Incomplete markets and wealth inequality

Lecture 3b

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The incomplete markets model with $T = \infty$

- Follows Bewley (1977, 1980), Huggett (1993), Aiyagari (1994, 1995).
- Consider a model where households face idiosyncratic income fluctuations which are not insurable.
- Instead households only trade a riskless asset with rate of return r.
- Continuum of households $i \in [0, 1]$ with preferences $\mathbb{E} \sum_{t \geq 0} \beta^t u(c_t^i)$.
- Labor income in period t is we_t^i with stochastic labor supply $e_t^i \in \{\bar{e}_1, \ldots, \bar{e}_n\}$ and $\bar{e}_1 < \ldots < \bar{e}_n$.
- Transitions are independent across households and across time with $\pi_i = Prob(\bar{e}_i)$. Generalizations to Markov processes are possible. Normalize such that $\mathbb{E}e = 1$.
- First consider a "partial equilibrium" environment where w and r are exogenous.

The borrowing constraint

- Flow budget constraint:

$$c_t^i + a_{t+1}^i = (1+r)a_t^i + we_t^i.$$

- Borrowing constraint $a_t^i \geq -\Phi$ where a_t^i are asset holdings.
- No default (by assumption).
- Since loans are riskfree, Φ should be smaller than the *natural debt limit*. This limit is the present value of the lowest possible income path (i.e. what can be repaid for sure when consumption is zero). That is,

$$\Phi \leq \sum_{t=1}^{\infty} \frac{w\bar{e}_1}{(1+r)^t} = \frac{w\bar{e}_1}{r} \ .$$

- Exercise: Write down the household problem in sequential form.

State and policy variables

- Rewrite the budget constraint

$$a_{t+1} = (1+r)a_t + we_t - c_t$$

as

$$\hat{a}_{t+1} = z_t - c_t$$

with $\hat{a}_t \equiv a_t + \Phi$ and
$z_t = (1+r)a_t + \Phi + we_t = (1+r)\hat{a}_t - r\Phi + we_t$ as "total disposable resources".

- With this notation, the model can be reduced to the single state variable z_t.

Recursive problem

- Household's recursive problem

$$V(z) = \max_{\hat{a}_+ \geq 0} \left\{ u[z - \hat{a}_+] + \beta \sum_{i=1}^{n} \pi_i V\left(w\bar{e}_i + (1+r)\hat{a}_+ - r\Phi\right) \right\} .$$

- Under standard conditions (bounded, strictly concave utility), unique value and policy function exist.
- Let $\hat{a}_+ = A(z)$ be the policy function for the recursive problem. Write $C(z) = z - A(z)$ for consumption.
- Finally, the policy function for next-period asset holdings is $a_+(z) = A(z) - \Phi$.

Recursive problem

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- Finally, the policy function for next-period asset holdings is $a_+(z) = A(z) - \Phi$.
- Exercise: Write the household's recursive problem in the general case where income process is Markov, i.e. $\pi_{i,i'} = Prob(e_{t+1} = \bar{e}_{i'} \mid e_t = \bar{e}_i)$.

Some immediate results

- The envelope condition and the first–order condition are

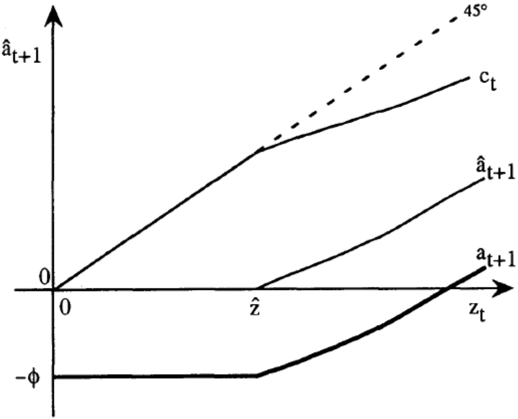
$$
\begin{aligned}
V'(z) &= u'(C(z)) \ , \\
u'(C(z)) &= \beta(1+r)\mathbb{E}V'[we + (1+r)A(z) - r\Phi] + \mu(z) \ ,
\end{aligned}
$$

  where $\mu(z)$ is the multiplier on the borrowing constraint $A(z) \geq 0$.

