}\mathbf{1}v_2'\right\} \end{aligned}$$

- Here $\mathbf{1}$ is the $n \times 1$ vector $\left[1\ 1 \cdots 1\right]'$. Furthermore, the max operator is applied *"row-wise"*.

Solving the household problem

- The two equations

$$
\begin{aligned}
tv_1 &= \max\left\{R_1 + \beta\pi_{11}\mathbf{1}v_1' + \beta\pi_{12}\mathbf{1}v_2'\right\} \\
tv_2 &= \max\left\{R_2 + \beta\pi_{21}\mathbf{1}v_1' + \beta\pi_{22}\mathbf{1}v_2'\right\}
\end{aligned}
$$

can be written compactly as

$$
\begin{bmatrix} tv_1 \\ tv_2 \end{bmatrix} = \max\left\{\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \beta\left(\Pi \otimes \mathbf{1}\right)\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}\right\}
$$

where $\otimes$ is the Kronecker product. Appendix

Solving the household problem

- The two equations

$$
\begin{aligned}
tv_1 &= \max\left\{R_1 + \beta\pi_{11}\mathbf{1}v_1' + \beta\pi_{12}\mathbf{1}v_2'\right\} \\
tv_2 &= \max\left\{R_2 + \beta\pi_{21}\mathbf{1}v_1' + \beta\pi_{22}\mathbf{1}v_2'\right\}
\end{aligned}
$$

  can be written compactly as

$$
\begin{bmatrix} \underbrace{tv_1}_{n\times 1} \\ \underbrace{tv_2}_{n\times 1} \end{bmatrix} = \max\left\{ \begin{bmatrix} \underbrace{R_1}_{n\times 1} \\ \underbrace{R_2}_{n\times 1} \end{bmatrix} + \beta \underbrace{\left(\Pi \otimes \mathbf{1}\right)}_{2n\times 2} \begin{bmatrix} \underbrace{v_1'}_{1\times n} \\ \underbrace{v_2'}_{1\times n} \end{bmatrix} \right\}
$$

  where $\otimes$ is the Kronecker product. Appendix

- The term in brackets is a $2n \times n$ matrix. After taking the max along the rows, it boils down to a $2n \times 1$ vector.

Solving the household problem

- To sum up, solving the household problem requires as **inputs**:
  - Predefined grids for state variables (a, s):
    $\mathcal{A} = \{a_1 < a_2 < \ldots < a_n\}$, $\mathcal{S} = \{e, u\}$.
  - Bond price q.
  - Discrete Markov chain for the exogenous state s, i.e. $\Pi_{2\times 2}$.
- After solving the household problem, the **outputs** are the value function $V(a, s)$ and the policy function for assets, $a' = g(a, s)$
  - Discretized value function: $V(i, j)$ is an $n \times 2$ matrix, for $i = 1, \ldots, n$, $j = 1, 2$
  - Discretized policy function $g(i, j)$ is an $n \times 2$ matrix, for $i = 1, \ldots, n$, $j = 1, 2$.
- All of this can be grouped into a **Matlab function**, e.g.

$$[V, g] = \text{solve\_household}\,[q, \Pi, \mathcal{A}, \mathcal{S}]$$

Solving for the invariant distribution

- As for the household problem, we use discretization.
- The unconditional prob. distrib. of (a_t, s_t) is denoted as $\mu_t(a, s) = P(a_t = a, s_t = s)$
- The Markov chain for s and the policy function $g(a, s)$ induce a law of motion for the distribution μ_t:

$$\mu_{t+1}(a', s') = \sum_a \sum_s P(a_{t+1} = a', s_{t+1} = s' | a_t = a, s_t = s) \cdot \mu_t(a, s)$$

- Observe that

$$P(a_{t+1} = a', s_{t+1} = s' | a_t = a, s_t = s) = \begin{cases} \pi(s'|s) & \text{if } g(a, s) = a' \\ 0 & \text{otherwise} \end{cases}$$

or, more compactly,

$$P(a_{t+1} = a', s_{t+1} = s' | a_t = a, s_t = s) = \mathbb{I}(a', a, s) \cdot \pi(s'|s)$$

where the indicator $\mathbb{I}(a', a, s) = 1$ if $g(a, s) = a'$ and 0 otherwise.

