Buera & Shin (2013) – Financial Friction and the Persistence of History: A Quantitative Exploration

Alessandro Di Nola and Robert Kirkby

June 2025

Introduction

Brief description of (incomplete) replication of Buera and Shin (2013). I denote the exogenous shock z, which they call e, and I also call γ what they call σ. I use μ as the agent distribution (while they had it as the distribution of just the exogenous process, they used G for the agent distribution).

The model is about solving a transition path in an infinite horizon model. Here I just write the steady-state problem.

Value Function Formulation

Buera and Shin (2013) present the household problem as a sequence problem. We rewrite it as a value function problem:

$$V(a,z) = \max_{c,a'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}[V(a',z')|z] \right\} \tag{1}$$

subject to

$$c + a' \leq \max\{w, \pi(a,z|r,w)\} + (1+r)a$$

and

$$a' \geq 0$$

where

$$\pi(a,z|r,w) = \max_{l,k} \{f(z,k,l) - wl - (r+\delta)k\} \text{ subject to } k \leq \lambda a$$

and

$$f(z,k,l) = z(k^{\alpha} l^{1-\alpha})^{1-\nu}$$

This reduces to a one-state dynamic programming problem since the profit part can be solved analytically. Solving the VFI problem delivers the policy functions $a'(a,z)$, $e = e(a,z)$, $e \in \{0,1\}$, where optimal occupation choice is

$$e(a,z) = \begin{cases} 1 & \text{if } \pi(a,z|r,w) > w \\ 0 & \text{otherwise.} \end{cases}$$

Analytical Solution

$$k_1 = \left[\left(\frac{1}{r+\delta} \alpha (1-\nu) z \right)^{1-(1-\alpha)(1-\nu)} \left(\frac{1}{w} (1-\alpha)(1-\nu) z \right)^{(1-\alpha)(1-\nu)} \right]^{1/\nu}$$

$$k^* = \min\{k_1, \lambda a\}$$

$$l^* = \left[\frac{1}{w} (1-\alpha)(1-\nu) z (k^*)^{\alpha(1-\nu)} \right]^{\frac{1}{1-(1-\alpha)(1-\nu)}}$$

$$\pi(a, z|r, w) = z(k^*)^{\alpha} (l^*)^{1-\alpha})^{1-\nu} - wl^* - (r+\delta)k^*$$

General Equilibrium Conditions

There are two GE prices: interest rate r and wage rate w.

Labor market clearing:

$$\int l^*(a,z) \cdot \mathbb{I}_{\{e(a,z)=1\}} \, d\mu(a,z) = \int \mathbb{I}_{\{e(a,z)=0\}} \, d\mu(a,z)$$

Capital market clearing:

$$\int k^*(a,z) \cdot \mathbb{I}_{\{e(a,z)=1\}} \, d\mu(a,z) = \int a \, d\mu(a,z)$$

Implied by Walras law:

$$\int c(a,z) d\mu(a,z) + \int a \, d\mu(a,z) = \int \left[\mathbb{I}_{\{e(a,z)=0\}} \cdot w + \mathbb{I}_{\{e(a,z)=1\}} \cdot \pi(a,z) \right] d\mu(a,z) + (1+r) \int a \, d\mu(a,z)$$

which simplifies to

$$\begin{aligned} C &= w \int \mathbb{I}_{\{e(a,z)=0\}} \, d\mu(a,z) + \int \mathbb{I}_{\{e(a,z)=1\}} \pi(a,z) d\mu(a,z) + rK \\ &= wL + Y - wL - (r+\delta) K + rK \\ &= Y - \delta K \end{aligned}$$

Calibration

External parameters. $\gamma = 1.5$, $\delta = 0.06$, $\alpha = 0.33$, $\lambda = \infty$
Internal calibration. Set v (span of control), η, ψ (dispersion and persistence of entrepreneurial ability) and β (discount rate) to match:

- Top 10% of employment (v, η)
- Top 5% of earnings (v, η)
- Entrepreneurs exit rate (ψ)

- Real interest rate (β)

Stochastic process. Entrepreneurial ability z follows a *truncated* Pareto distribution with density function

$$p(z) = \begin{cases} \eta z^{-(\eta+1)} & \text{if } z \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Each period, an individual keeps his old ability z with probability ψ. With complementary probability $1-\psi$ he draws a new ability z' from $p(z')$ given above. Clearly, the parameter ψ governs the persistence of the stochastic process for ability, whereas η governs the dispersion. Given these assumption for the stochastic process of entrepreneurial ability, the continuation value in (1) can be written more explicitely as follows:

$$E\left[V(a',z')|z\right] = \psi V(a',z) + (1-\psi)\int V(a',z')p(z')dz'.$$

In order to re-use the general routines for value function iteration and distribution iteration, it is convenient to *recast the stochastic process described above as a Markov chain*. First, we assume we have already discretized the Pareto distribution as $z \in \{z_1, z_2, \ldots, z_n\}$ and $p(z) = [p_1, \ldots, p_n]$ (a $n \times 1$ *column* vector such that $\sum_i p_i = 1$). Then, we observe that if the current ability shock is z_i, then next-period ability shock will be z_i with prob $\psi + (1-\psi)p_i$ and z_j with prob $(1-\psi)p_j$, for $j \neq i$. It is easy to check that $\sum \pi(z,z') = \psi + (1-\psi)p_i + \sum_{j\neq i}(1-\psi)p_j = 1$, i.e. each row i of the Markov chain sums to one, for all $i = 1, \ldots, n$. Therefore the $n \times n$ Markov chain for the shock (z, z') can be written as

$$\Pi_{n\times n} = \psi \mathbf{I}_n + (1-\psi)\mathbf{1}_n p^T$$

or, more in detail, as

$$\Pi_{n\times n} = \psi \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} + (1-\psi) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{1}_n$ is an $n \times 1$ *column* vector of 1s.

Note: For the replication I used the files `support.dat` and `dist.dat` kindly provided by the authors. If I replicate the discretization as described in their paper, I get values for z and $p(z)$ which are quite different from the ones used in the paper.

Appendix: Profit Maximization Derivations

$$\pi(a,z|r,w) = \max_{l,k}\left\{z(k^{\alpha}l^{1-\alpha})^{1-\nu} - wl - (r+\delta)k\right\}, \quad \text{s.t. } k \leq \lambda a$$

First Order Conditions (Unconstrained):

$$\frac{\partial \pi}{\partial l} = (1-\alpha)(1-\nu) z k^{\alpha(1-\nu)} l^{(1-\alpha)(1-\nu)-1} - w = 0$$

$$\frac{\partial \pi}{\partial k} = \alpha(1-\nu) z k^{\alpha(1-\nu)-1} l^{(1-\alpha)(1-\nu)} - (r+\delta) = 0$$

Solving gives:

$$l^* = \left[\frac{1}{w}(1-\alpha)(1-\nu) z k^{\alpha(1-\nu)}\right]^{\frac{1}{1-(1-\alpha)(1-\nu)}}$$

$$k_1 = \left[\left(\frac{1}{r+\delta}\alpha(1-\nu) z\right)^{1-(1-\alpha)(1-\nu)} \left(\frac{1}{w}(1-\alpha)(1-\nu) z\right)^{(1-\alpha)(1-\nu)}\right]^{1/\nu}$$

$$k^* = \min\{k_1, \lambda a\}$$