- If the borrowing constraint does not bind, differentiation of these conditions implies that $C'(z) > 0$ and $A'(z) > 0$.

- Further, there can exist some level $\widehat{z} > 0$ such that $A(z) = 0$ and $C(z) = z$ for all $z \leq \widehat{z}$.

Consumption and Assets as Functions of Total Resources



Source: Aiyagari (1994), Figure Ia

If $z_t \leq \widehat{z}$, it is optimal to consume all total resources, i.e. $c_t = z_t$ and set $\widehat{a}_{t+1} = 0 \implies a_{t+1} = -\Phi$.

Results

- Note that there is a lower bound on total resources:

$$z' = we + (1+r)A(z) - r\Phi \geq \underbrace{w\bar{e}_1 - r\Phi}_{z_{min}} \geq 0$$

We can summarize the above discussion in the following proposition.

Proposition

Consumption is strictly increasing in "total resources", i.e. $C'(z) \in (0, 1]$. Moreover, there exists $\hat{z} \geq z_{min}$ such that for all $z \leq \hat{z}$ we have $C(z) = z$ and $A(z) = 0$. If $z > \hat{z}$, $C'(z) \in (0, 1)$ and $A'(z) > 0$.

Dyanmics of Assets and Total Resources

- What can be said about the stochastic dynamics of z?
- What can be said about the stochastic dynamics of assets and consumption?

The non–stochastic case

- Let $e = 1$ so that labor income is constant at w.
- The Euler equation is

$$
\begin{aligned}
u'(c_t) &\geq \beta(1+r)u'(c_{t+1}) \quad \text{if } a_{t+1} = 0 \\
u'(c_t) &= \beta(1+r)u'(c_{t+1}) \quad \text{if } a_{t+1} > 0
\end{aligned}
$$

- Three cases to consider:

$$
\begin{aligned}
\text{Case I} &\quad \beta(1+r) > 1 \\
\text{Case II} &\quad \beta(1+r) = 1 \\
\text{Case III} &\quad \beta(1+r) < 1
\end{aligned}
$$

Non-stochastic income I

- Consider the case $\beta(1+r) > 1$.
- The consumer is very patient $\implies$ he prefers to save and postpone consumption.
- Borrowing constraint is never binding.
- Euler equation holds as equality:

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) > u'(c_{t+1})$$

- It follows that $c_t < c_{t+1}$ for all t, which implies $\lim_{t\to\infty} c_t = \infty$.
- This can only be achieved if $\lim_{t\to\infty} a_t = \infty$.
- **The asset space is therefore unbounded**.

Non-stochastic income II

- Consider the case $\beta(1+r)=1$.
- Euler equation implies that $c_{t+1}=c_t$.
- Solving forward the budget constraint,

$$a_0=(1+r)^{-1}\sum_{t=0}^{\infty}(1+r)^{-t}(c_t-w)$$

- Therefore consumption is constant at $c=ra_0+w$ and $a_t=a_0$ for all t.
- Optimal to consume the annuity value of lifetime income and roll over initial assets forever.

Non-stochastic income III

- Consider the case $\beta(1+r) < 1$.
- The consumer is very impatient $\implies$ consumption and asset holdings will converge.
- If $\beta(1+r) < 1$,

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) < u'(c_{t+1})$$

hence $c_t > c_{t+1}$ when consumer is not borrowing constrained.

- If $\Phi = \frac{w}{r}$, $c_t \to 0$ and $a_t \to -w/r$.
- If $\Phi < w/r$, the borrowing constraint binds eventually:

$$a_t = -\Phi \text{ and } c_t = w - r\Phi \ , \ t \geq T \ .$$

Non-stochastic income: a Summary

- Assets dynamics in income fluctuation problem: deterministic case.

	Deterministic Income
$\beta(1+r) > 1$	Diverging
$\beta(1+r) = 1$	Stationary
$\beta(1+r) < 1$	Stationary

Stochastic income

- Stochastic case: there is an additional motive for saving, the *precautionary motive*, due either to
  - prudence
  - borrowing constraint.
- Intuitively, conditions under which assets converge will be more stringent.
- Assets dynamics in income fluctuation problem: stochastic case.