Intuition

- Transition for states $(a, s) \to (a', s')$:

$$P\left(a_{t+1} = a', s_{t+1} = s' | a_t = a, s_t = s\right) = \begin{cases} \pi\left(s'|s\right) & \text{if } g\left(a, s\right) = a' \\ 0 & \text{otherwise} \end{cases}$$

- Q: "What is the probability that an agent with current assets a and current employment shock s ends up with assets a' tomorrow and shock s' tomorrow?"
- A1: The prob. that tomorrow's shock is s', given today's shock, is $\pi(s'|s)$.
- A2: The transition of assets is non-stochastic: either $g(a, s)$ is equal to a' or it is not.
- A1+A2: Hence the prob. of transition from (a, s) to (a', s') is $\pi(s'|s)$ if $g(a, s) = a'$ and zero if it is not equal to a'.

Solving for the invariant distribution

- Hence the equation

$$\mu_{t+1}\left(a',s'\right)=\sum_a\sum_s \underbrace{P\left(a_{t+1}=a',s_{t+1}=s|a_t=a,s_t=s\right)}_{I(a',a,s)\cdot\pi(s'|s)}\cdot\mu_t\left(a,s\right)$$

  boils down to

$$\begin{aligned}\mu_{t+1}\left(a',s'\right) &= \sum_a\sum_s \mathbb{I}\left(a',a,s\right)\cdot\pi\left(s'|s\right)\cdot\mu_t\left(a,s\right)\\ &= \sum_s \sum_{\{a:a'=g(a,s)\}} \pi\left(s'|s\right)\cdot\mu_t\left(a,s\right).\end{aligned}$$

- The term $\mathbb{I}\left(a',a,s\right)\cdot\pi\left(s'|s\right)$ defines a large, $2n\times 2n$ transition matrix (or, more generally, $mn\times mn$).

Solving for the invariant distribution

- Order the state variables in a single $1 \times 2n$ vector:

$$\left[\left(a_1, s_1\right), \left(a_2, s_1\right), \ldots, \left(a_n, s_1\right), \left(a_1, s_2\right), \ldots, \left(a_n, s_2\right)\right]$$

- The law of motion for the stationary distribution can be written in matrix notation as:

$$vec\left(\mu\right)' = Pvec\left(\mu\right)'$$

where

$$\underbrace{P}_{2n \times 2n} = \begin{bmatrix} \pi_{11} G_1 & \pi_{12} G_1 \\ \pi_{21} G_2 & \pi_{22} G_2 \end{bmatrix}$$

and for $j = 1, 2,$

$$\underbrace{G_j\left(a, a'\right)}_{n \times n} = \begin{cases} 1 \text{ if } g\left(a, s_j\right) = a' \\ 0 \text{ if } g\left(a, s_j\right) \neq a' \end{cases}$$

Solving for the invariant distribution

- Suppose $n = 2$.
- If $g(a_1, s_1) = g(a_2, s_1) = a_1$ and $g(a_1, s_2) = g(a_2, s_2) = a_2$, then

$$G_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

- Hence

$$P = \begin{bmatrix} \pi_{11} G_1 & \pi_{12} G_1 \\ \pi_{21} G_2 & \pi_{22} G_2 \end{bmatrix} = \begin{bmatrix} \pi_{11} & 0 & \pi_{12} & 0 \\ \pi_{11} & 0 & \pi_{12} & 0 \\ 0 & \pi_{21} & 0 & \pi_{22} \\ 0 & \pi_{21} & 0 & \pi_{22} \end{bmatrix}$$

Solving for the invariant distribution

- To sum up, solving for the invariant requires as **"inputs"**:
  - Discrete Markov chain Π for the exogenous state s, a 2×2 matrix.
  - Discretized policy function $g(i, j)$, an $n \times 2$ matrix, for $i = 1, \ldots, n$, $j = 1, 2$.
- After solving this problem, the **output** is the invariant distribution $\mu(a, s)$.
  - Discretized invariant distribution: $\mu(i, j)$ is an $n \times 2$ matrix, for $i = 1, \ldots, n$, $j = 1, 2$
- All of this can be grouped into a **Matlab function**, e.g.

$$[\mu] = \text{solve\_mu}\,[\Pi, g]$$

Updating the price

- The final step is finding the equilibrium price q^*.
- If $q = q^*$, then the assets market clears:

$$\sum_a \sum_s a\mu\left(a, s; q^*\right) = 0.$$

- Root finding problem. Let $ED\left(q\right) \equiv \sum_a \sum_s a\mu\left(a, s; q\right)$ and use a root-finding routine to find a zero of ED.
- In Matlab can use `fzero`.
- Can also code manually a bisection routine.

Updating the price: Bisection

- Guaranteed to converge in a finite number of iterations.
- Set initial interval
$$[q_{\min}, q_{\max}] = [\beta, 1]$$
- Check that ED has opposite signs on $[q_{\min}, q_{\max}]$, typically $ED(q_{\min}) > 0$ and $ED(q_{\max}) < 0$.
- Start with $q_0 = (q_{\min} + q_{\max})/2$. Compute $ED(q_0)$.
- If $ED(q_0) > 0$, then
  - set $q_{\min} = q_0$ and
  - update $q_1 = (q_0 + q_{\max})/2$
- Else, if $ED(q_0) < 0$
  - set $q_{\max} = q_0$ and
  - update $q_1 = (q_0 + q_{\min})/2$
- Stop if either $|q_1 - q_0| < \varepsilon_1$ or $|ED| < \varepsilon_2$.

Benchmark Calibration

- Recall our calibrated values for the model parameters:

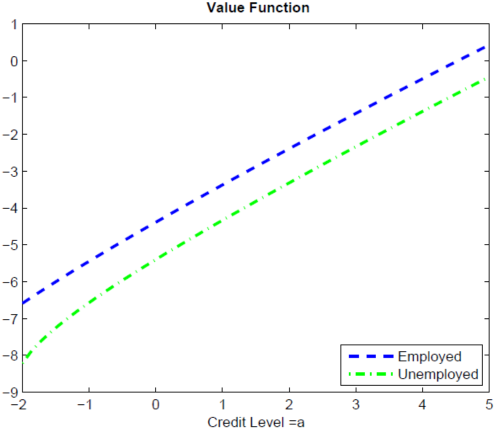
β	α	$y(e)$	$y(u)$	$\pi(e\|e)$	$\pi(u\|u)$	$\underline{a}$
0.994	1.5	1	0.5	0.97	0.5	-2

Results: Benchmark Economy

- Once we find an equilibrium, we have the following equilibrium objects:
- The equilibrium bond price is $q^* = 0.9951$.
- Other equilibrium objects are:
  - Value function $V(s, a; q^*)$
  - Policy function for assets $g(a; s; q^*)$
  - Invariant distribution $\mu(a, s; q^*)$.

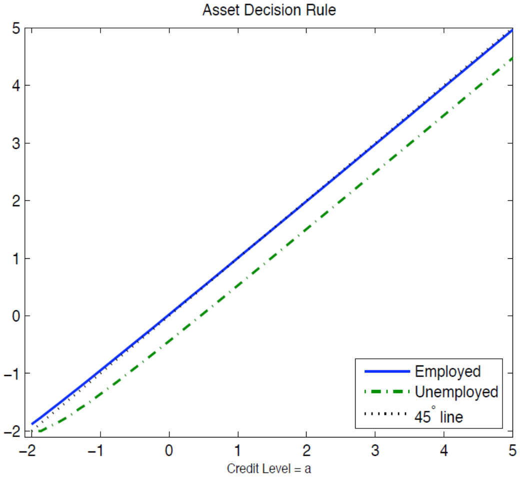
Results: Benchmark Economy

- Figure below plots the value function across assets of employed and unemployed agents: $V(a, e; q^*)$ and $V(a, u; q^*)$.