	Deterministic Income	Stochastic Income
$\beta(1+r) > 1$	Diverging	Diverging
$\beta(1+r) = 1$	Stationary	Diverging
$\beta(1+r) < 1$	Stationary	Stationary[1]

[1]Under mild conditions.

The stochastic case with $\beta(1+r) \geq 1$

- Recall the dynamic equation for z_t:

$$z_{t+1} = we_{t+1} + (1+r)A(z_t) - r\Phi.$$

- **Claim:** If $\beta(1+r) \geq 1$, z_t diverges to $+\infty$ almost surely.
- **Proof:** Suppose to the contrary that there exists z_{max} such that $z_t \leq z_{max} = w\bar{e}_n + (1+r)A(z_{max}) - r\Phi$ for all t when $z_0 \leq z_{max}$. Then,

$$\begin{aligned}
V'(z_{max}) &\geq \beta(1+r)\mathbb{E}V'[we + (1+r)A(z_{max}) - r\Phi] && (1)\\
&\geq \mathbb{E}V'[we + (1+r)A(z_{max}) - r\Phi] && (2)\\
&> V'[w\bar{e}_n + (1+r)A(z_{max}) - r\Phi] && (3)\\
&= V'[z_{max}]\ , && (4)
\end{aligned}$$

a contradiction.

The stochastic case with $\beta(1+r) \geq 1$

Economic Intuition

- Without uncertainty and $\beta(1+r) = 1$, consumer wants to keep constant consumption over time. With uncertainty, he has to accumulate ever-growing asset stock to self-insure against getting a sufficiently long sequence of bad income shocks.

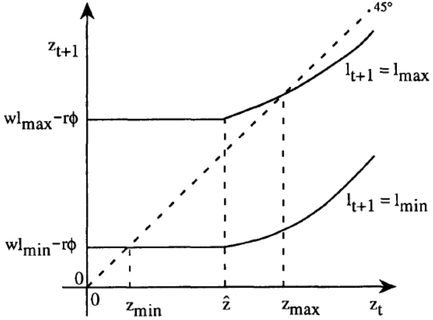
The stochastic case with $\beta(1+r) < 1$

- If $\beta(1+r) < 1$ and relative risk aversion is bounded, z_t stays bounded.
  - Specifically, Aiyagari (1994) proves that there exists a z_{max} such that for all $z_t \geq z_{max}$, $z_{t+1} \leq z_t$ with probability one.
- The same conditions also guarantee that there exists a unique invariant stationary distribution $G(z; \Phi, w, r)$ which behaves continuously w.r.t. parameters Φ, w, r.
- **Remark**: to have an invariant distribution, the asset space needs to be bounded. It is impossible for G to be invariant if there is mass being put on higher and higher z.

Evolution of Total Resources

Recall the dynamic equation for z_t: Note

$$z_{t+1} = wl_{t+1} + (1+r)A(z_t) - r\phi.$$



Source: Aiyagari (1994), Figure Ib.

- Left panel plots z_{t+1} as a function of z_t, for $l_{t+1} = l_{min}$ and $l_{t+1} = l_{max}$.
- $z_t \leq \hat{z} \implies A(z_t) = 0$ and $z_{t+1} = wl_{t+1} - r\phi$.
- The support of the distribution $G(z)$ is $[z_{min}, z_{max}]$.

The invariant distribution: Interpretation

- In a large economy (with a continuum of households), the distribution G can be interpreted in two ways:
  - Time-series interpretation: It is the invariant distribution of z for a single household across time.
  - Cross-sectional interpretation: It is the stationary cross–sectional distribution. That is, when the cross–sectional distribution of z_t^i is G, z_{t+1}^i will also have distribution G.
- For a given interest rate r, the population mean

$$\mathcal{A}(r) = \int A(z;r)dG(z;r) - \Phi$$

can be interpreted as:
  - Average asset level of a single consumer over a long time series, where the average is taken *across time*.
  - Average asset level held by the economy as a whole, where the average is taken *across consumers*.

General equilibrium

- **Aiyagari (1994)** considers the steady state of a growth model in which w is the equilibrium real wage and r is the equilibrium interest rate at which firms or households borrow.
- **Huggett (1993)** considers an endowment economy with pure consumption loans: w is an exogenous endowment, and r is the interest rate at which the market for one-period consumption loans clears.
- **Krusell and Smith (1998)** introduce aggregate shocks into the Aiyagari (1994) model. This is computationally very challenging.

The Aiyagari (1994) model

- There is a large number of firms producing output from capital and labor with technology $Y = F(K, L)$. δ is the capital depreciation rate.
- Aggregate labor supply is $\mathbb{E}(e)$. Equilibrium in the labor market requires $L = \mathbb{E}(e)$.
- For given aggregate capital, firms' optimization implies

$$\begin{aligned} r &= F_k(K, L) - \delta \\ w &= F_L(K, L). \end{aligned}$$

- It is useful to invert the first equation as $K = K(r)$ and to substitute it into the second, $w = w(K(r))$. Both are decreasing functions.

Asset demand

- Write $A(z; r, w)$ and $G(z; r, w)$ to express the dependence of the policy function and of the invariant distribution on (r, w).
- Aggregate asset demand:

$$\mathcal{A}(r) \equiv \int A(z; r, w(K(r)))dG(z; r, w(K(r))) - \Phi \ .$$

- $\mathcal{A}(r)$ is typically increasing (this is not necessary, however). However, $\mathcal{A}(r) \to \infty$ when r converges to $\frac{1}{\beta} - 1$ from below.
    - We proved before that if $r = \frac{1}{\beta} - 1$, then assets diverge to $+\infty$ almost surely.

Stationary equilibrium

A stationary competitive equilibrium is value function $V(z)$, policy function $A(z)$, distribution function $G(z)$, aggregate capital K, an interest rate r and real wage w such that

1. Given w and r, $V(\cdot)$ and $A(\cdot)$ are value and policy functions of the households' problem.
2. Given w and r, firms choose labor $L = \mathbb{E}(e)$ and capital K optimally: $K = K(r)$ and $w = w(K(r))$.
3. $G(\cdot)$ is an invariant distribution measure consistent with policy function $A(\cdot)$ and with exogenous distribution of shock e.
4. Capital market clearing: $K(r) = \mathcal{A}(r)$.
5. Goods market clearing: $C + \delta K = Y$, where C is aggregate consumption.

Equilibrium: Existence and Uniqueness

- Existence and uniqueness boils down to one equation in one unknown: $K(r) = \mathcal{A}(r)$.
- By Walras' law ignore goods market condition.
- Labor market equilibrium $L = \mathbb{E}(e)$ and $\bar{L} \equiv \mathbb{E}(e)$ is exogenously given.

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- By Walras' law ignore goods market condition.
- Labor market equilibrium $L = \mathbb{E}(e)$ and $\bar{L} \equiv \mathbb{E}(e)$ is exogenously given.
- Capital/asset market clearing condition:

$$K(r) = \int A(z; r) dG(z; r) - \Phi \equiv \mathcal{A}(r)$$

- Capital demand of firm $K(r)$ is defined implicitly as

$$r = F_k(K(r), \bar{L}) - \delta$$

- Given assumptions on $F(K, L)$, it follows that $K(r)$ is continuous, strictly decreasing on $r \in (-\delta, \infty)$ with

$$\lim_{r \to -\delta} K(r) = \infty, \quad \lim_{r \to \infty} K(r) = 0$$

Equilibrium: Existence and Uniqueness, cont'd

- Now characterization of capital supply (or aggregate savings) $\mathcal{A}(r)$
- $\mathcal{A}(r) \in [-\Phi, \infty]$ for all $r \in [-\delta, \frac{1}{\beta} - 1]$
- Under some restrictions, one can prove that the function $\mathcal{A}(r)$ is well-defined on $r \in [-\delta, \frac{1}{\beta} - 1)$. (See previous analysis).
- Furthermore,

$$\lim_{r \to -\delta} \mathcal{A}(r) < \infty, \quad \lim_{r \to \frac{1}{\beta} - 1} \mathcal{A}(r) = \infty.$$