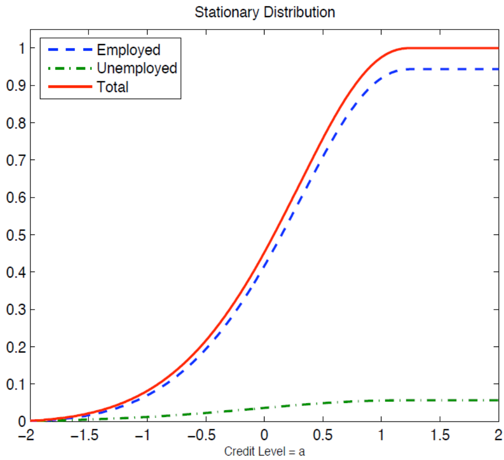
Results: Benchmark Economy

- The decision rules in figure below are consistent with the fact that there exists $\overline{a} = 1.0381$ where $g(\overline{a}, e) = \overline{a}$, which establishes an upper bound on the the invariant distribution.



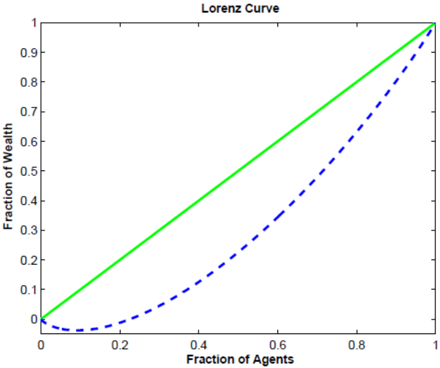
Results: Benchmark Economy

- Figure below graphs the cross-sectional wealth distribution (cumulative density function).



Results: Benchmark Economy

- The Gini coefficient with respect to total wealth for this economy is 0.3821.
- Figure below shows the Lorenz curve. The Lorenz curve is negative because in the model the sum of the wealth of agents in the first quintile is less than zero.



Results

- Results from steady state of incomplete markets benchmark:

	Data	Bench
Unemployment (target) %	5.66	5.66
Real Interest (target) %	2.00	2.00
Wealth Gini (untargeted)	0.80	0.38

- Comparing the benchmark with data, we see our Huggett model is only able to account for roughly half of the wealth inequality in the data.

Counterfactual Experiments

- Once we have solved our Huggett model under a given parameterization, we can conduct counterfactuals to assess the welfare benefits or costs of certain changes to the environment or policy.
- In particular, we will perform the following experiments:
  1. Tighter Financial Constraints (as in the Great Recession);
  2. Longer Duration of Unemployment (as in the Great Recession).

Experiment 1: Tighter Financial Constraints

- In the 2008-2009 financial crisis, lenders have tightened borrowing constraints.
- What happens if the borrowing limit is cut in half?
- Our experiment: replace $\underline{a} = -2$ with $\underline{a} = -1$.
- Effects:
  1. Households precautionary savings rise.
  2. Interest rate falls.
  3. Wealth inequality is lowered by roughly 50 percent.

Experiment 1: Tighter Financial Constraints

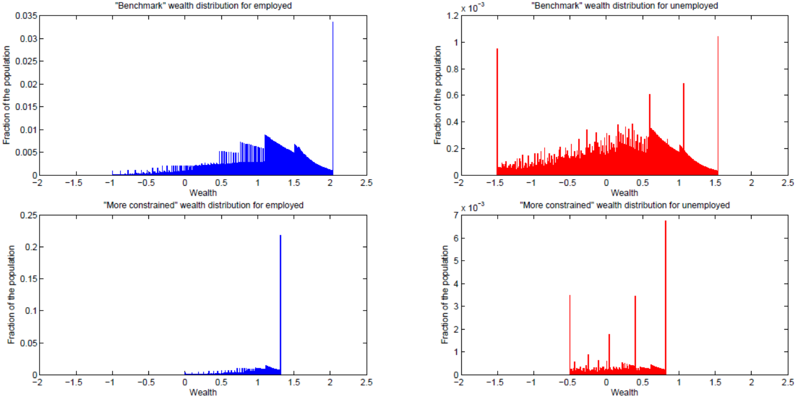
- Results from counterfactual experiment n.1:

	Data	Bench	$\underline{a} = -1$
Unemployment (target) %	5.66	5.66	5.66
Real Interest (target) %	2.00	2.00	0.82
Wealth Gini (untargeted)	0.80	0.38	0.18

Experiment 1: Tighter Financial Constraints

Wealth distribution

Figure plots new cross-sectional wealth distribution, which is more compressed at both ends (lower real rate discourages saving).