- Then there exists r^* such that

$$K(r^*) = \mathcal{A}(r^*)$$

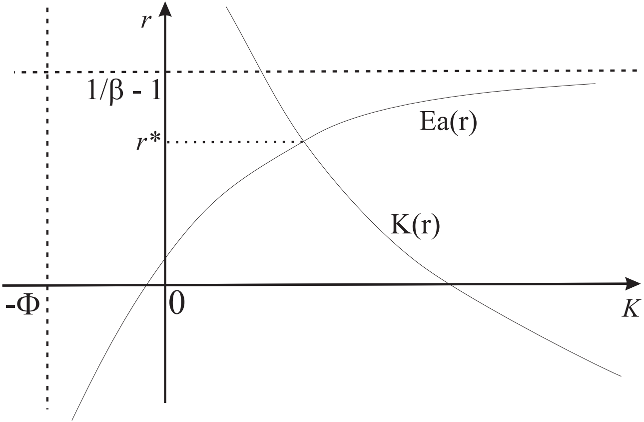
Equilibrium: Existence and Uniqueness, cont'd

- Define an excess demand function $ED(r) \equiv K(r) - \mathcal{A}(r)$.
- We proceed in three steps:
  1. There exists $\underline{r} < 0$ such that $ED(\underline{r}) > 0$. Indeed, for r sufficiently low, $K(r) \to \infty$ and $\mathcal{A}(r)$ is finite, hence **capital is in excess demand**.
  2. There exists $\bar{r} > 0$ such that $ED(\bar{r}) < 0$. Indeed, for $r \to \frac{1}{\beta} - 1$ from below, $K(r)$ is finite and $\mathcal{A}(r)$ becomes arbitrarily large, so that **capital is in excess supply**.
  3. Since $ED(\cdot)$ is continuous, by the Intermediate Value Theorem there exists r^* such that $ED(r^*) = 0$.
- Is market clearing r^* unique?

Equilibrium in the capital market

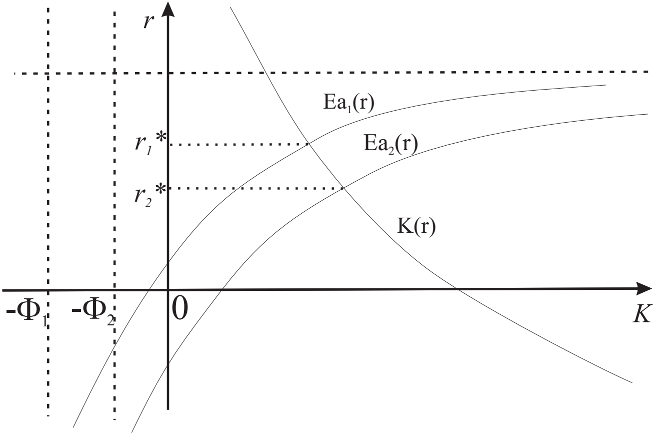
The equilibrium has a lower interest rate and a higher capital stock as the model with complete markets. *Notes:* $Ea(r)$ denotes the capital supply/aggregate savings function $\mathcal{A}(r)$. $K(r)$ is capital demand function.

$r^{CM} = \frac{1}{\beta} - 1 > r^*$



Comparative statics

- Tightening the borrowing constraint (reducing Φ), increases $\mathcal{A}(r)$, hence r falls and $K = K(r)$ rises.



Computation

1. Choose some initial guess for r_0 and set $j = 0$.
2. Solve the household's problem by value function iteration, given r_j and $w = \omega(r_j)$. This gives a discrete policy function $A : Z \to A$ where Z and A are finite grids.
3. Simulate the stochastic dynamics to obtain a frequency distribution for assets, $\mathcal{A}(r_j)$.
4. If $K(r_j) - \mathcal{A}(r_j) > 0$, adjust r_{j+1} upwards, otherwise downwards, and go back to step 2 until convergence is achieved.

Calibration

- CRRA utility $u(c) = \frac{c^{1-\mu}-1}{1-\mu}$ with values $\mu = \{1, 3, 5\}$.
- One model period equals 1 year.
- Pick $\beta = 0.96$ so that $r^{CM} = 4.17\%$.
- Cobb-Douglas production function with $\alpha = 0.36$.
- Depreciation rate $\delta = 8\%$.

Earnings profile

- In the theoretical analysis, e is i.i.d. shock.
- i.i.d. is unrealistic in that there is too little persistence, at odds with empirical micro data on labor earnings.
- In the quantitative analysis, Aiyagari assumes that e_t follows a stationary AR(1) process:

$$\log(e_{t+1}) = \rho \log(e_t) + \sigma(1-\rho^2)^{1/2}\varepsilon_{t+1}$$

where $\varepsilon_{t+1} \sim N(0,1)$ and

$$corr(\log(e_{t+1}), \log(e_t)) = \rho$$
$$Var(\log(e_{t+1})) = \sigma^2$$

- Discretize AR(1) into a discrete Markov chain, $N = 7$.

Calibration

- Average labor endowment is $\mathbb{E}(e) = 1$.
- Parameter values considered:

$$\rho \in \{0, 0.3, 0.6, 0.9\}$$
$$\sigma \in \{0.2, 0.4\}$$

- In his baseline simulation, Aiyagari sets $\Phi = 0$, i.e. borrowing is prohibited.
  - Another option: calibrate the borrowing limit to match the fraction of agents with *negative* net wealth (around 13% in the U.S. data).

Qualitative Results

- In the complete markets version of the Aiyagari model, consumption of all agents is constant in the steady state.
- Complete markets Euler equation $\implies \beta(1+r) = 1$, or $r^{CM} = \frac{1}{\beta} - 1$.
- Under *incomplete* markets, $r^* < r^{CM}$ and hence $K^* > K^{CM}$ (see Figure page 28).
- In other words,

$$S^* = \delta K^* > \delta K^{CM} = S^{CM}$$

  i.e. agents oversave in this model because of precautionary reasons.

Aiyagari (1994)'s main research question: **Is precautionary saving in general equilibrium quantitatively important?**

Complete Markets

- What is the aggregate savings rate under complete markets?

$$s^{CM} \equiv \frac{S^{CM}}{Y^{CM}} = \delta K^{1-\alpha}$$

$$r + \delta = \alpha * K^{\alpha-1}$$

- Combining the two equations above,

$$s = \frac{\alpha\delta}{r + \delta}$$

- Given $r^{CM} = 4.17\%$,

$$s^{CM} = 23.67\%$$

Results, cont.

- Table below shows how interest rate r and the aggregate saving rate change as functions of ρ (income persistence), σ (income st.dev.), μ (prudence).

TABLE II

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)			
$\rho \backslash \mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36
B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)			
$\rho \backslash \mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

Source: Aiyagari (1994), Table II.

Results, cont.

- Keeping prudence and dispersion fixed, increase in persistence of income shock (i.e. higher ρ) leads to increased precautionary saving, bigger overaccumulation of capital, compared to complete markets.
- Keeping fixed persistence and dispersion in income, an increase in prudence (i.e. higher μ) leads to more precautionary saving and more severe overaccumulation of capital.
- Keeping prudence and income persistence constant, an increase in the dispersion of the income process (i.e. higher σ) leads to more precautionary saving and more severe overaccumulation of capital.

Appendix

Jensen's inequality

- Given a real convex function f, for any $t \in [0,1]$

$$tf(x_1) + (1-t)f(x_2) \geq f\left(tx_1 + (1-t)x_2\right)$$

- Geometric interpretation: A secant line of a convex function lies above the graph
- In a probabilistic setting, if X is a r.v. and f is a convex function, then:

$$E\left[f(X)\right] \geq f\left(E\left[X\right]\right)$$

- The same holds for conditional expectation:

$$E\left[f(X) \mid Y\right] \geq f\left(E\left[X \mid Y\right]\right)$$

Back

A Remark on Notation

- Bear in mind that Aiyagari uses a slightly different notation in his 1994 paper (you can download the paper from ILIAS).
- He denotes the labor endowment shock as l_t instead of e_t as in these slides.
- Therefore he denotes the lowest labor shock as l_{min} instead of $\bar{e}_1$, and the highest labor shock as l_{max} instead of $\bar{e}_n$.
- Moreover he denotes the borrowing limit as ϕ instead of Φ.

Back