Experiment 2: Longer Duration of Unemployment

- In the 2008-2009 financial crisis, the duration of unemployment has risen.
- What happens if the duration of unemployment is doubled from 2 quarters to 1 year?
- Effects:
  1. Households precautionary savings rise..
  2. ..thereby lowering interest rates by roughly half.
  3. Longer spells of unemployment raises wealth inequality by about a quarter.

Experiment 2: Longer Duration of Unemployment

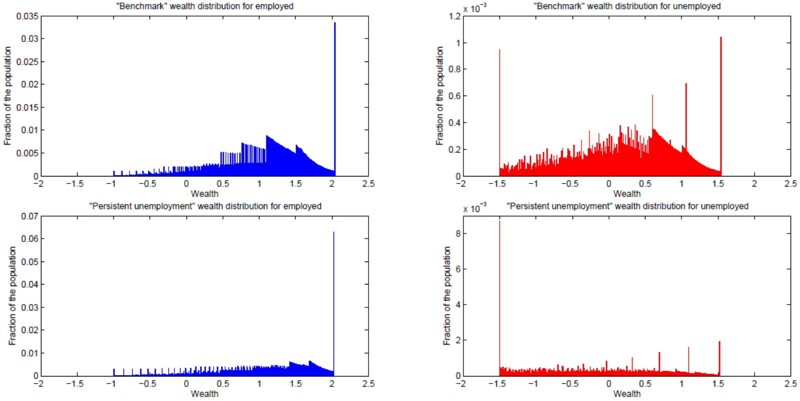
- Results from counterfactual experiment n.2:

	Data	Bench	$\pi(u\|u) = 0.75$
Unemployment (target) %	5.66	5.66	10.71
Real Interest (target) %	2.00	2.00	0.94
Wealth Gini (untargeted)	0.80	0.38	0.49

Experiment 2: Longer Duration of Unemployment

Wealth distribution

Figure below plots the new cross-sectional wealth distribution, which shows more mass at the tails, accounting for the higher Gini coefficient.



Appendix

Duration of Unemployment

Derivation

- Let T_u be the length of a spell of unemployment.
- We have to compute $D = \mathbb{E}(T_u)$ the duration of unemployment.
- $P(T_u = t)$ is the probability of being unemployed for t periods.
- It follows

$$P(T_u = t) = \pi\left(u|u\right)^{t-1}\left(1 - \pi\left(u|u\right)\right)$$

that is, the probability of remaining unemployed for $t-1$ periods and finding a job in period t.

Duration of Unemployment

Derivation

- The duration of unemployment $D = \mathbb{E}(T_u)$ can thus be computed as

$$\begin{aligned}
\mathbb{E}(T_u) &= \sum_{t=1}^{\infty} t \cdot P(T_u = t) \\
&= \sum_{t=1}^{\infty} t\pi\left(u|u\right)^{t-1}\left(1 - \pi\left(u|u\right)\right) \\
&= \left(1 - \pi(u|u)\right) \sum_{t=1}^{\infty} t\pi\left(u|u\right)^{t-1} \\
&= \left(1 - \pi(u|u)\right) \frac{1}{\left(1 - \pi(u|u)\right)^2} \\
&= \frac{1}{1 - \pi(u|u)}.
\end{aligned}$$

Back

Kronecker operator

- Let A be an $m \times n$ matrix and B a $p \times q$ matrix. Then the Kronecker product of A and B is defined as follows:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

  where $A \otimes B$ is an $mp \times nq$ matrix.
- For some useful properties of the Kronecker operator, see Hamilton (1994). Back

Vec operator

- If A is $m \times n$ matrix, then $vec(A)$ is an $mn \times 1$ column vector, obtained by stacking the columns of A, one below the other.
- For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

then

$$vec(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}.$$